Bounds for the dimensions of p-adic multiple L-value spaces

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ABSTRACT. First, we will define p-adic multiple L-values (p-adic MLV's), which are generalizations of Furusho's p-adic multiple zeta values (p-adic MZV's) in Section 2.

Next, we prove bounds for the dimensions of p-adic MLV-spaces in Section 3, assuming results in Section 4. The bounds come from the rank of K-groups of ring of S-integers of cyclotomic fields, and these are p-adic analogues of Goncharov-Terasoma's bounds for the dimensions of (complex) MZV-spaces and Deligne-Goncharov's bounds for the dimensions of (complex) MLV-spaces. In the case of MLV-spaces, the gap between the dimensions and the bounds is related to spaces of modular forms similarly as the complex case.

In Section 4, we define the crystalline realization of mixed Tate motives and show a comparison isomorphism, by using p-adic Hodge theory.

CONTENTS

1. Introduction. 1
2. p-adic Multiple L-values. 4
2.1. The Twisted p-adic Multiple Polylogarithm. 4
2.2. The p-adic Drinfel'd Associator for Twisted p-adic Multiple Polylogarithms. 5
3. Bounds for Dimensions of p-adic Multiple L-value spaces. 7
3.1. The Motivic Fundamental Groupoids of $\mathbb{U}_N$. 7
3.2. The p-adic MLV-space in the Sense of Deligne. 10
3.3. The Tannakian Interpretations of Two p-adic MLV's. 13
4. Crystalline Realization of Mixed Tate Motives. 16
4.1. Crystalline Realization. 16
4.2. Comparison Isomorphism. 19
4.3. Some Remarks. 20
References 24

1. INTRODUCTION.

For the multiple zeta values (MZV's)

$$\zeta(k_1, \ldots, k_d) := \sum_{n_1<\cdots<n_d} \frac{1}{n_1^{k_1}\cdots n_d^{k_d}} \left(= \lim_{\mathbb{C}\ni z\to 1} \mathrm{Li}_{k_1,\ldots,k_d}(z)\right)$$

$(k_1, \ldots, k_{d-1} \geq 1,\ k_d \geq 2)$, Zagier conjectures the dimension of the space of MZV's

$$Z_w := \langle \zeta(k_1, \ldots, k_d) \mid d \geq 1, k_1 + \cdots + k_d = w, k_1, \ldots, k_{d-1} \geq 1, k_d \geq 2\rangle_{\mathbb{Q}} \subset \mathbb{R},$$

and $Z_0 := \mathbb{Q}$ (Here, $\langle \cdots \rangle_{\mathbb{Q}}$ means the $\mathbb{Q}$-vector space spanned by $\cdots$) as follows.

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Conjecture 1. (Zagier) Let $D_{n+3} = D_{n+1} + D_n$, $D_0 = 1$, $D_1 = 0$, $D_2 = 1$ (that is, the generating function $\sum_{n=0}^{\infty} D_n t^n$ is $\frac{1}{1-t^2-t^3}$). Then, for $w \geq 0$ we have

$$\dim_{\mathbb{Q}} Z_w = D_w.$$

Terasoma, Goncharov, and Deligne-Goncharov proved the upper bound:

Theorem 1.1. *(Terasoma* [T], *Goncharov* [G1], *Deligne-Goncharov* [DG]*) For $w \geq 0$, we have*

$$\dim_{\mathbb{Q}} Z_w \leq D_w.$$

□

Deligne-Goncharov also proved an upper bound for dimensions of multiple L-value (MLV) spaces. ([DG])

On the other hand, Furusho defined p-adic MZV's [Fu1] by using Coleman's iterated integral theory:

$$\zeta_p(k_1, \ldots, k_d) := \lim_{\mathbb{C}_p \ni z \to 1}{}' \mathrm{Li}^a_{k_1,\ldots,k_d}(z).$$

where Li^a is the p-adic multiple polylogarithm defined by Coleman's iterated integral, and a is a branching parameter (For the notations $\lim'$, see [Fu1, Notation 2.12]). For $k_d \geq 2$, RHS converges, and the limit value is independent of a and lands in $\mathbb{Q}_p$ ([Fu1, Theorem 2.13, 2.18, 2.25]). Put

$$Z^p_w := \langle \zeta_p(k_1, \ldots, k_d) \mid d \geq 1, k_1 + \cdots + k_d = w, k_1, \ldots, k_{d-1} \geq 1, k_d \geq 2 \rangle_{\mathbb{Q}} \subset \mathbb{Q}_p,$$

and $Z^p_0 := \mathbb{Q}$. Note that for $k_d = 1$, p-adic MZV's may converge, however, these are $\mathbb{Q}$-linear combinations of p-adic MZV's corresponding to the same weight indices with $k_d \geq 2$ (See, [Fu1, Theorem 2.22]). The following conjecture is proposed.

Conjecture 2. (Furusho-Y.) Let $d_{n+3} = d_{n+1} + d_n$, $d_0 = 1$, $d_1 = 0$, $d_2 = 0$ (that is, the generating function $\sum_{n=0}^{\infty} d_n t^n$ is $\frac{1-t^2}{1-t^2-t^3}$). Then, for $w \geq 0$ we have

$$\dim_{\mathbb{Q}} Z^p_w = d_w.$$

From the fact $\zeta_p(2) = 0$ and the motivic point of views (see, Remark 3.7, p-adic analogue of Grothendieck's conjecture about an element of a motivic Galois group (Conjecture 3), and Proposition 3.12), it seems natural to conjecture as above.

Remark 1.2. The conjecture implies that $\dim_{\mathbb{Q}} Z^p_w$ is independent of p. On the other hand, $\zeta_p(2k+1) \neq 0$ is equivalent to the higher Leopoldt conjecture in the Iwasawa theory. For a regular prime p, or a prime p satisfying $(p-1) \mid 2k$, we have $\zeta_p(2k+1) \neq 0$. However, it is not known if $\zeta_p(2k+1)$ is zero or not in general. Thus, it is non-trivial that $\dim_{\mathbb{Q}} Z^p_w$ is independent of p (See also [Fu1, Example 2.19 (b)]). It seems that the above conjecture contains the "Leopoldt conjecture for higher depth". □

For Conjecture 2, we will prove the following result.

Theorem 1.3. *For $w \geq 0$, we have*

$$\dim_{\mathbb{Q}} Z^p_w \leq d_w.$$

□

We can also define p-adic multiple L-values for N-th roots of unity $\zeta_1, \dots, \zeta_d$ and $k_1, \dots, k_d \geq 1$, $(k_d, \zeta_d) \neq (1,1)$ and a prime ideal $\mathfrak{p} \nmid N$ above p in the cyclotomoic field $\mathbb{Q}(\mu_N)$,

$$L_{\mathfrak{p}}(k_1, \dots, k_d; \zeta_1, \dots, \zeta_d) \in \mathbb{Q}(\mu_N)_{\mathfrak{p}},$$

by Coleman's iterated integral as Furusho did for MZV's (See, Section 2.1). Here, $\mathbb{Q}(\mu_N)_{\mathfrak{p}}$ is the completion of $\mathbb{Q}(\mu_N)$ at the finite place $\mathfrak{p}$. Put

$$\begin{aligned} Z_w^{\mathfrak{p}}[N] :=& \langle L_{\mathfrak{p}}(k_1, \dots, k_d; \zeta_1, \dots, \zeta_d) \mid d \geq 1, k_1 + \cdots + k_d = w, k_1, \dots, k_d \geq 1, \\ & \zeta_1^N = \cdots = \zeta_d^N = 1, (k_d, \zeta_d) \neq (1,1) \rangle_{\mathbb{Q}} \subset \mathbb{Q}(\mu_N)_{\mathfrak{p}}, \end{aligned}$$

and $Z_0^{\mathfrak{p}}[N] := \mathbb{Q}$.

This $Z_w^p[1]$ is equal to the above Z_w^p. We will also prove bounds for the dimensions of p-adic MLV's.

Theorem 1.4. *For $w \geq 0$, we have*

$$\dim_{\mathbb{Q}} Z_w^{\mathfrak{p}}[N] \leq d[N]_w.$$

Here, $d[N]_w$ is defined as follows:

1. *For $N = 1$, $d[1]_{n+3} = d[1]_{n+1} + d[1]_n$ $(n \geq 0)$, $d[1]_0 = 1$, $d[1]_1 = 0$, $d[1]_2 = 0$, that is, the generating function is $\dfrac{1-t^2}{1-t^2-t^3}$ (This $d[1]_n$ is equal to the above d_n).*
2. *For $N = 2$, $d[2]_{n+2} = d[2]_{n+1} + d[2]_n$ $(n \geq 1)$, $d[2]_0 = 1$, $d[2]_1 = 1$, $d[2]_2 = 1$, that is, the generating function is $\dfrac{1-t^2}{1-t-t^2}$.*
3. *For $N \geq 3$, $d[N]_{n+2} = \left(\frac{\varphi(N)}{2} + \nu\right) d[N]_{n+1} - (\nu - 1) d[N]_n$ $(n \geq 0)$, $d[N]_0 = 1$, $d[N]_1 = \frac{\varphi(N)}{2} + \nu - 1$, that is, the generating function is $\dfrac{1-t}{1 - \left(\frac{\varphi(N)}{2} + \nu\right) t + (\nu - 1) t^2}$. Here, $\varphi(N) := \#(\mathbb{Z}/N\mathbb{Z})^\times$, and ν is the number of prime divisors of N.*

□

Remark 1.5. In the proof of the above bounds, we use some kinds of varieties, which are related to the algebraic K-theory. For $N > 4$, the above bounds are not best possible in general, because in the proof, we use smaller varieties in general than varieties, which give the above bounds. The gap of dimensions is related to the space of cusp forms of weight 2 on $X_1(N)$ if N is a prime. See also [DG, 5.27][G2]. □

First, we define the p-adic MLV's, twisted p-adic multiple polylogarithms (twisted p-adic MPL's), and p-adic Drinfel'd associator for twisted p-adic MPL's in Section 2. Next, assuming results of Section 4, we will show bounds for dimensions of p-adic MLV-spaces in the sense of Deligne [D1][DG], by using the motivic fundamental groupoid constructed in [DG] in Section 3.2. Lastly, we show bounds for dimensions of Furusho's p-adic MLV-spaces, by comparing the two p-adic MLV-spaces in the Tannakian interpretation in Section 3.3. In Section 4, we construct the crystalline realization of mixed Tate motives, and prove a comparison isomorphism, by using p-adic Hodge theory.

We fix conventions. We use the notation $\gamma'\gamma$ for a composition of paths, which means that γ followed by γ'. Similarly, we use the notation $g'g$ for a product of elements in a motivic Galois group, which means that the action of g followed by the one of g'.

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2. p-adic Multiple L-values.

In this section, we define twisted p-adic multiple polylogarithms (twisted p-adic MPL), p-adic multiple L-values (p-adic MLV), p-adic KZ-equation for twisted p-adic MPL, and p-adic Drinfel'd associator for twisted p-adic MPL, similarly as Furusho's definitions in [Fu1]. We discuss the fundamental properties of them.

Fix a prime ideal $\mathfrak{p}$ in $\mathbb{Q}(\mu_N)$, and an embedding $\iota_{\mathfrak{p}} : \mathbb{Q}(\mu_N) \hookrightarrow \mathbb{C}_p$. Put $S := \{0, \infty\} \cup \mu_N$, $\mathbb{U}_N := \mathbb{P}^1_{\mathbb{Q}(\mu_N)} \setminus S$, and $\overline{\mathbb{U}_N} := \mathbb{U}_N \otimes_{\mathbb{Q}(\mu_N)} \mathbb{C}_p$ (The variety $\mathbb{U}_N$ is defined over $\mathbb{Q}$, however, we use $\mathbb{U}_N$ over $\mathbb{Q}(\mu_N)$ for the purpose of bounding dimensions in the next section).

2.1. The Twisted p-adic Multiple Polylogarithm.

We use the same notations as in [Fu1]: the tube $]x[\subset \mathbb{P}^1_{\mathbb{C}_p}$ of $x \in (\mathbb{U}_N)_{\mathbb{F}_p}(\overline{\mathbb{F}_p})$, the algebra $A(U)$ of rigid analytic functions on U, and the algebra A^a_{Col} of Coleman functions on $\overline{\mathbb{U}_N}$ with a branching parameter a.

Definition 2.1. For $\mathfrak{p} \nmid N$, $k_1, \ldots, k_d \geq 1$, and $\zeta_1 \ldots, \zeta_d \in \mu_N$, we define the (one variable) *twisted p-adic multiple polylogarithm* (twisted p-adic MPL) $\mathrm{Li}^a_{(k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d)}(z) \in A^a_{\mathrm{Col}}$ attached to $a \in \mathbb{C}_p$ by the following integrals inductively:

$$\mathrm{Li}^a_{(1;\zeta_1)}(z) := -\log^a(\iota_{\mathfrak{p}}(\zeta_1) - z) := \int_0^z \frac{dt}{\iota_{\mathfrak{p}}(\zeta_1) - t},$$

$$\mathrm{Li}^a_{(k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d)}(z) := \begin{cases} \displaystyle\int_0^z \frac{1}{t}\mathrm{Li}^a_{(k_1,\ldots,(k_d-1);\zeta_1,\ldots,\zeta_d)}(t)dt & k_d \neq 1, \\ \displaystyle\int_0^z \frac{1}{\iota_{\mathfrak{p}}(\zeta_d) - t}\mathrm{Li}^a_{(k_1,\ldots,k_{(d-1)};\zeta_1,\ldots,\zeta_{d-1})}(t)dt & k_d = 1. \end{cases}$$

Here, $\log^a$ is the logarithm with a branching parameter a, which means $\log^a(p) = a$. □

Remark 2.2. For $|z|_p < 1$, it is easy to see that

$$\mathrm{Li}^a_{(k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d)}(z) = \sum_{0<n_1<\cdots<n_d} \frac{\iota_{\mathfrak{p}}(\zeta_1^{-n_1}\zeta_2^{n_1-n_2}\cdots\zeta_d^{n_{d-1}-n_d})z^{n_d}}{n_1^{k_1}\cdots n_d^{k_d}}.$$

□

Inductively, we can easily verify that $\mathrm{Li}^a_{(k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d)}(z)|_{]0[} \in A(]0[)$, $\mathrm{Li}^a_{(k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d)}(z)|_{]\infty[} \in A(]\infty[)[\log^a t^{-1}]$, and $\mathrm{Li}^a_{(k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d)}(z)|_{]\iota_{\mathfrak{p}}(\zeta)[} \in A(]\iota_{\mathfrak{p}}(\zeta)[)[\log^a(z - \iota_{\mathfrak{p}}(\zeta))]$ for $\zeta \in \mu_N$.

