Dessins d'enfants: Solving equations determining Belyi pairs

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Abstract

This paper deals with the Grothendieck dessins d'enfants, that is tamely embedded graphs on surfaces. We investigate combinatorics of systems of equations determining a Belyi pair corresponding to a dessin, that is a rational function with at most 3 critical values on an algebraic curve, such that its preimage is the dessin under consideration. Several properties of extra, or so-called parasitic, solutions of such systems are described.

1 Introduction

Theory of tamely embedded graphs on surfaces was originated by A. Grothendieck in [7]. He called these graphs *dessins d'enfants*, or just *dessins*, for their very simple combinatorial structure. Grothendieck put forward the correspondence between these dessins d'enfants and algebraic curves together with non-constant rational functions with at most 3 critical values on these curves which gives rise to plenty of new and non-trivial interrelations between different structures in category theory, algebra, algebraic geometry, complex analysis, topology, etc. For example, this correspondence establishes an approach to a visualization of algebraic curves over number fields and to an interpretation of the action of general Galois group $\mathrm{Aut}(\overline{\mathbb{Q}})$ on the set of their isomorphism classes, here $\overline{\mathbb{Q}}$ denotes the field of algebraic numbers. The origins of the theory can be found in the special volumes [14, 16] and the modern development of the theory and its numerous applications are described in the detailed and self-contained surveys [13, 19].

The main problem of the theory is to calculate a particular Belyi pair, corresponding to a given dessin. There are several particular solutions, see [2, 5, 6, 12, 10, 15, 17, 18, 20], but to solve this question more or less generally is still an open problem even for dessins of genus zero. In this paper we provide a general system of equations in the Hurwitz space, which determines the Belyi functions of all dessins of the prescribed combinatorial type and investigate some combinatorial properties of such systems. Note that even for the dessins of genus zero (in this case the system may be written more explicitly) there is no direct way to get a solution of a general system of this type within reasonable time limits since the degrees appear to be too high. We develop some special technique to predict and investigate extra, or so-called parasitic, solutions of such systems, which gives a possibility to choose the "right" normalization leading to the equations of minimal possible degrees.

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2 Definitions and Notations

2.1 Dessins d'enfants

Definition 2.1. A graph, Γ, is called *bicolored* if all its vertices are colored in two different colors, say black and white, in such a way that each edge has two vertices of different colors.

Definition 2.2. A *dessin d'enfant* D is a compact connected smooth oriented surface M together with a bicolored graph Γ on it such that the complement $M \setminus \Gamma$ is homeomorphic to a disjoint union of open discs. Such a disc is called a *face* of the dessin. *Vertices* and *edges* of the dessin are vertices and edges of the corresponding graph.

We denote by $\alpha(D), \omega(D), n(D), \gamma(D)$ the number of black vertices, white vertices, edges, and faces of D, correspondingly. We will write just $\alpha, \omega, n, \gamma$ if the dessin D is clear from the context.

Definition 2.3. A *valency of a vertex* of a dessin is a number of edges incident to this vertex. A *valency of a face* is defined to be the number of edges incident to this face divided by 2, note that if both sides of an edge are incident to a given face then it should counted twice.

Definition 2.4. A sequence of numbers $\langle a_1, \ldots, a_\alpha | w_1, \ldots, w_\omega | c_1, \ldots, c_\gamma \rangle$ is called *a combinatorial type* if it can be realized as a list of valencies of all vertices and faces of a certain dessin.

Definition 2.5. Two dessins are said to be *isomorphic* if there exists a homeomorphism between corresponding surfaces under which one dessin is transformed into another.

Remark 2.6. It is straightforward to see that isomorphic dessins have the same combinatorial types. However, combinatorial type does not determine a dessin up to isomorphism, see Figure 1, for example.

Figure 1. Non-isomorphic dessins with the same combinatorial type $\langle 3, 2, 1|3, 1, 1, 1|6\rangle$.

