Open-closed field algebras, operads and tensor categories

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Abstract

We introduce the notions of open-closed field algebra and open-closed field algebra over a vertex operator algebra V. In the case that V satisfies certain finiteness and reductivity conditions, we show that an open-closed field algebra over V canonically gives an algebra over a $\mathbb{C}$-extension of the Swiss-cheese partial operad. We also give a tensor categorical formulation and categorical constructions of open-closed field algebras over V.

Contents

0 Introduction

Open-closed (or boundary) conformal field theories were first developed in physics by Cardy [C1]-[C4][CL] soon after the fundamental work of Belavin, Polyakov and Zamolodchikov [BPZ] on conformal field theories appeared. They describe certain critical phenomena on surfaces with boundaries in condensed matter physics. They are also powerful tools for the study of the still mysterious objects called "D-branes" in string theory [Po1][Po2].

By generalizing Kontsevich and Segal's celebrated definition of conformal field theory [S1][S2][1], Huang [H7] and Hu and Kriz [HKr2] gave mathematical definitions of open-closed conformal field theory. Roughly speaking, an open-closed conformal field theory is a monoidal functor between two monoidal categories. The domain category has objects being ordered sets of finitely many copies of S^1 (a circle) and $[0,1]$ (an interval), and has morphisms being conformal equivalent classes of Riemann surfaces with some oriented, ordered and parametrized boundary components and oriented, ordered and parametrized line segments of other boundary components. The target category is the monoidal category of Hilbert spaces with morphisms being the projective spaces of the spaces of multilinear continuous maps.

When the monoidal functor only depends on the topological structures (instead of the conformal structures) of Riemann surfaces, it is called 2-dimensional open-closed topological field theory, a study of which was carried out by Lazaroiu [La] and Moore and Segal [Mo1][Mo2][S3][MS]. They proved the following Theorem:

Theorem 0.1. *A reduced 2-dimensional open-closed topological field theories is equivalent to a finite dimensional commutative Frobenius algebra H_{cl} and a finite dimensional Frobenius algebra H_{op} together with an algebra map $\iota_*: H_{cl} \to Z(H_{op})$, where $Z(H_{op})$ is the center of H_{op}, satisfying an additional condition called Cardy condition.*

A systematic study of open-closed conformal field theories is much more difficult. In [HKo1][HKo2][Ko1], Huang and the author carried out a few first steps in this direction, based on the theory of vertex operator algebra [B1][FLM][FHL][FB][LL], the construction of genus-zero and genus-one chiral closed conformal field theories by Huang [H1]-[H13] and the tensor category theory from vertex operator algebras developed by Huang and Lepowsky [HL2]-[HL6][H3][H10]. In particular, in [HKo1], Huang and the author studied the pure open-string part of genus-zero open-closed conformal field theory, in terms of the so-called open-string vertex operator algebra. In [HKo2][Ko1][HKo3], we studied the pure closed-string part of genus-zero and genus-one open-closed conformal field theory, in terms of conformal full field algebra.

In this work, we will study interactions between closed strings and open strings in genus-zero open-closed conformal field theories. More precisely, we will study interactions that fuse arbitrary number of in-coming open strings with arbitrary number of in-coming closed strings into a single out-going open string, as shown in Figure 1. Other situations in which more than one outgoing open strings and an open-string loop appear will be studied in [Ko2].

In [V], Voronov introduced the so-called "Swiss-cheese operad" to describe the interactions among open strings and closed strings depicted in Figure 1. In order to incorporate the full conformal symmetry which is not included in Voronov's Swiss-cheese operad, Huang and the author introduce a notion called Swiss-cheese partial operad in [HKo1]. The Swiss-cheese partial operad consists of disks with strips and tubes, which

[1]It is further rigorized by Hu and Kriz in [HKr1].

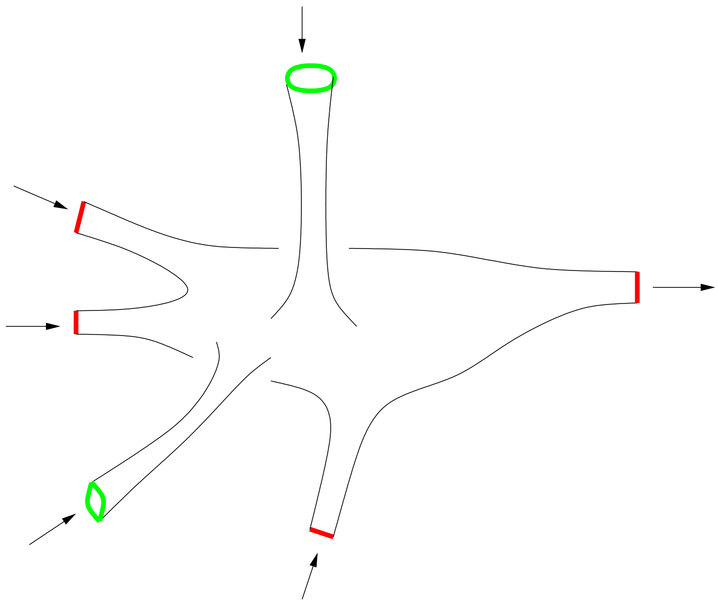

Figure 1: open strings interact with closed strings

are conformal equivalent classes of disk with oriented punctures in the interior of the disk (tubes) and on the boundary of the disk (strips), together with local coordinate map around each puncture.

On the moduli space of disks with strips and tubes, which is denoted as $\mathfrak{S}$, there are two kinds of sewing operations. One is called boundary sewing operation, which sews two elements in $\mathfrak{S}$ along two strips with opposite orientations. The other one is called interior sewing operation, which sews a disk with strips and tubes with a sphere with tubes [H4] along two oppositely oriented tubes.

Disks without interior punctures are closed under boundary sewing operations. This closed structure is nothing but the operad of disks with strips, which is introduced and studied in [HKo1] and denoted as Υ. In [HKo1], Huang and the author showed that open-string vertex operator algebras of central charge c are algebras over $\tilde{\Upsilon}^c$, which is a $\mathbb{C}$-extension of Υ [HKo1]. In an open-closed conformal field theory, an open-string vertex operator algebra, denoted as V_{op}, plays a similar role as that of the associative algebra H_{op} in Theorem 0.1.

The moduli space of spheres with tubes equipped with sewing operations has a structure of partial operad, called sphere partial operad and denoted as K. It is well understood by the works of Huang [H1][H2][H4]. Similar to the study of Υ, we can study $\mathfrak{S}$ in the framework of sphere partial operad K by embedding $\mathfrak{S}$ into K via a doubling map δ [A][C1][HKo1]. Then the interior sewing operations on $\mathfrak{S}$ correspond to a double-sewing "action" of K on the image of δ (see (2.24)). As a consequence, the closed-string theory in an open-closed conformal field theory must contain both chiral part and anti-chiral part. Such closed theories were studied in [HKo2][Ko1][HKo3] in terms of the so-called (conformal) full field algebra and variants of it. More precisely, we showed in [Ko1] that conformal full field algebras over $V^L \otimes V^R$, where V^L and V^R are vertex operator algebras of central charge c^L and c^R respectively and satisfy certain finiteness and reductivity conditions, are algebras over $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$, which is a $\mathbb{C}$-extension of K [HKo1][Ko1]. In an open-closed conformal field theory, a conformal full field algebra, denoted as V_{cl}, plays a similar role as the commutative associative algebra H_{cl} in Theorem 0.1.

We are interested in algebras over $\tilde{\mathfrak{S}}^c$, which is a $\mathbb{C}$-extension of $\mathfrak{S}$. There is a natural action of $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$ on $\tilde{\mathfrak{S}}^c$ induced by interior sewing operations [HKo1]. Therefore, it is natural to expect that an algebra over $\tilde{\mathfrak{S}}^c$ with certain natural properties should contain a conformal full field algebra V_{cl} and an open-string vertex operator algebra V_{op}. Moreover, V_{cl} should "acts" on V_{op} according to the action of $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$ on $\tilde{\mathfrak{S}}^c$. This requires certain compatibilities between V_{cl} and V_{op}. One of the compatibility conditions is the so-called conformal invariant boundary condition. It roughly means that the two vertex operator algebras generated by the left and right Virasoro elements in V_{cl} should match in some way with that generated from the Virasoro element in V_{op}. In this case, we also call it a boundary condition preserving conformal symmetry. In this work, we only study boundary conditions that preserve an enlarged symmetry given by a vertex operator algebra U (see for example [FS1][FS2] and references therein for symmetry-broken situations). Such structure is formalized by a notion called open-closed field algebra over U. We will introduce this notion in Section 2. When U satisfies conditions in Theorem 0.2, we will show that an open-closed field algebra over U canonically gives an algebra over $\tilde{\mathfrak{S}}^c$.

Open-closed field algebras over U are difficult to study in general. The following Theorem proved by Huang in [H8] is important for us.

Theorem 0.2. *Let V be a vertex operator algebra with central charge c satisfying the following conditions:*

1. *Every $\mathbb{C}$-graded generalized V-module is a direct sum of $\mathbb{C}$-graded irreducible V-modules,*
2. *There are only finitely many inequivalent $\mathbb{C}$-graded irreducible V-modules,*
3. *Every $\mathbb{R}$-graded irreducible V-module satisfies the C_1-cofiniteness condition.*

Then the direct sum of all (in-equivalent) irreducible V-modules has a natural structure of intertwining operator algebra and the category of V-modules, denoted as $\mathcal{C}_V$, has a natural structure of vertex tensor category. In particular, $\mathcal{C}_V$ has natural structure of braided tensor category.

Assumption 0.3. In this work, we fix a vertex operator algebra V, which is assumed to satisfy the conditions in Theorem 0.2 without further announcement.

The notion of open-closed field algebra over V can be described by very few data and axioms in the framework of intertwining operator algebra (see Theorem 1.28 for a precise statement). Moreover, such algebra has a very simple categorical formulation. In Section 3, we introduce a notion called open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra. We show that the category of open-closed field algebras over V is isomorphic to the category of open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebras. Once the categorical formulation is known, some categorical constructions are easy to obtain. We discuss some simple categorical constructions in Section 3.

One of recent important developments on open-closed conformal field theories is a series of works [FFFS][FS3][FRS1]-[FRS4] [FjFRS][FFRS] by Felder, Fröhlich, Fuchs, Runkel, Schweigert, Fjelstad, on rational open-closed conformal field theories using the theories of modular tensor category and 3-dimensional topological field theory [RT1][RT2][T][BaKi]. Assuming the existence of the structure of a modular tensor category on the category of modules for a vertex operator algebra and the existence of conformal blocks with monodromies compatible with the modular tensor category and all the necessary convergence properties, they constructed conformal blocks for open-closed conformal field theories of all genus and proved their factorization properties and invariance properties under the actions of mapping class groups. Our approach can be viewed as the complement of theirs. In [HKo1]-[HKo3][Ko1][Ko2] and this work, based on Huang's fundamental works [H1]-[H13], Huang and the author replace above assumptions by some easy-to-varify conditions on vertex operator algebras and construct explicitly the genus-zero and genus-one correlation functions of open-closed conformal field theories. We hope to combine these two approaches together in the future to give a rather complete picture of open-closed conformal field theory.

The layout of this work is as follow. In Section 1, we review some old notions such as open-string vertex (operator) algebra, (conformal) full field algebra and variants of them. We also introduce the notions of open-closed field algebra and open-closed field algebra over U, and study their basic properties. In Section 2, we gives an operadic formulation of open-closed field algebra over V. In particular, we show that an open-closed field algebra over V canonically gives an algebra over $\tilde{\mathfrak{S}}^c$. In Section 3, we discuss a categorical formulation and some categorical constructions of open-closed field algebras over V.

Convention of notations: $\mathbb{N}, \mathbb{Z}, \mathbb{Z}_+, \mathbb{R}, \mathbb{R}_+, \mathbb{C}$ denote the set of natural numbers, integers, positive integers, real numbers, positive real numbers, complex numbers, respectively. Let $\mathbb{H} = \{z \in \mathbb{C} | \mathrm{Im} z > 0\}$, $\overline{\mathbb{H}} = \{z \in \mathbb{C} | \mathrm{Im} z < 0\}$, $\tilde{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}$, $\tilde{\overline{\mathbb{H}}} = \overline{\mathbb{H}} \cup \mathbb{R}$. We use $\hat{\mathbb{R}}, \hat{\mathbb{H}}, \hat{\overline{\mathbb{H}}}$ to denote the one point compactification of $\mathbb{R}$, $\tilde{\mathbb{H}}$, $\tilde{\overline{\mathbb{H}}}$ respectively. The following notations will also be used: $\forall n \in \mathbb{Z}_+$,

$$\Lambda^n := \{(r_1, \dots, r_n) \in \mathbb{R}^n | r_1 > \cdots > r_n \geq 0\}, \tag{0.1}$$

$$M^n_{\mathbb{H}} := \{(z_1, \dots, z_n) \in \mathbb{H}^n | z_i \neq z_j, \text{ for } i, j = 1, \dots, n \text{ and } i \neq j\}, \tag{0.2}$$

$$M^n_{\mathbb{C}} := \{(z_1, \dots, z_n) \in \mathbb{C}^n | z_i \neq z_j, \text{ for } i, j = 1, \dots, n \text{ and } i \neq j\}. \tag{0.3}$$

The ground field is always assumed to be $\mathbb{C}$.

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1 Open-closed field algebras

In Section 1.1, we introduce the notion of boundary field algebra, and recall the definition of open-string vertex (operator) algebra [HKo1]. In Section 1.2, we recall the notion of full field algebra [HKo2] and variants of this notion. In Section 1.3, we introduce the notion of open-closed field algebra. In Section 1.4, we introduce the notion of analytic open-closed field algebra and study its basic properties. In Section 1.5, we introduce the notion of open-closed conformal field algebra over a vertex operator algebra U.

Let G be an abelian group. For any G-graded vector space $F = \oplus_{n\in G} F_{(n)}$ and any $n \in G$, we shall use P_n to denote the projection from F or $\overline{F} = \prod_{n\in G} F_{(n)}$ to $F_{(n)}$. We give F and its graded dual $F' = \oplus_{n\in G} F^*_{(n)}$ the topology induced from the pairing between F and F'. We also give $\mathrm{Hom}(F, \overline{F})$ the topology induced from the linear functionals on $\mathrm{Hom}(F, \overline{F})$ given by $f \mapsto \langle v', f(v)\rangle$ for $f \in \mathrm{Hom}(F, \overline{F})$, $v \in F$ and $v' \in F'$.

For any vector space F and any set S, $F^{\otimes 0} = \mathbb{C}$ and S^0 is an one-point set $\{*\}$.

1.1 Boundary field algebras and open-string vertex algebras

We first introduce a notion called *boundary field algebra*.

Definition 1.1. A *boundary field algebra* is a $\mathbb{R}$-graded vector space V_{op} with grading operator being $\mathbf{d}_{op}$, together with a correlation-function map for each $n \in \mathbb{N}$:

$$\begin{aligned} m_{op}^{(n)} : V_{op}^{\otimes n} \times \Lambda^n &\to \overline{V}_{op} \\ (u_1 \otimes \cdots \otimes u_n, (r_1, \dots, r_n)) &\mapsto m_{op}^{(n)}(u_1, \dots, u_n; r_1, \dots, r_n) \end{aligned}$$

and an operator $D_{op} \in \mathrm{End}\, V_{op}$, satisfying the following axioms:

1. For each $n \in \mathbb{Z}_+$, $m_{op}^{(n)}(u_1, \dots, u_n; r_1, \dots, r_n)$ is linear in $u_1, \dots, u_n$ and smooth in $r_1, \dots, r_n$.

2. $\forall u \in V_{op}$, $m_{op}^{(1)}(u; 0) = u$ and $\mathbf{1}_{op} := m_{op}^{(0)}(1) \in (V_{op})_{(0)}$.