Proposition 2.3. *Fix $k_1, \ldots, k_d \geq 1$, and N-th roots of unity $\zeta_1, \ldots, \zeta_d \in \mu_N$. Then the convergence of* $\lim'_{\mathbb{C}_p \ni z \to 1} \mathrm{Li}^a_{(k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d)}(z)$ *is independent of branches $a \in \mathbb{C}_p$. Moreover, if it converges in $\mathbb{C}_p$, the limit value is independent of branches $a \in \mathbb{C}_p$ and lands in $\mathbb{Q}(\mu_N)_{\mathfrak{p}}$ (For the notation* $\lim'$, *see* [Fu1, Notation 2.12]*).* □

Proof. The same as [Fu1, Theorem 2.13, Theorem 2.25]. □

Definition 2.4. When the limit $\lim'_{\mathbb{C}_p\ni z\to 1}\mathrm{Li}^a_{(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d)}(z)$ converges, we define the corresponding *p-adic multiple L-value* to be its limit value:

$$L_{\mathfrak{p}}(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d) := \lim_{\mathbb{C}_p\ni z\to 1}{}'\mathrm{Li}^a_{(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d)}(z)$$

□

For example, $L_{\mathfrak{p}}(1;\zeta) = -\log^a(\iota_{\mathfrak{p}}(\zeta)-1)$ $(1\neq\zeta\in\mu_N)$ is independent of a, since $\log^a(z)$ does not depend on a for $|z|=1$. (Recall that we assume $\mathfrak{p}\nmid N$.)

2.2. **The p-adic Drinfel'd Associator for Twisted p-adic Multiple Polylogarithms.** Let $A^{\wedge}_{\mathbb{C}_p} := \mathbb{C}_p\langle\langle A, B_\zeta \mid \zeta\in\mu_N\rangle\rangle$ be the non-commutative formal power series ring with $\mathbb{C}_p$ coefficients generated by variables A and B_ζ for $\zeta\in\mu_N$. For a word W consisting of A and $\{B_\zeta\}_{\zeta\in\mu_N}$, we call the sum of all exponents of A and $\{B_\zeta\}_{\zeta\in\mu_N}$ the weight of W, and the sum of all exponents of $\{B_\zeta\}_{\zeta\in\mu_N}$ the depth of W.

Definition 2.5. Fix a prime ideal $\mathfrak{p}$ above p in $\mathbb{Q}(\mu_N)$ and an embedding $\iota_{\mathfrak{p}}:\mathbb{Q}(\mu_N)\hookrightarrow\mathbb{C}_p$. The *$p$-adic Knizhnik-Zamolodchikov equation* (p-adic KZ-equation) is the differential equation

$$\frac{dG}{dz}(z) = \left(\frac{A}{z} + \sum_{\zeta\in\mu_N}\frac{B_\zeta}{z-\iota_{\mathfrak{p}}(\zeta)}\right)G(z),$$

where $G(z)$ is an analytic function in variable $z\in\overline{\mathbb{U}_N}$ with values in $A^{\wedge}_{\mathbb{C}_p}$. Here, $G=\sum_W G_W(z)W$ is 'analytic' means each of whose coefficient $G_W(z)$ is locally p-adically analytic. □

Proposition 2.6. *Fix $a\in\mathbb{C}_p$. Then, there exist unique solutions $G^a_0(z), G^a_1(z)\in A^a_{\mathrm{Col}}\widehat{\otimes}A^{\wedge}_{\mathbb{C}_p}$, which are locally analytic on $\mathbb{P}^1(\mathbb{C}_p)\setminus S$ and satisfy $G^a_0(z)\approx z^A$ $(z\to 0)$, and $G^a_1(z)\approx(1-z)^{B_1}$ $(z\to 1)$.* □

Here, the notations u^A means $\sum_{n=0}^{\infty}\frac{1}{n!}(A\log^a u)^n$. Note that it depends on a. For the notations $G^a_0(z)\approx z^A$ $(z\to 0)$, see [Fu1, Theorem 3.4].

Remark 2.7. We do not have the symmetry $z\mapsto 1-z$ on $\overline{\mathbb{U}_N}$. Thus, we do not have a simple relation between $G^a_0(z)$ and $G^a_1(z)$ as in [Fu1, Proposition 3.8]. On the other hand, we have the symmetry $z\mapsto z^{-1}$ on $\overline{\mathbb{U}_N}$. Thus, we have a unique locally analytic solution $G^a_\infty(z)$ with $G^a_\infty(z)\approx(z^{-1})^{-A-\sum_{\zeta\in\mu_N}B_\zeta}$ $(z\to\infty)$, and have a relation

$$G^a_\infty(A,\{B_\zeta\}_{\zeta\in\mu_N})(z) = G^a_0(-A-\sum_{\zeta\in\mu_N}B_\zeta,\{B_{\zeta^{-1}}\}_{\zeta\in\mu_N})(z^{-1}).$$

However, when we define a Drinfel'd associator by using G^a_0 and G^a_∞ similarly as below (Definition 2.8), there appears

$$\lim_{\mathbb{C}_p\in z\to\infty}{}'\mathrm{Li}^a_{(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d)}(z)$$

in the coefficient of that Drinfel'd associator. What we want is $\lim_{\mathbb{C}_p\in z\to 1}{}'$. Thus, we use the boundary condition at $z=1$. □

Proof. The uniqueness is easy. In [Fu1], he cites Drinfel'd's paper [Dr] for the existence of a solution of the KZ-equation. Here, we give an alternative proof of the existence without using the quasi-triangular quasi-Hopf algebra theory and the quasi-tensor category theory. In fact, we put $G^a_0(z)$ to be $\sum_W(-1)^{\mathrm{depth}(W)}\mathrm{Li}^a_W(z)W$. Here, for a word W, we define $\mathrm{Li}^a_W(z)$

inductively as following: $\mathrm{Li}^a_{A^n}(z) := \frac{1}{n!}(\log^a z)^n$, $\mathrm{Li}^a_{AW}(z) := \int_0^z \frac{1}{t}\mathrm{Li}^a_W(t)dt$, for $W \neq A^n$ $(n \geq 0)$, $\mathrm{Li}^a_{B_\zeta W}(z) := \int_0^z \frac{1}{\iota_{\mathfrak{p}}(\zeta)-t}\mathrm{Li}^a_W(t)dt$, for $\zeta \in \mu_N$. It is easy to verify that $\sum_W (-1)^{\mathrm{depth}(W)}\mathrm{Li}^a_W(z)W$ satisfies the p-adic KZ-equation. As for the boundary condition $G^a_0(z) \approx z^A$ $(z \to 0)$, it is easy to show that

$$\sum_{W : W \neq W'A, W' \neq \emptyset} (-1)^{\mathrm{depth}(W)}\mathrm{Li}^a_W(z)W$$

satisfies the above boundary condition.

Thus, it remains to show that $\mathrm{Li}^a_{W'A^n}(z) \to 0$ $(z \to 0)$ for $n > 0$, $W' \neq \emptyset$. For $\mathrm{Li}^a_{B_\zeta A^n}$,

$$\mathrm{Li}^a_{B_\zeta A^n}(z) = \int_0^z \frac{1}{\iota_{\mathfrak{p}}(\zeta)-t}\mathrm{Li}^a_{A^n}(t)dt = \frac{1}{n!}\int_0^z \zeta^{-1}\sum_{k=0}^{\infty}(\zeta^{-1}t)^k(\log^a t)^n dt,$$

in $|z| < 1$. Since $\int_0^z t^k \log^a t dt = \frac{z^{k+1}}{k+1}\log^a z - \frac{z^{k+1}}{(k+1)^2}$, we have $\int_0^z t^k \log^a t dt \to 0$ $(z \to 0)$. Inductively, we have $\int_0^z t^k(\log^a t)^n dt \to 0$ $(z \to 0)$. Thus, we showed $\mathrm{Li}^a_{B_\zeta A^n}(z) \to 0$ $(z \to 0)$. For general $\mathrm{Li}^a_{W'A}(z)$'s, we can inductively show $\mathrm{Li}^a_{W'A}(z) \to 0$ $(z \to 0)$ by using the following fact for $f(z) = \mathrm{Li}^a_{**}(z)$: For a locally analytic function $f(z)$ satisfying $f(0) = 0$, we have $\int_0^z \frac{1}{t}f(t)dt \to 0$ $(z \to 0)$, $\int_0^z \frac{1}{\iota_{\mathfrak{p}}(\zeta)-t}f(t)dt \to 0$ $(z \to 0)$.

As for $G^a_1(z)$, the same argument works, by replacing $\mathrm{Li}^a_{A^n}(z) := \frac{1}{n!}(\log^a z)^n$ by $\mathrm{Li}^a_{B_1^n}(z) := \frac{1}{n!}(\log^a(1-z))^n$, and $\int_0^z$ by $\int_1^z$. □

Definition 2.8. We define the *p-adic Drinfel'd associator for twisted p-adic multiple polylogarithms* to be $\Phi^{\mathfrak{p}}_{\mathrm{KZ}}(A, \{B_\zeta\}_{\zeta\in\mu_N}) := G^a_1(z)^{-1}G^a_0(z)$. It is in $A^\wedge_{\mathbb{C}_p} = \mathbb{C}_p\langle\langle A, \{B_\zeta\}_{\zeta\in\mu_N}\rangle\rangle$, and independent of a by the same argument in [Fu1, Remark 3.9, Theorem 3.10]. □

By the same arguments as in [Fu1], we can show the following propositions.

Proposition 2.9. $\lim_{\mathbb{C}_p \ni z \to 1}{}'\mathrm{Li}^a_{(k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d)}(z)$ *converges when* $(k_d, \zeta_d) \neq (1, 1)$. □

Proof. See, [Fu1, Theorem 2.18] for the case where $N = 1$. □

For W in $A \cdot A^\wedge_{\mathbb{C}_p} \cdot B_\zeta$ or $B_{\zeta'} \cdot A^\wedge_{\mathbb{C}_p} \cdot B_\zeta$ $(\zeta' \neq 1)$, we define $L_{\mathfrak{p}}(W)$ to be $\lim_{\mathbb{C}_p \ni z\to 1}{}'\mathrm{Li}^a_W(z)$.

Proposition 2.10. *(Explicit Formulae) The coefficient $I_{\mathfrak{p}}(W)$ of W in the p-adic Drinfel'd associator for twisted p-adic MPL's is the following: When W is written as $B_1^r V A^s$ for $(r, s \geq 0)$, V is in $A \cdot A^\wedge_{\mathbb{C}_p} \cdot B_\zeta$ or $B_{\zeta'} \cdot A^\wedge_{\mathbb{C}_p} \cdot B_\zeta$ $(\zeta' \neq 1)$,*

$$I_{\mathfrak{p}}(W) = (-1)^{\mathrm{depth}(W)}(-1)^{a+b}\sum_{0\leq a\leq r, 0\leq b\leq s} L_{\mathfrak{p}}(f(B_1^a \circ B_1^{r-a}VA^{s-b} \circ A^b)).$$

In particular, when W is in $A \cdot A^\wedge_{\mathbb{C}_p} \cdot B_\zeta$ or $B_{\zeta'} \cdot A^\wedge_{\mathbb{C}_p} \cdot B_\zeta$ $(\zeta' \neq 1)$, $I_{\mathfrak{p}}(W) = (-1)^{\mathrm{depth}(W)}L_{\mathfrak{p}}(W)$. Here, $f : A^\wedge_{\mathbb{C}_p} \to A^\wedge_{\mathbb{C}_p}$ is the composition of $A^\wedge_{\mathbb{C}_p} \twoheadrightarrow A^\wedge_{\mathbb{C}_p}/(B_1 \cdot A^\wedge_{\mathbb{C}_p} + A^\wedge_{\mathbb{C}_p} \cdot A)$, $A^\wedge_{\mathbb{C}_p}/(B_1 \cdot A^\wedge_{\mathbb{C}_p} + A^\wedge_{\mathbb{C}_p} \cdot A) \xrightarrow{\sim} \mathbb{C}_p \cdot 1 + A \cdot A^\wedge_{\mathbb{C}_p} \cdot B_1$, and $\mathbb{C}_p \cdot 1 + A \cdot A^\wedge_{\mathbb{C}_p} \cdot B_1 \hookrightarrow A^\wedge_{\mathbb{C}_p}$. □

For the definition of the shuffle product $\circ$, see [Fu0, Definition 3.2.2].