In some cases it is more convenient to consider non-bicolored graphs embedded into surfaces in such a way that the complement is homeomerphic to a disjoint union of open discs. In this case we will speak about *non-bicolored dessins*. It is easy to see that non-bicolored dessins can be obtained from bicolored ones by forgetting the coloring. Conversely, for any non-bicolored dessin d'enfant we can add a vertex of the other color in the middle of each edge to get a dessin d'enfant in the sense of Definition 2.2. Note that non-bicolored dessins can possess with loops and multiple edges. In the non-bicolored context we say that a valency of a face is the number of edges incident to this face (without dividing by 2).

2.2 Belyi pairs

Definition 2.7. A *Belyi pair* $(\mathcal{X}, \beta)$ is an algebraic curve $\mathcal{X}$ together with a non-constant rational function $\beta : \mathcal{X} \to \mathbb{CP}^1$, which has at most three critical values. Function β is usually called a *Belyi function*.

Remark 2.8. Up to the linear-fractional transformation of $\mathbb{CP}^1$ we may and we do fix the critical values of β, $crit(\beta) \subseteq \{0, 1, \infty\}$.

Definition 2.9. A Belyi pair is called *clean* if all ramifications of β over 1 has the order 2.

Remark 2.10. It is straightforward to see that if $(\mathcal{X}, \beta)$ is a Belyi pair, then $(\mathcal{X}, 4\beta(1-\beta))$ is a clean Belyi pair.

Belyi pairs are in some sense widely spread objects among all algebraic curves and rational functions on them, namely the following is true:

Theorem 2.11. [4] *Let $\mathcal{X}$ be a smooth complete irreducible algebraic curve over $\mathbb{C}$. Then the following statements are equivalent:*

1. *$\mathcal{X}$ is isomorphic to the complexification of a curve defined over a number field;*

2. *There exists a Belyi function on $\mathcal{X}$.*

It is easy to check, that if $(\mathcal{X}, \beta)$ is a Belyi pair, then $\beta^{-1}([0,1])$ is a dessin d'enfant on the topological model of $\mathcal{X}$ whose edges are $\{\beta^{-1}((0,1))\}$, black vertices are $\{\beta^{-1}(0)\}$, and white vertices are $\{\beta^{-1}(1)\}$. If $(\mathcal{X}, \beta)$ is a clean Belyi pair, then $\beta^{-1}([0,1])$ is a non-bicolored dessin d'enfant on the topological model of $\mathcal{X}$ whose edges are $\{\beta^{-1}((0,1])\}$ and vertices are $\{\beta^{-1}(0)\}$.

Moreover, the following result is true.

Theorem 2.12. ([7],[17]) *Any non-bicolored dessin d'enfant can be obtained by the above construction from some complex clean Belyi pair. This pair is defined uniquely up to an isomorphism.*

Similar result for bicolored dessins and (not necessarily clean) Belyi pairs holds, see [1].

Moreover it is possible to define categories of dessins d'enfants and Belyi pairs (correspondingly, non-bicolored dessins and clean Belyi pairs) and to prove their equivalence, see [17, 1].

2.3 Determining systems and parasitic solutions

Theorem 2.13. *Let $Val := \langle a_1, \ldots, a_\alpha | w_1, \ldots, w_\omega | c_1, \ldots, c_\gamma \rangle$ be a certain combinatorial type. Let $\mathcal{X}$ be a smooth irreducible curve of genus $g = \frac{1}{2}(2 - \alpha - \omega + n - \gamma)$ over $\mathbb{C}$. Assume that there exist pairwise distinct points $A_1, \ldots, A_\alpha$, $W_1, \ldots, W_\omega$, $C_1, \ldots, C_\gamma \in \mathcal{X}$ and a function $\beta \in \mathbb{C}(\mathcal{X})$ such that the following equalities for divisors hold*

$$(\beta) = \sum_{j=1}^{\alpha} a_j A_j - \sum_{j=1}^{\gamma} c_j C_j,$$
$$(\beta - 1) = -\sum_{j=1}^{\gamma} c_j C_j + \sum_{j=1}^{\omega} w_j W_j.$$

Then $(\mathcal{X}, \beta)$ is a Belyi pair.