3. *Convergence property*: For $n \in \mathbb{Z}_+, k \in \mathbb{N}$ and $i = 1, \dots, n$, $u_1, \dots, u_n, u_1^{(i)}, \dots, u_k^{(i)} \in V_{op}$, the following series[2]

$$\sum_{s\in\mathbb{R}} m_{op}^{(n)}(u_1, \dots, u_{i-1}, P_s m_{op}^{k}(u_1^{(1)}, \dots, u_k^{(i)}, r_1^{(i)}, \dots, r_k^{(i)}), u_{i+1}, \dots, u_n; r_1, \dots, r_n) \qquad (1.1)$$

converges absolutely, whenever $r_1^{(i)} < |r_i - r_j|$ for all $j \neq i$, to

$$m_{op}^{(n+k-1)}(u_1, \dots, u_{i-1}, u_1^{(i)}, \dots, u_k^{(i)}, u_{i+1}, \dots, u_n; r_1, \dots, r_{i-1}, r_i + r_1^{(i)}, \dots, r_i + r_k^{(i)}, r_{i+1}, \dots, r_n). \qquad (1.2)$$

[2]That the number of nonvanishing terms in the sum is countable is automatically assumed.

4. $\mathbf{d}_{op}$*-bracket property*:

$$e^{a\mathbf{d}_{op}} m_{op}^{(n)}(u_1,\dots,u_n;r_1,\dots,r_n) = m_{op}^{(n)}(e^{a\mathbf{d}_{op}}u_1,\dots,e^{a\mathbf{d}_{op}}u_n;e^a r_1,\dots,e^a r_n). \quad (1.3)$$

for $n\in\mathbb{Z}_+, r_1>\cdots>r_n\geq 0, r\in\mathbb{R}$ and $u_1\dots,u_n\in V_{op}$.

5. D_{op}*-property*: For $u_1,\dots,u_n\in V_{op}$, $r_1>\cdots>r_n\geq 0$ and $r_n+a\geq 0$,

$$e^{aD_{op}} m_{op}^{(n)}(u_1,\dots,u_n;r_1,\dots,r_n) = m_{op}^{(n)}(u_1,\dots,u_n,r_1+a,\dots,r_n+a). \quad (1.4)$$

We denote such a boundary field algebra as $(V_{op}, m_{op}, \mathbf{d}_{op}, D_{op})$. *Homomorphisms*, *isomorphisms* and *subalgebras* of boundary field algebras are defined in the obvious way.

Let the map $Y_{op}: V_{op}^{\otimes(2)}\times\mathbb{R}_+\to\overline{V}_{op}$ be defined by

$$Y_{op}: \ (u\otimes v, r)\mapsto Y_{op}(u,r)v = m_2(u,v;r,0). \quad (1.5)$$

Then by the convergence property, we have

$$\begin{aligned} Y_{op}(\mathbf{1}_{op}, r) &= \mathrm{id}_F, & (1.6)\\ \lim_{r\to 0} Y_{op}(u,r)\mathbf{1}_{op} &= u, & (1.7)\end{aligned}$$

for $u\in V_{op}$. (1.7) implies that the map $u\mapsto Y_{op}(u,r)$ is one-to-one. By (1.3), we have

$$e^{a\mathbf{d}_{op}} Y_{op}(u,r) e^{-a\mathbf{d}_{op}} = Y_{op}(e^{a\mathbf{d}_{op}}u, ar) \quad (1.8)$$

for $u\in V_{op}, r\in\mathbb{R}_+$ and $a\in\mathbb{R}$. By (1.4), we also have

$$e^{aD_{op}} Y_{op}(u,r) e^{-aD_{op}} = Y_{op}(u, r+a) \quad (1.9)$$

for $u\in V_{op}, r>0, r+a>0$. Moreover, using (1.7) and (1.4), we also have

$$Y_{op}(v,r)\mathbf{1}_{op} = m_{op}^{(1)}(v;r) = e^{rD_{op}}v. \quad (1.10)$$

If we assume some analytic properties on $m_{op}^{(n)}$, it is possible to use Y_{op} to generate all $m_{op}^{(n)}$. This motivate us to introduce the notion of open-string vertex algebra in [HKo1]. The definition given here is a refinement of that in [HKo1].

Definition 1.2. An *open-string vertex algebra* is an $\mathbb{R}$-graded vector space $V_{op} = \oplus_{n\in\mathbb{R}}(V_{op})_{(n)}$ (graded by *weights*) equipped with a *vertex map*:

$$\begin{aligned} Y_{op}: (V_{op}\otimes V_{op})\times\mathbb{R}_+ &\to \overline{V}_{op}\\ (u\otimes v, r) &\mapsto Y_{op}(u,r)v, \end{aligned} \quad (1.11)$$

a *vacuum* $\mathbf{1}_{op}\in V_{op}$ and an operator $D_{op}\in \mathrm{End}\, V_{op}$ of weight 1, satisfying the following conditions:

1. *Vertex map weight property*: For $s_1,s_2\in\mathbb{R}$, there exists a finite subset $S(s_1,s_2)\subset\mathbb{R}$ such that the image of $(\oplus_{s\in s_1+\mathbb{Z}}(V_{op})_{(s)})\otimes(\oplus_{s\in s_1+\mathbb{Z}}(V_{op})_{(s)})$ under the vertex map Y_{op} is in $\prod_{s\in S(s_1,s_2)+\mathbb{Z}}(V_{op})_{(s)}$.

2. *Vacuum properties*:

   (a) *identity property*: For any $r \in \mathbb{R}_+$, $Y_{op}(\mathbf{1}_{op}, r) = \mathrm{id}_{V_{op}}$,

   (b) *creation property*: $\forall u \in V_{op}$, $\lim_{r\to 0} Y_{op}(u, r)\mathbf{1}_{op} = u$.

3. *Convergence properties*:

   For $n \in \mathbb{N}$, $u_1, \dots, u_n, v \in V_{op}$ and $v' \in V'_{op}$, the series

$$\begin{aligned}&\langle v', Y_{op}(u_1, r_1) \dots Y_{op}(u_n, r_n)v\rangle \\ &= \sum_{m_1,\dots,m_{n-1}} \langle v', Y_{op}(u_1, r_1)P_{m_1} Y_{op}(u_2, r_2) \dots P_{m_{n-1}} Y_{op}(u_n, r_n)v\rangle\end{aligned}$$

   converges absolutely when $r_1 > \cdots > r_n > 0$ and is a restriction to the domain $\{r_1 > \cdots > r_n > 0\}$ of an (possibly multivalued) analytic function in $(\mathbb{C}^\times)^n$ with only possible singularities at $r_i = r_j$ for $1 \leq i, j \leq n$ and $i \neq j$.

   (a) For $u_1, u_2, v \in V_{op}, v' \in V'_{op}$, the series

$$\langle v', Y_{op}(Y_{op}(u_1, r_0)u_2, r_2)v\rangle = \sum_m \langle v', Y_{op}(P_m Y_{op}(u_1, r_0)u_2, r_2)v\rangle$$

   converges absolutely when $r_2 > r_0 > 0$.

4. *Associativity*: For $u_1, u_2, v \in V_{op}$ and $v' \in V'_{op}$,

$$\langle v', Y_{op}(u_1, r_1)Y_{op}(u_2, r_2)v\rangle = \langle v', Y_{op}(Y_{op}(u_1, r_1 - r_2)u_2, r_2)v\rangle$$

   for $r_1, r_2 \in \mathbb{R}$ satisfying $r_1 > r_2 > r_1 - r_2 > 0$.

5. $\mathbf{d}_{op}$*-bracket property*: Let $\mathbf{d}_{op}$ be the grading operator on V_{op}. For $u \in V_{op}$ and $r \in \mathbb{R}_+$,

$$[\mathbf{d}_{op}, Y_{op}(u, r)] = Y_{op}(\mathbf{d}_{op}u, r) + r\frac{d}{dr}Y_{op}(u, r). \tag{1.12}$$

6. D_{op}*-derivative property*: We still use D_{op} to denote the natural extension of D_{op} to $\mathrm{Hom}(\overline{V}_{op}, \overline{V}_{op})$. For $u \in V_{op}$, $Y_{op}(u, r)$ as a map from $\mathbb{R}_+$ to $\mathrm{Hom}(V_{op}, \overline{V}_{op})$ is differentiable and

$$\frac{d}{dr}Y_{op}(u, r) = [D_{op}, Y_{op}(u, r)] = Y_{op}(D_{op}u, r). \tag{1.13}$$

Homomorphisms, *isomorphisms*, *subalgebras* of open-string vertex algebras are defined in the obvious way.

We denote such algebra as $(V_{op}, Y_{op}, \mathbf{1}_{op}, \mathbf{d}_{op}, D_{op})$ or simply V_{op}. For $u \in V_{op}$, it was shown in [HKo1] that there is a formal vertex operator

$$Y^f_{op}(u, x) = \sum_{n\in\mathbb{R}} u_n x^{-n-1}, \tag{1.14}$$

where $u_n \in \text{End } V_{op}$, so that

$$Y_{op}^f(u, x)|_{x=r} = Y_{op}(u, r). \tag{1.15}$$

We can also replace x by complex variable z if we choose a branch cut. For any $z \in \mathbb{C}^\times$ and $n \in \mathbb{R}$, we define

$$\log z := \log |z| + \arg z, 0 \leq \arg z < 2\pi. \tag{1.16}$$

But for power functions, we distinguish two types of complex variables, z (or $z_1, z_2, \ldots$) and $\bar{z}_1, \zeta$ (or $\bar{z}_1, \zeta_1, \bar{z}_2, \zeta_2, \ldots$). We define

$$z^n := e^{n \log z}, \quad \bar{z}^n := e^{n \overline{\log z}}, \quad \zeta^n := e^{n \overline{\log \zeta}}. \tag{1.17}$$

Proposition 1.3. *An open-string vertex algebra canonically gives a boundary field algebra.*

Proof. The proof is standard. We omit it here. ∎

Definition 1.4. An *open-string vertex operator algebra* is an open-string vertex algebra, together with a conformal element ω_{op}, satisfying the following conditions:

7. *grading-restriction conditions*: For all $n \in \mathbb{R}$, $\dim(V_{op})_{(n)} < \infty$ and $(V_{op})_{(n)} = 0$ when n is sufficiently negative.

8. *Virasoro relations*: The vertex operator associated to ω_{op} has the following expansion:
$$Y_{op}(\omega_{op}, r) = \sum_{n \in \mathbb{Z}} L(n) r^{-n-2}$$
where $L(n)$ satisfying the following condition: $\forall m, n \in \mathbb{Z}$,
$$[L(m), L(n)] = (m - n)L(m + n) - \frac{c}{12}(m^3 - m)\delta_{m+n,0},$$
for some $c \in \mathbb{C}$.

9. *Commutator formula for Virasoro operators and formal vertex operators (or component operators)*: For $v \in V_{op}$, $Y_{op}^f(\omega_{op}, x)v$ involves only finitely many negative powers of x and
$$[Y_{op}^f(\omega_{op}, x_1), Y_{op}^f(v, x_2)] = \text{Res}_{x_0} x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y_{op}^f(Y_{op}^f(\omega_{op}, x_0)v, x_2).$$

10. $L(0)$-*grading property* and $L(-1)$-*derivative property*: $L(0) = \mathbf{d}_{op}$ and $L(-1) = D_{op}$.

We shall denote the open-string vertex operator algebra defined above by

$$(V_{op}, Y_{op}, \mathbf{1}_{op}, \omega_{op})$$

or simply V_{op}. The complex number c in the definition is called the *central charge* of the algebra.

The *meromorphic center of* V_{op} is defined as

$$C_0(V_{op}) = \Big\{ u \in \oplus_{n\in\mathbb{Z}}(V_{op})_{(n)} \;\Big|\; Y^f_{op}(u,x) \in (\mathrm{End}\, V_{op})[[x, x^{-1}]], \\ Y^f_{op}(v,x)u = e^{xD}Y^f_{op}(u,-x)v, \ \forall v \in V_{op} \Big\}.$$

It was shown in [HKo1] that the meromorphic center of a grading-restricted open-string vertex (operator) algebra is a grading-restricted vertex (operator) algebra.

Let U be a grading-restricted vertex (operator) algebra. If there is a monomorphism $\iota_{op} : U \hookrightarrow C_0(V_{op})$ of grading-restricted vertex (operator) algebra, we call V_{op} an open-string vertex (operator) algebra over U, and denote it by $(V_{op}, Y_{op}, \iota_{op})$ or simply by V_{op}. In this case, the formal vertex operator Y^f_{op} is an intertwining operator of type $\binom{V_{op}}{V_{op}V_{op}}$, where V_{op} is a U-module.

Remark 1.5. Open-string vertex operator algebra can be viewed as a noncommutative generalization of vertex operator algebra. Other noncommutative generalizations of vertex operator algebra are also studied in the literature [B2][BaKa][L1]-[L3]. However, many interesting examples of open-string vertex operator algebras are not covered in other axiomatic frameworks.

1.2 Full field algebras

Definition 1.6. A $\mathbb{R}\times\mathbb{R}$-graded *full field algebra* is an $\mathbb{R}\times\mathbb{R}$-graded vector space $V_{cl} = \coprod_{m,n\in\mathbb{R}}(V_{cl})_{(m,n)}$ (graded by *left weight* wt^L and *right weight* wt^R with left and right grading operators $\mathbf{d}^L$ and $\mathbf{d}^R$), equipped with *correlation-function maps*

$$\begin{array}{rrcl} m^{(n)}_{cl} : & V_{cl}^{\otimes n} \times M^n_{\mathbb{C}} & \to & \overline{V}_{cl} \\ & (u_1\otimes\cdots\otimes u_n, (z_1,\dots,z_n)) & \mapsto & m^{(n)}_{cl}(u_1,\dots,u_n; z_1,\bar{z}_1,\dots,z_n,\bar{z}_n), \end{array}$$

for each $n \in \mathbb{N}$, and operators D^L and D^R of weights $(1,0)$ and $(0,1)$ respectively, satisfying the following axioms:

1. *Single-valuedness property*: $e^{2\pi i(\mathbf{d}^L-\mathbf{d}^R)} = \mathrm{id}_{V_{cl}}$.
2. For $n\in\mathbb{Z}_+$, $m^{(n)}_{cl}(u_1,\dots,u_n; z_1,\bar{z}_1,\dots,z_n,\bar{z}_n)$ is linear in $u_1,\dots,u_n$ and smooth in the real and imaginary parts of $z_1,\dots,z_n$.
3. *Identity properties*: $\forall u\in V_{cl}$, $m_1(u;0,0) = u$ and $\mathbf{1}_{cl} := m^{(0)}_{cl}(1) \in (V_{cl})_{(0,0)}$.

4. *Convergence property*: For $k, l_1, \dots, l_k \in \mathbb{Z}_+$ and $u_1, \dots, u_n, u_1^{(i)}, \dots, u_k^{(i)} \in V_{cl}$ and $i = 1, \dots, n$, the series

$$\sum_{p,q\in\mathbb{R}} m_{cl}^{(n)}(u_1, \dots, u_{i-1}, P_{(p,q)} m_{cl}^{(k)}(v_1, \dots, v_k; z_1^{(i)}, \bar{z}_1^{(i)}, \dots, z_k^{(i)}, \bar{z}_k^{(i)}), \\ u_{i+1}, \dots, u_n; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n) \tag{1.18}$$

converges absolutely to

$$m_{cl}^{(n+k-1)}(u_1, \dots, u_{i-1}, v_1, \dots, v_k, u_{i+1}, \dots, u_n; z_1, \bar{z}_1, \dots, z_{i-1}, \bar{z}_{i-1}, \\ z_i + z_1^{(i)}, \bar{z}_i + \bar{z}_1^{(i)} \dots, z_i + z_k^{(i)}, \bar{z}_i + \bar{z}_k^{(i)}, z_{i+1}, \bar{z}_{i+1}, \dots, z_n, \bar{z}_n) \tag{1.19}$$

whenever $|z_p^{(i)}| < |z_i - z_j|$ for all $j = 1, \dots, n$, $i \neq j$ and for $p = 1, \dots, k$.