Proof. See, [Fu1, Theorem 3.28] for the case where $N = 1$. Note we use $G^a_i(A - \alpha, B_1 - \beta, \{B_\zeta\}_{\zeta\in\mu_N,\zeta\neq 1})(z) = z^{-\alpha}(1-z)^{-\beta}G^a_i(A, \{B_\zeta\}_{\zeta\in\mu_N})(z)$ for $i = 0, 1$. □

Proposition 2.11. *Suppose* $\lim_{\mathbb{C}_p\ni z\to 1}{}'\mathrm{Li}^a_{(k_1,\ldots,k_{d-1},1;\zeta_1,\ldots,\zeta_{d-1},1)}(z)$ *converges. Then, the limit value is a p-adic regularized MLV, that is, $L_{\mathfrak{p}}(k_1, \ldots, k_{d-1}, 1; \zeta_1, \ldots, \zeta_{d-1}, 1) = (-1)^{\mathrm{depth}(W)}I_{\mathfrak{p}}(W)$. In particular, $L_{\mathfrak{p}}(k_1, \ldots, k_{d-1}, 1; \zeta_1, \ldots, \zeta_{d-1}, 1)$ can be written as a $\mathbb{Q}$-linear combination of p-adic MLV's corresponding to the same weight indices with $(k_d, \zeta_d) \neq (1, 1)$.* □

Proof. See, [Fu1, Theorem 2.22] for the case where $N = 1$. $\square$

Definition 2.12. We define the *p-adic multiple L-value space* of weight w $Z^{\mathfrak{p}}_w[N]$ to be the finite dimensional $\mathbb{Q}$-linear subspace of $\mathbb{Q}(\mu_N)_{\mathfrak{p}}$ generated by the all p-adic MLV's of indices of weight w, $\zeta_1^N = \cdots = \zeta_d^N = 1$. Put $Z^{\mathfrak{p}}_0[N] := \mathbb{Q}$. We define $Z^{\mathfrak{p}}_{\bullet}[N]$ to be the formal direct sum of $Z^{\mathfrak{p}}_w[N]$ for $w \geq 0$. $\square$

Remark 2.13. By Proposition 2.11, we see that

$$\begin{aligned} Z^{\mathfrak{p}}_w[N] :=& \langle L_{\mathfrak{p}}(k_1, \ldots, k_d; \zeta_1, \ldots, \zeta_d) \mid d \geq 1, k_1 + \cdots + k_d = w, k_1, \ldots, k_d \geq 1, \\ & \zeta_1^N = \cdots = \zeta_d^N = 1, (k_d, \zeta_d) \neq (1,1) \rangle_{\mathbb{Q}} \\ =& \langle I_{\mathfrak{p}}(W) \mid \text{the weight of } W \text{ is } w \rangle_{\mathbb{Q}} \subset \mathbb{Q}(\mu_N)_{\mathfrak{p}}. \end{aligned}$$

$\square$

Proposition 2.14. *We have $\Delta(\Phi^{\mathfrak{p}}_{\mathrm{KZ}}) = \Phi^{\mathfrak{p}}_{\mathrm{KZ}} \widehat{\otimes} \Phi^{\mathfrak{p}}_{\mathrm{KZ}}$. In particular, the graded $\mathbb{Q}$-vector space $Z^{\mathfrak{p}}_{\bullet}[N]$ has a $\mathbb{Q}$-algebra structure, that is, $Z^{\mathfrak{p}}_a[N] \cdot Z^{\mathfrak{p}}_b[N] \subset Z^{\mathfrak{p}}_{a+b}[N]$ for $a, b \geq 0$.* $\square$

Proof. See, [Fu1, Proposition 3.39, Theorem 2.28] for the case where $N = 1$. $\square$

Proposition 2.15. *(Shuffle Product Formulae) For $W, W' \in (A \cdot A^{\wedge}_{\mathbb{C}_p} \cdot B_{\zeta}) \cup \cup_{\zeta' \neq 1} (B_{\zeta'} \cdot A^{\wedge}_{\mathbb{C}_p} \cdot B_{\zeta})$, we have*

$$L_{\mathfrak{p}}(W \circ W') = L_{\mathfrak{p}}(W) L_{\mathfrak{p}}(W').$$

$\square$

Proof. This follows from Proposition 2.10 and Proposition 2.14. See, [Fu1, Corollary 3.42] for the case where $N = 1$. $\square$

3. Bounds for Dimensions of p-adic Multiple L-value spaces.

In this section, we show Theorem 1.4, by the method of Deligne-Goncharov [DG], assuming results of Section 4. First, we recall some facts about the motivic fundamental groupoids in [DG]. Next, we show that bounds for dimensions of p-adic MLV-spaces in the sense of Deligne [D1][DG]. Lastly, we show that p-adic MLV-spaces in the previous section is equal to p-adic MLV-spaces in the sense of Deligne by the Tannakian interpretations.

3.1. **The Motivic Fundamental Groupoids of $\mathbb{U}_N$.** Deligne-Goncharov constructed the category $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$ of mixed Tate motives over $\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}]$, the fundamental $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-group $\pi_1^{\mathcal{M}}(\mathbb{U}_N, x)$ and the fundamental $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-groupoid $P^{\mathcal{M}}_{y,x}$ for $\mathbb{U}_N$ not only for rational base points x, y, but also for tangential base points x, y [DG, Theorem 4.4, Proposition 5.11]. Here, $w \mid N$ runs through primes w dividing N, and ζ_w is a w-th root of unity (Since $\mathbb{U}_N$ is defined over $\mathbb{Q}$, $\pi_1^{\mathcal{M}}(\mathbb{U}_N, x)$, $P^{\mathcal{M}}_{y,x}$ are also $\mathrm{MAT}(\mathbb{Q}(\mu_N)/\mathbb{Q})$-schemes. However, we do not use this fact. Here, $\mathrm{MAT}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ is the category of mixed Artin-Tate motives for $\mathbb{Q}(\mu_N)/\mathbb{Q})$. For $\mathcal{T}$-schemes, $\mathcal{T}$-group schemes, and $\mathcal{T}$-groupoids for a Tannakian category $\mathcal{T}$, see [D1, §5, §6], [D2, 7.8], and [DG, 2.6].

First, we recall some facts about them. Let

$$G := \pi_1(\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])) \in \text{pro-}\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$$

be the fundamental $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-group [D1, §6][D2, Definition 8.13]. Then, by its action on $\mathbb{Q}(1)$, we have a surjection $G \twoheadrightarrow \mathbb{G}_m$ (Here, we regard $\mathbb{G}_m$ as an $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-group). The kernel U of the map $G \to \mathbb{G}_m$ is a pro-unipotent group. Then, we have an isomorphism [DG, 2.8.2]:

$$\mathrm{Lie}(U^{\mathrm{ab}}) \cong \prod_n \mathrm{Ext}^1_{\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])}(\mathbb{Q}(0), \mathbb{Q}(n))^\vee \otimes \mathbb{Q}(n) \in \text{pro-}\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}]).$$

The extension group is related to the algebraic K-theory [DG, 2.1.3]:

$$\mathrm{Ext}^1_{\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])}(\mathbb{Q}(0), \mathbb{Q}(n)) = \begin{cases} 0 & n \leq 0, \\ \mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}]^\times \otimes_{\mathbb{Z}} \mathbb{Q} & n = 1, \\ K_{2n-1}(\mathbb{Q}(\mu_N)) \otimes_{\mathbb{Z}} \mathbb{Q} & n \geq 2. \end{cases}$$

Let ω be the canonical fiber functor $\omega : \mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}]) \to \mathrm{Vect}_{\mathbb{Q}}$, which sends a motive M to $\oplus_n \mathrm{Hom}(\mathbb{Q}(n), \mathrm{Gr}^W_{-2n}(M))$. Here, $W_m(M)$ is the weight filtration of M. Let $G_\omega := \omega(G) = \underline{\mathrm{Aut}}^\otimes(\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}]), \omega)$ be the motivic Galois gruop of $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$ with respect to the canonical fiber functor ω (For the de Rham realization M_{dR} of a motive $M \in \mathrm{MT}(\mathbb{Q}(\mu_N))$, we have $M_{\mathrm{dR}} = \omega(M) \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_N)$ [DG, Proposition 2.10]). Then, the ω-realization of the exact sequence $0 \to U \to G \to \mathbb{G}_m \to 0$ is split by the action of $\mathbb{G}_m$, which gives the grading by weights,

$$G_\omega = \mathbb{G}_m \ltimes U_\omega.$$

Here, $U_\omega := \omega(U)$. Let τ denote the splitting $\mathbb{G}_m \to G_\omega$. The pro-unipotent group U_ω is equipped with the grading $\{(U_\omega)_n\}_n$. Put $(\mathrm{Lie}U_\omega)^{\mathrm{gr}} := \oplus_n (\mathrm{Lie}U_\omega)_n$. Then, $(\mathrm{Lie}U_\omega)^{\mathrm{gr}}$ is a free Lie algebra, since we have $\mathrm{Ext}^2_{\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])}(\mathbb{Q}(0), \mathbb{Q}(n)) = K_{2n-2}(\mathbb{Q}(\mu_N)) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ [DG, Proposition 2.3]. Thus, the generating function of the universal envelopping algebra of $(\mathrm{Lie}U_\omega)^{\mathrm{gr}}$ is $\sum_{n=0}^\infty f(t)^n$, where

$$f(t) = \begin{cases} t^3 + t^5 + t^7 + \cdots = \frac{t^3}{1-t^2} & N = 1, \\ t + t^3 + t^5 + \cdots = \frac{t}{1-t^2} & N = 2, \\ \left(\frac{\varphi(N)}{2} + \nu - 1\right) t + \frac{\varphi(N)}{2} t^2 + \frac{\varphi(N)}{2} t^3 + \cdots = \frac{\varphi(N)}{2} \frac{t}{1-t} + (\nu - 1)t & N \geq 3. \end{cases}$$

Therefore, we have

$$\sum_{n=0}^\infty f(t)^n = \frac{1}{1 - f(t)} = \begin{cases} \dfrac{1-t^2}{1-t^2-t^3} & N = 1, \\ \dfrac{1-t^2}{1-t-t^2} & N = 2, \\ \dfrac{1-t}{1 - \left(\frac{\varphi(N)}{2} + \nu\right) t + (\nu - 1)t^2} & N \geq 3. \end{cases}$$

That is the generating function of $d[N]_n$'s in Section 1.

Let $P^{\mathcal{M}}_{y,x}$ be the fundamental $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-groupoid of $\mathbb{U}_N$ at (tangential) base points x and y. We consider only tangential base points λ_x at $x \in S := \{0, \infty\} \cup \mu_N$ with tangent vectors λ in roots of unity under the identification the tangent space at x with $\mathbb{G}_a$. Then, $P^{\mathcal{M}}_{\lambda'_y, \lambda_x}$ depends only on x and y, by the triviality of a Kummer $\mathbb{Q}(1)$-torsor [DG, 5.4]. Let $P^{\mathcal{M}}_{y,x}$ denote $P^{\mathcal{M}}_{\lambda'_y, \lambda_x}$. We have the following structures of the system of $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-schemes $\{P^{\mathcal{M}}_{y,x}\}_{x,y \in S}$ [DG, 5.5, 5.7]:

[The system of groupoids in the level of motives]

$(1)^{\mathcal{M}}$ The Tate object $\mathbb{Q}(1)$,
$(2)^{\mathcal{M}}$ For $x, y \in S$, the fundamental $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-groupoid $P^{\mathcal{M}}_{y,x}$,
$(3)^{\mathcal{M}}$ The composition of paths,
$(4)^{\mathcal{M}}$ For $x \in S$, a morphism of $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-group scheme (the local monodromy around x):

$$\mathbb{Q}(1) \to P^{\mathcal{M}}_{x,x},$$

$(5)^{\mathcal{M}}$ An equivariance under the dihedral group $\mathbb{Z}/2\mathbb{Z} \ltimes \mu_N$.

By applying a fiber functor F to the category of K-vector spaces, we get the following structure [DG, 5.8]:

[The system of groupoids under the fiber functor F]

$(1)^F$ A vector space $K(1)$ of dimension 1,
$(2)^F$ For $x, y \in S$, a scheme $P^F_{y,x}$ over K,
$(3)^F$ a system of morphisms of schemes $P^F_{z,y} \times P^F_{y,x} \to P^F_{z,x}$ making $P^F_{y,x}$'s a groupoid. The group schemes $P^F_{x,x}$ are pro-unipotent,
$(4)^F$ For $x \in S$, a morphism

$$(\text{additive group } K(1)) \to P^F_{x,x}.$$

That is equivalent to giving $K(1) \to \mathrm{Lie} P^F_{x,x}$,
$(5)^F$ An $\mathbb{Z}/2\mathbb{Z} \ltimes \mu_N$-equivariance.

In particular, we take the canonical fiber functor ω as F, and we consider the following weakened structure (forgetting the conditions at infinity) [DG, 5.8]. Note that in the realization ω, the weight filtrations split and give the grading, and that all $\pi_1^{\omega}(\mathbb{U}_N, x)$-groupoid is trivial since $H^1(\mathbb{U}_N, \mathcal{O}_{\mathbb{U}_N}) = 0$. Let $\mathcal{L}$ be the Lie algebra freely generated by symbols A, and $\{B_\zeta\}_{\zeta \in \mu_N}$. Let Π be the pro-unipotent group

$$\Pi := \varprojlim_n \exp(\mathcal{L}/\text{degree } \geq n)$$

[The (weakened) system of groupoids under the canonical fiber functor ω]

$(1)^\omega$ The vector space $\mathbb{Q}$,
$(2)^\omega$ A copy $\Pi_{0,0}$ of Π, and the trivial $\Pi_{0,0}$-torsor $\Pi_{1,0}$. The twist of $\Pi_{0,0}$ by this torsor is a new copy of Π, denoted by $\Pi_{1,1}$,
$(3)^\omega$ The group law of Π,
$(4)^\omega$ The morphism

$$\mathbb{Q} \to \mathcal{L}^\wedge : 1 \mapsto A, \ \mathbb{Q} \to \mathcal{L}^\wedge : 1 \mapsto B_1.$$

for $x = 0, 1$ respectively. Here, $\mathcal{L}^\wedge := \varprojlim_n \mathcal{L}/(\text{degree } \geq n)$,
$(5)^\omega$ The action μ_N on $\Pi_{0,0}$, which induces on the Lie algebra $B_\zeta \mapsto B_{\sigma\zeta}$.

Let H_ω be the group scheme of automorphisms of $\mathbb{Q}$ and Π preserving the above structure $(1)^\omega$-$(5)^\omega$. The action of H_ω on the one dimensional vector space $(1)^\omega$ gives a morphism $H_\omega \twoheadrightarrow \mathbb{G}_m$. Let V_ω be the kernel. The grading gives a splitting,

$$H_\omega = \mathbb{G}_m \ltimes V_\omega.$$

Also let τ denote the splitting $\mathbb{G}_m \to V_\omega$. The action G_ω on the above structure factors through H_ω, which sends U_ω to V_ω.

$$\begin{tikzcd}
1 \arrow[r] & U_\omega \arrow[r] \arrow[d] & G_\omega \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & V_\omega \arrow[r] & H_\omega \arrow[r] & \mathbb{G}_m \arrow[r] & 1.
\end{tikzcd}$$

Let ι denote both of $G_\omega \to H_\omega$, and $U_\omega \to V_\omega$. The above diagram comes from $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-schemes (splitting does not come from $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-schemes), however we do not use this fact (see, [DG, 5.12.1]). By the Proposition 5.9 in [DG], the map

$$\eta : V_\omega \to \Pi_{1,0} \ (v \mapsto v(\gamma_{\mathrm{dR}}))$$

is bijective. Here, γ_{dR} is the neutral element of $\Pi_{1,0}$, that is, γ_{dR} is the canonical path from 0 to 1 in the realization of ω.

3.2. **The p-adic MLV-space in the Sense of Deligne.** We will discuss the crystalline realization of mixed Tate motives, and now *we assume the results of Section* 4 (See, Remark 4.8). We use the word "crystalline", not "rigid" for the purpose of fixing terminologies.