Proof. Let us assume that β has a critical value $V \notin \{0, 1, \infty\}$. In this case the preimage of V is effective (since ramification over ∞ is already counted).

Therefore,

$$(\mathrm{d}\,\beta) = \sum_{j=1}^{\alpha}(a_j - 1)A_j + \sum_{j=1}^{\omega}(w_j - 1)W_j - \sum_{j=1}^{\gamma}(c_j + 1)C_j + V.$$

Let us compute the degrees of divisors on both sides of this equality. On the left hand side we have

$$\deg\,(\mathrm{d}\,\beta) = 2g - 2.$$

On the right hand side, since all points A_i, W_j, C_k are pair-wise distinct, we have

$$\deg\,(\sum_{j=1}^{\alpha}(a_j - 1)A_j + \sum_{j=1}^{\omega}(w_j - 1)W_j - \sum_{j=1}^{\gamma}(c_j + 1)C_j + V) =$$

$$= \sum_{j=1}^{\alpha}(a_j - 1) + \sum_{j=1}^{\omega}(w_j - 1) - \sum_{j=1}^{\gamma}(c_j + 1) + \deg W =$$

$$= \sum_{j=1}^{\alpha} a_j - \alpha + \sum_{j=1}^{\omega} w_j - \omega - \sum_{j=1}^{\gamma} c_j - \gamma + \deg V.$$

Since Val is a combinatorial type, we have that $\sum_{j=1}^{\alpha} a_j = \sum_{j=1}^{\omega} w_j = \sum_{j=1}^{\gamma} c_j = n$. Thus $\deg\,(\mathrm{d}\,\beta) = 2g - 2 + \deg V$. It follows that $\deg V = 0$. ■

Definition 2.14. For any sequence of positive integers

$$Val := \langle a_1, \ldots, a_\alpha | w_1, \ldots, w_\omega | c_1, \ldots, c_\gamma \rangle$$

we consider the following system of equations in the Hurwitz space $\mathcal{H}_{g,n}(\mathbb{C})$, where $n = \sum_{j=1}^{\alpha} a_j$, $g = \frac{1}{2}(2 + n - \alpha - \omega - \gamma)$:
There exists a curve $\mathcal{X}$, points $A_1, \ldots, A_\alpha$, $W_1, \ldots, W_\omega$, $C_1, \ldots, C_\gamma \in \mathcal{X}$, and a rational function $\beta : \mathcal{X} \to \mathbb{CP}^1$ such that the following equalities for the divisors hold

$$\begin{cases} (\beta) = \sum_{j=1}^{\alpha} a_j A_j - \sum_{j=1}^{\gamma} c_j C_j, \\ (\beta - 1) = -\sum_{j=1}^{\gamma} c_j C_j + \sum_{j=1}^{\omega} w_j W_j. \end{cases} \tag{2.1}$$

This system is denoted by $\mathcal{S}(Val)$.

Remark 2.15. It follows from Theorem 2.13 that if Val is a combinatorial type, then system (2.1) has a solution and this solution is a Belyi pair, corresponding to one of the realizations of Val.

If $g = 0$ we may re-write system (2.1) in the following, more explicit, form:

Definition 2.16. Let Val be a certain combinatorial type, $g := \sum_{j=1}^{\alpha} a_j = 0$. Then the system of polynomial equations $\mathcal{S}_0(Val)$, obtained from the formal polynomial equality

$$\begin{aligned} K_1(z-A_1)^{a_1}\cdots(z-A_\alpha)^{a_\alpha} - (z-C_1)^{c_1}\cdots(z-C_\gamma)^{c_\gamma} = \\ = K_2(z-W_1)^{w_1}\cdots(z-W_\omega)^{w_\omega} \quad . \end{aligned} \tag{2.2}$$

by considering the coefficients at the same degrees of the variable z on both sides of this equality, is called the *determining system* for Belyi functions corresponding to different realizations of Val.