5. *Permutation property*: For $n \in \mathbb{Z}_+$ and $\sigma \in S_n$, we have

$$m_{cl}^{(n)}(u_1, \dots, u_n; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n) \\ = m_{cl}^{(n)}(u_{\sigma(1)}, \dots, u_{\sigma(n)}; z_{\sigma(1)}, \bar{z}_{\sigma(1)}, \dots, z_{\sigma(n)}, \bar{z}_{\sigma(n)}) \tag{1.20}$$

for $u_1, \dots, u_n \in V_{cl}$ and $(z_1, \dots, z_n) \in M_{\mathbb{C}}^n$.

6. $\mathbf{d}^L$ *and* $\mathbf{d}^R$ *property*: For $u_1, \dots, u_n \in V_{cl}$ and $a \in \mathbb{C}$,

$$e^{a\mathbf{d}^L} e^{\bar{a}\mathbf{d}^R} m_{cl}^{(n)}(u_1, \dots, u_n; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n) \\ = m_{cl}^{(n)}(e^{a\mathbf{d}^L} e^{\bar{a}\mathbf{d}^R} u_1, \dots, e^{a\mathbf{d}^L} e^{\bar{a}\mathbf{d}^R} u_n, e^a z_1, e^{\bar{a}} \bar{z}_1, \dots, e^a z_n, e^{\bar{a}} \bar{z}_n). \tag{1.21}$$

7. D^L *and* D^R *property*: $[D^L, D^R] = 0$ and for $u_1, \dots, u_n \in V_{cl}$ and $a \in \mathbb{C}$,

$$e^{aD^L} e^{\bar{a}D^R} m_{cl}^{(n)}(u_1, \dots, u_n; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n) \\ = m_{cl}^{(n)}(u_1, \dots, u_n, z_1 + a, \bar{z}_1 + \bar{a}, \dots, z_n + a, \bar{z}_n + \bar{a}). \tag{1.22}$$

We denote the $\mathbb{R} \times \mathbb{R}$-graded full field algebra defined above by

$$(V_{cl}, m_{cl}, \mathbf{d}^L, \mathbf{d}^R, D^L, D^R)$$

or simply by V_{cl}.

Let $\mathbb{Y} : V_{cl}^{\otimes 2} \times \mathbb{C}^\times \to \overline{V}_{cl}$ be so that

$$\mathbb{Y} : (u \otimes v, z) \mapsto \mathbb{Y}(u; z, \bar{z})v := m_2(u \otimes v; z, \bar{z}, 0, 0). \tag{1.23}$$

Then by the convergence property, it is easy to see that

$$\mathbb{Y}(\mathbf{1}_{cl}; z, \bar{z}) = \mathrm{id}_{V_{cl}}, \tag{1.24}$$

$$\lim_{z\to 0} \mathbb{Y}(u; z, \bar{z})\mathbf{1}_{cl} = u, \qquad \forall u \in V_{cl}. \tag{1.25}$$

Moreover, it is also not hard to show the following two properties of $\mathbb{Y}$:

1. *The* $\mathbf{d}^L$- *and* $\mathbf{d}^R$-*bracket properties*:

$$\begin{aligned}\left[\mathbf{d}^L, \mathbb{Y}(u; z, \bar{z})\right] &= z\frac{\partial}{\partial z}\mathbb{Y}(u; z, \bar{z}) + \mathbb{Y}(\mathbf{d}^L u; z, \bar{z}), &(1.26)\\ \left[\mathbf{d}^R, \mathbb{Y}(u; z, \bar{z})\right] &= \bar{z}\frac{\partial}{\partial \bar{z}}\mathbb{Y}(u; z, \bar{z}) + \mathbb{Y}(\mathbf{d}^R u; z, \bar{z}). &(1.27)\end{aligned}$$

2. The D^L- *and* D^R-*derivative property*:

$$\left[D^L, \mathbb{Y}(u; z, \bar{z})\right] = \mathbb{Y}(D^L u; z, \bar{z}) = \frac{\partial}{\partial z}\mathbb{Y}(u; z, \bar{z}), \tag{1.28}$$

$$\left[D^R, \mathbb{Y}(u; z, \bar{z})\right] = \mathbb{Y}(D^R u; z, \bar{z}) = \frac{\partial}{\partial \bar{z}}\mathbb{Y}(u; z, \bar{z}). \tag{1.29}$$

It was shown in [HKo2] that we have the following expansion:

$$\mathbb{Y}(u; z, \bar{z}) = \sum_{r,s\in\mathbb{R}} \mathbb{Y}_{l,r}(u)e^{(-l-1)\log z}e^{(-r-1)\overline{\log z}} \tag{1.30}$$

where $\mathbb{Y}_{l,r}(u) \in \text{End}\, F$ with $\text{wt}^L\mathbb{Y}_{l,r}(u) = \text{wt}^L u - l - 1$ and $\text{wt}^R\mathbb{Y}_{l,r}(u) = \text{wt}^R u - r - 1$. Moreover, the expansion above is unique. Let x and $\bar{x}$ be independent and commuting formal variables. We define the *formal full vertex operator* $\mathbb{Y}_f$ associated to $u \in V_{cl}$ by

$$\mathbb{Y}_f(u; x, \bar{x}) = \sum_{l,r\in\mathbb{R}} \mathbb{Y}_{l,r}(u)x^{-l-1}\bar{x}^{-r-1}. \tag{1.31}$$

These formal full vertex operators give a *formal full vertex operator map*

$$\mathbb{Y}_f : V_{cl} \otimes V_{cl} \mapsto V_{cl}\{x, \bar{x}\}.$$

For nonzero complex numbers z and ζ, we can substitute $e^{r\log z}$ and $e^{s\,\overline{\log\zeta}}$ for x^r and $\bar{x}^s$, respectively, in $\mathbb{Y}_f(u; x, \bar{x})$ to obtain a map

$$\mathbb{Y}_{\text{an}}(u; z, \zeta) : V_{cl}^{\otimes 2} \times (\mathbb{C}^\times)^2 \to \overline{V}_{cl}$$

called *analytic full vertex operator map.*

Definition 1.7. An $\mathbb{R}\times\mathbb{R}$-graded full field algebra $(V_{cl}, m_{cl}, \mathbf{d}^L, \mathbf{d}^R, D^L, D^R)$ is called *grading-restricted* if it satisfies the following grading-restriction conditions:

1. There exists $M \in \mathbb{R}$ such that $(V_{cl})_{(m,n)} = 0$ if $n < M$ or $m < M$.

2. $\dim(V_{cl})_{(m,n)} < \infty$ for $m, n \in \mathbb{R}$.

We say that V_{cl} is *lower-truncated* if V_{cl} satisfies the first grading restriction condition.

In this case, for $u \in V_{cl}$ and $k \in \mathbb{R}$, we have

$$\sum_{l+r=k} \mathbb{Y}_{l,r}(u) \in \text{End}\, V_{cl}$$

with total weight wt $u-k-2$. We denote $\sum_{l+r=k}\mathbb{Y}_{l,r}(u)$ by $\mathbb{Y}_{k-1}(u)$. Then we have the expansion

$$\mathbb{Y}_f(u;x,x)=\sum_{k\in\mathbb{R}}\mathbb{Y}_k(u)x^{-k-1}, \tag{1.32}$$

where wt $\mathbb{Y}_k(u)=$ wt $u-k-1$. For given $u,v\in v_{cl}$, we have $\mathbb{Y}_k(u)w=0$ for sufficiently large k.

Let $(V^L,Y^L,\mathbf{1}^L,\omega^L)$ and $(V^R,Y^R,\mathbf{1}^R,\omega^R)$ be vertex operator algebras. Let ι_{cl} be an injective homomorphism from the full field algebra $V^L\otimes V^R$ to V_{cl}. Then we have $\mathbf{1}_{cl}=\iota_{cl}(\mathbf{1}^L\otimes\mathbf{1}^R)$, $\mathbf{d}^L\circ\iota_{cl}=\iota_{cl}\circ(L^L(0)\otimes I_{V^R})$, $\mathbf{d}^R\circ\iota_{cl}=\iota_{cl}\circ(I_{V^L}\otimes L^R(0)))$, $D^L\circ\iota_{cl}=\iota_{cl}\circ(L^L(-1)\otimes I_{V^R})$ and $D^R\circ\iota_{cl}=\iota_{cl}\circ(I_{V^L}\otimes L^R(-1))$. Moreover, V_{cl} has a *left conformal element* $\iota_{cl}(\omega^L\otimes\mathbf{1}^R)$ and an *right conformal element* $\iota_{cl}(\mathbf{1}^L\otimes\omega^R)$. We have the following operators on V_{cl}:

$$\begin{aligned}
L^L(0) &= \mathrm{Res}_x\mathrm{Res}_{\bar{x}}\bar{x}^{-1}\mathbb{Y}_f(\iota_{cl}(\omega^L\otimes\mathbf{1}^R);x,\bar{x}),\\
L^R(0) &= \mathrm{Res}_x\mathrm{Res}_{\bar{x}}x^{-1}\mathbb{Y}_f(\iota_{cl}(\mathbf{1}^L\otimes\omega^L);x,\bar{x}),\\
L^L(-1) &= \mathrm{Res}_x\mathrm{Res}_{\bar{x}}x\bar{x}^{-1}\mathbb{Y}_f(\iota_{cl}(\omega^L\otimes\mathbf{1}^R);x,\bar{x}),\\
L^R(-1) &= \mathrm{Res}_x\mathrm{Res}_{\bar{x}}x^{-1}\bar{x}\mathbb{Y}_f(\iota_{cl}(\mathbf{1}^L\otimes\omega^L);x,\bar{x}).
\end{aligned}$$

Since these operators are operators on V_{cl}, it should be easy to distinguish them from those operators with the same notation but acting on V^L or V^R.

Definition 1.8. Let $(V^L,Y^L,\mathbf{1}^L,\omega^L)$ and $(V^R,Y^R,\mathbf{1}^R,\omega^R)$ be vertex operator algebras. A *full field algebra over* $V^L\otimes V^R$ is a grading-restricted $\mathbb{R}\times\mathbb{R}$-graded full field algebra $(V_{cl},m_{cl},\mathbf{d}^L,\mathbf{d}^R,D^L,D^R)$ equipped with an injective homomorphism ι_{cl} from the full field algebra $V^L\otimes V^R$ to V_{cl} such that $\mathbf{d}^L=L^L(0)$, $\mathbf{d}^R=L^R(0)$, $D^L=L^L(-1)$ and $D^R=L^R(-1)$. A full field algebra over $V^L\otimes V^R$ equipped with left and right conformal elements $\iota_{cl}(\omega^L\otimes\mathbf{1}^R)$ and $\iota_{cl}(\mathbf{1}^L\otimes\omega^R)$ is called *conformal full field algebra over* $V^L\otimes V^R$.

We shall denote the (conformal) full field algebra over $V^L\otimes V^R$ defined above by $(V_{cl},m_{cl},\iota_{cl})$ or simply by V_{cl}. From now on, we will not distinguish V^L with $V^L\otimes\mathbf{1}^R$ and $\iota_{cl}(V^L\otimes\mathbf{1}^R)$. Similarly for V^R with $\mathbf{1}^L\otimes V^R$ and $\iota_{cl}(\mathbf{1}^L\otimes V^R)$. For $u^L\in V^L$, $\mathbb{Y}_{an}(u^L;z,\zeta)$ is independent of ζ. So we simply denote it as $\mathbb{Y}_{an}(u^L,z)$. Similarly, we denote $\mathbb{Y}_{an}(u^R;z,\zeta)$ as $\mathbb{Y}_{an}(u^R,\zeta)$ for $u^R\in V^R$.

1.3 Open-closed field algebras

Definition 1.9. An open-closed field algebra consists of a full field algebra

$$(V_{cl},m_{cl},\mathbf{d}^L_{cl},\mathbf{d}^R_{cl},D^L_{cl},D^R_{cl})$$

and a $\mathbb{R}$-graded vector space V_{op} with the grading operator $\mathbf{d}_{op}$ and an additional operator $D_{op}\in\mathrm{End}\ V_{op}$, together with a map for each pair of $n,l\in\mathbb{N}$:

$$\begin{aligned}
m^{(l;n)}_{cl-op}:V_{cl}^{\otimes l}\otimes V_{op}^{\otimes n}\times M^l_{\mathbb{H}}\times\Lambda^n &\to \overline{V}_{op}\\
(u_1\otimes\cdots\otimes u_l\otimes v_1\otimes\cdots\otimes v_n,(z_1,\dots,z_l;r_1,\dots,r_n)) &\mapsto\\
&\quad m^{(l;n)}_{cl-op}(u_1,\dots,u_l;v_1,\dots,v_n;z_1,\bar{z}_1\dots,z_l,\bar{z}_l;r_1,\dots,r_n),
\end{aligned}$$

satisfying the following axioms:

1. $m^{(l;n)}_{cl-op}(u_1,\dots,v_n;z_1,\bar{z}_1,\dots,z_l,\bar{z}_l;r_1,\dots,r_n)$ is linear in $u_1,\dots,v_n$ and smooth in $r_1,\dots,r_n,z_1,\dots,z_l$.

2. *Identity properties*: $m^{(0;1)}_{cl-op}(v;0)=v, \forall v\in V_{op}$ and $\mathbf{1}_{op}:=m^{(0;0)}_{cl-op}(1)\in (V_{op})_{(0)}$.

3. *Convergence properties*:

   (a) For $u_1,\dots,u_l,\tilde{u}_1,\dots,\tilde{u}_k\in V_{cl}$, $v_1,\dots,v_n,\tilde{v}_1,\dots,\tilde{v}_m\in V_{op}$ and $i=1,\dots,n$, the following series

$$\sum_{n_1\in\mathbb{R}} m^{(l;n)}_{cl-op}(u_1,\dots,u_l;v_1,\dots,v_{i-1},P_{n_1}m^{(k;m)}_{cl-op}(\tilde{u}_1,\dots,\tilde{u}_k; \tilde{v}_1,\dots,\tilde{v}_m;z^{(i)}_1,\overline{z^{(i)}_1},\dots,z^{(i)}_k,\overline{z^{(i)}_k};r^{(i)}_1,\dots,r^{(i)}_m); v_{i+1},\dots v_n;z_1,\bar{z}_1,\dots,z_l,\bar{z}_l;r_1,\dots,r_n) \tag{1.33}$$

   converges absolutely when $|z_s-r_i|,|r_t-r_i|>|z^{(i)}_p|, r^{(i)}_q\geq 0$ for all $s=1,\dots,l$, $t=1,\dots,n$, $t\neq i$, $p=1,\dots,k$ and $q=1,\dots,m$ to

$$m^{(l+k;n+m-1)}_{cl-op}(u_1,\dots,u_l,\tilde{u}_1,\dots,\tilde{u}_k;v_1,\dots,v_{i-1},\tilde{v}_1,\dots,\tilde{v}_m, v_{i+1},\dots,v_n;z_1,\bar{z}_1,\dots,z_l,\bar{z}_l,z^{(i)}_1,\overline{z^{(i)}_1},\dots,z^{(i)}_k,\overline{z^{(i)}_k}; r_1,\dots,r_{i-1},r_i+r^{(i)}_1,\dots,r_i+r^{(i)}_m,r_{i+1},\dots,r_n). \tag{1.34}$$

   (b) For $u_1,\dots,u_l,\tilde{u}_1,\dots,\tilde{u}_k\in V_{cl}$, $v_1,\dots,v_n\in V_{op}$ and $i=1,\dots,n$, the following series

$$\sum_{n_1,n_2\in\mathbb{R}} m^{(l;n)}_{cl-op}(u_1,\dots,u_{i-1},P_{(n_1,n_2)}m^{(k)}_{cl}(\tilde{u}_1,\dots,\tilde{u}_k;z^{(i)}_1,\overline{z^{(i)}_1},\dots,z^{(i)}_k,\overline{z^{(i)}_k}), u_{i+1},\dots,u_l;v_1,\dots v_n;z_1,\bar{z}_1,\dots,z_l,\bar{z}_l;r_1,\dots,r_n) \tag{1.35}$$

   converges absolutely, when $|z_s-z_i|,|r_t-z_i|>|z^{(i)}_p|$ for all $s=1,\dots,l$, $s\neq i$, $t=1,\dots,n$, $p=1,\dots,k$, to

$$m^{(l+k-1;n)}_{cl-op}(u_1,\dots,u_{i-1},\tilde{u}_1,\dots,\tilde{u}_k,u_{i+1},\dots,u_l;v_1,\dots v_n; z_1,\bar{z}_1,\dots,z_{i-1},\bar{z}_{i-1},z_i+z^{(i)}_1,\overline{z_i+z^{(i)}_1},\dots, z_i+z^{(i)}_k,\overline{z_i+z^{(i)}_k},z_{i+1},\bar{z}_{i+1},\dots,z_l,\bar{z}_l;r_1,\dots,r_n). \tag{1.36}$$