In [D1], Deligne has found the p-adic zeta values (i.e., the p-adic MZV's of depth 1), and the p-adic differential equation of p-adic polylogarithms in the study of crystalline aspects of the fundamental group of $\mathbb{U}_N$ modulo depth ≥ 2 [D1, 19.6]. Deligne-Goncharov proposed that the coefficients of the image of

$$\varphi_{\mathfrak{p}} := F_{\mathfrak{p}}^{-1}\tau(q)^{-1} \in U_\omega(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$$

by the map

$$\eta \cdot \iota : U_\omega(\mathbb{Q}(\mu_N)_{\mathfrak{p}}) \to V_\omega(\mathbb{Q}(\mu_N)_{\mathfrak{p}}) \xrightarrow{\sim} \Pi(\mathbb{Q}(\mu_N)_{\mathfrak{p}}) \subset \mathbb{Q}(\mu_N)_{\mathfrak{p}}\langle\langle A, \{B_\zeta\}_{\zeta\in\mu_N}\rangle\rangle$$

"seem" to be p-adic analogies of MZV's [DG, 5.28]. Here, $F_{\mathfrak{p}}$ is the Frobenius endomorphism at $\mathfrak{p}$, and q is the cardinality of the residue field at $\mathfrak{p}$. Note that we have the Frobenius endomorphism on $M_\omega \otimes \mathbb{Q}(\mu_N)_{\mathfrak{p}} \cong M_{\mathrm{crys}}$ for $M \in \mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$ by Remark 4.8. Here, M_{crys} is the crystalline realization of M.

Definition 3.1. We define the *p-adic multiple L-values in the sense of Deligne* of weight w to be the coefficients $I_{\mathfrak{p}}^{\mathrm{D}}(W)$ of words W of wight w in $\eta\iota(\varphi_{\mathfrak{p}}) \in \Pi(\mathbb{Q}(\mu_N)_{\mathfrak{p}}) \subset \mathbb{Q}(\mu_N)_{\mathfrak{p}}\langle\langle A, \{B_\zeta\}_{\zeta\in\mu_N}\rangle\rangle$. We define the *$p$-adic L-value spaces in the sense of Deligne* of weight w $Z_w^{\mathfrak{p},\mathrm{D}}[N]$ to be the finite dimensional $\mathbb{Q}$-linear subspace of $\mathbb{Q}(\mu_N)_{\mathfrak{p}}$ generated by the all p-adic MLV's in the sense of Deligne of indices of weight w. By the definition, we have $Z_0^{\mathfrak{p},\mathrm{D}}[N] = \mathbb{Q}$. We define $Z_\bullet^{\mathfrak{p},\mathrm{D}}[N]$ to be the formal direct sum of $Z_w^{\mathfrak{p},\mathrm{D}}[N]$ for $w \geq 0$. $\square$

On the othe hand, we call p-adic MLV's defined in Section 2.1 p-adic MLV's in the sense of Furusho.

Remark 3.2. If we calculate the action of Frobenius $F_{\mathfrak{p}}^{-1}$ on $(P_{1,0})_\omega$, we get the following KZ-like p-adic differential equation by the same arguments as in [D1, 19.6]:

$$dG(t) = -qG(t)\left(\frac{dt}{t}A + \sum_{\zeta\in\mu_N}\frac{dt}{t-\iota_{\mathfrak{p}}(\zeta)}\zeta(\Phi_D^{\mathfrak{p}})^{-1}B_\zeta\zeta(\Phi_D^{\mathfrak{p}})\right) + \left(\frac{d(t^q)}{t^q}A + \sum_{\zeta\in\mu_N}\frac{d(t^q)}{t^q-\iota_{\mathfrak{p}}(\zeta)}B_\zeta\right)G(t).$$

Here, $\zeta(\Phi_D^{\mathfrak{p}})$ means the action of ζ on $\Phi_D^{\mathfrak{p}}$ determined by $\zeta(A) = A$ and $\zeta(B_{\zeta'}) = B_{\zeta\zeta'}$. Here, $\Phi_D^{\mathfrak{p}}$ is the Deligne associator (See, the subsection of Tannakian interpretions, and Proposition 3.10).

The coefficient of a word W in the solution of the above p-adic differential equation is $q^{w(W)}I^{\mathrm{D}}_{\mathfrak{p}}(W)$ in the limit $t \to 1$, that is, p-adic MLV's in the sense of Deligne (multiplied by $q^{w(W)}$). (More precisely, we have to consider the effect $(1-t)^{-B_1}$ of the tangential base point in taking the limit). The first term in RHS is multiplied by G from the left, and the second term in RHS is multiplied by G from the right. Thus, the inductive procedure of determining coefficients is more complicated.

In [D1, 19.6], Deligne calculated the Frobenius action on $\pi_1^{\omega}(\mathbb{U}_N, 1_0) = (P_{1,0})_\omega$ modulo depth ≥ 2, however, we get the above p-adic differential equation by the same arguments. Here we give a sketch. We use some notations in [D1]. The above equation arises from the horizontality of Frobenius ([D1, 19.6.2]):

$$F_{\mathfrak{p}}^{-1}(e^{-1}\nabla e) = G^{-1}\nabla G.$$

Here, e is the identity element. The above $F_{\mathfrak{p}}^{-1}$ and G are F_* and v in [D1] respectively. On the LHS, we have [D1, 12.5, 12.12, 12.15]

$$e^{-1}\nabla e = -\alpha = -\left(\frac{dt}{t}A + \sum_{\zeta\in\mu_N}\frac{dt}{t-\iota_{\mathfrak{p}}(\zeta)}B_\zeta\right).$$

Here, α is the Maurer-Cartan form ([D1, 12.5.5]). On the RHS, since the connection is the one of $\tilde{F}^*(P_{1,0})_\omega$, we have $\nabla e = -\tilde{F}^*\alpha$, where $\tilde{F}^*$ means the Frobenius lift $t \mapsto t^q$. Combining these and $\nabla G = dG + (\nabla e)G$, we get

$$-qG\left(\frac{dt}{t}A + \sum_{\zeta\in\mu_N}\frac{dt}{t-\iota_{\mathfrak{p}}(\zeta)}F_{\mathfrak{p}}^{-1}(B_\zeta)\right) = dG - \left(\frac{d(t^q)}{t^q}A + \sum_{\zeta\in\mu_N}\frac{d(t^q)}{t^q-\iota_{\mathfrak{p}}(\zeta)}B_\zeta\right)G.$$

This gives the equation (For $F_{\mathfrak{p}}^{-1}(B_\zeta)$, see the proof of Proposition 3.10). □

Example 1. From the p-adic differential equation in the above Remark 3.2, the coefficient of $A^{k-1}B$ in $\eta\iota(F_p^{-1}\tau(p)^{-1})$ in the case where $N=1$ is the limit value at $z=1$ of the p-adic analytic continuation of the following analytic function on $|z|_p < 1$ [D1, 19.6]:

$$\sum_{p\nmid n}\frac{z^n}{n^k}.$$

That limit value is $(1-p^{-k})\zeta_p(k)$. From the condition $p \nmid n$ in the summation, we lose the Euler factor at p for p-adic MZV's of depth 1 in the sense of Deligne.

Proposition 3.3. *For $a, b \geq 0$, we have*

$$Z_a^{\mathfrak{p},\mathrm{D}}[N]\cdot Z_b^{\mathfrak{p},\mathrm{D}}[N] \subset Z_{a+b}^{\mathfrak{p},\mathrm{D}}[N].$$

□

Proof. The group $\Pi(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$ is the subgroup of group-like elements in $\mathbb{Q}(\mu_N)_{\mathfrak{p}}\langle\langle A, \{B_\zeta\}_{\zeta\in\mu_N}\rangle\rangle$, and $\eta\iota(\varphi_{\mathfrak{p}})$ is an element of $\Pi(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$ by the definition. Thus, we have $\Delta(\eta\iota(\varphi_{\mathfrak{p}})) = \eta\iota(\varphi_{\mathfrak{p}})\widehat{\otimes}\eta\iota(\varphi_{\mathfrak{p}})$. This implies the proposition. □

Proposition 3.4. *For $w \geq 0$, we have*

$$\dim_{\mathbb{Q}} Z_w^{\mathfrak{p},\mathrm{D}}[N] \leq d[N]_w.$$

□

Proof. Let $U_\omega = \mathrm{Spec}R$ and $\eta\iota(U_\omega) = \mathrm{Spec}S$. The algebras $R = \prod_n R^n$ and $S = \prod_n S^n$ are graded algebras over $\mathbb{Q}$. Here, the grading of R and S come from the grading of U_ω. Then, $\eta\iota(\varphi_\mathfrak{p}) \in \eta\iota(U_\omega)(\mathbb{Q}(\mu_N)_\mathfrak{p})$ gives a homomorphism $\psi_\mathfrak{p} : S \to \mathbb{Q}(\mu_N)_\mathfrak{p}$. The coefficients of $\eta\iota(\varphi_\mathfrak{p})$ of weight w are contained in $\psi_\mathfrak{p}(S^w)$. Thus, we have $Z_w^{\mathfrak{p},\mathrm{D}}[N] \subset \psi_\mathfrak{p}(S^w)$. By the surjection $\iota : U_\omega \twoheadrightarrow \iota(U_\omega)(\subset V_\omega \overset{\eta}{\cong} \Pi)$, the dimension of S^w is at most the one of the w-th graded part of the universal envelopping algebra of $(\mathrm{Lie}U_\omega)^{\mathrm{gr}}$. That dimension is $d[N]_w$. We are done. □

Remark 3.5. As remarked in [DG, 5.27], $\iota : \mathrm{Lie}U_\omega \to \mathrm{Lie}V_\omega$ is not injective for $N > 4$ in general. Thus, the above bounds are not best possible for $N > 4$ in general. The kernel is related to the space of cusp forms of weight 2 on $X_1(N)$ if N is a prime. See also [G2]. □

Remark 3.6. In the complex case [DG], $\mathrm{dch}(\sigma)$ is in $(P_{1,0})_\omega \otimes \mathbb{C} = \Pi(\mathbb{C}) \overset{\sim}{\leftarrow} V_\omega(\mathbb{C})$. (Here, $\mathrm{dch}(\sigma)$ is the "droit chemin" from 0 to 1 in the Betti realization with respect to $\sigma : \mathbb{Q}(\mu_N) \hookrightarrow \mathbb{C}$.) Thus, Deligne-Goncharov relate $\mathrm{dch}(\sigma)$ to the motivic Galois group U_ω for the purpose of bounds for the dimensions in [DG, Proposition 5.18, 5.19, 5.20, 5.21, 5.22]. (The point is that V_ω is too big, and U_ω is small enough.) However, in the p-adic situation, $\varphi_\mathfrak{p}$ is contained a priori in a small enough variety, i.e., we have $\varphi_\mathfrak{p} \in U_\omega(\mathbb{Q}(\mu_N)_\mathfrak{p})$ by the definition. Thus, the bounds from K-theory of p-adic MLV's in the sense of Deligne is almost trivial. □

We give remarks on $\zeta_p(2)$.

Remark 3.7. By Proposition 3.4 and Example 1, we have $\zeta_p(2) = 0$, since $\dim_\mathbb{Q} Z_2^{p,\mathrm{D}}[1] = 0$. It is another proof of that well-known fact. To bound dimensions, Deligne-Goncharov used $\iota(U_\omega) \times \mathbb{A}^1$ in the complex case [DG, 5.20, 5.21, 5.22, 5.23, 5.24, 5.25]. This affine line corresponds to "π^2", and we need this affine line simply because π^2 is not in $\mathbb{Q}$. In the p-adic case, we do not need such an affine line, simply because the image of F_p^{-1} in $(\mathbb{G}_m)_\omega$ (i.e., p) is in $\mathbb{Q}$. This gives a motivic interpretation of $\zeta_p(2) = 0$. □

Remark 3.8. It is well-known that $\zeta_p(2m) = 0$. However, it is non-trivial because we do not know how to show directly

$$\text{``} \sum_{\mathbb{C}_p \ni z \to 1} \frac{z^n}{n^{2m}} = 0\text{''}$$

(We add a double quotation in the above, since we have to take p-adic analytic continuation). The well-known proof of $\zeta_p(2m) = 0$ is following (also see, [Fu1, Example 2.19(a)]): By the Coleman's comparison [C], we have $\lim_{\mathbb{C}_p \ni z \to 1} \mathrm{Li}_k^a(z) = (1 - p^{-k})^{-1} L_p(k, \omega^{1-k})$ for $k \geq 2$. Here, L_p is the p-adic L-function of Kubota-Leopoldt, ω is the Teichmüller character. This is the values of the p-adic L-function at *positive* integers. On the other hand, the p-adic L-function interpolates the values of usual L-functions at *negative* integers, thus, $L_p(z, \omega^{1-k})$ is constantly zero for even k. Therefore, we have $\zeta_p(2m) = 0$. That proof is indirect.

Furusho informed to the author that 2-, and 3-cycle relations induce $\zeta_p(2m) = 0$ similarly as in [D1, §18] (In the notations in [D1, §18], we can take γ =(the unique Frobenius invariant path from 0 to 1) (see, the next subsection,) and $x = 0$). These relations come from the geometry of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Thus, it seems that it comes from "the same origin" that '$\zeta_p(2) = 0$ from cycle relations' and '$\zeta_p(2) = 0$ from the bounds by K-theory'. Furusho also comments that we may translate '$\zeta_p(2m) = 0$ from cycle relations' into '$\zeta_p(2m) = 0$ from p-adic differential equation', i.e., we may show that $\zeta_p(2k) = 0$ directly from the p-adic analytic function $\sum_{n \geq 1} \frac{z^n}{n^{2m}}$. □

3.3. **The Tannakian Interpretations of Two p-adic MLV's.** Besser proved that there exists a unique Frobenius invariant path in the fundamental groupoinds of p-adic analytic spaces [B, Corollary 3.2]. Furthermore, Besser showed the existence of Frobenius invariant path on p-adic analytic spaces is equivalent to the Coleman's integral theory [B, §5].