Remark 2.17. In the cases of $g = 1, 2$ the system $\mathcal{S}(Val)$ also can be written explicitly. In addition, there are special families of dessins of positive genus, for which $\mathcal{S}$ can be written more or less explicitly.

Definition 2.18. Let Val be a combinatorial type. A solution of the system $\mathcal{S}(Val)$ is called *parasitic* if one of the following conditions is not satisfied:

1. the genus of $\mathcal{X}$ is not $\frac{1}{2}(2 + \sum_{j=1}^{\alpha} a_j - \alpha - \omega - \gamma)$;
2. $\deg \beta \neq \sum_{j=1}^{\alpha}$ (the number of edges of the dessin);
3. β is not a Belyi function;
4. β is a Belyi function for a certain dessin, with combinatorial type different from Val.

Definition 2.19. A parasitic solution is called *geometrical* if $(\mathcal{X}, \beta)$ is a Belyi pair, and *non-geometrical* otherwise.

Several attempts to work with determining systems and parasitic solution for dessins of genus 0 where made in [3, 10, 11].

3 A necessary condition for parasitic solution

It follows from Theorem 2.13 that parasitic solutions can appear only in the case of junction of some of the points A_i, W_j, C_k with each other. In order to "predict" parasitic solutions we have to find what particular points can, or can not, coincide.

Theorem 3.1. *Let Val be a combinatorial type of genus 0 ($2 + \sum_{i=1}^{\alpha} a_i - \alpha - \omega - \gamma = 0$). Assume that $\mathcal{S}_0(Val)$ has a geometric parasitic solution. Then there exist indices i, j, $1 \leq i \leq \alpha, 1 \leq j \leq \gamma$, such that $A_i = C_j$.*

Proof. We denote the obtained solution by σ. Since this solution is geometric, there is a dessin d'enfants corresponding to σ, denote this dessin by D_σ. Also since Val is a combinatorial type, we denote by D a certain dessin corresponding to this type. Assume that for any i, j $A_i \neq C_j$ and consider the following cases:

1. There exists a pair (i, j), $i \neq j$ such that $A_i = A_j$. Since there is no cancellation, the sum of the degrees in the numerator of σ is equal to the sum of degrees in the numerator of β. Thus the number of edges of D_σ is equal to the number of edges of D, but the number of black vertices of D_σ is at most $\alpha - 1$. However D_σ has to satisfy a Euler formula, which implies that either the number of faces or the number of white vertices should be greater than the corresponding number for D. Thus σ is not a solution of $\mathcal{S}_0(Val)$. A contradiction.

2. There exist i, j such that $C_i = C_j$, or there exists i, j such that $W_i = W_j$. These cases can be considered similar with the previous.

3. There exist i, j such that $A_i = W_j$. Now writing $-(z - C_1)^{c_1} \cdots (z - C_\gamma)^{c_\gamma} = -K_1(z - A_1)^{a_1} \cdots (z - A_\alpha)^{a_\alpha} + K_2(z - W_1)^{w_1} \cdots (z - W_\omega)^{w_\omega}$ and substituting $B_j = A_i$ we get that the right hand side of this equality is divisible by $(z - A_i)$. Then by the Main Theorem of Algebra there exists k such that $(z - C_k)$ is divisible by $(z - A_i)$. Then $C_k = A_i$ which contradicts to our assumptions. ∎

