Remarks on compact shrinking Ricci solitons of dimension four

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Abstract. In this paper, we study the topological restriction of gradient shrinking Ricci solitons (M, g) of dimension 4. Let s be the scalar curvature of the metric g. Then we have

$$\int_M s dv_g = 4\rho vol(M),$$

where $\rho > 0$ is the shrinking constant and $vol(M)$ is the volume of (M, g). We also have two kinds of topology results. 1). If we assume that

$$\int_M s^2 \leq 24\rho^2 vol(M),$$

then

$$2\chi(M) \pm 3\tau(M) \geq 0.$$

2). If (M, g) is a natural oriented Kahler surface, then we have

$$2\chi(M) + 3\tau(M) = \frac{\rho^2 vol(M)}{2\pi^2}.$$

Actually, we shall show that the assumption in 1) above is equivalent to the fact that

$$\int_M \sigma_2(Rc - \frac{s}{6}g) \geq 0.$$

Here $\sigma_2(A)$ is the 2nd symmetric function of the eigenvalues of the matrix $A := Rc - \frac{s}{6}g$.

1. Introduction

We study the topological restriction of gradient shrinking Ricci solitons (M, g) of dimension 4. Roughly speaking, Ricci solitons are self-similar solutions to Ricci flow. Special examples of Ricci solitons are Einstein metrics, which are fixed points of Ricci flow. One of the most

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important topic in Ricci flow is to classify the Ricci solitons in dimension four. Just like in Einstein metric case, we can expect that there are some topological restrictions to Ricci solitons and there is some type of Hitchin-Thorpe inequality for Ricci solitons. Our aim here is to show this true for some cases. We write by s the scalar curvature of the metric g. We point out here that the dimension four case is special that we do have compact non-trivial Ricci solitons like Cao and Koiso solitons, see [6]. In dimension two and three, there is no compact non-trivial Ricci solitons, and we have non-compact Ricci solitons like Bryant soliton in dimension three and cigar soliton of R.Hamilton (also called Witten black hole) in dimension two, see [5]. We show that

Main Theorem . *Let (M,g) be a gradient shrinking Ricci soliton of dimension 4 with shrinking constant $\rho > 0$. Then we have*

$$\int_M s dv_g = 4\rho vol(M).$$

We have the following two topology results. 1). If we assume that

$$\int_M s^2 \leq 24\rho^2 vol(M), \tag{1}$$

then

$$2\chi(M) \pm 3\tau(M) \geq 0.$$

2). If (M,g) is a natural oriented Kahler surface, then we have

$$2\chi(M) + 3\tau(M) = \frac{\rho^2 vol(M)}{2\pi^2}.$$

Here the word "natural oriented" means that the Kahler form is self-dual.

This result is a generalization of Hitchin-Thorpe inequality (see A.Besse's book [4]) for Einstein metrics. We basically show that the assumption in 1) above is equivalent to the fact that

$$\int_M \sigma_2(Rc - \frac{s}{6}g) \geq 0.$$

Here $\sigma_2(A)$ is the 2nd symmetric function of the eigenvalues of the matrix $A := Rc - \frac{s}{6}g$. Then the result follows from Chern-Gauss-Bonnet Theorem. There is a large room to be done for more results. For example, it is expected that there are some close relationship between Einstein metrics and Ricci solitons on compact Kahler manifolds. One also likes to see some applications of Ricci-Kahler solitons to algebraic geometry [20])or conformal field theory. Using the Seiberg-Witten invariant method of C.Lebrun [16], one may sharpen the inequality above for

Ricci solitons. One may also prove some results as done by N.Hitchin [10] and M.Anderson [1] for complete manifolds with asymptotic flat or hyperbolic geometry. There should be a nice relationship between Ricci solitons and Yamabe constants, see the work of M.Gursky and C.Lebrun [13] for the Einstein metric cases.

2. PRELIMINARY

We recall first some basic properties about Ricci solitons [15].

Given a compact Riemannian manifold (M^n, g) of dimension $n \geq 3$. We say that (M, g) is a gradient shrinking Ricci soliton if for the metric g, there is a smooth function $f$and a constant $\rho > 0$ such that

$$Rc = \rho g + D^2 f, \tag{2}$$

where Rc is the Ricci curvature of the metric g, and $D^2 f$ is the Hessian of the potential function f on (M, g). We normalize f such that

$$\int_M f dv = 0,$$

where dv is the volume element of the metric g.