Let γ_{crys} be the unique Frobenius invariant path in $(P_{1,0})_{\mathrm{crys}}$. To a differential form ω, the path γ_{crys} associates the Colman integration $\int_0^1 \omega$. Let $\gamma_{\mathrm{dR}} \in (P_{1,0})_\omega$ be the canonical path from 0 to 1 under the realization ω. Furusho proved the path $\alpha_F := \gamma_{\mathrm{dR}}^{-1}\gamma_{\mathrm{crys}} \in \pi_1^{\mathrm{crys}}(\mathbb{U}_N, 1_0)$ is equal to the p-adic Drinfel'd associator Φ_{KZ}^{p} for p-adic MZV's, that is, for $N = 1$ in [Fu2]. By the same argument, we can verify that $\alpha_F = \Phi_{\mathrm{KZ}}^{\mathfrak{p}}$ for p-adic MLV's. Briefly, we review the argument. For details, see [Fu2] (See also [Ki, Proposition 4]). The coefficient of a word $A^{k_d-1}B_{\zeta_d}\cdots A^{k_1-1}B_{\zeta_1}$ in $\alpha_F = \gamma_{\mathrm{dR}}^{-1}\gamma_{\mathrm{crys}} \in \pi_1^{\mathrm{crys}}(\mathbb{U}_N, 1_0) \subset \mathbb{Q}(\mu_N)_{\mathfrak{p}}\langle\langle A, \{B_\zeta\}_{\zeta\in\mu_N}\rangle\rangle$ for $(k_d, \zeta_d) \neq (1,1)$ is an iterated integral

$$\int_0^1 \frac{dt}{t}\cdots\int_0^t \frac{dt}{t}\int_0^t \frac{dt}{t-\iota_{\mathfrak{p}}(\zeta_d)}\int_0^t \frac{dt}{t}\cdots\int_0^t \frac{dt}{t}\int_0^t \frac{dt}{t-\iota_{\mathfrak{p}}(\zeta_1)}$$

by the characterization of γ_{crys} with respect to Coleman's integration theory (Here, the succesive numbers of dt/t are $k_d - 1, k_{d-1} - 1, \cdots, k_2 - 1$ and $k_1 - 1$). For words beginning from A or ending B_1, the coefficients are regularized p-adic MLV's, because the coefficients in α_F are the one in $\lim'_{\mathbb{C}_p\ni z\to 1}(1-z)^{-B_1}G_0(z)$ by using the tangential base point. Thus, α_F is the p-adic Drinfel'd associator $\Phi_{\mathrm{KZ}}^{\mathfrak{p}}$ for twisted p-adic MPL's in Section 2.2:

$$\alpha_F := \gamma_{\mathrm{dR}}^{-1}\gamma_{\mathrm{crys}} = \Phi_{\mathrm{KZ}}^{\mathfrak{p}} = \sum_W I_{\mathfrak{p}}(W)W.$$

On the other hand, $\eta\iota(\varphi_{\mathfrak{p}}) \in \Pi_{0,0}(\mathbb{Q}(\mu_N)_{\mathfrak{p}}) = \pi_1^{\mathrm{crys}}(\mathbb{U}_N, 1_0)$ is $\gamma_{\mathrm{dR}}^{-1}\varphi_{\mathfrak{p}}(\gamma_{\mathrm{dR}})$ by the definition (Recall that $V_\omega \overset{\eta}{\cong} \Pi_{1,0}$ and $\Pi_{0,0} \cong \Pi_{1,0} : 1 \mapsto \gamma_{\mathrm{dR}}$). Briefly, p-adic MLV's in the sense of Furusho come from $\alpha_F = \gamma_{\mathrm{dR}}^{-1}\gamma_{\mathrm{crys}}$, and p-adic MLV's in the sense of Deligne come from $\alpha_D := \gamma_{\mathrm{dR}}^{-1}\varphi_{\mathfrak{p}}(\gamma_{\mathrm{dR}})$. That is the Tannakian interpretations of p-adic MLV's. In [Fu2], he calls $\Phi_D^{\mathfrak{p}} := \gamma_{\mathrm{dR}}^{-1}F_{\mathfrak{p}}^{-1}(\gamma_{\mathrm{dR}})$ the Deligne associator.

Remark 3.9. In both of complex and p-adic cases, the iterated integrals appear in the theory of MZV's. However, the iterated integrals come from different origins in the complex case and the p-adic case.

In the complex case, the iterated integrals appear in the comparison map between the Betti fundamental group $\pi_1^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$ tensored by $\mathbb{C}$ of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the de Rham fundamental group $\pi_1^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$ tensored by $\mathbb{C}$ of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The difference between the $\mathbb{Q}$-structure π_1^{B} and the $\mathbb{Q}$-structure π_1^{dR} under the comparison $\pi_1^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C} \cong \pi_1^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$ is expressed by iterated integrals.

In the p-adic case, iterated integrals do not appear in the comparison map between the de Rham fundamental group $\pi_1^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ tensored by $\mathbb{Q}_p$ and the crystalline fundamental group π_1^{crys}. Furthermore, there is no $\mathbb{Q}$-structure on π_1^{crys}. For p-adic MZV's in the sense of Deligne, iterated integrals appear in the difference between the $\mathbb{Q}$-structure π_1^{dR} and the $\mathbb{Q}$-structure $F_{\mathfrak{p}}^{-1}(\pi_1^{\mathrm{dR}})$ in $P_{1,0}^{\mathrm{crys}}$ under the comparison $P_{1,0}^{\mathrm{crys}} \cong P_{1,0}^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{Q}_p = \pi_1^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{Q}_p$. For p-adic MZV's in the sense of Furusho, they appear in the difference between $\mathbb{Q}$-structure π_1^{dR} and the $\mathbb{Q}$-structure $\alpha\pi_1^{\mathrm{dR}}$ in π_1^{crys} under the comparison $\pi_1^{\mathrm{crys}} \cong \pi_1^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Here, α is a unique element in π_1^{crys} such that $\gamma_{\mathrm{dR}} \cdot \alpha \in P_{1,0}^{\mathrm{crys}}$ is invariant under the Frobenius (Thus, α is equal to α_F).

From this, it seems difficult to find a "motivic Drinfel'd associator", which is an origin of both complex and p-adic MZV's, and a motivic element, which is an origin of linear relations of both complex and p-adic MZV's. Note also that roughly speaking, the complex Drinfel'd associator is the differenc between Betti and de Rham realizations ([DG, 5.19]), and the p-adic Drinfel'd associator is the Frobenius element at p. □

Example 2. (1) (Kummer torsor) Let $K(x)_\omega$ be the fundamental groupoid from 1 to x on $\mathbb{G}_m$ with respect to the realization ω. Deligne calculated in [D1, 2.10] the action of F_p^{-1} on $K(x)_\omega \subset K(x)_{\mathrm{crys}}$:

$$F_p^{-1}(\gamma_{\mathrm{dR}}) = \gamma_{\mathrm{dR}} + \log^a x^{1-p}.$$

Here, γ_{dR} is the canonical de Rham path from 1 to x, and $+$ means the right action of $\pi_1^{\mathrm{crys}}(\mathbb{G}_m, 1) = \mathbb{Q}(1)_{\mathrm{crys}} = \mathbb{Q}_p(1)$ on $K(x)_{\mathrm{crys}}$. From this, we have

$$F_p^{-1}(\gamma_{\mathrm{dR}} + \log^a x) = \gamma_{\mathrm{dR}} + \log^a x^{1-p} + p\log^a x = \gamma_{\mathrm{dR}} + \log^a x.$$

Thus, $\gamma_{\mathrm{dR}} + \log^a x$ is Frobenius invariant, that is, the unique crystalline path γ_{crys} from 1 to x.

(2) (Polylogarithm torsor) Let $P_{1,k}(\zeta)_\omega$ be the k-th polylogarithm torsor with respect to the realization ω for $\zeta \in \mu_N$ (see, [D1, Definition 16.18]). The polylogarithm torsors are not fundamental groupoids, but quotients of fundamental groupoids. However, we use the terminology "$\mathbb{Z}(k)$-torsor of $\mathbb{Z}(k)$-paths from 0 to ζ" in [D1, 13.15]. Here, we consider as $\mathbb{Q}(k)_\omega$-torsor not as $\mathbb{Z}(k)_\omega$-torsor, and we do not multiply $\frac{1}{(k-1)!}$ on the integral structure unlike as [D1]. Deligne calculated in [D1, 19.6, 19.7] the action of F_p^{-1} on $P_{1,k}(\zeta)_\omega \subset P_{1,k}(\zeta)_{\mathrm{crys}}$:

$$F_p^{-1}(\gamma_{\mathrm{dR}}) = \gamma_{\mathrm{dR}} + p^k(1-p^{-k})N^{k-1}\mathrm{Li}_k^a(\zeta)$$

(That is, $F_p^{-1}\tau(p)^{-1}(\gamma_{\mathrm{dR}}) = \gamma_{\mathrm{dR}} + (1-p^{-k})N^{k-1}\mathrm{Li}_k^a(\zeta)$). Here, $+$ means the right action of $\mathbb{Q}(k)_{\mathrm{crys}} = \mathbb{Q}_p(k)$ on $P_{1,k}(\zeta)_{\mathrm{crys}}$. From this, we have

$$\begin{aligned} F_p^{-1}(\gamma_{\mathrm{dR}} - N^{k-1}\mathrm{Li}_k^a(\zeta)) &= \gamma_{\mathrm{dR}} + p^k(1-p^{-k})N^{k-1}\mathrm{Li}_k^a(\zeta) - p^kN^{k-1}\mathrm{Li}_k^a(\zeta) \\ &= \gamma_{\mathrm{dR}} - N^{k-1}\mathrm{Li}_k^a(\zeta). \end{aligned}$$

Thus, $\gamma_{\mathrm{dR}} - N^{k-1}\mathrm{Li}_k^a(\zeta)$ is Frobenius invariant, that is, the unique crystalline path γ_{crys} from 0 to ζ.

(3) In the case where $N = 1$, the coefficient of $A^{k-1}B$ in $\Phi_{\mathrm{KZ}}^{\mathfrak{p}}$ is $-\zeta_p(k)$ and the one of $A^{k-1}B$ in $\eta\iota(F_p^{-1}\tau(p)^{-1})$ is $(1-p^{-k})\zeta_p(k)$, from the above example.

(4) (Furusho) The coefficient of $A^{b-1}BA^{a-1}B$ in $F_p^{-1}\tau(p)^{-1}$ in the case where $N = 1$ is

$$\begin{aligned} &\left(\frac{1}{p^{a+b}} - 1\right)\zeta_p(a,b) - \left(\frac{1}{p^a} - 1\right)\zeta_p(a)\zeta_p(b) \\ &+ \sum_{r=0}^{a-1}(-1)^r\left(\frac{1}{p^{b+r}} - 1\right)\binom{b-1+r}{b-1}\zeta_p(a-r)\zeta_p(b+r) \\ &+ (-1)^{a+1}\sum_{s=0}^{b-1}\left(\frac{1}{p^{a+s}} - 1\right)\binom{a-1+s}{a-1}\zeta_p(a+s)\zeta_p(b-s), \end{aligned}$$

for $b > 1$.

The following proposition combined with Proposition 3.4 gives a proof of Theorem 1.4. The author learned the following proposition from Furusho's caluculation Example 2(4).

Proposition 3.10. *For $w \geq 0$, we have*
$$Z^{\mathfrak{p}}_w[N] = Z^{\mathfrak{p},\mathrm{D}}_w[N].$$
□

Proof. The effect of $\tau(q)$ is the multiplication by q^w on p-adic MLV's of weight w in the sense of Deligne. Thus, $Z^{\mathfrak{p},\mathrm{D}}_w[N]$ is not changed when we replace $F_{\mathfrak{p}}^{-1} \in G_\omega(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$ by $\varphi_{\mathfrak{p}} = F_{\mathfrak{p}}^{-1}\tau(q)^{-1} \in G_\omega(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$ in $\alpha_D = \gamma_{\mathrm{dR}}^{-1}\varphi_{\mathfrak{p}}(\gamma_{\mathrm{dR}})$. Let $J^{\mathrm{D}}_{\mathfrak{p}}(W)$ be the coefficient of a word W in $\Phi^{\mathfrak{p}}_D := \gamma_{\mathrm{dR}}^{-1}F_{\mathfrak{p}}^{-1}(\gamma_{\mathrm{dR}})$. We have
$$Z^{\mathfrak{p},\mathrm{D}}_w[N] = \langle J^{\mathrm{D}}_{\mathfrak{p}}(W) \mid \text{the weight of } W \text{ is } w\rangle_{\mathbb{Q}} \subset \mathbb{Q}(\mu_N)_{\mathfrak{p}}$$
(We recall that the coefficient of a word W in α_F is $I_{\mathfrak{p}}(W)$). We have
$$\begin{aligned}\alpha_F = \gamma_{\mathrm{dR}}^{-1}\gamma_{\mathrm{crys}} &= \gamma_{\mathrm{dR}}^{-1}F_{\mathfrak{p}}^{-1}(\gamma_{\mathrm{dR}}) \cdot (F_{\mathfrak{p}}^{-1}(\gamma_{\mathrm{dR}}))^{-1}F_{\mathfrak{p}}^{-1}(\gamma_{\mathrm{crys}}) = \Phi^{\mathfrak{p}}_D F_{\mathfrak{p}}^{-1}(\alpha_F)\\ &= \left(\sum_W J^{\mathrm{D}}_{\mathfrak{p}}(W)W\right)\left(\sum_W I_{\mathfrak{p}}(W)F_{\mathfrak{p}}^{-1}(W)\right)\end{aligned}$$
(By a theorem of Besser [B, Theorem 3.1], we see that α_F and α_D determine each other from the above formula).