4. *Permutation axiom*: For $u_1,\dots,u_l\in V_{cl}$, $v_1,\dots,v_n\in V_{op}$ and $\sigma\in S_l$,

$$m^{(l;n)}_{cl-op}(u_1,\dots,u_l;v_1,\dots,v_n;z_1,\bar{z}_1,\dots,z_l,\bar{z}_l;r_1,\dots,r_n) = m^{(l;n)}_{cl-op}(u_{\sigma(1)},\dots,u_{\sigma(l)};v_1,\dots,v_n; z_{\sigma(1)},\bar{z}_{\sigma(1)},\dots,z_{\sigma(l)},\bar{z}_{\sigma(l)};r_1,\dots,r_n). \tag{1.37}$$

5. $\mathbf{d}_{op}-$, $\mathbf{d}^L_{cl}-$ *and* $\mathbf{d}^R_{cl}$*-property*: For $u_1,\dots,u_l\in V_{cl}$, $v_1,\dots,v_n\in V_{op}$ and $a\in\mathbb{R}$,

$$e^{a\mathbf{d}_{op}}m^{(l;n)}_{cl-op}(u_1,\dots,u_l;v_1,\dots,v_n;z_1,\bar{z}_1,\dots,z_l,\bar{z}_l;r_1,\dots,r_n;) = m^{(l;n)}_{cl-op}(e^{a(\mathbf{d}^L_{cl}+\mathbf{d}^R_{cl})}u_1,\dots,e^{a(\mathbf{d}^L_{cl}+\mathbf{d}^R_{cl})}u_l;e^{a\mathbf{d}_{op}}v_1,\dots,e^{a\mathbf{d}_{op}}v_n; e^az_1,e^a\bar{z}_1,\dots,e^az_l,e^a\bar{z}_l;e^ar_1,\dots,e^ar_n). \tag{1.38}$$

6. D_{op}-*property*: For $u_1, \dots, u_l \in V_{cl}$, $v_1, \dots, v_n \in V_{op}$ and $r_n + a \geq 0$,

$$\begin{aligned} &e^{aD_{op}} m_{cl-op}^{(l;n)}(u_1, \dots, u_l; v_1, \dots, v_n; z_1, \bar{z}_1, \dots, z_l, \bar{z}_l; r_1, \dots, r_n) \\ &\quad = m_{cl-op}^{(l;n)}(u_1, \dots, u_l; v_1, \dots, v_n; \\ &\qquad\qquad z_1 + a, \bar{z}_1 + a, \dots, z_l + a, \bar{z}_l + a; r_1 + a, \dots, r_n + a). \end{aligned} \tag{1.39}$$

We denote such algebra by $(V_{cl}, V_{op}, m_{cl-op})$ for simplicity. *Homomorphisms*, *isomorphisms*, *subalgebras* of open-closed field algebras are defined in the obvious way.

Remark 1.10. From above definition, it is clear that an open-closed field algebra includes a boundary field algebra as a substructure.

We discuss a few results which follow immediately from the definition. By the identity properties and (1.39), we have, for $a \geq 0$,

$$m_{cl-op}^{(0;1)}(v; a) = e^{aD_{op}} m_{cl-op}^{(0;1)}(v; 0) = e^{aD_{op}} v. \tag{1.40}$$

By (1.40) and convergence property, for $i = 1, \dots, n$ and $a \in \mathbb{R}$, we obtain

$$\begin{aligned} &m_{cl-op}^{(l;n)}(u_1, \dots, u_l; v_1, \dots, v_{i-1}, e^{aD_{op}} v_i, v_{i+1}, \dots, v_n; \\ &\qquad\qquad z_1, \bar{z}_1, \dots, z_l, \bar{z}_l; r_1, \dots, r_n) \\ &\quad = m_{cl-op}^{(l;n)}(u_1, \dots, u_l; v_1, \dots, v_{i-1}, m_{cl-op}^{(0;1)}(v_i; a), \\ &\qquad\qquad v_{i+1}, \dots, v_n; z_1, \bar{z}_1, \dots, z_l, \bar{z}_l; r_1, \dots, r_n) \\ &\quad = m_{cl-op}^{(l;n)}(u_1, \dots, u_l; v_1, \dots, v_n; z_1, \bar{z}_1, \dots, z_l, \bar{z}_l \\ &\qquad\qquad r_1, \dots, r_{i-1}, r_i + a, r_{i+1}, \dots, r_n) \end{aligned} \tag{1.41}$$

when $|r_j - r_i|, |z_k - r_i| > |a|$ for $j \neq i$ and $k = 1, \dots, l$. By (1.22) and the convergence property, we also have, for $j = 1, \dots, l$ and $b \in \mathbb{C}$,

$$\begin{aligned} &m_{cl-op}^{(l;n)}(u_1, \dots, u_{j-1}, e^{bD_{cl}^L + \bar{b}D_{cl}^R} u_j, u_{j+1}, \dots, u_l; v_1, \dots, v_n; \\ &\qquad\qquad z_1, \bar{z}_1, \dots, z_l, \bar{z}_l; r_1, \dots, r_n) \\ &\quad = m_{cl-op}^{(l;n)}(u_1, \dots, u_{j-1}, m_{cl}^{(1)}(u_j; b, \bar{b}), u_{j+1}, \dots, u_l; \\ &\qquad\qquad v_1, \dots, v_n; z_1, \bar{z}_1, \dots, z_l, \bar{z}_l; r_1, \dots, r_n) \\ &\quad = m_{cl-op}^{(l;n)}(u_1, \dots, u_l; v_1, \dots, v_n; z_1, \bar{z}_1, \dots, z_{j-1}, \bar{z}_{j-1}, \\ &\qquad\qquad z_j + b, \overline{z_j + b}, z_{j+1}, \bar{z}_{j+1}, \dots, z_l, \bar{z}_l; r_1, \dots, r_n) \end{aligned} \tag{1.42}$$

when $|z_i - z_j|, |r_k - z_j| > |b|$ for $i = 1, \dots, l, i \neq j$ and $k = 1, \dots, n$.

Let $m_{op}^{(n)} := m_{cl-op}^{(0;n)}$. The definition of open-closed field algebra immediately implies that $(V_{op}, m_{op}, \mathbf{d}_{op}, D_{op})$ is a boundary field algebra. In particular, we have the map Y_{op} defined in (1.5) satisfying (1.6), (1.7), (1.8) and (1.9).

We also have the map $\mathbb{Y}$ defined in (1.23) satisfying (1.24), (1.25), (1.26), (1.27), (1.28), (1.29), (1.30) and (1.31).

In an open-closed field algebra $(V_{cl}, V_{op}, m_{cl-op})$, there is an additional vertex operator map:

$$\begin{aligned}\mathbb{Y}_{cl-op} : (V_{cl} \otimes V_{op}) \times \mathbb{H} &\to \overline{V}_{op} \\ (u \otimes v, (z, \bar{z})) &\mapsto \mathbb{Y}_{cl-op}(u; z, \bar{z})v,\end{aligned}$$

defined by

$$\mathbb{Y}_{cl-op}(u; z, \bar{z})v := m^{(1;1)}_{cl-op}(u; v; z, \bar{z}; 0). \tag{1.43}$$

By the convergence property, we have the following identity property:

$$\begin{aligned}\mathbb{Y}_{cl-op}(\mathbf{1}_{cl}; z, \bar{z})v &= m^{(1;1)}_{cl-op}(m^{(0)}_{cl}(1); v; z, \bar{z}; 0) \\ &= m^{(0;1)}_{cl-op}(v; 0) \\ &= v. \end{aligned} \tag{1.44}$$

Proposition 1.11. *For $u \in V_{cl}$, we have*

$$\begin{aligned}[\mathbf{d}_{op}, \mathbb{Y}_{cl-op}(u; z, \bar{z})] &= \mathbb{Y}_{cl-op}((\mathbf{d}^L_{cl} + \mathbf{d}^R_{cl})u; z, \bar{z}) \\ &\quad + \left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} \right) \mathbb{Y}_{cl-op}(u; z, \bar{z}). \end{aligned} \tag{1.45}$$

Proof. Applying $\frac{\partial}{\partial a}|_{a=0}$ to the both sides of (1.38) when $n = 1, l = 1$ and $r_1 = 0$, we obtain (1.45) immediately. ∎

Proposition 1.12. *For $u \in V_{cl}$, we have*

$$[D_{op}, \mathbb{Y}_{cl-op}(u; z, \bar{z})] = \mathbb{Y}_{cl-op}((D^L_{cl} + D^R_{cl})u; z, \bar{z}), \tag{1.46}$$

$$\mathbb{Y}_{cl-op}(D^L_{cl}u; z, \bar{z}) = \frac{\partial}{\partial z}\mathbb{Y}_{cl-op}(u; z, \bar{z}), \tag{1.47}$$

$$\mathbb{Y}_{cl-op}(D^R_{cl}u; z, \bar{z}) = \frac{\partial}{\partial \bar{z}}\mathbb{Y}_{cl-op}(u; z, \bar{z}). \tag{1.48}$$

Proof. Applying $\frac{\partial}{\partial b}|_{b=0}$ ($\frac{\partial}{\partial \bar{b}}|_{\bar{b}=0}$) to the both sides of (1.42) when $n = 1, l = 1$ and $r_1 = 0$, we obtain (1.47) and (1.48).

Applying $\frac{\partial}{\partial a}|_{a=0}$ to the both sides of (1.39) when $n = 1, l = 1$ and $r_1 = 0$ and using (1.47) and (1.48) we obtain the first identity in (1.46) immediately. ∎

1.4 Analytic open-closed field algebras

The notion of open-closed field algebra introduced in the last subsection is very general. There are not much to say about open-closed field algebras in such generality. In this subsection, we study those open-closed field algebras satisfying some nice analytic properties. In these cases, the whole structures can be reconstructed from some simple ingredients.

Definition 1.13. An open-closed field algebra $(V_{cl}, V_{op}, m_{cl-op})$ is called *analytic* if it satisfies the following conditions:

1. $\mathbb{Y}_{cl-op}$ can be extended to a map $V_{cl}\otimes V_{op}\times\mathbb{H}\times\overline{\mathbb{H}}\to\overline{V}_{op}$ such that for $z\in\mathbb{H},\zeta\in\overline{\mathbb{H}}$
$$\mathbb{Y}_{cl-op}(u;z,\bar{z})=\mathbb{Y}_{cl-op}(u;z,\zeta)|_{\zeta=\bar{z}}. \tag{1.49}$$
2. For $n\in\mathbb{N}$, $v_1,\cdots,v_{n+1}\in V_{op}, v'\in(V_{op})'$ and $u_1,\cdots,u_n\in V_{cl}$, the series
$$\langle v',\mathbb{Y}_{cl-op}(u_1,z_1,\zeta_1)Y_{op}(v_1,r_1)\cdots\mathbb{Y}_{cl-op}(u_n,z_n,\zeta_n)Y_{op}(v_n,r_n)v_{n+1}\rangle$$
is absolutely convergent when $|z_1|,|\zeta_1|>r_1>\cdots>|z_n|,|\zeta_n|>r_n>0$ and can be extended to a (possibly multivalued) analytic function on
$$\{(z_1,\zeta_1,r_1,\dots,z_n,\zeta_n,r_n)\in M^{3n}_{\mathbb{C}}\}.$$
3. For $n\in\mathbb{N}$, $v',u_1,\dots,u_{n+1}\in V_{cl}$, the series
$$\langle v',\mathbb{Y}_{an}(u_1;z_1,\zeta_1)\dots\mathbb{Y}_{an}(u_n;z_n,\zeta_n)u_{n+1}\rangle$$
is absolutely convergent when $|z_1|>\cdots>|z_n|>0$ and $|\zeta_1|>\cdots>|\zeta_n|>0$ and can be extended to an analytic function on $M^{2n}_{\mathbb{C}}$.
4. For $v'\in V'_{op}, v_1,v_2\in V_{op}, u\in V_{cl}$, the series
$$\langle v',Y_{op}(\mathbb{Y}_{cl-op}(u;z,\zeta)v_1,r)v_2\rangle$$
is absolutely convergent when $r>|z|,|\zeta|>0$.
5. For $v'\in V'_{op}, v\in V_{op}, u_1,u_2\in V_{cl}$, the series
$$\langle v',\mathbb{Y}_{cl-op}(\mathbb{Y}_{an}(u_1;z_1,\zeta_1)u_1;z_2,\zeta_2)v\rangle,$$
converges absolutely when $|z_2|>|z_1|>0$, $|\zeta_2|>|\zeta_1|>0$ and $|z_1|+|\zeta_1|<|z_2-\zeta_2|$.

By the convergence properties of open-closed field algebra, $m^{(l;n)}_{cl-op}$ can be expressed as products of $\mathbb{Y}_{cl-op},Y_{op}$ on a dense subdomain. $m^{(l;n)}_{cl-op}$ on the complement of this dense subdomain is completely determined by analytic extension. Therefore, $m^{(l;n)}_{cl-op}, l,n\in\mathbb{N}$ are completely determined by $\mathbf{1}_{op}$, Y_{op} and $\mathbb{Y}_{cl-op}$. Similarly, $m^{(n)}_{cl}, n\in\mathbb{N}$ is completely determined by $\mathbf{1}_{cl}$ and $\mathbb{Y}_{an}$. Therefore, we also denote an analytic open-closed conformal field algebra as
$$((V_{cl},\mathbb{Y},\mathbf{1}_{cl}),(V_{op},Y_{op},\mathbf{1}_{op}),\mathbb{Y}_{cl-op}).$$
or $(V_{cl},V_{op},\mathbb{Y}_{cl-op})$ for simplicity.

Two immediate consequences of the definition of analytic open-closed field algebra are given in the following two Lemmas.

Lemma 1.14. D_{op}-bracket properties*: $u\in V_{cl}$, we have*
$$\begin{aligned}[D_{op},\mathbb{Y}_{cl-op}(u;z,\zeta)] &= \mathbb{Y}_{cl-op}((D^L_{cl}+D^R_{cl})u;z,\zeta),\\ \mathbb{Y}_{cl-op}(D^L_{cl}u;z,\zeta) &= \frac{\partial}{\partial z}\mathbb{Y}_{cl-op}(u;z,\zeta),\\ \mathbb{Y}_{cl-op}(D^R_{cl}u;z,\zeta) &= \frac{\partial}{\partial \zeta}\mathbb{Y}_{cl-op}(u;z,\zeta).\end{aligned} \tag{1.50}$$

Proof. By (1.46),(1.47) and (1.48), for any fixed $z \in \mathbb{H}$, all three identities holds when $\zeta = \bar{z}$. Replacing u by $(D_{cl}^R)^k u$ for any $k \in \mathbb{N}$ in (1.46), (1.47) and (1.48), we obtained that all derivatives $(\frac{\partial}{\partial \zeta})^k|_{\zeta=\bar{z}}, \forall k \in \mathbb{N}$ of the both sides of above three identities equal respectively. By the property of analytic function and the fact that $\overline{\mathbb{H}}$ is a simply connected domain, we obtain that the both sides of above three identities, viewed as analytic functions for a fixed $z \in \mathbb{H}$ and any $\zeta \in \overline{\mathbb{H}}$, must equal respectively. ∎

Lemma 1.15. $\mathbf{d}_{op}$*-bracket properties : $u \in V_{cl}$, we have*

$$[\mathbf{d}_{op}, \mathbb{Y}_{cl-op}(u; z, \zeta)] = \mathbb{Y}_{cl-op}((\mathbf{d}_{cl}^L + \mathbf{d}_{cl}^R)u; z, \zeta) + \left(z\frac{\partial}{\partial z} + \zeta \frac{\partial}{\partial \zeta} \right) \mathbb{Y}_{cl-op}(u; z, \zeta). \tag{1.51}$$

Proof. The proof is the same as that of Lemma 1.14. ∎

An analytic open-closed field algebra $(V_{cl}, V_{op}, \mathbb{Y}_{cl-op})$ satisfies two associativities and two commutativities. These properties are very important for later formulations.