Theorem 3.2. *Let Val be a combinatorial type of genus 0 ($2 + \sum_{i=1}^{\alpha} a_i - \alpha - \omega - \gamma = 0$). Assume that $\mathcal{S}_0(Val)$ has a parasitic solution. Then there exist indices i, j, $1 \leq i \leq \alpha, 1 \leq j \leq \gamma$, such that $A_i = C_j$.*

Proof. Let $n = \sum_{j=1}^{\alpha} a_j$ be the number of edges of a dessin corresponding to Val. We assume that

$$(K_1, K_2; A_1, \ldots, A_\alpha; C_1, \ldots, C_\gamma; W_1, \ldots, W_\omega)$$

is a solution of $\mathcal{S}_0(Val)$ and for all i, j, $i = 1, \ldots, \alpha$, $j = 1, \ldots, \gamma$ it holds that $A_i \neq C_j$. Then the degree of the function

$$\sigma = K_1 \frac{(z - A_1)^{a_1} \cdots (z - A_\alpha)^{a_\alpha}}{(z - C_1)^{c_1} \cdots (z - C_\gamma)^{c_\gamma}}$$

is equal to $2n$. Hence, its derivative has exactly $4n-2$ zeros. At the same time, it follows from the system $\mathcal{S}(Val)$ that σ has at least

$$\sum_{i=1}^{\alpha} a_i - \alpha + \sum_{j=1}^{\gamma} c_j - \gamma + 2n - n = 2n - \alpha + 2n - \gamma + n = 4n - 2$$

critical values. Thus σ has no critical points except $A_1, \ldots, A_\alpha; C_1, \ldots, C_\gamma; W_1, \ldots, W_\omega$. However, the values of σ in these points are determined by the equations $\mathcal{S}_0(Val)$ and lie in the set $\{0, 1, \infty\}$. Therefore, σ is a Belyi function. It follows by the definition that if σ is parasitic solution now, then it is a geometrical parasitic solution. Now Theorem 3.1 completes the proof. ∎

4 Combinatorics of parasitic solutions

Theorem 4.1. *Let $Val := \langle a_1, \ldots, a_\alpha | w_1, \ldots, w_\omega | c_1, \ldots, c_\gamma \rangle$ be a certain sequence of positive integers, not necessary a combinatorial type. The set of different (up to a linear-fractional transformation) geometrical parasitic solutions of $\mathcal{S}(Val)$ is finite.*

Proof. Let D_σ be a certain dessin corresponding to parasitic solution σ. Since D_σ satisfies $\mathcal{S}(Val)$ it follows that $\alpha(D_\sigma) \leq \alpha$, $\omega(D_\sigma) \leq \omega$, $\gamma(D_\sigma) \leq \gamma$, $n(D_\sigma) \leq \sum_{j=1}^{\alpha} a_j$. Also the combinatorial type of D_σ has the upper bound Val. By [9] it follows that only finite number of abstract graphs can appear. Any abstract graph can be embedded in a surface in finite number of ways, and hence gives rise to just finite number of dessins d'enfants. By the equivalence theorem for Belyi pairs and dessins d'enfants, see [1, 17], there is only a finite set, B, of different, up to linear-fractional transformations, Belyi functions corresponding to a given finite set of dessins d'enfants. Thus all solutions of $\mathcal{S}(Val)$ are contained in a finite set of Belyi functions corresponding to dessins whose combinatorial type is coordinatewise less than or equal to Val. Then $\mathcal{S}(Val)$ has at most finite number of geometrical parasitic solutions. ∎

Remark 4.2. Note that even in the simplest examples we may have an infinite number of non-geometrical parasitic solutions.

Example 4.3. Let $D_{0,2,0}$ be the following dessin d'enfant:

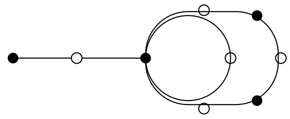

Figure 2.