We compute the action $F_{\mathfrak{p}}^{-1}$ on a word W. Let $\gamma_{\mathrm{dR},\zeta}$ be the canonical path from 0 to ζ under the realization ω, that is, $\gamma_{\mathrm{dR},1} = \gamma_{\mathrm{dR}}$, $\gamma_{\mathrm{dR},\zeta} = \zeta(\gamma_{\mathrm{dR},1})$. Here, $\zeta(\gamma_{\mathrm{dR},1})$ is the action of $\zeta \in \mu_N$ on Π. Then, $B_\zeta = (\gamma_{\mathrm{dR},\zeta})^{-1}A \cdot \gamma_{\mathrm{dR},\zeta}$. Thus, we have $F_{\mathfrak{p}}^{-1}(A) = qA$ and
$$\begin{aligned}F_{\mathfrak{p}}^{-1}(B_\zeta) &= (F_{\mathfrak{p}}^{-1}(\gamma_{\mathrm{dR},\zeta}))^{-1}qAF_{\mathfrak{p}}^{-1}(\gamma_{\mathrm{dR},\zeta}) = q\zeta(\Phi^{\mathfrak{p}}_D)^{-1}B_\zeta\zeta(\Phi^{\mathfrak{p}}_D)\\ &= q\left(\sum_W J^{\mathrm{D}}_{\mathfrak{p}}(\zeta^{-1}(W))W\right)^{-1} B_\zeta \left(\sum_W J^{\mathrm{D}}_{\mathfrak{p}}(\zeta^{-1}(W))W\right).\end{aligned}$$
Here, the action of $\zeta \in \mu_N$ on words is given by $\zeta(A) = A$, and $\zeta(B_{\zeta'}) = B_{\zeta\zeta'}$. From the above formula about α_F, we have
$$\begin{aligned}\alpha_F = \Phi^{\mathfrak{p}}_D F_{\mathfrak{p}}^{-1}(\alpha_F) &= \left(\sum_W J^{\mathrm{D}}_{\mathfrak{p}}(W)W\right)\left(\sum_W I_{\mathfrak{p}}(W)F_{\mathfrak{p}}^{-1}(W)\right)\\ &= \left(\sum_W J^{\mathrm{D}}_{\mathfrak{p}}(W)W\right)\Bigg[\sum_{W=A^{k_d}B_{\zeta_d}\cdots A^{k_1}B_{\zeta_1}A^{k_0}} q^{k_0+\cdots+k_d+d} I_{\mathfrak{p}}(W)A^{k_d}\\ &\quad \cdot\left(\sum_W J^{\mathrm{D}}_{\mathfrak{p}}(\zeta_d^{-1}(W))W\right)^{-1} B_{\zeta_d}\left(\sum_W J^{\mathrm{D}}_{\mathfrak{p}}(\zeta_d^{-1}(W))W\right)\cdots\\ &\quad \cdot\left(\sum_W J^{\mathrm{D}}_{\mathfrak{p}}(\zeta_1^{-1}(W))W\right)^{-1} B_{\zeta_1}\left(\sum_W J^{\mathrm{D}}_{\mathfrak{p}}(\zeta_1^{-1}(W))W\right)A^{k_0}\Bigg],\end{aligned}$$
There, by using Proposition 2.14 and Proposition 3.3, for a word W of weight w we have
$$(1-q^w)I_{\mathfrak{p}}(W) - J^{\mathrm{D}}_{\mathfrak{p}}(W) \in \sum_{w=w'+w'':w'<w,w''<w} Z^{\mathfrak{p}}_{w'} \cdot Z^{\mathfrak{p},\mathrm{D}}_{w''}.$$
By induction, we have $Z^{\mathfrak{p}}_w = Z^{\mathfrak{p},\mathrm{D}}_w$. □

Finally, we remark on some conjectures. The following conjecture is a p-adic analogue of Grothendieck's conjecture [DG, 5.20], which says that $a_\sigma \in G_\omega(\mathbb{C})$ is $\mathbb{Q}$-Zariski dense (weakly, $a_\sigma^0 := a_\sigma \tau(2\pi\sqrt{-1})^{-1} \in U_\omega(\mathbb{C})$ is $\mathbb{Q}$-Zariski dense). Here, a_σ is the "difference" between the Betti realization with respect to σ and the de Rham realization (For elements a_σ and a_σ^0, see [DG, Proposition 2.12] and [D1, 8.10 Proposition]).

Conjecture 3. The element $\varphi_{\mathfrak{p}} \in U_\omega(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$ is $\mathbb{Q}$-Zariski dense. That means that if a subvariety X of U_ω over $\mathbb{Q}$ satisfies $\varphi_{\mathfrak{p}} \in X(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$, then $X = U_\omega$.

Remark 3.11. We have the Chebotarev density theorem for usual Galois groups. So, the author expects that there may be "Chebotarev density like" theorem for the Frobenius element in the motivic Galois group varying the prime number p. It will be interesting to study for this "Chebotarev density like" theorem varying p, adèle valued points of the motivic Galois group, and possible relations among "Chebotarev density like" theorem varying p, Grothendieck's conjecture about the motivic element, and the above p-adic analogue of Grothendieck's conjecture about the Frobenius element. □

The following conjecture in the case $N = 1$ (i.e. p-adic MZV's) is proposed by Furusho (non published).

Conjecture 4. All linear relations among p-adic MLV's are linear combinations of linear relations among p-adic MLV's with same weights.

The following proposition is obvious (cf. [DG, 5.27]).

Proposition 3.12. *We consider the following statements:*

(1) *The inequality in Theorem 1.4 is an equality (For $N = 1$, this is Conjecture 2).*
(2) *The map $\iota : U_\omega \to V_\omega$ is injective.*
(3) *Conjecture* 3.
(4) *Conjecture* 4.

Then, (1) *is equivalent to the combination of* (2) *and* (3)*, and implies* (4)*.* □

Remark 3.13. The statement (2) is true for $N = 2, 3, 4$. For $N > 4$, the statement (2) is false in general. The kernel is related to the space of cusp forms of weight 2 on $X_1(N)$ if N is a prime. See, [DG, 5.27][G2]. □

4. Crystalline Realization of Mixed Tate Motives.

In this section, we consider the construction of the crystalline realization of mixed Tate motives, and Berthelot-Ogus isomorphism for the de Rham and crystalline realizations of mixed Tate motives.

4.1. Crystalline Realization.

Let k be a number field, v be a finite place of k, and G_k be the absolute Galois group of k. First, we define the crystalline inertia group at v. Let p be a prime divided by v. Let $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_k)$, and $\underline{\mathrm{Rep}}^{\mathrm{crys},v}_{\mathbb{Q}_p}(G_k)$ be the category of finite dimensional representations of G_k over $\mathbb{Q}_p$, and the subcategory of crystaline representations of G_k at v.

Definition 4.1. (crystalline inertia group) The inclusion $\underline{\mathrm{Rep}}^{\mathrm{crys},v}_{\mathbb{Q}_p}(G_k) \hookrightarrow \underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_k)$ induces the map of Tannaka dual groups with respect to the forgetful fiber functor. We define a *crystalline inertia group* $I_v^{\mathrm{crys}} (\subset G_{k,p} := \underline{\mathrm{Aut}}^{\otimes}(\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_k)))$ at v to be its kernel. □

Here, $G_{k,p}$ is the (algebraic group over $\mathbb{Q}_p$)-closure of G_k. The group I_v^{crys} is a pro-algebraic group over $\mathbb{Q}_p$. Note that by the definition, the action of G_k on M_p is crystalline at v if and only if the action of I_v^{crys} on M_p is trivial.

We recall Bloch-Kato's group H^1_f. Let $O_{(v)}$ be the localization at v of the ring of integers of k, and k_v be the completion of k with respect to v. For a finite dimensional representation V of G_{k_v} over $\mathbb{Q}_\ell$, they defined [BK, §3]

$$H^1_f(k_v, V) := \begin{cases} \ker(H^1(k_v, V) \to H^1(k_v^{\text{ur}}, V)) & v \nmid \ell, \\ \ker(H^1(k_v, V) \to H^1(k_v, B_{\text{crys}} \otimes V)) & v \mid \ell. \end{cases}$$

Here, k_v^{ur} is the maximal unramified extension of k_v, and B_{crys} is the Fontaine's p-adic period ring (See, [Fo1]). For a prime ℓ not divided by v, the monodromy action $\mathbb{Q}_\ell(m) \to \mathbb{Q}_\ell(m+n)$ of $I_v \twoheadrightarrow \mathbb{Z}_\ell(1)$ is trivial for $n \geq 2$ (Here, I_v is the usual inertia at v). Thus, we have

$$H^1_f(k_v, \mathbb{Q}_\ell(n)) = \begin{cases} O_{(v)}^\times \otimes \mathbb{Q}_\ell & n = 1, \\ H^1(k_v, \mathbb{Q}_\ell(n)) & n \geq 2. \end{cases}$$

In the crystalline case, conversely from the calculations

$$H^1_f(k_v, \mathbb{Q}_p(n)) = \begin{cases} O_{(v)}^\times \otimes \mathbb{Q}_p & n = 1, \\ H^1(k_v, \mathbb{Q}_p(n)) & n \geq 2, \end{cases} \tag{4.1}$$

(See, [BK, Example 3.9]), we will get monodromy informaions of I_v^{crys} on mixed Tate motives. We recall that the fact $H^1_f(k_v, \mathbb{Q}_p(n)) = H^1(k_v, \mathbb{Q}_p(n))$ for $n \geq 2$, $v \mid p$ follows from

$$\begin{aligned} \dim_{\mathbb{Q}_p} H^1_f(k_v, \mathbb{Q}_p(n)) &= \dim_{\mathbb{Q}_p} D_{\text{dR}}(\mathbb{Q}_p(n))/\text{Fil}^0 D_{\text{dR}}(\mathbb{Q}_p(n)) + \dim_{\mathbb{Q}_p} H^0(k_v, \mathbb{Q}_p(n)) \\ &= [k_v : \mathbb{Q}_p] + 0 = -\chi(\mathbb{Q}_p(n)) = \dim_{\mathbb{Q}_p} H^1(k_v, \mathbb{Q}_p(n)) \end{aligned}$$

(See, [BK, Corollary 3.8.4, Example 3.9]). Here, D_{dR} is the Fontaine's functor ([Fo2]), and $\chi(V)$ is the Euler characteristic of V. Thus, it holds without assuming that k_v is unramified over $\mathbb{Q}_p$. Let $H^1_f(k, V)$ be the inverse image of $H^1_f(k_v, V)$ via the restriction map $H^1(k, V) \to H^1(k_v, V)$.

Theorem 4.2. *(cf.* [DG, Proposition 1.8]*) Let k be a number field, and v be a finite place of k. Take a mixed Tate motive M in* $\mathrm{MT}(k)$*. Then, the following statements are equivalent.*

(1) *The motive M is unramified at v, that is, $M \in \mathrm{MT}(O_{(v)})$.*
(2) *For a prime ℓ not divided by v, the ℓ-adic realization M_ℓ of M is an unramified representation at v.*
(3) *For all prime ℓ not divided by v, the ℓ-adic realization M_ℓ of M is an unramified representation at v.*
(4) *For the prime p divided by v, the p-adic realization M_p of M is a crystalline representation at v.*

□

Proof. The equivalence of (1), (2), and (3) is proved in [DG, Proposition 1.8]. We show that (1) is equivalent to (4). The proof is a crystalline analogue of [DG, Proposition 1.8]. The Kummer torsor $K(a)$ for $a \in k^\times \otimes \mathbb{Q}$ is crystalline at v, if and only if $a \in O_{(v)}^\times \otimes \mathbb{Q}$ (See, the isomorphism (4.1) $H^1_f(k_v, \mathbb{Q}_p(1)) \cong O_{(v)}^\times \otimes \mathbb{Q}_p$).

Since Kummer torsors generate $\mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(1))$, it suffices to show that the following statement: For a mixed Tate motive $M \in \mathrm{MT}(k)$, the action of I_v^{crys} on M_p is trivial if the action of I_v^{crys} on $W_{-2n}M_p/W_{-2(n+2)}M_p$ is trivial for each $n \in \mathbb{Z}$. Assume that the action of

I_v^{crys} on $W_{-2n}M_p/W_{-2(n+2)}M_p$ is trivial for each $n \in \mathbb{Z}$. We show that the action of I_v^{crys} on $W_{-2n}M_p/W_{-2(n+r)}M_p$ is trivial by the induction on r. For $r = 2$, it is the hypothesis. For $r > 2$, the induction hypothesis assure that the action of I_v^{crys} is trivial on $W_{-2n}/W_{-2(n+r-1)}$ and $W_{-2(n+1)}/W_{-2(n+r)}$. Thus, the action of $\sigma \in I_v^{\mathrm{crys}}$ is of the form $1 + \nu(\sigma)$, where $\nu(\sigma)$ is the composite:

$$W_{-2n}/W_{-2(n+r)} \twoheadrightarrow \mathrm{Gr}^W_{-2n} \xrightarrow{\mu(\sigma)} \mathrm{Gr}^W_{-2(n+r-1)} \hookrightarrow W_{-2n}/W_{-2(n+r)}.$$

We have $\mu(\sigma_1\sigma_2) = \mu(\sigma_1) + \mu(\sigma_2)$. This μ is compatible with the action of $G_{k,p}$. It suffices to show that the map $\mu(\sigma) : \mathrm{Gr}^W_{-2n} \to \mathrm{Gr}^W_{-2(n+r-1)}$ is trivial. This follows from

$$\begin{aligned}
&\mathrm{Hom}_{G_{k,p}}(I_v^{\mathrm{crys}}, \mathrm{Hom}(\mathbb{Q}_p(0), \mathbb{Q}_p(r-1))) \\
&\cong \mathrm{Ext}^1_{\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(I_v^{\mathrm{crys}})}(\mathbb{Q}_p(0), \mathbb{Q}_p(r-1))^{G_{k,p}/I_v^{\mathrm{crys}}} \\
&\cong \mathrm{Ext}^1_{\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_{k,p})}(\mathbb{Q}_p, \mathbb{Q}_p(r-1))/\mathrm{Ext}^1_{\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_{k,p}/I_v^{\mathrm{crys}})}(\mathbb{Q}_p, \mathbb{Q}_p(r-1)) \\
&\cong \mathrm{Ext}^1_{\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_k)}(\mathbb{Q}_p, \mathbb{Q}_p(r-1))/\mathrm{Ext}^1_{\underline{\mathrm{Rep}}^{\mathrm{crys},v}_{\mathbb{Q}_p}(G_k)}(\mathbb{Q}_p, \mathbb{Q}_p(r-1)) \\
&\cong H^1(k, \mathbb{Q}_p(r-1))/H^1_f(k, \mathbb{Q}_p(r-1)) = 0.
\end{aligned}$$

The second isomorphism follows from the fact that $\mathrm{Ext}^2_{\underline{\mathrm{Rep}}^{\mathrm{crys},v}_{\mathbb{Q}_p}(G_k)} = 0$, and the action of I_v^{crys} on $\mathbb{Q}_p(r-1)$ is trivial, and the last equality follows from the isomorphism (4.1). (We have $\mathrm{Ext}^2_{\underline{\mathrm{Rep}}^{\mathrm{crys},v}_{\mathbb{Q}_p}(G_k)} = 0$ from the elemental theory of the category of filtered φ-modules. In fact, RHom is calculated by a complex, which is concentrated only in degree 0 and 1.) □

Remark 4.3. If we have a full sub-Tannakian category $\mathrm{MT}(O_{(v)})^{\mathrm{good}}$ of $\mathrm{MT}(k)$ satisfying

$$\mathrm{Ext}^1_{\mathrm{MT}(O_{(v)})^{\mathrm{good}}}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong \begin{cases} O_{(v)}^{\times} \otimes \mathbb{Q}, & n = 1, \\ \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)), & n \geq 2, \end{cases}$$

and

$$\mathrm{Ext}^2_{\mathrm{MT}(O_{(v)})^{\mathrm{good}}}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0 \text{ for any } n,$$

then by introducing the "motivic inertia group" at v

$$I_v^{\mathcal{M}} := \ker\{\underline{\mathrm{Aut}}^{\otimes}(\omega_{\mathrm{MT}(k)}) \to \underline{\mathrm{Aut}}^{\otimes}(\omega_{\mathrm{MT}(O_{(v)})^{\mathrm{good}}})\},$$

we can prove the similar result for $\mathrm{MT}(O_{(v)})^{\mathrm{good}}$, that is, M is in $\mathrm{MT}(O_{(v)})$ if and only if M is in $\mathrm{MT}(O_{(v)})^{\mathrm{good}}$ by the "motivic analogue" of the above proof.