Proposition 1.16 (Associativity I). *For $u \in V_{cl}, v_1, v_2 \in V_{op}, v' \in V_{op}'$ and $z \in \mathbb{H}, \zeta \in \overline{\mathbb{H}}$, we have*

$$\langle v', \mathbb{Y}_{cl-op}(u; z, \zeta) Y_{op}(v_1, r)v_2 \rangle = \langle v', Y_{op}(\mathbb{Y}_{cl-op}(u; z-r, \zeta - r)v_1, r)v_2 \rangle \tag{1.52}$$

when $|z|, |\zeta| > r > 0$ and $r > |r-z|, |r-\zeta| > 0$.

Proof. We abbreviate the "left (right) hand side" as "LHS (RHS)" . For $z \in \mathbb{H}$,

$$\begin{aligned} \text{LHS of (1.52) } |_{\zeta=\bar{z}} &= m_{cl-op}^{(1;1)}(u; m_{op}^{(2)}(v_1, v_2; r, 0); z, \bar{z}; 0) \\ &= m_{cl-op}^{(1;2)}(u; v_1, v_2; z, \bar{z}; r, 0) \end{aligned} \tag{1.53}$$

when $|z| > r > 0$, and

$$\begin{aligned} \text{RHS of (1.52) } |_{\zeta=\bar{z}} &= m_{cl-op}^{(0;2)}(m_{cl-op}^{(1;1)}(u; v_1; z-r, \bar{z}-r; 0), v_2; r, 0) \\ &= m_{cl-op}^{(1;2)}(u; v_1, v_2; z, \bar{z}; r, 0) \end{aligned} \tag{1.54}$$

when $r > |r-z| > 0$. Therefore (1.52) holds when $\zeta = \bar{z}$ and $|z| > r > |r-z| > 0, z \in \mathbb{H}$. Now replace u in (1.52) by $(D_{cl}^R)^k u$, $k \in \mathbb{N}$, we obtain:

$$\left.\frac{\partial^k}{\partial \zeta^k}\right|_{\zeta=\bar{z}} \text{LHS of (1.52)} = \left.\frac{\partial^k}{\partial \zeta^k}\right|_{\zeta=\bar{z}} \text{RHS of (1.52)} \tag{1.55}$$

when $z \in \mathbb{H}$ and $|z| > r > |r-z| > 0$. Then by the properties of analytic function, it is clear that (1.52) holds for all $z \in \mathbb{H}, \zeta \in \overline{\mathbb{H}}$ and $|z|, |\zeta| > r > 0$ and $r > |r-z|, |r-\zeta| > 0$. ∎

Proposition 1.17 (Associativity II). *For $u_1, u_2 \in V_{cl}, v_1, v_2 \in V_{op}, v' \in V_{op}'$ and $z_1, z_2 \in \mathbb{H}, \zeta_1, \zeta_2 \in \overline{\mathbb{H}}$, we have*

$$\begin{aligned} &\langle w', \mathbb{Y}_{cl-op}(u_1; z_1, \zeta_1)\mathbb{Y}_{cl-op}(u_2; z_2, \zeta_2)v_2 \rangle \\ &\quad = \langle v', \mathbb{Y}_{cl-op}(\mathbb{Y}(u_1; z_1 - z_2, \zeta_1 - \zeta_2)u_2; z_2, \zeta_2)v_2 \rangle \end{aligned} \tag{1.56}$$

when $|z_1|, |\zeta_1| > |z_2|, |\zeta_2|$ and $|z_2| > |z_1 - z_2| > 0, |\zeta_2| > |\zeta_1 - \zeta_2| > 0$ and $|z_2 - \zeta_2| > |z_1 - z_2| + |\zeta_1 - \zeta_2|$.

Proof. The proof is similar to that of (1.52). So we will only sketch the difference here. First, it is easy to show that (1.56) is true for $z_i \in \mathbb{H}$, $\zeta_i = \bar{z}_i$, $i = 1, 2$ in a proper domain. Then replacing u_1, u_2 by $(D^R_{cl})^p u_1$, $(D^R_{cl})^q u_2$ for $p, q \in \mathbb{N}$ and using (1.29) and (1.50), we obtain that

$$\frac{\partial^p}{\partial \zeta_1^p}\frac{\partial^q}{\partial \zeta_2^q}\bigg|_{\zeta_i=\bar{z}_i} \text{LHS of (1.56)} = \frac{\partial^p}{\partial \zeta_1^p}\frac{\partial^q}{\partial \zeta_2^q}\bigg|_{\zeta_i=\bar{z}_i} \text{RHS of (1.56)} \tag{1.57}$$

in a proper domain. By the property of analytic function again, the analytic extension of both sides of (1.56) to the following simply connected domain

$$D_0 = \{(z_1, \zeta_1, z_2, \zeta_2) | z_i \in \mathbb{H}, \zeta_i \in \overline{\mathbb{H}}, |z_1|, |\zeta_1| > |z_2|, |\zeta_2|\}$$

must be identical. Moreover, the additional restrictions of domain in the statement of Associativity II guarantee the absolute convergence of both sides of (1.56). ∎

For an analytic open-closed field algebra, the map $\mathbb{Y}_{cl-op}$ can be uniquely extended to $V_{cl} \otimes V_{op} \times R$ where

$$R := \{(z, \zeta) \in \mathbb{C}^2 | z \in \mathbb{H} \cup \mathbb{R}_+, \zeta \in \overline{\mathbb{H}} \cup \mathbb{R}_+, z \neq \zeta\}.$$

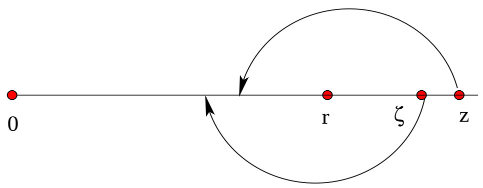
Proposition 1.18 (Commutativity I). *For $u \in V_{cl}, v_1, v_2 \in V_{op}$ and $v' \in V'_{op}$,*

$$\langle v', \mathbb{Y}_{cl-op}(u; z, \zeta) Y_{op}(v_1, r) v_2 \rangle, \tag{1.58}$$

which is absolutely convergent when $z > \zeta > r > 0$, and

$$\langle v', Y_{op}(v_1, r) \mathbb{Y}_{cl-op}(u; z, \zeta) v_2 \rangle, \tag{1.59}$$

which is absolutely convergent when $r > z > \zeta > 0$, are analytic continuation of each other along the following path [3].



(1.60)

Proof. When $z \in \mathbb{H}$, $\zeta = \bar{z}$ and $|z| > r > 0$, (1.58) is absolutely convergent to $m^{(1;2)}_{cl-op}(u; v_1, v_2; z, \bar{z}; r, 0)$. Hence, if we analytic extend the analytic function (1.58) to

$$D_1 := \{(z, \zeta) | z \in \mathbb{H}, \zeta = \bar{z}, |z| = r\}.$$

then its value on D_1 must equal to $m^{(1;2)}_{cl-op}(u; v_1, v_2; z, \bar{z}; r, 0)$ when $z \in \mathbb{H}, |z| = r$ by the continuity of $m^{(1;2)}_{cl-op}$.

Similarly the unique extension of (1.59) from $r > |z|, |\zeta| > 0$ to D_1 also equal to $m^{(1;2)}_{cl-op}(u; v_1, v_2; z, \bar{z}; r, 0)$ when $z \in \mathbb{H}, |z| = r$.

[3]Our extended domain is simply connected for fixed $r > 0$, all possible paths of analytic continuation are homotopically equivalent.

Both (1.58) and (1.59) can be uniquely extended to two analytic functions on $\mathbb{H}\times\overline{\mathbb{H}}$. By the discussion above, these two extended analytic functions of (z,ζ) take same value on D_1 which is of lower dimension.

Now we replace u by $(D^L_{cl})^m(D^R_{cl})^n u, m,n\in\mathbb{N}$ and repeat the arguments in the proof of Proposition 1.16, we obtain that all the derivatives of the two extended analytic functions also match on D_1. By the properties of analytic functions, these two extended functions are identical on $\mathbb{H}\times\overline{\mathbb{H}}$. Then their extensions to $(z,\zeta)\in R$ are also identical. Thus we have proved the first commutativity. ∎

Proposition 1.19 (Commutativity II)**.** *For $u_1,u_2\in V_{cl}, v\in V_{op}$ and $v'\in V'_{op}$,*

$$\langle v', \mathbb{Y}_{cl-op}(u_1;z_1,\zeta_1)\mathbb{Y}_{cl-op}(u_2;z_2,\zeta_2)v\rangle, \tag{1.61}$$

which is absolutely convergent when $z_1>\zeta_1>z_2>\zeta_2>0$, and

$$\langle v', \mathbb{Y}_{cl-op}(u_2;z_2,\zeta_2)\mathbb{Y}_{cl-op}(u_1;z_1,\zeta_1)v\rangle, \tag{1.62}$$

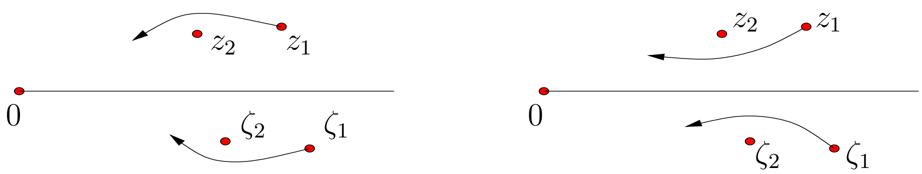
which is absolutely convergent when $z_2>\zeta_2>z_1>\zeta_1>0$, are analytic continuation of each other along the following paths.

(1.63)

Proof. Commutativity II follows directly from the Associativity II (1.56) and the skew-symmetry of the full field algebra V_{cl} [HKo2]. Here, we give a more direct proof which is similar to that of Proposition 1.18. The unique extension of (1.61) from $|z_1|,|\zeta_1|>|z_2|,|\zeta_2|>0$ to

$$D_2 := \{(z_1,\zeta_1,z_2,\zeta_2)|z_1,z_2\in M^2_{\mathbb{H}}, |z_1|=|z_2|, \zeta_i=\bar{z}_i, i=1,2\}$$

equal to $m^{(2;1)}_{cl-op}(u_1,u_2;v;z_1,\bar{z}_1,z_2,\bar{z}_2;0)$, the unique extension of (1.62) from $|z_2|,|\zeta_2|>|z_1|,|\zeta_1|>0$ to D_2 also match with $m^{(2;1)}_{cl-op}(u_1,u_2;v;z_1,\bar{z}_1,z_2,\bar{z}_2;0)$. By the similar argument as the proof of the first commutativity, we see that (1.61) in $|z_1|,|\zeta_1|>|z_2|,|\zeta_2|>0$ and (1.61) in $|z_2|,|\zeta_2|>|z_1|,|\zeta_1|>0$ are analytic continuation of each other along the following paths.



(1.64)

Then it is obvious to see that the unique extension of both (1.61) and (1.62) to the subdomain of $\{(z_1,\zeta_1,z_2,\zeta_2)\in\mathbb{R}^4_+\}$, where z_1,ζ_1,z_2,ζ_2 have distinct values, are analytic extension of each other along the paths (1.63). ∎

1.5 Open-closed field algebras over U

Now we gradually add more structures on an analytic open-closed field algebra such that this process eventually leads us to a notion called open-closed field algebra over a vertex operator algebra U. Let $(V^L, Y_{V^L}, \mathbf{1}^L)$ and $(V^R, Y_{V^R}, \mathbf{1}^R)$ be two vertex algebras. We now consider an analytic open-closed field algebra $(V_{cl}, V_{op}, \mathbb{Y}_{cl-op})$ such that V_{cl} is a full field algebra over $V^L \otimes V^R$. For such full field algebra V_{cl}, we will not distinguish V^L with $V^L \otimes \mathbf{1}^R \subset V_{cl}$ and V^R with $\mathbf{1}^L \otimes V^R \subset V_{cl}$.

Lemma 1.20. *For $u^L \in V^L$ and $u^R \in V^R$, $\mathbb{Y}_{cl-op}(u^L; z, \zeta)$ is independent of ζ and $\mathbb{Y}_{cl-op}(u^R; z, \zeta)$ is independent of z.*

Proof. For $u^L \in V^L, w \in V_{op}, w' \in (V_{op})'$, using the associativity (1.56), we have

$$\begin{aligned} \langle w', \mathbb{Y}_{cl-op}(u^L; z, \zeta) w\rangle &= \langle w', \mathbb{Y}_{cl-op}(u^L; z, \zeta)\mathbb{Y}_{cl-op}(\mathbf{1}_{cl}; z_1, \zeta_1) w\rangle \\ &= \langle w', \mathbb{Y}_{cl-op}(\mathbb{Y}_{an}(u^L; z - z_1, \zeta - \zeta_1)\mathbf{1}_{cl}; z_1, \zeta_1) w\rangle \quad (1.65) \end{aligned}$$

when $|z_1| > |z - z_1| > 0$, $|\zeta_1| > |\zeta - \zeta_1| > 0$ and $|z - z_1| + |\zeta - \zeta_1| < |z_1 - \zeta_1|$. The right hand side of (1.65) is independent of ζ and the left hand side of (1.65) is analytic in ζ. Hence $\mathbb{Y}_{cl-op}(u^L; z, \zeta)$ is independent of ζ for all $z \in \mathbb{H}$. Similarly, $\mathbb{Y}_{cl-op}(u^R; z, \zeta)$ is independent of z for all $\zeta \in \overline{\mathbb{H}}$ and $u^R \in V^R$. ∎

In order to emphasis these ζ- or z-independence properties, we denote them simply as $\mathbb{Y}_{cl-op}(u^L, z)$ and $\mathbb{Y}_{cl-op}(u^R, \zeta)$ for $u^L \in V^L$ and $u^R \in V^R$ respectively.

Replace u in (1.51) by $u^L \in V^L$ and $u^R \in V^R$ respectively, we obtain

$$\begin{aligned} [\mathbf{d}_{op}, \mathbb{Y}_{cl-op}(u^L, z)] &= \mathbb{Y}_{cl-op}(\mathbf{d}^L_{cl} u^L, z) + z\frac{\partial}{\partial z}\mathbb{Y}_{cl-op}(u^L, z), \\ \left[\mathbf{d}_{op}, \mathbb{Y}_{cl-op}(u^R, \zeta)\right] &= \mathbb{Y}_{cl-op}(\mathbf{d}^R_{cl} u^R, \zeta) + \zeta\frac{\partial}{\partial \zeta}\mathbb{Y}_{cl-op}(u^R, \zeta). \quad (1.66) \end{aligned}$$

As a consequence, we have

$$\begin{aligned} \mathbb{Y}_{cl-op}(u^L, z) &= \sum_{n\in\mathbb{R}} u^L_n z^{-n-1}, \\ \mathbb{Y}_{cl-op}(u^R, \zeta) &= \sum_{n\in\mathbb{R}} u^R_n \zeta^{-n-1}, \quad (1.67) \end{aligned}$$

where $u^L_n, u^R_n \in \text{End}\, V_{op}$ and $\text{wt}\, u^L_n = \text{wt}^L\, u^L - n - 1$ and $\text{wt}\, u^R_n = \text{wt}^R\, u^R - n - 1$, and $z^n = e^{n \log z}$ and $\zeta^n = e^{n\overline{\log \zeta}}$.