Taking the trace of both sides of (2), we have

$$s = n\rho + \Delta f. \tag{3}$$

Then we have

$$\int_M s = n\rho vol(M) > 0,$$

and

$$\int_M s^2 = n^2\rho^2 vol(M) + \int_M (\Delta f)^2.$$

Hence, the Yamabe constant in the conformal class of the metric g is positive, and the scalar curvature s is positive somewhere. Using the maximum principle, one has (see Proposition 1 in [11]) that the scalar curvature is positive everywhere in M.

Take a point $x \in X$. In local normal coordinates (x^i) of the Riemannian manifold (X, g) at a point x, we write the metric g as (g_{ij}). The corresponding Riemannian curvature tensor and Ricci tensor are denoted by $Rm = (R_{ijkl})$ and $Rc = (R_{ij})$ respectively. Hence,

$$R_{ij} = g^{kl} R_{ikjl}$$

and

$$s = g^{ij} R_{ij}.$$

We write the covariant derivative of a smooth function f by $Df = (f_i)$, and denote the Hessian matrix of the function f by $D^2 f = (f_{ij})$, where D the covariant derivative of g on M. The higher order covariant derivatives are denoted by f_{ijk}, etc. Similarly, we use the $T_{ij,k}$ to denote the covariant derivative of the tensor (T_{ij}). We write $T^i_j = g^{ik}T_{jk}$. Then the Ricci soliton equation is

$$R_{ij} = f_{ij} + \rho g_{ij}.$$

Taking covariant derivative, we get

$$f_{ijk} = R_{ij,k}.$$

So we have

$$f_{ijk} - f_{ikj} = R_{ij,k} - R_{ik,j}.$$

By the Ricci formula we have that

$$f_{ijk} - f_{ikj} = R^l_{ijk} f_l.$$

Hence we obtain that

$$R_{ij,k} - R_{ik,j} = R^l_{ijk} f_l.$$

Recall that the contracted Bianchi identity is

$$R_{ij,j} = \frac{1}{2} s_i.$$

Upon taking the trace of the previous equation we get that

$$\frac{1}{2} s_i + R^k_i f_k = 0,$$

i.e.,

$$s_k = -2R^j_k f_j. \tag{4}$$

Then at x,

$$D_k(|Df|^2 + R + 2\rho f) = 2f_j(f_{jk} - R_{jk} + 2cg_{jk}) = 0.$$

So,

$$|Df|^2 + s + 2\rho f = \Lambda, \tag{5}$$

where Λ is a constant. We normalize f such that

$$\int_M f = 0.$$

Then we have

$$\Lambda vol(M) = \int_M |Df|^2 + \int_M s = \int_M |Df|^2 + n\rho vol(M).$$

This gives us that

$$\int_M |Df|^2 = (\Lambda - n\rho) vol(M). \tag{6}$$

This implies that $\Lambda \geq n\rho$.

3. Basic facts on 4-dimensional geometry

In this section, we assume that $n = 4$. The basic references for the four dimensional geometry are the papers of Atiyah, Hitchin and Singer [2], and the books of A.Besse [4], Freed and Uhlenbeck [12] and Donaldson and Kronheimer [9]. Let W be the Weyl tensor, which is a conformal invariant, and let A be the Weyl-Schouten tensor

$$A = Rc - \frac{s}{2(n-1)}g.$$

Then

$$Rm = W \bigoplus \frac{1}{n-2} A \bigodot g.$$

Here $\bigodot$ is the Kulkarni-Nomizu product (see [[4], 1.110]). Define $\sigma_2(A)$ the 2nd symmetric function of the eigenvalues of the matrix A. Then we have

$$\sigma_2(A) = \frac{1}{2}(|tr(A)|^2 - |A|^2) = \frac{1}{6}s^2 - \frac{1}{2}|Rc|^2.$$

Assume that $n = 4$. Then by using the Chern-Gauss-Bonnet formula, the Euler number of M is given by the formula:

$$8\pi^2\chi(M) = \int_M (\sigma_2(A) + |W|^2)dv.$$

By this formula, we see that the integral $\int_M \sigma_2(A)$ is a conformal invariant.