Theorem 4.4. *A combinatorial type* $Val_{0,2,0} := \langle 5,2,2,1|2,2,2,2,2|5,4,1\rangle$, *whose unique realization is the dessin* $D_{0,2,0}$ *from Example 4.3 has a 1-parametric family of non-geometrical parasitic solutions.*

Proof. We will use the following notations for the complex coordinates of critical points of β. Let A_0 be the coordinate of the vertex of valency 1 of $D_{0,2,0}$, A_1, A_2 be the coordinates of the black vertices of valencies 2, A be the coordinate of the vertex of valency 5; C_1, C_2, C_3 be the coordinates of the centers of faces of valencies 1, 4, 5, correspondingly; K_1 be a proportion coefficient. Thus the direct substitution shows that the values $A_0 = -1$, $A = C_2 = 0$, $A_1 = t$, $A_2 = -\frac{1}{8}\frac{8t+5}{2t+1}$, $C_1 = \frac{1}{8}\frac{(1+12t+16t^2)^2}{(4t+3)(4t+1)(2t+1)}$, $C_3 = \infty$, $K = 8^3\frac{2t+1}{(4t+3)(4t+1)}$ provides a 1-parametric system of parasitic solutions

$$\sigma = 8^3\frac{2t+1}{(4t+3)(4t+1)}\frac{z(z+1)(z-t)^2(z+\frac{8t+5}{16t+8})^2}{z-\frac{(1+12t+16t^2)^2}{8(4t+3)(4t+1)(2t+1)}}.$$

Then by taking the derivatives and comparing the values in their roots, one can see that if $t = 0$ or $t = -\frac{5}{8}$ then σ is a Belyi function and for all other values of t the function σ is not a Belyi function. ∎

Corollary 4.5. *There exist combinatorial types that possess with the infinitely many parasitic solutions.*

Until the end of this section let a dessin will be a *non-bicolored* dessin.

Definition 4.6. A *dual* dessin d'enfants to a dessin D, denoted by D^*, is the dessin with the same surface as D, whose vertices are the centers of the faces of D. It is said that vertices C, C' of D^* are incident to the same edge if and only if the corresponding faces are incident to the same edge and C, C' are connected by the same number of common edges as many common edges have the corresponding faces. These edges are embedded into the surface in such a way that any edge of D intersects exactly one edge of D^* and any edge of D^* intersects exactly one edge of D.

Example 4.7. Below we give an example of a pair of dual dessins:

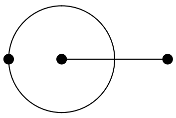

Figure 3.

Remark 4.8. It is straightforward to check that $(D^*)^* = D$.

This notion is vital in connection with triangularization of a surface and in connection with the bigraphs and Euler bridge problem, see [8].

For parasitic solutions corresponding to dual dessins we have the following classification result:

Theorem 4.9. *Let* $Val := \langle a_1, \ldots, a_\alpha | c_1, \ldots, c_\gamma \rangle$ *be a certain combinatorial type of a non-bicolored dessin (so we do not have white vertices, hence, their valencies as well). The system* $\mathcal{S}(Val)$ *has a parasitic solution if and only if the corresponding system for the dual dessin has a parasitic solution. Moreover, a parasitic solution of the first system is geometrical if and only if the corresponding parasitic solution of the second system is geometrical.*

Proof. Note that the combinatorial type of the dual dessin is $Val^* = \langle c_1, \ldots, c_\gamma | a_1, \ldots, a_\alpha \rangle$. Also if β is a Belyi function for a dessin D then there is a normalization under which the Belyi function for D^* is $\frac{1}{\beta}$, see [17]. The result follows now from the fact that $\mathcal{S}(Val)$ is homogeneous. ∎

5 Series of graphs with geometrical parasitic solutions

Theorem 5.1. *Let*

$$Val := \langle a_1, \ldots, a_\varphi, a, \underbrace{2, \ldots, 2}_{2\mu} | w_1, \ldots, w_\psi, \underbrace{2, \ldots, 2}_{2\mu+2} | c_1 \ldots, c_\xi, 2\mu + 2 \rangle$$

be a combinatorial type of a certain genus-0 dessin D *having the fragment*

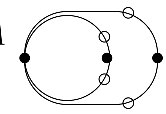

Figure 4.