Moreover, we have

$$\begin{aligned} [D_{op}, \mathbb{Y}_{cl-op}(u^L, z)] &= \mathbb{Y}_{cl-op}(D^L_{cl} u^L, z) = \frac{\partial}{\partial z}\mathbb{Y}_{cl-op}(u^L, z), \quad (1.68) \\ \left[D_{op}, \mathbb{Y}_{cl-op}(u^R, \zeta)\right] &= \mathbb{Y}_{cl-op}(D^R_{cl} u^R, \zeta) = \frac{\partial}{\partial \zeta}\mathbb{Y}_{cl-op}(u^R, \zeta), \end{aligned}$$

which further implies that

$$\begin{aligned} e^{aD_{op}}\mathbb{Y}_{cl-op}(u^L, z)e^{-aD_{op}} &= \mathbb{Y}_{cl-op}(u^L, z + a) \\ e^{aD_{op}}\mathbb{Y}_{cl-op}(u^L, \zeta)e^{-aD_{op}} &= \mathbb{Y}_{cl-op}(u^R, \zeta + a) \quad (1.69) \end{aligned}$$

for $|z| > |a|, z + a \in \mathbb{H}$ and $|\zeta| > |a|, \zeta + a \in \overline{\mathbb{H}}$ respectively.

Lemma 1.21. *For any $u^L \in V^L \otimes \mathbf{1}^R$ and $u^R \in \mathbf{1}^L \otimes V^R$, the following two limits:*

$$\lim_{z\to 0} \mathbb{Y}_{cl-op}(u^L, z)\mathbf{1}_{op}, \qquad \lim_{z\to 0} \mathbb{Y}_{cl-op}(u^R, \zeta)\mathbf{1}_{op}$$

exist in V_{op}.

Proof. By the associativity and the creation property of open-string vertex algebra, we have

$$\begin{aligned}\mathbb{Y}_{cl-op}(u^L, z+r)\mathbf{1}_{op} &= \mathbb{Y}_{cl-op}(u^L, z+r)Y_{op}(\mathbf{1}_{op}, r)\mathbf{1}_{op} \\ &= Y_{op}(\mathbb{Y}_{cl-op}(u^L, z)\mathbf{1}_{op}, r)\mathbf{1}_{op} \\ &= e^{rD_{op}}\mathbb{Y}_{cl-op}(u, z)\mathbf{1}_{op} \qquad (1.70)\end{aligned}$$

when $|z+r| > r > |z| > 0$. For fixed $z \in \mathbb{H}$, the left hand side of (1.70) is an analytic function valued in $\overline{V}_{op}$ on the domain $\{r \in \mathbb{C} | z+r \neq 0\}$. The right hand side of (1.70), as a power series of r, is absolutely convergent when $|r| > |z| > 0$. By the general property of power series, the right hand side of (1.70) is absolutely convergent for all $r \in \mathbb{C}$ to a singlevalued analytic function. Because both sides of (1.70) are analytic functions, the equality (1.70) must hold for all $r \in \mathbb{C}$. In particular, $\lim_{r\to -z} \mathbb{Y}_{cl-op}(u^L, z+r)\mathbf{1}_{op}$ exists. Equivalently, $\lim_{z\to 0} \mathbb{Y}_{cl-op}(u^L, z)\mathbf{1}_{op}$ exists. By the expansion (1.67), we must have $u_n^L\mathbf{1}_{op} = 0$ for all $n > -1$. Moreover, it is also easy to see that $u_n^L\mathbf{1}_{op} = 0$ for any $n \notin -\mathbb{Z}_+$ by (1.68) (see the proof of Proposition 1.8 in [HKo1]). Therefore, we have

$$\lim_{z\to 0} \mathbb{Y}_{cl-op}(u^L, z)\mathbf{1}_{op} = u_{-1}^L\mathbf{1}_{op} \in V_{op}.$$

The proof of the existence of the second limit is entirely same. ∎

By the Lemma above, we can define two maps $h^L : V^L \to V_{op}$ and $h^R : V^R \to V_{op}$ as follow: for all $u^L \in V^L$ and $u^R \in V^R$,

$$\begin{aligned}h^L : u^L &\mapsto \lim_{z\to 0} \mathbb{Y}_{cl-op}(u^L, z)\mathbf{1}_{op}, \\ h^R : u^R &\mapsto \lim_{\zeta\to 0} \mathbb{Y}_{cl-op}(u^R, \zeta)\mathbf{1}_{op}. \qquad (1.71)\end{aligned}$$

Notice also that h^L, h^R preserve the weights. Namely

$$\begin{aligned}\text{wt } h^L(u^L) &= \text{wt } (u_{-1}^L\mathbf{1}_{op}) = \text{wt}^L\ u^L, \\ \text{wt } h^R(u^R) &= \text{wt } (u_{-1}^R\mathbf{1}_{op}) = \text{wt}^R\ u^R.\end{aligned}$$

Therefore both h^L and h^R can be naturally extended to maps $\overline{V^L} \to \overline{V_{op}}$ and $\overline{V^R} \to \overline{V_{op}}$. We still denote the extended maps as h^L and h^R respectively.

Lemma 1.22. *For $u^L \in V^L, u^R \in V^R$, we have*

$$\begin{aligned}\mathbb{Y}_{cl-op}(u^L, z)\mathbf{1}_{op} &= e^{zD_{op}}h^L(u^L), \\ \mathbb{Y}_{cl-op}(u^R, \zeta)\mathbf{1}_{op} &= e^{\zeta D_{op}}h^R(u^R). \qquad (1.72)\end{aligned}$$

Proof. Since we have shown that (1.70) holds for all $r \in \mathbb{C}, z \in \mathbb{H}$ and both sides of (1.70) are analytic for all $r, z \in \mathbb{C}$, if we take the limit $\lim_{z\to 0}$ on both sides of (1.70), the equality should still holds. Thus we obtain the first identity in (1.72).

The proof of the second identity is entirely same. ∎

Proposition 1.23. *h^L and h^R are homomorphisms from V^L and V^R respectively to their images, viewed as graded vertex algebras.*

Proof. We have shown that h^L, h^R preserve gradings. By the identity property of open-closed field algebra, we have

$$h^L(\mathbf{1}_{cl}) = \mathbf{1}_{op}.$$

Next, for $u^L \in V^L$ and $r > 0$, we define

$$\begin{aligned} \mathbb{Y}_{cl-op}(u^L, r)w &:= \lim_{z\to r} \mathbb{Y}_{cl-op}(u^L, z)w \\ &= \lim_{z\to r} \mathbb{Y}_{cl-op}(u^L, z)Y_{op}(\mathbf{1}_{op}, r)w \end{aligned} \tag{1.73}$$

where the limit is taken along a path from a fixed initial point in $\mathbb{H}$ to $r > 0$. Since $\mathbb{Y}_{cl-op}(u^L, z)w$ is analytic in $\mathbb{C}^\times$, the limit is independent of the path we choose. So we choose a path in the domain $\{z \in \mathbb{H} | |z| > r > |z - r| > 0\}$. In this domain, we can apply the associativity (1.52) to the right hand side of (1.73). We obtain

$$\begin{aligned} \mathbb{Y}_{cl-op}(u^L, r)w &= \lim_{z\to r} Y_{op}(\mathbb{Y}_{cl-op}(u^L, z - r)\mathbf{1}_{op}, r)w \\ &= Y_{op}(h^L(u^L), r)w. \end{aligned}$$

For $|z| > r > 0$, by (1.72), we have,

$$\mathbb{Y}_{cl-op}(\mathbb{Y}(u^L; r, r)v^L, z)\mathbf{1}_{op} = e^{zD_{op}} h^L(\mathbb{Y}(u^L; r, r)v^L), \tag{1.74}$$

the right hand side of which is absolutely convergent for all $z \in \mathbb{C}$. And both sides are analytic in z. Therefore $\mathbb{Y}_{cl-op}(\mathbb{Y}(u^L; r, r)v^L, z)\mathbf{1}_{op}$ is absolutely convergent for all $z \in \mathbb{C}$ and (1.74) holds for all $z \in \mathbb{C}$. By the associativity, we have

$$\mathbb{Y}_{cl-op}(\mathbb{Y}(u^L; r, r)v^L, z)\mathbf{1}_{op} = \mathbb{Y}_{cl-op}(u^L, r + z)\mathbb{Y}_{cl-op}(v^L, z)\mathbf{1}_{op} \tag{1.75}$$

for $|r + z| > |z| > r > 0$. Again both sides of (1.75) are analytic in z. Hence the left hand side of (1.75) defined for all $z \in \mathbb{C}$ is the analytic extension of the right hand side of (1.75), which is defined on $\{|r + z| > |z|\}$. Since the extension is free of singulairity on entire $\mathbb{C}$, the right hand side of (1.75) must be well-defined on entire $\mathbb{C}$. Therefore, we must have

$$\lim_{z\to 0} \mathbb{Y}_{cl-op}(\mathbb{Y}(u^L, r)v^L, z)\mathbf{1}_{op} = \lim_{z\to 0} \mathbb{Y}_{cl-op}(u^L, r + z)\mathbb{Y}_{cl-op}(v^L, z)\mathbf{1}_{op}. \tag{1.76}$$

Combining above results, we have

$$\begin{aligned} h^L(\mathbb{Y}(u^L, r)v^L) &= \lim_{z\to 0} e^{zD_{op}} h^L(\mathbb{Y}(u^L, r)v^L) \\ &= \lim_{z\to 0} \mathbb{Y}_{cl-op}(\mathbb{Y}(u^L, r)v^L, z)\mathbf{1}_{op} \\ &= \lim_{z\to 0} \mathbb{Y}_{cl-op}(u^L, r + z)\mathbb{Y}_{cl-op}(v^L, z)\mathbf{1}_{op} \\ &= \mathbb{Y}_{cl-op}(u^L, r)h^L(v^L) \\ &= Y_{op}(h^L(u^L), r)h^L(v^L). \end{aligned}$$

Thus h^L is a vertex algebra homomorphism. The proof for h^R is entirely same. ∎

Let $(U, Y, \mathbf{1}, \omega)$ be a vertex operator algebra with central charge c. U and $U \otimes U$ naturally give an analytic open-closed field algebra, in which $\mathbb{Y}_{cl-op}(\cdot; z, \zeta)\cdot$ is given by

$$\begin{aligned} \mathbb{Y}_{cl-op}(u \otimes v; z, \zeta)w &= Y(u, z)Y(v, \zeta)w, \qquad |z| > |\zeta| > 0, \\ &= Y(v, \zeta)Y(u, z)w, \qquad |\zeta| > |z| > 0 \end{aligned} \tag{1.77}$$

for $u, v, w \in U$. In this case, $h^L : u \otimes \mathbf{1} \mapsto u$ and $h^R : \mathbf{1} \otimes u \mapsto u$. We denote this open-closed field algebra as $(U \otimes U, U)$.

In general, let $\rho^L, \rho^R \in \mathrm{Aut}(U)$ where $\mathrm{Aut}(U)$ is the set of automorphisms of U as vertex operator algebra. We can obtain a new action of $U \otimes U$ on U by composing (1.77) with the automorphism $\rho^L \otimes \rho^R : U \otimes U \to U \otimes U$. Namely, there exists another open-closed field algebra structure on U and $U \otimes U$, in which $\mathbb{Y}_{cl-op}(u \otimes v; z, \zeta)w$, for $u, v, w \in U$, is given by

$$\begin{aligned} &Y(\rho^L(u), z)Y(\rho^R(v), \zeta)w, \qquad \text{for } |z| > |\zeta| > 0, \\ &Y(\rho^R(v), \zeta)Y(\rho^L(u), z)w, \qquad \text{for } |\zeta| > |z| > 0. \end{aligned} \tag{1.78}$$

In this case, $h^L : u \otimes \mathbf{1} \mapsto \rho^L(u)$ and $h^R : \mathbf{1} \otimes u \mapsto \rho^R(u)$. We denote such open-closed field algebra as $(U \otimes U, U, \rho^L, \rho^R)$. In particular, $(U \otimes U, U, \mathrm{id}_U, \mathrm{id}_U)$ is just $(U \otimes U, U)$.

Remark 1.24. $(U \otimes U, U, \rho^L, \rho^R)$ for general automorphisms ρ^L and ρ^R is very interesting in physics. But it adds some technical subtleties in later formulations. So we postpone its study to future publications. In this work, we focus on $(U \otimes U, U)$.

Definition 1.25. Let $(U, Y, \mathbf{1}, \omega)$ be a vertex operator algebra. An *open-closed field algebra over U* is an analytic open-closed field algebra

$$((V_{cl}, m_{cl}, \iota_{cl}), (V_{op}, Y_{op}, \iota_{op}), \mathbb{Y}_{cl-op}),$$

where $(V_{cl}, m_{cl}, \iota_{cl})$ is a conformal full field algebra over $U \otimes U$ and $(V_{op}, Y_{op}, \iota_{op})$ is an open-string vertex operator algebra over U, satisfying the following conditions:

1. *U-invariant boundary condition*: $h^L = h^R = \iota_{op}$.

2. *Chirality splitting property*: $\forall u \in V_{cl}$, $u = u^L \otimes u^R \in W^L \otimes W^R \subset V_{cl}$ for some U-modules W^L, W^R. There exist U-modules W_1, W_2 and intertwining operators $\mathcal{Y}^{(1)}, \mathcal{Y}^{(2)}, \mathcal{Y}^{(3)}, \mathcal{Y}^{(4)}$ of type $\binom{V_{op}}{W^L W_1}$, $\binom{W_1}{W^R V_{op}}$, $\binom{V_{op}}{W^R W_2}$, $\binom{W_2}{W^L V_{op}}$ respectively[4], such that
$$\langle w', \mathbb{Y}_{cl-op}(u, z, \zeta)w\rangle = \langle w', \mathcal{Y}^{(1)}(u^L, z)\mathcal{Y}^{(2)}(u^R, \zeta)w\rangle \tag{1.79}$$
when $|z| > |\zeta| > 0$ (recall the convention (1.16) and (1.17)), and
$$\langle w', \mathbb{Y}_{cl-op}(u, z, \zeta)w\rangle = \langle w', \mathcal{Y}^{(3)}(u^R, \zeta)\mathcal{Y}^{(4)}(u^L, z)w\rangle \tag{1.80}$$
when $|\zeta| > |z| > 0$ for all $u \in V_{cl}, w \in V_{op}, w' \in V_{op}$.

[4]It was proved in [HKo1] that V_{op} is a U-modules, and in [HKo2] that V_{cl} is a $U \otimes U$-modules.

In the case, U is generated by ω, i.e. $U = \langle \omega \rangle$. The $\langle \omega \rangle$-invariant boundary condition is simply called *conformal invariant boundary condition*. We also call open-closed field algebra over $\langle \omega \rangle$ in this case simply as *open-closed conformal field algebra*.

Remark 1.26. The U-invariant boundary condition actually says that the open-closed field algebra over U contains $(U \otimes U, U)$ as a subalgebra. If we only want to construct algebras over Swiss-cheese partial operad, the U-invariant boundary condition in Definition 1.25 can be weaken to the *conformal invariant boundary condition*:

$$h^L|_{\langle \omega \rangle} = h^R|_{\langle \omega \rangle} = \iota_{op}|_{\langle \omega \rangle}. \tag{1.81}$$

These situations appear in physics in the study of the so-called symmetry breaking boundary conditions (see for example [FS1][FS2] and references therein). All examples studied in this work and [Ko2] satisfy the U-invariant boundary condition. We leave the study of general symmetry-broken situations to the future.

Remark 1.27. The chirality splitting property is a very natural condition because the interior sewing operation of $\tilde{\mathfrak{S}}^c$ is defined by a double sewing operation as given in (2.26). Unfortunately, we do not know whether this chirality splitting property is necessary for general constructions of algebras over $\tilde{\mathfrak{S}}^c$.

For an open-closed field algebra over V, there are three Virasoro elements, $\omega_{op} := \iota_{op}(\omega)$ and $\omega^L := \iota_{cl}(\omega \otimes \mathbf{1})$ and $\omega^R := \iota_{cl}(\mathbf{1} \otimes \omega)$, and we have

$$\begin{aligned} Y_{op}(\omega_{op}, r) &= \sum_{n \in \mathbb{Z}} L(n) r^{-n-2}, \\ \mathbb{Y}(\omega^L, z) &= \sum_{n \in \mathbb{Z}} L^L(n) z^{-n-2}, \\ \mathbb{Y}(\omega^R, \zeta) &= \sum_{n \in \mathbb{Z}} L^R(n) \zeta^{-n-2}. \end{aligned}$$

where $L^L(n) = L(n) \otimes 1$ and $L^R(n) = 1 \otimes L(n)$ for $n \in \mathbb{Z}$.

