Rigidity theorem for expanding Gradient Ricci Solitons

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Abstract. In this paper, we study the rigidity problem for expanding gradient Ricci soliton equation on a complete conformally compact Riemannian manifold. We show that under a natural condition on the Ricci curvature and the scalar curvature, the expanding Ricci soliton is Poincare-Einstein.

1. Introduction

Ricci solitons are important objects in the understanding the geometric structure of manifolds (see [25], [35], [27],[31], [11], [12], and [32]). We study the uniformization problem of conformally compact expanding Ricci solitons with non-positive Ricci curvature. Let (M, g) be a Riemannian manifold of dimension $n > 2$. Let Rc be the Ricci tensor of the metric g. The equation for a homothetic Ricci soliton is

$$Rc = cg + L_V g$$

where c is a homothetic constant, V is a smooth vector field on X, and $L_V g$ is the Lie derivative of the metric g. When $c = 0$, the soliton is steady. For $c > 0$ the soliton is shrinking, and one can consider the Ricci flow on the sphere as such an example. For $c < 0$ the soliton is expanding. When V is the gradient of a smooth function, we call such solitons *Gradient Homothetic Ricci Solitons*. In this case, we write the equation as

$$Rc = cg + D^2 f, \tag{1}$$

where f is a smooth function on M. Let s be the scalar curvature of g.

Our intention of this paper is to show that under some natural conditions, a conformally compact expanding Ricci soliton is Poincare-Einstein metric in the sense of Fefferman and Graham [19].

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Here we recall the concept of Poincare-Einstein metric. Following C.Fefferman and C.R.Graham ([19] and [20]), we introduce the Poincare metric on conformally compact manifolds as follows. A Riemannian metric g in the interior of manifold M^n with boundary ∂M is said to be conformally compact if $\bar{g} = r^2 g$ extends smoothly as a metric to the closure $\bar{M}$, where r is a defining function for ∂M in the sense that $r > 0$ in M and $r = 0$,$dr \neq 0$ on ∂M. It is called the *Poincare metric* if $|dr|^2 = 1$ on $r = 0$. It is shown in [20] that the metric can written as

$$g = r^{-2}\bar{g} = r^{-2}(dr^2 + g_r).$$

If the conformally compact Poincare metric satisfies that

$$Rc(g) = -(n-1)g,$$

we say g a *Poincare-Einstein metric* on M. It is shown in [22] that Poincare-Einstein metrics are rich. See also M.Anderson's work [4]. For more material about Einstein metrics, one may see the beautiful book of A.Besse [8].

Our main result is the following rigidity result.

Theorem 1. *Let (M, g) be an expanding gradient Ricci soliton so that there is a smooth function f satisfying (1). Assume that the metric g is conformally compact Poincare metric with non-positive Ricci curvature and its scalar curvature function s no less that -n(n-1) everywhere on M. Then f is a constant and g is a Poincare-Einstein on M.*

Let's see why our assumptions are nature. Based on a beautiful observation and the argument of E.Witten [39], Min-Oo [33] proved a scalar curvature rigidity theorem on spin manifolds which are asymptotic to the hyperbolic space in a strong sense. Later, Anderson and Dahl [1] refined Min-Oo's method. However, their assumptions are still more restrictive. Xiaodong Wang [38] (see also P.Chrusciel and M.Herslich's work [15]) extended their works by introducing a scalar invariant provided the boundaries of the conformally compact manifolds are spheres. Wang's theorem is still for spin manifolds. Motivated by Schoen and Yau's positive mass theorem [37], people believe that there are should be some scalar curvature rigidity results for general asymptotically hyperbolic manifolds. This is clearly open if one can follow Schoen-Yau's argument since the minimal surfaces in this case are non-compact even though there is such a nice minimal surface theory developed by M.Anderson [2] (see also F.H.Lin [28]). Interestingly, Listing [29] made an attempt in another direction by using the Bochner type argument like E.Witten did in [39]. He proved a sectional curvature rigidity of asymptocally locally hyperbolic manifolds.