Using the Hodge star operator we can write by W into self-dual and anti-self-dual part:

$$W = W_+ + W_-.$$

Then we have the following Hirzebruch formula [4] for signature of the manifold M:

$$12\pi^2\tau(M) = \int_M (|W_+|^2 - |W_-|^2)dv.$$

Therefore, we have

$$2\chi(M) \pm 3\tau(M) = \frac{1}{4\pi^2}\int_M (\sigma_2(A) + 2|W_\pm|^2)dv_g.$$

We make two remarks below. One remark is that people often write the Chern-Gauss-Bonnet formula in dimension 4 (see [4]) as

$$8\pi^2\chi(M) = \int_M (\frac{s^2}{24} + |W|^2 - \frac{|B|^2}{2})dv,$$

where B is the trace-free part of the Ricci curvature Rc. For our gradient shrinking Ricci soliton, we then have, by an elementary computation that

$$\int_M |B|^2 = \frac{1}{4}\int_M |\Delta f|^2.$$

The other remark is that the assumption (1) is equivalent to

$$\int_M (\Delta f)^2 \leq 8\rho^2 vol(M).$$

4. Proof of Main Theorem

Assume that (M, g) is a gradient shrinking Ricci soliton of dimension four. Then we have

$$\begin{aligned}\int_M |Rc|^2 &= \int_M |D^2 f + \rho g|^2 \\ &= \int_M (|D^2 f|^2 + 2\rho\Delta f) + 4\rho^2 vol(M) \\ &= \int_M |D^2 f|^2 + 4\rho^2 vol(M).\end{aligned}$$

Using integration by part, (5),(6), and the Ricci formula, we have

$$\begin{aligned}\int_M |D^2 f|^2 &= -\int_M f_{iji} f_j \\ &= -\int_M ((\Delta f)_j f_j + R_{ij} f_i f_j) \\ &= \int_M (\Delta f)^2 - \int_M (f_{ij} f_i f_j + \rho |Df|^2) \\ &= \int_M (\Delta f)^2 + \frac{1}{2}\int_M |Df|^2 \Delta f - \int_M \rho |Df|^2 \\ &= \int_M (\Delta f)^2 + \frac{1}{2}\int_M (\Lambda - s - 2\rho f)\Delta f - \int_M \rho |Df|^2 \\ &= \int_M (\Delta f)^2 + \frac{1}{2}\int_M (4\rho - s)\Delta f \\ &= \frac{1}{2}\int_M (\Delta f)^2 \\ &= \frac{1}{2}\int_M (s - 4\rho)^2 \\ &= \frac{1}{2}\int_M s^2 - 8\rho^2 vol(M).\end{aligned}$$

This also implies that

$$\int_M s^2 \geq 16\rho^2 vol(M).$$

Then,

$$\int_M |Rc|^2 = \frac{1}{2}\int_M s^2 - 4\rho^2 vol(M).$$

Hence, by

$$\sigma_2(A) = \frac{1}{6}s^2 - \frac{1}{2}|Rc|^2,$$

we have

$$\int_M \sigma_2(A) = -\frac{1}{12}\int_M s^2 + 2\rho^2 vol(M). \tag{7}$$

Recall our assumption (1) is

$$\int_M s^2 \leq 24\rho^2 vol(M).$$

From the computation above, we have the following inequality

$$\int_M \sigma_2(A) \geq 0.$$

Then we have

$$2\chi(M) \pm 3\tau(M) \geq \frac{1}{4\pi^2}\int_M \sigma_2(A) \geq 0.$$

Assume that (M^4, g) is a natural oriented Kahler surface. Then it is well-known (see Proposition 16.62 in [4] or see [8]) that

$$|W_+|^2 = \frac{1}{24}s^2.$$

Hence, we have

$$2\chi(M) + 3\tau(M) = \frac{1}{4\pi^2}\int_M (\sigma_2(A) + \frac{1}{12}s^2).$$

It is clear from the formulae above that if (M, g) is a natural oriented Kahler surface, then we have by (7)

$$2\chi(M) + 3\tau(M) = \frac{\rho^2 vol(M)}{2\pi^2}.$$

This completes the proof of the Main Theorem.

We remark that it is interesting to know which closed manifolds admit Riemannian metrics with inequality

$$\int_M \sigma_2(A) dv \geq 0.$$

One may see the work of A.Chang, P.Yang, and M.Gurskey [7] for related topic.

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