When $U = V$ a vertex operator algebra satisfying the conditions in Theorem 0.2, we have a very simple description of open-closed field algebra over V given in the following Theorem.

Theorem 1.28. *An open-closed field algebra over V is equivalent to the following structure: $(V_{op}, Y_{op}, \iota_{op})$ an open-string vertex operator algebra over V and $(V_{cl}, m_{cl}, \iota_{cl})$ a conformal full field algebra over $V \otimes V$, together with a vertex map $\mathbb{Y}_{cl-op}(\cdot; z, \zeta)\cdot$ given by intertwining operators $\mathcal{Y}^{(i)}, i = 1, 2, 3, 4$ as in (1.79) and (1.80), satisfying the unit property:*

$$\mathbb{Y}_{cl-op}(\mathbf{1}_{cl}; z, \bar{z}) v = v, \qquad \forall v \in V_{op}, \tag{1.82}$$

the associativity (1.52) and (1.56) and the commutativity given in Proposition 1.18.

Proof. It is clear that an open-closed field algebra over V gives the data and properties included in the statement of the Theorem. We only need to show that such data is sufficient to reconstruct an open-closed field algebra over V. Moreover, such open-closed field algebra over V with the given data is unique.

Since V satisfies the condition in Theorem 0.2, the conditions listed in the Definition 1.13 are all automatically satisfied. In particular, for $v_i \in V_{op}, u_i \in V_{cl}, i = 1, \dots, n$ and $v' \in V'_{op}$, the following series:

$$\langle v', \mathbb{Y}_{cl-op}(u_1, z_1, \zeta_1) Y_{op}(v_1, r_1) \cdots \mathbb{Y}_{cl-op}(u_n, z_n, \zeta_n) Y_{op}(v_n, r_n) \mathbf{1}_{op} \rangle. \tag{1.83}$$

is absolutely convergent when $z_1 > \zeta_1 > r_1 > \cdots > z_n > \zeta_n > r_n > 0$, and can be extended by analytic continuation to a multi-valued analytic function for variables $z_i, \zeta_i \in \mathbb{C}^\times, i = 1, \dots, n$ with possible singularities only when two of z_i, ζ_j, r_k are equal. Using this property of (1.83), we can define a non-analytic but single-valued smooth function in $\Lambda^n \times M^n_{\mathbb{H}}$ as follow.

Let $(r_1, \dots, r_n) \in \Lambda^n$ and $(\xi_1, \dots, \xi_n) \in M^n_{\mathbb{H}}$, $n \in \mathbb{Z}_+$. Let γ_1 be a smooth path from $(3n, 3(n-1), \dots, 1)$ to $(\xi_1, \dots, \xi_n)$ such that $\gamma_1((0,1)) \subset M^n_{\mathbb{H}}$. Let $\gamma_0 : [0,1] \to \mathbb{R}^n_+$ be so that

$$\gamma_0(t) = ((1-t)(3n-1) + t3n, \dots, (1-t)2 + t3).$$

Clearly, γ_0 is the straight line from $(3n-1, 3(n-1)-1, \dots, 2)$ to $(3n, 3(n-1), \dots, 3)$ in $\mathbb{R}^n_+$. Then we define a path $\gamma_2 : [0,1] \in (\overline{\mathbb{H}} \cup \mathbb{R}_+)^n$ to be the composition $\gamma_2 = \bar{\gamma}_1 \circ \gamma_0$ where $\bar{\gamma}_1$ is the complex conjugation of γ_1. It is clear that γ_2 is a path from $(3n-1, 3(n-1)-1, \dots, 2)$ to $(\bar{\xi}_1, \dots, \bar{\xi}_n)$. Let $\gamma_3 : [0,1] \to \mathbb{R}^n_+$ be so that

$$\gamma_3(t) = ((1-t)(3n-2) + tr_1, \dots, (1-t)1 + tr_n).$$

So γ_3 is the straight line from $(3n-2, 3(n-1)-2, \dots, 1)$ to $(r_1, \dots, r_n) \in \Lambda^n$.

Combining $\gamma_1, \gamma_2, \gamma_3$, we obtain a path γ in $\mathbb{C}^{3n}$ from the initial point $(3n, 3n-1, 3n-2, \dots, 3, 2, 1) \in \mathbb{R}^{2n}_+$ to the final point $(\xi_1, \bar{\xi}_1, r_1, \dots, \xi_n, \bar{\xi}_n, r_n)$ in the obvious way.

Then we define

$$\langle v', m^{(n;n)}_{cl-op}(u_1, \dots, u_n, v_1, \dots, v_n; \xi_1, \bar{\xi}_1, \dots, \xi_n, \bar{\xi}_n, r_1, \dots, r_n) \rangle$$

to be the value obtained from the value of (1.83) at the initial point $(3n, 3n-1, 3n-2, \dots, 3, 2, 1)$ by analytic continuation along the path γ. Following a similar argument as in the proof of Theorem 2.11 in [HKo2], it is easy to show, that such defined

$$m^{(n;n)}_{cl-op}(u_1, \dots, u_n; v_1, \dots, v_n; \xi_1, \bar{\xi}_1, \dots, \xi_n, \bar{\xi}_n, r_1, \dots, r_n), \tag{1.84}$$

is independent of the choice of γ_1 and its initial points. Moreover, such defined (1.84) is single-valued and smooth in $\mathbb{H} \times \overline{\mathbb{H}}$.

For $n > l \geq 0$, we define

$$\begin{aligned} &m^{(l;n)}_{cl-op}(u_1, \dots, u_l; v_1, \dots, v_n; \xi_1, \bar{\xi}_1, \dots, \xi_l, \bar{\xi}_l; r_1, \dots, r_n), \\ &\quad := m^{(n;n)}_{cl-op}(u_1, \dots, u_l, \mathbf{1}_{cl}, \dots, \mathbf{1}_{cl}; v_1, \dots, v_n; \xi_1, \bar{\xi}_1, \dots, \xi_n, \bar{\xi}_n; r_1, \dots, r_n), \end{aligned}$$

and for $l > n \geq 0$, we define

$$\begin{aligned} &m^{(l;n)}_{cl-op}(v_1, \dots, v_l; w_1, \dots, w_n; \xi_1, \bar{\xi}_1, \dots, \xi_l, \bar{\xi}_l; r_1, \dots, r_n), \\ &\quad := m^{(l;l)}_{cl-op}(u_1, \dots, u_l; v_1, \dots, v_n, \mathbf{1}_{op}, \dots, \mathbf{1}_{op}; \xi_1, \bar{\xi}_1, \dots, \xi_l, \bar{\xi}_l; r_1, \dots, r_l), \end{aligned} \tag{1.85}$$

and for $n = l = 0$, we define $m^{(0;0)}_{cl-op}(1) = \mathbf{1}_{op} \in V_{op}$.

Immediately following from the construction of $m^{(l;n)}_{cl-op}$, we have, for all $v \in V_{op}$,

$$
\begin{aligned}
m^{(0;1)}_{cl-op}(v;0) &= m^{(1;1)}_{cl-op}(\mathbf{1}_{cl}; v; z; 0) \\
&= \mathbb{Y}_{cl-op}(\mathbf{1}_{cl}; z, \bar{z})v \\
&= v.
\end{aligned}
\tag{1.86}
$$

Now we show the permutation axiom for $m^{(l;n)}_{cl-op}$. This is enough to just consider adjacent permutations $(ii+1), i = 1, \dots, l-1$ because they generate the whole permutation group. We can just consider (12) because all the other cases are exactly same. This is amount to show that

$$
\begin{aligned}
&m^{(l;n)}_{cl-op}(v_1, v_2, \dots, v_l; w_1, \dots, w_n; \xi_1, \bar{\xi}_1, \xi_2, \bar{\xi}_2, \dots, \xi_l, \bar{\xi}_l; r_1, \dots, r_n), \\
&\quad = m^{(l;n)}_{cl-op}(v_2, v_1, \dots, v_l; w_1, \dots, w_n; \xi_2, \bar{\xi}_2, \xi_1, \bar{\xi}_1, \dots, \xi_l, \bar{\xi}_l; r_1, \dots, r_n).
\end{aligned}
\tag{1.87}
$$

By our construction, the only difference of the two sides of (1.87) is that they are obtained by analytic continuation along paths with different initial points. The initial points of the path for the left hand side of (1.87) is $z_1 > \zeta_1 > r_1 > z_2 > \zeta_2 > r_2 > \dots$, that for the right hand side of (1.87) is $z_2 > \zeta_2 > r_1 > z_1 > \zeta_1 > r_2 > \dots$. But by the commutativity, the value of (1.83) at these two initial points are analytic continuation of each other along the paths given in the commutativity axiom of open-closed field algebra. Hence the equality (1.87) follows.

((1.38) and (1.39)) can be proved by first proving similar properties of (1.83), which is obvious by the properties of intertwining operators. Then those properties (1.38) and (1.39) of $m^{(l;n)}_{cl-op}$ follow from analytic continuations.

Since $\mathcal{Y}^{(i)}, i = 1, 2, 3, 4$ are intertwining operators of V, we have, for $u \in V$,

$$
\begin{aligned}
h^L(u) &= \lim_{z\to 0} \mathbb{Y}_{cl-op}(u, z)\mathbf{1}_{op} \\
&= \lim_{z\to 0} \lim_{z_1\to 0} \mathcal{Y}^{(1)}(Y(u, z_1)\mathbf{1}, z)\mathcal{Y}^{(2)}(\mathbf{1}, \bar{z})\iota_{op}(\mathbf{1}) \\
&= \lim_{z\to 0} \lim_{z_1\to 0} Y_{V_{op}}(u, z + z_1)\mathcal{Y}^{(1)}(\mathbf{1}, z)\mathcal{Y}^{(2)}(\mathbf{1}, \bar{z})\iota_{op}(\mathbf{1}) \\
&= \lim_{z\to 0} \lim_{z_1\to 0} Y_{V_{op}}(u, z + z_1)\mathbb{Y}_{cl-op}(\mathbf{1} \otimes \mathbf{1}; z, \bar{z})\iota_{op}(\mathbf{1}) \\
&= \lim_{z\to 0} \lim_{z_1\to 0} Y_{V_{op}}(u, z + z_1)\iota_{op}(\mathbf{1}) \\
&= \lim_{z\to 0} \lim_{z_1\to 0} \iota_{op}(Y(u, z + z_1)\mathbf{1}) \\
&= \iota_{op}(u).
\end{aligned}
\tag{1.88}
$$

Similarly, one can show that $h^R(u) = \iota_{op}(u), \forall u \in V$. Thus we have proved the V-invariant boundary condition $h^L = h^R = \iota_{op}$.

It remains to show the convergence properties of open-closed field algebra. For the first convergence property ((1.33) and (1.34)), one first consider cases when $z^{(i)}_j, r^{(i)}_p, j = 1, \dots, k, p = 1, \dots, m$ in (1.33) have distinct absolute values and $z_j, r_p, j = 1, \dots, l, p = 1, \dots, n$ in (1.33) have distinct absolute values. In these cases, one can express $m^{(l;n)}_{cl-op}$ and $m^{(k;m)}_{cl-op}$ as products of $\mathbb{Y}_{cl-op}$ and Y_{op}. Then by using the associativity (1.52) and that of open-string vertex operator algebra, it is easy to show that (1.33) converges absolutely to (1.34) in the required domain.

The rest cases can all be reduced to above cases by using (1.39) (see the proof of Theorem 2.11 in [HKo2] for reference). More precisely, for $z_1^{(i)}, \dots, z_k^{(i)}$, $z_1, \dots, z_l \in \mathbb{H}$; $r_1^{(i)} > \cdots > r_m^{(i)} \geq 0$, $r_1 > \cdots > r_n > 0$[5], there always exists $a \in \mathbb{R}_+$ small enough so that both of the following sets

$$\{z_1^{(i)} + a, \dots, z_k^{(i)} + a, r_1^{(i)} + a, \dots, r_k^{(i)} + a\},$$
$$\{z_1 - a, \dots, z_l - a, r_1 - a, \dots, r_n - a\}$$

are sets whose elements have distinct absolute values. Then (1.33) equals to the following iterate series:

$$\sum_{n_2}\sum_{n_1} m_{cl-op}^{(l;n)}(u_1, \dots, u_l; v_1, \dots, v_{i-1}, P_{n_2} e^{-aD_{op}} P_{n_1} m_{cl-op}^{(m;k)}(\tilde{u}_1, \dots, \tilde{u}_k;$$
$$\tilde{v}_1, \dots, \tilde{v}_m; z_1^{(i)} + a, \overline{z_1^{(i)}} + a, \dots, z_k^{(i)} + a, \overline{z_k^{(i)}} + a; r_1^{(i)} + a, \dots, r_m^{(i)} + a),$$
$$v_{i+1}, \dots v_n; z_1, \bar{z}_1, \dots, z_l, \bar{z}_l; r_1, \dots, r_n) a_1^{n_1} a_2^{n_2} \tag{1.89}$$

when $a_1 = a_2 = 1$. Hence we first switch the order of above iterate sum. Then using (1.41) and the analytic extension properties of (1.83), we can easily show that the iterate series (1.89) with opposite summing order is absolutely convergent when $1 \geq |a_1|, |a_2|$. Then we can switch the order of iterate sum in (1.89) freely without changing the value of the double sum (1.89). By (1.41), we have reduced all the remaining cases to the previous cases.

The proof of the second convergence property ((1.35) and (1.36)) is entirely same except that one use the associativity (1.56) and (1.42). We omit the details.

It is also clear that such open-closed field algebra over V is unique because products of $\mathbb{Y}_{cl-op}$ and Y_{op} determine a dense subset of the set of all cases. The rest cases are uniquely determined by continuity. ∎

2 Operadic formulation

In this section, we first review the notion of 2-colored partial operad and algebra over it. Then we recall the notion of Swiss-cheese partial operad $\mathfrak{S}$, its relation to sphere partial operad K and its $\mathbb{C}$-extension $\tilde{\mathfrak{S}}^c$ [HKo1]. In the end, we prove that an open-closed field algebra over V canonically gives an algebra over $\tilde{\mathfrak{S}}^c$.

2.1 2-colored partial operads

We recall the notion of 2-colored (partial) operad [V][Kont] and algebras over it. The 2-colored operad is called *relative operad* by Voronov in [V].

We first recall some basic notions from [H4]. The notion of (partial) operad can be defined in any symmetric monoidal category [MSS]. In this work, we only work in

[5] If $r_n = 0$, we can further introduce another small real variable b, using (1.39) to move $r_n = 0$ to some $r_n' > 0$. We omit the detail.

the category of small sets. We will use the definition of (partial) operad given in [H4]. We denote a (partial) operad as a triple $(\mathcal{P}, I_{\mathcal{P}}, \gamma_{\mathcal{P}})$, where $\mathcal{P} = \{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ is a family of sets, $I_{\mathcal{P}}$ the identity element and $\gamma_{\mathcal{P}}$ the substitution map. Note that the definition of (partial) operad in [H4] is slightly different from that in [MSS] for the appearance of $\mathcal{P}(0)$ which is very important for the study of conformal field theory. If a triple $(\mathcal{P}, I_{\mathcal{P}}, \gamma_{\mathcal{P}})$ satisfies all the axioms of a (partial) operad except the associativity of $\gamma_{\mathcal{P}}$, then it is called *(partial) nonassociative operad*.