In [36], J.Qing proved a non-spin rigidity. In all these works, people used the conformally compact and scalar curvature lower bound conditions. Conformally compact conditions have been used to develop the Hodge theory on complete Riemannian manifolds. One may see R.Mazzeo's work [30]. Of course, we may try to use invariants to understand complete conformally compact manifolds. This is exactly the beautiful idea from C.Fefferman and Graham [19]. In [21], C.R.Graham computed the volume and area renormalizations for conformally compact Einstein metrics. In [3], M.Anderson used Chern-Gauss-Bonnet theorem and a L^2 conformal curvature condition to express the Euler number of an AHE manifolds of dimension four. It is should be very interesting to relate metric invariants defined by M.Gromov in [24] to complete non-compact Ricci solitons. Actually one hope that Ricci flow may also preserve some Mass invariants. As we showed in [17], the Ricci flow does preserved the mass in locally asymptotic flat manifolds. In [26], N.Hitchin computed Euler number for ALE manifolds of dimension four. The study of geometry and topology of open 4-manifolds is still a new field ([16], [23], and [18]).

As we mentioned at the very beginning, people want to classify Ricci solitons. Then one may use the Ricci flow or mean curvature flow method to attack geometric rigidity problems. The interesting new problem arises. That is that one needs to prove the uniformization theorems for Ricci solitons of expanding type. Clearly, it is also interesting to study this kind problem for steady solitons which are asymptotic flat manifolds. We made a small progress in our work [31] by using a result done by S.Bando, A.Kasue, and H.Nakajima [6]. Our main result in [31] says that if (M^n, g) (n¿2) is a complete non-compact gradient Ricci soliton which is quasi-isometric to Euclidean space at infinity such that $\int_M |Rc|^{n/2} < +\infty$, then (M, g) is Ricci-flat and ALE of order $n-1$. We believe that this result (see Theorem 3 in [31]) can be improved by using the weighted Sobolev spaces method. For compact gradient shrinking Ricci solitons, R.Hamilton made a conjecture that such a soliton with positive curvature operator must be Einstein. One may see the recent work of Xiaodong Cao [14] for interesting identities in this area.

Kahler-Ricci solitons are special cases from Ricci solitons. It is a generally belief that Kahler-Ricci solitons are more rigid. In fact, there are also many interesting results on uniformization problems on Kahler-Ricci solitons. The existence results of Kahler-Ricci solitons was first discovered by Koiso and H.D.Cao [14]. In [10], Chau and Tam studied the uniformization theorem for gradient Kahler-Ricci solitons. Their result says that if (M^n, g) is a complete non-compact gradient Kahler-Ricci soliton of complex dimension n, which is either steady with positive Ricci curvature so that the sacalar curvature attains its maximum at some point, or expanding with non-negative Ricci curvature, then M^n is biholomorphic ro C^n. Notice that the assumptions of this result are almost opposite to our main result above. In [9], R.Bryant found more results for Kahler-Ricci solitons extending a result of Cao-Hamilton [13]. L.Ni also found interesting results for Kahler-Ricci solitons [34]. For the Kahler-Ricci solitons, one may improve our theorem above.

2. Some facts about expanding Ricci solitons

In local coordinates (x^i) of the Riemannian manifold (M, g), we can write the metric

$$g = g_{ij}dx^i dx^j.$$

We let $(g^{ij}) = (g_{ij})^{-1}$ the inverse matrix of (g_{ij}). The corresponding Riemannian curvature tensor and Ricci tensor are denoted by $Rm = (R_{ijkl})$ and $Rc = (R_{ij})$ respectively. Hence,

$$R_{ij} = g^{kl}R_{ikjl}$$

and

$$s = g^{ij}R_{ij}.$$

We write the covariant derivative of a smooth function f by $Df = (f_i)$, and denote the Hessian matrix of the function f by $D^2 f = (f_{ij})$, where D the covariant derivative of g on M. The higher order covariant derivatives are denoted by f_{ijk}, etc. Similarly, we use the $T_{ij,k}$ to denote the covariant derivative of the tensor (T_{ij}). We write $T^i_j = g^{ik}T_{jk}$. Then the expanding Ricci soliton equation is

$$R_{ij} = f_{ij} + cg_{ij},$$

where $c < 0$. Taking the trace we find that

$$s = \Delta f + nc, \tag{2}$$

where Δ is the Laplacian of the metric g.