In a naive way, we cannot define "$M \otimes_{O_{(v)}} k(v)$" the reduction at v of an object M in $\mathrm{MT}(O_{(v)})$, since $\mathrm{MT}(O_{(v)})$ is not defined by a "geometrical way". So, the author hopes that this remark will be useful to construct "the reduction at v" of object in $\mathrm{MT}(O_{(v)})$. If we "geometrically" construct a full sub-Tannakian category $\mathrm{MT}(O_{(v)})^{\mathrm{good}}$ of $\mathrm{MT}(k)$ satisfying the above conditions, then we can get a good definition of "the reduction at v". Here, the word "geometrically" means that returning the definition of Voevodsky's category $DM(k)$. See also the proof of Theorem 4.6. □

Definition 4.4. For a mixed Tate motive $M \in \mathrm{MT}(O_{(v)})$ unramified at v, we define the *crystalline realization* $M_{\mathrm{crys},v}$ to be $D_{\mathrm{crys}}(M_p)$. Here D_{crys} is the Fontaine's functor $(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} -)^{G_{k_v}}$, and M_p is the p-adic realization of M. □

Note that M_p is a crystalline representaion of G_{k_v} by Theorem 4.2, so we have $\dim_{k_{0,v}} M_{\mathrm{crys},v} = \dim_{\mathbb{Q}_p} M_p = \dim_{\mathbb{Q}} M_\omega$. Here, $k_{0,v}$ is the fraction field of the ring of Witt vectors with coefficients in the residue field $k(v)$ of $O_{(v)}$. Note also that the pair $(M_{\mathrm{crys},v}, M_{crys,v} \otimes_{k_{0.v}} k_v)$ gives an admissible filtered φ-module in the sense of Fontaine ([Fo1], [Fo2]). The crystalline realization is functorial, and defines a fiber functor $\mathrm{MT}(O_{(v)}) \to \mathrm{Vect}_{k_{0,v}}$, which factors through the category of admissible filtered φ-modules $\mathrm{MF}^{\mathrm{ad}}_{k_{0,v}}(\varphi)$.

Remark 4.5. By using the fact that $H^1_{\mathrm{st}}(k_v, \mathbb{Q}_p(1)) = H^1(k_v, \mathbb{Q}_p(1))$ and introducing "semistable inertia group" at v, we can show that M_p is a semistable representation of G_{k_v} for any mixed Tate motive M in $\mathrm{MT}(k)$, similarly as the proof of Theorem 4.2. Thus, we can define the crystalline realization (or semistable realization) $M_{\mathrm{crys},v}$ (or $M_{\mathrm{st},v}$) to be $D_{\mathrm{st}}(M_p) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} M_p)^{G_{k_v}}$ for all $M \in \mathrm{MT}(k)$, and get a functor $\mathrm{MT}(k) \to \mathrm{MF}^{\mathrm{ad}}_{k_{0,v}}(\varphi, N)$ to the category of admissible filtered (φ, N)-modules. $\square$

4.2. **Comparison Isomorphism.** In this subsection, we prove a "Berthelot-Ogus like" comparison isomorphism between the crystalline realization and the de Rham realization. We defined the crystalline realization by using Fontaine's functor, so we need another "geometrical" construction of the crystalline realization to compare it with the de Rham realization (it is not obvious that the other construction is functorial).

For preparing the following theorem, we briefly recall that Voevodsky's category $\mathrm{DM}(k)$ (see, [V]), Levine's category $\mathrm{MT}(k)$ (see, [L]), and Deligne-Goncharov's category $\mathrm{MT}(O_{(v)})$ (see, [DG]). Let k be a field. First, let $\mathrm{SmCor}(k)$ be the additive category whose objects are smooth separated scheme over k, and morphisms $\mathrm{Hom}(X, Y)$ are free abelian group generated by reduced irreducible closed subschemes Z of $X \times Y$, which are finite over X and dominate a connected component of X. Then, Voevodsky's tensor triangulated category $\mathrm{DM}(k)$ is constructed from the category of bounded complexes $K^b(\mathrm{SmCor}(k))$ of $\mathrm{SmCor}(k)$ by localizing the thick subcategory generated by $[X \times \mathbb{A}^1] \to [X]$ (homotopy invariance), and $[U \cap V] \to [U] \oplus [V] \to [X]$ for $X = U \cup V$ (Mayer-Vietoris), adding images of direct factors of idempotents, and inverting formally $\mathbb{Z}(1)$.

Let k be a number field. Then, the vanishing conjecture of Beilinson-Soulé holds for k. From the vanishing conjecture of Beilinson-Soulé, Levine constructed the Tannakian category of mixed Tate motives $\mathrm{MT}(k)$ from $\mathrm{DMT}(k)$ by taking a heart with respect to a t-structure. Here, $\mathrm{DMT}(k)$ is the sub-tensor triangulated category of $\mathrm{DM}(k)_{\mathbb{Q}}$ generated by $\mathbb{Q}(n)$'s.

For a finite place v of k, let $O_{(v)}$ denote the localization of k at v. Deligne-Goncharov defined the full subcategory $\mathrm{MT}(O_{(v)})$ of mixed Tate motives unramified at v in $\mathrm{MT}(k)$, whose objects are mixed Tate motives M in $\mathrm{MT}(k)$ such that for each subquotient E of M, which is an extension of $\mathbb{Q}(n)$ by $\mathbb{Q}(n+1)$, the extension class of E in

$$\mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(n), \mathbb{Q}(n+1)) \xleftarrow{\cong} \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong k^\times \otimes \mathbb{Q}$$

is in $O^\times_{(v)} \otimes \mathbb{Q} (\subset k^\times \otimes \mathbb{Q})$.

Theorem 4.6. *(Berthelot-Ogus isomorphism) For any mixed Tate motive M in $\mathrm{MT}(O_{(v)})$, we have a canonical isomorphism*

$$k_v \otimes_{k_{0,v}} M_{\mathrm{crys},v} \cong k_v \otimes_k M_{\mathrm{dR}}.$$

$\square$

Remark 4.7. (Hyodo-Kato isomorphism) After choosing a uniformizer π of k_v, we can prove a canonical isomorphism

$$k_v \otimes_{k_{0,v}} M_{\mathrm{crys},v} \cong k_v \otimes_k M_{\mathrm{dR}}$$

for any mixed Tate motive M in $\mathrm{MT}(k)$ by the same way (cf. Remark 4.5). $\square$

Remark 4.8. From the functorial isomorphism $M_{\mathrm{crys},v} \otimes_{k_{0,v}} k_v \cong M_{\mathrm{dR}} \otimes_k k_v$, we have $G_\omega \otimes_{\mathbb{Q}} k_v \cong G_{\mathrm{crys}} \otimes_{k_{0,v}} k_v$. Here, $G := \pi_1(\mathrm{MT}(O_{(v)})) \in \text{pro-}\mathrm{MT}(O_{(v)})$ is the fundamental $\mathrm{MT}(O_{(v)})$-group (See, [D1, §6][D2, Definition 8.13]). Thus, we can consider the Frobenius element $F_{\mathfrak{p}}^{-1} \in G_\omega(k_v)$ if $k_{0,v} = k_v$ (For example, in the case where k is $\mathbb{Q}(\mu_N)$ and v is a prime ideal not dividing (N)). $\square$

Proof. For any finite extension k' of k and a prime ideal w over v, we define an additive category $\mathrm{SmCor}(O_{(w)})$ as follows. The objects are pairs of a proper smooth scheme $\mathcal{X}$ over $O_{(w)}$ and a horizontal simple normal crossing divisor $\mathcal{D}$ of $\mathcal{X}$, which is also normal crossing to the special fiber of $\mathcal{X}$. The set of morphisms $\mathrm{Hom}((\mathcal{X}, \mathcal{D}), (\mathcal{Y}, \mathcal{E}))$ is the free abelian group generated by horizontal smooth irreducible closed subschemes $\mathcal{Z}$ of $\mathcal{X} \times \mathcal{Y}$, which are finite over $\mathcal{X}$, dominate a connected component of $\mathcal{X}$, and are transversal to $\mathcal{X} \times \mathcal{E} \cup \mathcal{D} \times \mathcal{Y}$.

By using de Jong's alterations, there exists a finite extension k' of k, a bounded complex $(\mathcal{X}^\bullet, \mathcal{D}^\bullet)$ in $K^b(\mathrm{SmCor}(O_{(w)}))$, an idempotent f in $K^b(\mathrm{SmCor}(O_{(w)}))$, and an integer $n \in \mathbb{Z}$ such that $f(\mathcal{X}^\bullet \otimes k \setminus \mathcal{D}^\bullet \otimes k)(n)$ represents $M \otimes_k k'$ in $\mathrm{MT}(k')$. We call such a triple $\{(\mathcal{X}^\bullet, \mathcal{D}^\bullet), f, n\}$ a good reduction model of M at v' over k'. We can define the log-crystalline cohomology $H_{\mathrm{crys},w}(M) := H_{\mathrm{crys}}(M/W(k(w))) \otimes k_{0,w}$ for this triple $\{(\mathcal{X}^\bullet, \mathcal{D}^\bullet), f, n\}$. Note that the homotopy invariance holds for the log-crystalline cohomology. Choose a uniformizer π' of k'_w. By the semistable conjecture for open varieties proved in [Y1] (or [Fa]), we have a canonical isomorphism $M_{\mathrm{crys},w} \cong H_{\mathrm{crys},w}(M)$. Note that the compatiblity of the comparison isomorphism for algebraic correspondences is proved in [Y1], [Y2]. By using Hyodo-Kato isomorphism for open varieties in [Y1], we have an isomorphism $k'_w \otimes_{k_{0,w}} H_{\mathrm{crys},w}(M) \cong k'_w \otimes_k M_{\mathrm{dR}}$. Now, M_p is crystalline at v by Thoerem 4.2. So, we have $M_{\mathrm{crys},w} \cong k_{0,w} \otimes_{k_{0,v}} M_{\mathrm{crys},v}$. Therefore, we have an isomorphism $k'_w \otimes_{k_{0,v}} M_{\mathrm{crys},v} \cong k'_w \otimes_k M_{\mathrm{dR}}$. In general, for any element $\tau \in \mathrm{Gal}(k'_w/k_v)$, we have an isomorphism $k'^{,\tau}_w \otimes_{k_{0,v}} M_{\mathrm{crys},v} \cong k'^{,\tau}_w \otimes_k M_{\mathrm{dR}}$ by using the triple $\{(\mathcal{X}^{\bullet,\tau}, \mathcal{D}^{\bullet,\tau}), f^\tau, n\}$. Thus, we have an isomorphism $k_v \otimes_{k_{0,v}} M_{\mathrm{crys},v} \cong k_v \otimes_k M_{\mathrm{dR}}$ by the descent. Since M_p is crystalline at v, this isomorphism does not depend on the choice of π', and we can show that this isomorphism does not depend on the choice of good reduction models by using the standard product argument. $\square$

4.3. **Some Remarks.** The crystalline realization to the category of φ-modules (*not* to the category of admissible filtered φ-modules) is split, because we have $\mathrm{Ext}^1_{\mathrm{MT}(O_{(v)})}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$ for $n \leq 0$ and $\mathrm{Ext}^1_{\mathrm{Mod}_{k_{0,v}}(\varphi)}(k_{0,v}(0), k_{0,v}(n)) = 0$ for $n > 0$.

So, we can expect that the crystalline realization $\mathrm{MT}(O_{(v)}) \to \mathrm{Vect}_{k_{0,v}}$ factors through $\mathrm{MT}(k(v))$. Note that the weight filtration of mixed Tate motives over a finite field is split by Quillen's calculations of K-groups of finite fields ([Q]). Thus, they are sums of $\mathbb{Q}(n)$'s.

The weight filtration is motivic, and both of the de Rham realization and the crystalline realization are split. However, the splittings do not coincide, that is, the splitting of the crystalline realization does not coincide to the splitting of the de Rham realization via the Berthelot-Ogus isomorphism of Theorem 4.6. The iterated integrals and p-adic MLV's appear in the difference of these splittings. See also Remark 3.9.

Remark 4.9. We have $\mathrm{Ext}^1_{\mathrm{Mod}_{k_{0,v}}(\varphi)}(k_{0,v}(0), k_{0,v}(0)) \cong \mathbb{Q}_p \neq 0$, and this gap corresponds to the "near critical strip case" of Beilinson's conjecture and Bloch-Kato's Tamagawa number conjecture, that is, we need not only regulator maps, but also Chow groups to formulate these conjectures near the critical strip case (that is, the case where the weight of motive is 0 or -2). In this case, this corresponds to the "dual" of the fact that the image of the Dirichlet regulator is not a lattice of $\mathbb{R}^{r_1+r_2}$, but a lattice of a hyperplane of $\mathbb{R}^{r_1+r_2}$. The author does not know a direct proof of the fact that the non-trivial extension in $\mathrm{Ext}^1_{\mathrm{Mod}_{k_{0,v}}(\varphi)}(k_{0,v}(0), k_{0,v}(0)) = \mathbb{Q}_p$ does not occur in the crystalline realization. $\square$

Example 3. (Kummer torsor) Let K be a finite extension of $\mathbb{Q}_p$, K_0 be the fraction field of the ring of Witt vectors with coefficient in the residue field of K. Let $z \in 1 + \pi O_K$. Let

$$0 \to \mathbb{Q}_p(1) \to V(z)_p \to \mathbb{Q}_p(0) \to 0$$

be the extension of p-adic realization corresponding to z. Fix e_0 a generator of $\mathbb{Q}_p(1)$ corresponding $\{\zeta_n\}_n$, and e_1 the generator of $\mathbb{Q}_p(0)$ corresponding 1. Then, the action of Galois group is the following:

$$\begin{cases} ge_0 = \chi(g)e_0, \\ ge_1 = e_1 + \psi_z(g)e_0. \end{cases}$$

Here, χ is the p-adic cyclotomic character, and ψ_z is characterized by $g(z^{1/p^n}) = \zeta_n^{\psi_z(g)} z^{1/p^n}$.