with the valency of the vertex in the point A *is equal to* $a \geq 5$ *and such that each of two loops in this fragment contains* μ *black points of valency 2 and* μ *white points of valency 2 and the valency of the face between these two loops is equal to* $2\mu + 2$*, which means that there is no other fragments of the* *dessin inside this face. Then the system $\mathcal{S}_0(Val)$ has a geometrical parasitic solution σ corresponding to the dessin D_σ with the combinatorial type*

$$Val_\sigma := \langle a_1, \ldots, a_\varphi, a-2, \underbrace{2, \ldots, 2}_{\mu} | w_1, \ldots, w_\psi, \underbrace{2, \ldots, 2}_{\mu+1} | c_1 \ldots, c_\xi \rangle$$

which can be obtained from D by the deleting of the face of valency $(2\mu+2)$.

Proof. The system $\mathcal{S}(Val)$ is given by the following formal equality of polynomials

$$\begin{array}{c} -K(z-C_1)^{c_1} \cdots (z-C_\xi)^{c_\xi}(z-C)^{2\mu+2}+ \\ +(z-A_1)^{a_1} \cdots (z-A_\varphi)^{a_\varphi}(z-A)^a(z-M_1)^2 \cdots (z-M_{2\mu})^2 = \\ = (K-1)(z-W_1)^{w_1} \cdots (z-W_\psi)^{w_\psi}(z-N_1)^2 \cdots (z-N_{2\mu+2})^2 \end{array} \tag{5.1}$$

here the letters C_i denote the complex numbers giving the coordinates of the centers of faces of the dessin D, the letters A_i, M_i denote the coordinates of the black vertices, and W_i, N_i denote the coordinates of the white vertices of D.

Let us consider now the dessin d'enfant D_σ which is essentially the same as D but instead of the part on the Figure 4 it contains the following one loop:

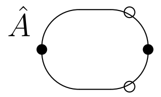

Figure 5.

where the valency of the vertex in point $\hat{A}$ is $(a-2)$ and there are μ black and $\mu+1$ white vertices of valency 2 on the loop. It is easy to check that the combinatorial type of D_σ is equal to Val_σ. Thus there exist complex numbers

$$\hat{K}; \hat{A}_1, \ldots, \hat{A}_\varphi, \hat{A}, \hat{M}_1, \ldots, \hat{M}_\mu; \hat{C}_1, \ldots, \hat{C}_\xi \in \mathbb{CP}^1,$$

such that the rational function

$$\sigma(z) = \hat{K} \frac{(z-\hat{A}_1)^{a_1} \cdots (z-\hat{A}_\varphi)^{a_\varphi}(z-\hat{A})^{a-2}(z-\hat{M}_1)^2 \cdots (z-\hat{M}_\mu)^2}{(z-\hat{C}_1)^{c_1} \cdots (z-\hat{C}_\xi)^{c_\xi}}$$

is a Belyi function for D_σ. Thus there exist complex numbers

$$\hat{W}_1, \ldots, \hat{W}_\psi, \hat{N}_1, \ldots, \hat{N}_{\mu+1} \in \mathbb{CP}^1$$

such that the following equality holds for all $z \in \mathbb{CP}^1$:

$$\sigma(z) - 1 = (\hat{K}-1) \frac{(z-\hat{W}_1)^{w_1} \cdots (z-\hat{W}_\psi)^{w_\psi}(z-\hat{N}_1)^2 \cdots (z-\hat{N}_{\mu+1})^2}{(z-\hat{C}_1)^{c_1} \cdots (z-\hat{C}_\xi)^{c_\xi}}. \tag{5.2}$$