We now do computation in normal coordinates at some point p. Taking covariant derivative we get

$$f_{ijk} = R_{ij,k}.$$

So we have

$$f_{ijk} - f_{ikj} = R_{ij,k} - R_{ik,j}.$$

By the Ricci formula we have that

$$f_{ijk} - f_{ikj} = R^l_{ijk}f_l.$$

Hence we obtain that

$$R_{ij,k} - R_{ik,j} = R^l_{ijk} f_l.$$

Recall that the contracted Bianchi identity is

$$R_{ij,j} = \frac{1}{2} s_i.$$

Upon taking the trace of the previous equation we get that

$$\frac{1}{2} s_i + R^k_i f_k = 0,$$

i.e.,

$$s_k = -2R^j_k f_j. \tag{3}$$

Then

$$D_k(|Df|^2 + s + 2cf) = 2f_j(f_{jk} + cg_{jk} - R_{jk}) = 0.$$

So, $|Df|^2+s+2cf$ is a constant, which is denoted by A. In other term, we have

$$s + |\nabla f|^2 + 2cf = A. \tag{4}$$

We now formulate some well-known formulae for conformal metrics (see [**?**] and [8]). Let $\bar{s}$ be the scalar curvature and let $\bar{\nabla}$ the covariant derivative of $\bar{g} = r^2 g$ respectively. Set

$$u = r^{-(n-2)/2}$$

Then $g = u^{4/(n-2)}\bar{g}$ and

$$s = u^{-\frac{n+2}{n-2}}(-\frac{4(n-1)}{n-2}\bar{\Delta}u + \bar{s}), \tag{5}$$

where $\bar{\Delta}$ is the Lapacian of the metric $\bar{g}$ (with the sign $\Delta = d^2/dx^2$ on the real line $\mathbf{R}$). By direct computation, we have from (5) that

$$s = 2(n-1)r\bar{\Delta}r - n(n-1)|\bar{\nabla}r|^2 + \bar{s}r^{(n+2)/2}, \tag{6}$$

which implies that

$$s = -n(n-1), \tag{7}$$

at $r = 0$ since $|\bar{\nabla}r|^2 = 1$.

By an elementary computation we have

$$\Delta f = (2-n) < \bar{\nabla}r, \bar{\nabla}f > +r^2\bar{\Delta}f. \tag{8}$$

3. Proof of main result

We now prove our main theorem. By our assumption, we can extend the function f to the boundary $r = 0$. We may assume that $f = 0$ at some point x_0 on $r = 0$. We first determine the constants c and A. Combining (8), (2) with (6), we find that

$$(2-n) < \bar{\nabla} r, \bar{\nabla} f > + r^2 \bar{\Delta} f + nc = s.$$

Using (7), we obtain that

$$c = -(n-1).$$

Using (4) and (7) again, and the fact $f = 0$ at x_0 on the boundary $r = 0$, we have that

$$A = -n(n-1).$$

We now argue by contradiction. We assume that f is not a constant. Otherwise, we are done. Recall that

$$\Delta f = s + n(n-1) \geq 0, on \qquad M$$

that is to say, f is a subharmonic function on M. By the strong Maximum Principle ([5]) we know that f can not attain its maximum in the interior of M. So we have

$$f \leq 0, on \qquad M.$$

We now consider any integral curve $\gamma(t)$ of the gradient vector fields ∇f near the conformal boundary $r = 0$ such that $\gamma(t) \to \partial M$ as $t \to +\infty$. Recall our assumption that $Rc \leq 0$ near $r = 0$. Using (3), we have that

$$\frac{d}{dt} s(\gamma(t)) = ds(\nabla f) = -2Rc(\nabla f, \nabla f) \geq 0.$$

That is to say that s is non-decreasing along $\gamma(t)$ as $t \to +\infty$. However, by our assumption that $s \geq -n(n-1)$, we should have $s = -n(n-1)$ on $\gamma(t)$ for any t. Using the strong Maximum principle again we must that $s = -n(n-1)$ on M. This implies that f is a harmonic function on M with boundary condition $f = 0$ at $r = 0$. Hence, $f = 0$ on M. This is absurd. Therefore, g is a Poincare-Einstein metric. This proves our main result.

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