Then, $V(z)_{\mathrm{crys}} \cong (B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V(z)_p)^{G_K}$ has the following basis:

$$\begin{cases} t^{-1} \otimes e_0 =: x_0, \\ e_1 - t^{-1} \log[\underline{z}] \otimes e_0 =: x_1. \end{cases}$$

Here, $t := \log[\underline{\zeta}], \log[\underline{z}] \in B_{\mathrm{crys}}$. Thus, the Frobenius action is the following:

$$\begin{cases} \phi(x_0) = \frac{1}{p} x_0, \\ \phi(x_1) = x_1. \end{cases}$$

The filtration after $K \otimes_{K_0}$ is the following:

$$\begin{cases} \mathrm{Fil}^{-1} V(z)_{\mathrm{dR}} = V(z)_{\mathrm{dR}} = \langle x_0, x_1 \rangle_K, \\ \mathrm{Fil}^0 V(z)_{\mathrm{dR}} = \langle x_1 + (\log z) x_0 \rangle_K, \\ \mathrm{Fil}^1 V(z)_{\mathrm{dR}} = 0 \end{cases}$$

(In B_{dR}, we have $t^{-1} \log \frac{z}{[\underline{z}]} \in \mathrm{Fil}^0 B_{\mathrm{dR}}$). Thus, we have splittings:

$$V(z)_{\mathrm{crys}} = \langle x_0 \rangle_{K_0} \oplus \langle x_1 \rangle_{K_0} = K_0(1) \oplus K_0(0),$$

$$V(z)_{\mathrm{dR}} = \langle x_0 \rangle_K \oplus \langle x_1 + (\log z) x_0 \rangle_K = K(1) \oplus K(0).$$

These splittings do not coincide in general.

We will recover the calculation $\phi^{-1}(0) = \log z^{1-p}$ in [D1, 2.9, 2.10]. In this case, we assume $K = K_0$. By the above calculation, the Kummer torsor $K(z)_{\mathrm{dR}}$ is

$$K(z)_{\mathrm{dR}} = -(x_1 + (\log z) x_0) + K x_0$$

(For the purpose of making satisfy $\nabla(u) = du - \frac{dz}{z}$ in [D1, 2.10], we use the above sign convention). Then, we have

$$\begin{aligned}\phi^{-1}(0) &\leftrightarrow \phi^{-1}(-(x_1 + (\log z)x_0) + 0) = -(x_1 + p(\log z)x_0) \\ &= -(x_1 + (\log z)x_0) + (1-p)(\log z)x_0 = -(x_1 + (\log z)x_0) + (\log z^{1-p})x_0 \\ &\leftrightarrow \log z^{1-p}.\end{aligned}$$

This coincides the calculation in [D1, 2.10]. Here, $\leftrightarrow$ is the identification via $K(z)_{\mathrm{dR}} = -(x_1 + (\log z)x_0) + Kx_0 \cong K$.

Next, we define polylogarithm extensions. In the following, we consider the case where k is a cyclotimic field $\mathbb{Q}(\mu_N)$ for $N \geq 1$. For $\zeta \in \mu_N$, let $U_\zeta \in$ pro-MT$(\mathbb{Q}(\mu_N))$ be the kernel of $\pi_1^{\mathcal{M}}(\mathbb{P}^1 \setminus \{0, 1\infty\}, \zeta) \to \pi_1^{\mathcal{M}}(\mathbb{G}_m, \zeta)$. We define $\mathcal{L}og_\zeta$ to be the abelianization of U_ζ Tate-twisted by (-1). We define $\mathcal{P}ol_\zeta$ with Tate twist (1) to be the push-out in the following diagram (see also, [D1, §16]):

$$\begin{tikzcd} 0 \arrow[r] & U_\zeta \arrow[r] \arrow[d] & \pi_1^{\mathcal{M}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \zeta) \arrow[r] \arrow[d] & \pi_1^{\mathcal{M}}(\mathbb{G}_m, \zeta) \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & \mathcal{L}og_\zeta(1) \arrow[r] & \mathcal{P}ol_\zeta(1) \arrow[r] & \mathbb{Q}(1) \arrow[r] & 0. \end{tikzcd}$$

For $n \geq 1$, we also define $\mathcal{P}ol_{n,\zeta}$ to be the push-out under $\mathcal{L}og_\zeta = \Pi_{n \geq 0}\mathbb{Q}(n) \to \mathbb{Q}(n)$ (see also, [D1, §16]):

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}og_\zeta \arrow[r] \arrow[d] & \mathcal{P}ol_\zeta \arrow[r] \arrow[d] & \mathbb{Q}(0) \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & \mathbb{Q}(n) \arrow[r] & \mathcal{P}ol_{n,\zeta} \arrow[r] & \mathbb{Q}(0) \arrow[r] & 0. \end{tikzcd}$$

The extension class $[\mathcal{P}ol_{n,\zeta}]$ lives in $\mathrm{Ext}^1_{\mathrm{MT}(\mathbb{Q}(\mu_N))}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong K_{2n-1}(\mathbb{Q}(\mu_N))_{\mathbb{Q}}$. Let μ_N^0 be the group of primitive N-th roots of unity. Recall that Huber-Wildeshaus constructed motivic polylogarithm *classes* $\mathrm{pol}_\zeta \in \prod_{n \geq 2} K_{2n-1}(\mathbb{Q}(\mu_N))_{\mathbb{Q}}$ (not *extensions* of motives) in [HW].

Proposition 4.10. *Let n be an integer greater than or equal to 2, and ζ be an N-th root of unity. Then, the n-th component of Huber-Wildeshaus' motivic polylogarithm class* pol_ζ *(see,* [HW, Definition 9.4]*) is equal to $(-1)^{n-1}\frac{n!}{N^{n-1}}[\mathcal{P}ol_{n,\zeta}]$ under the identification*

$$K_{2n-1}(\mathbb{Q}(\mu_N))_{\mathbb{Q}} \cong \mathrm{Ext}^1_{\mathrm{MT}(\mathbb{Q}(\mu_N))}(\mathbb{Q}(0), \mathbb{Q}(n)).$$

In particular, the extension classes $\{[\mathcal{P}ol_{n,\zeta}]\}_{\zeta \in \mu_N^0}$ generate $K_{2n-1}(\mathbb{Q}(\mu_N))_{\mathbb{Q}}$. □

Proof. It is sufficient to show the equality after taking the Hodge realization. This follows from [D1, §3, §16, §19] and [HW, Theorem 9.5, Corolary 9.6]. Note that we consider as $\mathbb{Q}(n)_\omega$-torsor not as $\mathbb{Z}(n)_\omega$-torsor, and we do not multiply $\frac{1}{(n-1)!}$ on the integral structure unlike as [D1] (See also Example (2, 2)). □

Fix a place $v \nmid N$ of $\mathbb{Q}(\mu_N)$. Put $K := \mathbb{Q}(\mu_N)_v$. Let p be the prime devied by v. Note that K is unramified over $\mathbb{Q}_p$. Let σ denote the Frobenius endomorphism on K. For a mixed Tate motive $[0 \to \mathbb{Q}(n) \to M \to \mathbb{Q}(0) \to 0] \in \mathrm{Ext}^1_{\mathrm{MT}(O_{(v)})}(\mathbb{Q}(0), \mathbb{Q}(n))$, the pair $M_{\mathrm{syn}} := (M_{\mathrm{crys},v}, M_{\mathrm{dR}} \otimes_{\mathbb{Q}(\mu_N)} K)$ defines a extension of filtered φ-modules:

$$0 \to K(n) \to M_{\mathrm{syn}} \to K(0) \to 0.$$

Here, $K(i)$ is the Tate object in the category of filtered φ-modules over K. Thus, we have a map

$$r_n : K_{2n-1}(O_{(v)})_{\mathbb{Q}} \cong \mathrm{Ext}^1_{\mathrm{MT}(O_{(v)})}(\mathbb{Q}(0), \mathbb{Q}(n)) \to \mathrm{Ext}^1_{\mathrm{MF}^f_K}(K(0), K(n)) \cong H^1_{\mathrm{syn}}(K, K(n)).$$

See, [B] for the last isomorphism. We call r_n the n-th *syntomic regulator map*. Recall that H^1_{syn} is a finite dimensional $\mathbb{Q}_p$-vector space, not a K-vector space.

We fix an isomorphism $H^1_{\mathrm{syn}}(K, K(n)) \cong K$ as $\mathbb{Q}_p$-vector spaces for $n \geq 1$ as follows.

$$\begin{aligned} H^1_{\mathrm{syn}}(K, K(n)) &\cong \mathrm{coker}(K(n)_{\mathrm{crys}} \overset{a\mapsto(\bar{a},(1-\varphi)(a))}{\longrightarrow} (K(n)_{\mathrm{dR}}/\mathrm{Fil}^0 K(n)_{\mathrm{dR}}) \oplus K(n)_{\mathrm{crys}}) \\ &\cong \mathrm{coker}(K \overset{a\mapsto(a,(1-p^{-n}\sigma)(a))}{\longrightarrow} K \oplus K) \\ &\overset{[(a,b)]\mapsto b-(1-p^{-n}\sigma)(a)}{\cong} K. \end{aligned}$$

In general, note that for a filtered φ-module D and for

$$[(x,y)] \in \mathrm{coker}(D \overset{a\mapsto(\bar{a},(1-\varphi_D)(a))}{\longrightarrow} (D/\mathrm{Fil}^0 D) \oplus D) \cong \mathrm{Ext}^1_{\mathrm{MF}^f_K}(K(0), D),$$

the corresponding extension E of $K(0)$ by D is the following: $E = D \oplus Ke_0$

$$\begin{cases} \mathrm{Fil}^i E = \mathrm{Fil}^i D + \langle x + e_0 \rangle_K & \text{for } i \leq 0, \\ \mathrm{Fil}^i E = \mathrm{Fil}^i D & \text{for } i > 0, \end{cases}$$

$$\begin{cases} \varphi_E(a) = \varphi_D(a) & \text{for } a \in D, \\ \varphi_E(e_0) = e_0 + y. \end{cases}$$

Proposition 4.11. *The syntomic regulator map*

$$r_1 : K_1(O_{(v)})_{\mathbb{Q}} \cong O^{\times}_{(v)} \otimes \mathbb{Q} \to H^1_{\mathrm{syn}}(K, K(1)) \cong K$$

is given by $z \mapsto -(1-\frac{1}{p}) \log z$. *For* $n \geq 2$, *the syntomic regulator map*

$$r_n : K_{2n-1}(\mathbb{Q}(\mu_N))_{\mathbb{Q}} \to H^1_{\mathrm{syn}}(K, K(n)) \cong K$$

sends $[\mathcal{P}ol_{n,\zeta}]$ *to* $-N^{n-1}(1-\frac{1}{p^n})\mathrm{Li}^a_n(\zeta)$. □

Note that Coleman's p-adic polylogarithm $(1-\frac{1}{p^n})\mathrm{Li}^a_n(\zeta)$ is often written by $\ell^{(p)}_n(\zeta)$, and does not depend on the chice of a.

Remark 4.12. If we use an identification

$$\mathrm{coker}(K \overset{a\mapsto(a,(1-p^{-n}\sigma)(a))}{\longrightarrow} K \oplus K) \overset{[(a,b)]\mapsto a-(1-p^{-n}\sigma)^{-1}(b)}{\cong} K$$

(note that $1 - p^{-n}\sigma$ is a bijection on K for $n \geq 1$), then the above formula changes as the following: the map

$$r_1 : K_1(O_{(v)})_{\mathbb{Q}} \cong O^{\times}_{(v)} \otimes \mathbb{Q} \to H^1_{\mathrm{syn}}(K, K(1)) \cong K$$

is given by $z \mapsto \log z$. For $n \geq 2$, the map

$$r_n : K_{2n-1}(\mathbb{Q}(\mu_N))_{\mathbb{Q}} \to H^1_{\mathrm{syn}}(K, K(n)) \cong K$$

sends $[\mathcal{P}ol_{n,\zeta}]$ to $N^{n-1}\mathrm{Li}^a_n(\zeta)$. □

Proof. The first assertion follows from Example (3). The second assertion follows from the following structure of $(\mathcal{P}ol_{n,\zeta})_{\mathrm{syn}} = ((\mathcal{P}ol_{n,\zeta})_{\mathrm{crys}}, (\mathcal{P}ol_{n,\zeta})_{\mathrm{dR}})$: $(\mathcal{P}ol_{n,\zeta})_{\mathrm{crys}} = \langle x_0, x_1\rangle_K$

$$\begin{cases} \varphi(x_0) = \frac{1}{p^n} x_0, \\ \varphi(x_1) = x_1 - N^{n-1}(1-p^{-n})\mathrm{Li}^a_n(\zeta), \end{cases}$$

$$\begin{cases} \mathrm{Fil}^{-n}(\mathcal{P}ol_{n,\zeta})_{\mathrm{dR}} = \langle x_0, x_1\rangle_K, \\ \mathrm{Fil}^{i}(\mathcal{P}ol_{n,\zeta})_{\mathrm{dR}} = \langle x_1\rangle_K \text{ for } -n < i \leq 0, \\ \mathrm{Fil}^{1}(\mathcal{P}ol_{n,\zeta})_{\mathrm{dR}} = 0. \end{cases}$$

This structure follows from Example (2). □

Remark 4.13. We have an isomorphism

$$B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} (P^{\mathcal{M}}_{y,x})_p \cong B_{\mathrm{crys}} \otimes_{K_0} (P^{\mathcal{M}}_{y,x})_{\mathrm{crys}}.$$

Here, $P^{\mathcal{M}}_{y,x}$ is a fundamental groupoid of $\mathbb{P}^1 \setminus \{0,\infty\} \cup \mu_N$. This induces an isomorphism

$$B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} (\mathcal{P}ol_{\zeta})_p \cong B_{\mathrm{crys}} \otimes_{K_0} (\mathcal{P}ol_{\zeta})_{\mathrm{crys}}.$$

Thus, we have the following commutative diagram for $n \geq 2$:

$$\begin{tikzcd} K_{2n-1}(\mathbb{Q}(\mu_N))_{\mathbb{Q}} \arrow[r] \arrow[dr] & H^1(K, \mathbb{Q}_p(n)) \arrow[d, "\cong"] \\ & H^1_{\mathrm{syn}}(K, K(n)) \end{tikzcd} \qquad \begin{tikzcd} {[\mathcal{P}ol_{n,\zeta}]} \arrow[r, mapsto] \arrow[dr, mapsto] & {[(\mathcal{P}ol_{n,\zeta})_p]} \arrow[d, mapsto] \\ & {[(\mathcal{P}ol_{n,\zeta})_{\mathrm{syn}}].} \end{tikzcd}$$

Here, K denotes $\mathbb{Q}_p(\mu_N)$, ζ is in μ_N, and p does not divide N. The horizontal map sends the extension class $[\mathcal{P}ol_{n,\zeta}]$ to the one $[(\mathcal{P}ol_{n,\zeta})_p]$, and the oblique map sends the extension class $[\mathcal{P}ol_{n,\zeta}]$ to the one $[(\mathcal{P}ol_{n,\zeta})_{\mathrm{syn}}]$. □

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