Let us check that the values

$$\begin{gathered} K=\hat{K}; A_1=\hat{A}_1,\dots,A_\varphi=\hat{A}_\varphi, A=\hat{A},\\ M_1=\hat{M}_1,\dots,M_\mu=\hat{M}_\mu, M_{\mu+1}=\hat{A},\dots,M_{2\mu}=\hat{A};\\ C_1=\hat{C}_1,\dots,C_\psi=\hat{C}_\psi, C=\hat{A};\\ W_1=\hat{W}_1,\dots,W_\psi=\hat{W}_\psi;\\ N_1=\hat{N}_1,\dots,N_{\mu+1}=\hat{N}_{\mu+1}, N_{\mu+2}=\hat{A},\dots,N_{2\mu+2}=\hat{A} \end{gathered} \tag{5.3}$$

satisfy the system of equations determined by (5.1). Indeed, it follows from (5.2) that the following formal polynomial equality holds:

$$\begin{gathered} -\hat{K}(z-\hat{C}_1)^{c_1}\cdots(z-\hat{C}_\psi)^{c_\psi}+\\ +(z-\hat{A}_1)^{a_1}\cdots(z-\hat{A}_\varphi)^{a_\varphi}(z-\hat{A})^{a-2}(z-\hat{M}_1)^2\cdots(z-\hat{M}_\mu)^2=\\ =(\hat{K}-1)(z-\hat{W}_1)^{w_1}\cdots(z-\hat{W}_\psi)^{w_\psi}(z-N_1)^2\cdots(z-N_{\mu+1})^2. \end{gathered} \tag{5.4}$$

Multiplying this with $(z-\hat{A})^{2\mu+2}$ on both sides we get the equality for any $z\in\mathbb{C}$. However, it is easy to see that by substituting the values (5.3) into the equation (5.1) we obtain (5.4) multiplied by $(z-\hat{A})^{2\mu+2}$ on both sides, i.e., the equality. Thus the sequence (5.3) satisfy the system $\mathcal{S}_0(Val)$. The fact that this solution is geometrical and satisfy the conditions of the theorem follows directly from its construction. ∎

Let us note that there are infinitely many combinatorial types satisfying this theorem.

Corollary 5.2. *Let $D_{\lambda\mu}$ be a genus-0 dessin d'enfants of the following form:*

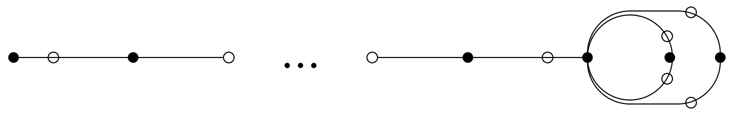

Figure 6.

which consists of the handle with λ white and λ black vertices of valency 2, and two loops with the common vertex and μ white and μ black vertices of valency 2 on each of the loops. For all $\lambda\ge 0$ and $\mu\ge 0$ the system $\mathcal{S}_0(Val)$, where

$$Val=\langle 5,\underbrace{2,\dots,2}_{\lambda+2\mu},1|\underbrace{2,\dots,2}_{\lambda+2\mu+3}|2\lambda+\mu+3,2\mu+2,\mu+1\rangle$$

with the unique realization $D_{\lambda\mu}$ has a geometrical parasitic solution, corresponding to the dessin $\hat{D}_{\lambda\mu}$, which is drown at Figure 7:

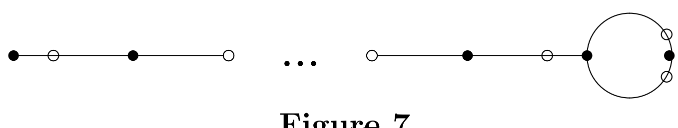

Figure 7.

Remark 5.3. We would like to note that in the above example the coloring does not play a role, in particular, similar result can be obtained if the most left vertex is white as far the vertex of valency 5 is still black.

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