We consider an important example of partial nonassociative operad. Let $U = \oplus_{n\in J} U_{(n)}$ be a vector space graded by an index set J and $E_U = \{E_U(n)\}_{n\in\mathbb{N}}$ a family of vector spaces, where

$$E_U(n) = \mathrm{Hom}_{\mathbb{C}}(U^{\otimes n}, \overline{U}).$$

For $k, n_1, \ldots, n_k \in \mathbb{N}$, $f \in E_U(k), g_i \in E_U(n_i), i = 1, \ldots, k$ and $v_j \in U, j = 1, \ldots, n_1 + \cdots + n_k$,

$$\begin{aligned}
&\gamma_{E_U}(f; g_1, \ldots, g_k)(v_1 \otimes \cdots \otimes v_{n_1+\cdots+n_k}) \\
&\quad := \sum_{s_1,\ldots,s_k\in J} f(P_{s_1} g_1(v_1 \otimes \cdots \otimes v_{n_1}) \otimes \cdots \\
&\qquad\qquad \otimes P_{s_k} g_k(v_{n_1+\cdots+n_{k-1}+1} \otimes \cdots \otimes v_{n_1+\cdots+n_k}))
\end{aligned} \tag{2.1}$$

is well-defined if the sum is countable and absolutely convergent. γ_{E_U} is not associative in general because an iterate series may converge in both order but may not converge to the same value. It is clear that $(E_U, \mathrm{id}_U, \gamma_{E_U})$ is a partial nonassociative operad. We sometimes denote it simply as E_U. If J is the set of equivalent classes of irreducible modules over a group G and $U_{(n)}$ is a direct sum of irreducible G-modules of equivalent class $n \in J$, we denote this partial nonassociative operad as E_U^G.

Definition 2.1. An *algebra over a partial operad* $(\mathcal{P}, I_{\mathcal{P}}, \gamma_{\mathcal{P}})$, or simply a $\mathcal{P}$-algebra, is a graded vector space U, together with a partial nonassociative operad homomorphism $\nu : \mathcal{P} \to E_U$.

Definition 2.2. Given a partial operad $(\mathcal{P}, I_{\mathcal{P}}, \gamma_{\mathcal{P}})$, a subset G of $\mathcal{P}(1)$ is called *rescaling group* for $\mathcal{P}$ if

1. For any $n \in \mathbb{N}$, $P_i \in G, i = 0, \ldots, n$ and $P \in \mathcal{P}(n)$, $\gamma_{\mathcal{P}}(P; P_1, \ldots, P_n)$ and $\gamma_{\mathcal{P}}(P_0; P)$ are well-defined.

2. $I_{\mathcal{P}} \in G$ and G together with the identity $I_{\mathcal{P}}$ and multiplication map $G \times G \xrightarrow{\gamma_{\mathcal{P}}} G$ is a group.

Definition 2.3. A partial operad $(\mathcal{P}, I_{\mathcal{P}}, \gamma_{\mathcal{P}})$ is called *G-rescalable*, if for $P_i \in \mathcal{P}(n_i), i = 1, \ldots, k$ and $P_0 \in \mathcal{P}(k)$, then there exists $g_i \in G, i = 1, \ldots, k$ such that

$$\gamma_{\mathcal{P}}(\gamma_{\mathcal{P}}(P_0; g_1, \ldots, g_k); P_1, \ldots, P_k)$$

is well-defined.

Definition 2.4. An *algebra over a G-rescalable partial operad* $(\mathcal{P}, I_{\mathcal{P}}, \gamma_{\mathcal{P}})$, or a *$G$-rescalable $\mathcal{P}$-algebra*, consists of a completely reducible G-modules $U = \oplus_{n\in J} U_{(n)}$, where J is the set of equivalent classes of irreducible G-modules and $U_{(n)}$ is a direct sum of irreducible G-modules of equivalent class n, and a partial nonassociative operad homomorphism $\nu : \mathcal{P} \to E_U^G$ such that $\nu : G \to \text{End } U_{(n)}$ coincides with G-module structure on $U_{(n)}$.

We denote such algebra as (U, ν).

For $m \in \mathbb{Z}_+$, let $m = m_1 + \cdots + m_n$ be an ordered partition and $\sigma \in S_n$. The block permutation $\sigma_{(m_1,\ldots,m_n)} \in S_m$ is the permutation acts on $\{1, \ldots, m\}$ by permuting n intervals of lengths $m_1, \ldots, m_n$ in the same way that σ permute $1, \ldots, n$. Let $\sigma_i \in S_{m_i}, i = 1, \ldots, n$, we view the element $(\sigma_1, \ldots, \sigma_n) \in S_{m_1} \times \cdots \times S_{m_n}$ naturally as an element in S_m by the canonical embedding $S_{m_1} \times \cdots \times S_{m_n} \hookrightarrow S_m$.

Definition 2.5. Let $(\mathcal{Q}, I_{\mathcal{Q}}, \gamma_{\mathcal{Q}})$ be an operad. A right module over $\mathcal{Q}$, or a right $\mathcal{Q}$-module, is a family of sets $\mathcal{P} = \{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ with actions of permutation groups, equipped with maps:

$$\mathcal{P}(k) \times \mathcal{Q}(n_1) \times \cdots \times \mathcal{Q}(n_l) \xrightarrow{\gamma} \mathcal{P}(n_1 + \cdots + n_k)$$

such that

1. For $c \in \mathcal{P}(k)$, we have
$$\gamma(c; I_{\mathcal{Q}}, \ldots, I_{\mathcal{Q}}) = c. \qquad (2.2)$$

2. γ is associative. Namely, for $c \in \mathcal{P}(k)$, $d_i \in \mathcal{Q}(p_i), i = 1, \ldots, k$, $e_j \in \mathcal{Q}(q_j), j = 1, \ldots, p_1 + \cdots + p_k$, we have
$$\gamma(\gamma(c; d_1, \ldots, d_k); e_1, \ldots, e_{p_1+\cdots+p_k}) = \gamma(c; f_1, \ldots, f_k), \qquad (2.3)$$
where
$$f_s = \gamma_{\mathcal{Q}}(d_s; e_{p_1+\cdots+p_{s-1}+1}, \ldots, e_{p_1+\cdots+p_s}).$$

3. For $c \in \mathcal{P}(k; l)$, $d_i \in \mathcal{Q}(p_i), i = 1, \ldots, l$, $\sigma \in S_l$ and $\tau_j \in S_{p_j}, j = 1, \ldots, l$,
$$\gamma(\sigma(c); d_1, \ldots, d_l) = \sigma_{(p_1,\ldots,p_l)}(\gamma(c; d_1, \ldots, d_l)), \qquad (2.4)$$
$$\gamma(c; \tau_1(d_1), \ldots, \tau_l(d_l)) = (\tau_1, \ldots, \tau_l)(\gamma(c; d_1, \ldots, d_l)). \qquad (2.5)$$

Homomorphisms and isomorphisms between right $\mathcal{Q}$-modules are naturally defined.

The right module over a partial operad can be similarly defined.

Definition 2.6. A right module $\mathcal{P}$ over a partial operad $\mathcal{Q}$, or a right $\mathcal{Q}$-module, is called *G-rescalable* if $\mathcal{Q}$ is G-rescalable and for any $c \in \mathcal{P}(k)$, $d_i \in \mathcal{Q}(n_i), i = 1, \ldots, k$, there exist $g_i \in G$, $i = 1, \ldots, k$ such that

$$\gamma(\gamma(c; g_1, \ldots, g_k); d_1, \ldots, d_k)$$

is well-defined.

Definition 2.7. A 2-*colored operad* consists of an operad $(\mathcal{Q}, I_{\mathcal{Q}}, \gamma_{\mathcal{Q}})$, a family of sets $\mathcal{P}(m;n)$ equipped with an $S_m \times S_n$-action for $m, n \in \mathbb{N}$, a distinguished element $I_{\mathcal{P}} \in \mathcal{P}(1,1)$ and substitution maps γ_1, γ_2 given as follow:

$$\begin{aligned} &\mathcal{P}(k;l) \times \mathcal{P}(m_1;n_1) \times \cdots \times \mathcal{P}(m_k;n_k) \\ &\qquad\qquad\qquad \xrightarrow{\gamma_1} \mathcal{P}(m_1 + \ldots + m_k; l + n_1 + \ldots + n_k), \\ &\mathcal{P}(k;l) \times \mathcal{Q}(p_1) \times \cdots \times \mathcal{Q}(p_l) \xrightarrow{\gamma_2} \mathcal{P}(k; p_1 + \cdots + p_l), \end{aligned} \tag{2.6}$$

satisfying the following axioms:

1. The family of sets $\mathcal{P} := \{\cup_{n\in\mathbb{N}} \mathcal{P}(m;n)\}_{m\in\mathbb{N}}$ equipped with the natural S_m-action on $\mathcal{P}(m) = \cup_{n\in\mathbb{N}} \mathcal{P}(m;n)$, together with identity element $I_{\mathcal{P}}$ and substitution maps γ_1 is an operad.

2. γ_2 gives each $\mathcal{P}(k)$ a right $\mathcal{Q}$-module structure for $k \in \mathbb{N}$.

We denote it as $(\mathcal{P}|\mathcal{Q}, (\gamma_1, \gamma_2))$ or $\mathcal{P}$ for simplicity.

Remark 2.8. The substitution map γ_1, γ_2 can be combined into a single substitution map $\gamma = (\gamma_1, \gamma_2)$:

$$\begin{aligned} &\mathcal{P}(k;l) \times \mathcal{P}(m_1;n_1) \times \ldots \times \mathcal{P}(m_k;n_k) \times \mathcal{Q}(p_1) \times \cdots \times \mathcal{Q}(p_l) \\ &\qquad\qquad \xrightarrow{\gamma} \mathcal{P}(m_1 + \ldots + m_k; n_1 + \ldots + n_k + p_1 + \cdots + p_l). \end{aligned} \tag{2.7}$$

For this reason, we also denote $(\mathcal{P}|\mathcal{Q}, (\gamma_1, \gamma_2))$ as $(\mathcal{P}|\mathcal{Q}, \gamma)$. For some examples we encounter later, γ_1, γ_2 are often defined all together in terms of γ.

Definition 2.9. 2-colored partial operad is defined similarly as that of 2-colored operad except that γ_1, γ_2 are only partially defined and $\mathcal{Q}, \mathcal{P}$ are partial operads and (2.3) holds whenever both sides exist. If only the associativities of $\gamma_{\mathcal{Q}}, \gamma_1, \gamma_2$ do not hold, then it is called 2-*colored nonassociative (partial) operad*.

We give an important example of 2-colored partial nonassociative operad. Let J_1, J_2 be two index sets. $U_1 = \oplus_{n\in J_1} (U_1)_{(n)}, U_2 = \oplus_{n\in J_2} (U_2)_{(n)}$ be two graded vector spaces. Consider two families of vector spaces,

$$\begin{aligned} E_{U_2}(n) &= \mathrm{Hom}_{\mathbb{C}}(U_2^{\otimes n}, \overline{U}_2), \\ E_{U_1|U_2}(m;n) &= \mathrm{Hom}_{\mathbb{C}}(U_1^{\otimes m} \otimes U_2^{\otimes n}, \overline{U}_1). \end{aligned} \tag{2.8}$$

We denote both of the projection operators $U_1 \to (U_1)_{(n)}$, $U_2 \to (U_2)_{(n)}$ as P_n for $n \in J_1$ or J_2. For any $f \in E_{U_1|U_2}(k;l)$, $g_i \in E_{U_1|U_2}(m_i;n_i), i = 1, \ldots, k$, and $h_j \in E_{U_2}(p_j)$, $j = 1, \ldots, l$, we say that

$$\begin{aligned} &\Gamma(f; g_1, \ldots, g_n; h_1, \ldots, h_l) := \\ &\qquad \sum_{s_1,\ldots,s_k \in J_1; t_1,\ldots,t_l \in J_2} f(P_{s_1} g_1(u_1^{(1)}, \ldots, u_{m_1}^{(1)}, v_1^{(1)}, \ldots, v_{n_1}^{(1)}), \\ &\qquad\qquad\qquad \ldots, P_{s_k} g_k(u_1^{(k)}, \ldots, u_{m_k}^{(k)}, v_1^{(k)}, \ldots, v_{n_k}^{(k)}); \\ &\qquad\quad P_{t_1} h_1(w_1^{(1)}, \ldots, w_{n_1}^{(1)}), \ldots, P_{t_l} h_l(w_1^{(l)}, \ldots, w_{n_l}^{(l)})) \end{aligned}$$

where $u_j^{(i)} \in U_1, v_j^{(i)} \in U_2$, is well-defined if the multiple sum is absolutely convergent. This gives arise to a partially defined substitution map Γ:

$$E_{U_1|U_2}(k;l) \times E_{U_1|U_2}(m_1;n_1) \times \cdots \times E_{U_1|U_2}(m_k;n_k) \times E_{U_2}(p_1) \times \cdots \times E_{U_2}(p_l)$$
$$\xrightarrow{\Gamma} E_{U_1|U_2}(m_1 + \cdots + m_k, n_1 + \cdots + n_k + p_1 + \cdots + p_l).$$

Γ does not satisfy the associativity in general. Let $E_{U_1|U_2} = \{E_{U_1|U_2}(m;n)\}_{m,n\in\mathbb{N}}$ and $E_{U_2} = \{E_{U_2}(n)\}_{n\in\mathbb{N}}$. It is obvious that $(E_{U_1|U_2}|E_{U_2}, \Gamma)$ is a 2-colored nonassociative partial operad.

Let U_1 be a completely reducible G_1-modules and U_2 a completely reducible G_2-modules. Namely, $U_1 = \oplus_{n_1\in J_1}(U_1)_{(n_1)}, U_2 = \oplus_{n_2\in J_2}(U_2)_{(n_2)}$ where J_i is the set of equivalent classes of irreducible G_i-modules and $(U_i)_{(n_i)}$ is a direct sum of irreducible G_i-modules of equivalent class n_i for $i = 1, 2$. In this case, we denote $E_{U_1|U_2}$ by $E_{U_1|U_2}^{G_1|G_2}$.

Definition 2.10. A homomorphism between two 2-colored (partial) operads $(\mathcal{P}_i|\mathcal{Q}_i, \gamma_i), i = 1, 2$ consists of two (partial) operad homomorphisms:

$$\nu_{\mathcal{P}_1|\mathcal{Q}_1} : \mathcal{P}_1 \to \mathcal{P}_2, \qquad \text{and} \qquad \nu_{\mathcal{Q}_1} : \mathcal{Q}_1 \to \mathcal{Q}_2$$

such that $\nu_{\mathcal{P}_1|\mathcal{Q}_1} : \mathcal{P}_1 \to \mathcal{P}_2$, where $\mathcal{P}_2$ is a right $\mathcal{Q}_1$-module by $\nu_{\mathcal{Q}_1}$, is also a right $\mathcal{Q}_1$-module homomorphism.

Definition 2.11. An *algebra over a 2-colored partial operad* $(\mathcal{P}|\mathcal{Q}, \gamma)$, or a $\mathcal{P}|\mathcal{Q}$*-algebra* consists of two graded vector spaces U_1, U_2 and a homomorphism $(\nu_{\mathcal{P}|\mathcal{Q}}, \nu_{\mathcal{Q}})$ from $(\mathcal{P}|\mathcal{Q}, \gamma)$ to $(E_{U_1|U_2}|E_{U_2}, \Gamma)$. We denote this algebra as $(U_1|U_2, \nu_{\mathcal{P}|\mathcal{Q}}, \nu_{\mathcal{Q}})$.

Definition 2.12. If a 2-colored partial operad $(\mathcal{P}|\mathcal{Q}, \gamma)$ is so that $\mathcal{P}$ is a G_1-rescalable partial operad and a G_2-rescalable right $\mathcal{Q}$-module, then it is called $G_1|G_2$*-rescalable*.

Definition 2.13. A $G_1|G_2$*-rescalable* $\mathcal{P}|\mathcal{Q}$*-algebra* $(U_1|U_2, \nu_{\mathcal{P}|\mathcal{Q}}, \nu_{\mathcal{Q}})$ is a $\mathcal{P}|\mathcal{Q}$-algebra so that $\nu_{\mathcal{P}|\mathcal{Q}} : \mathcal{P} \to E_{U_1|U_2}^{G_1|G_2}$ and $\nu_{\mathcal{Q}} : \mathcal{Q} \to E_{U_2}^{G_2}$; moreover, $\nu_{\mathcal{P}|\mathcal{Q}} : G_1 \to \mathrm{End}\ U_1$ coincides with the G_1-module structure on U_1 and $\nu_{\mathcal{Q}} : G_2 \to \mathrm{End}\ U_2$ coincides with the G_2-module structure on U_2.

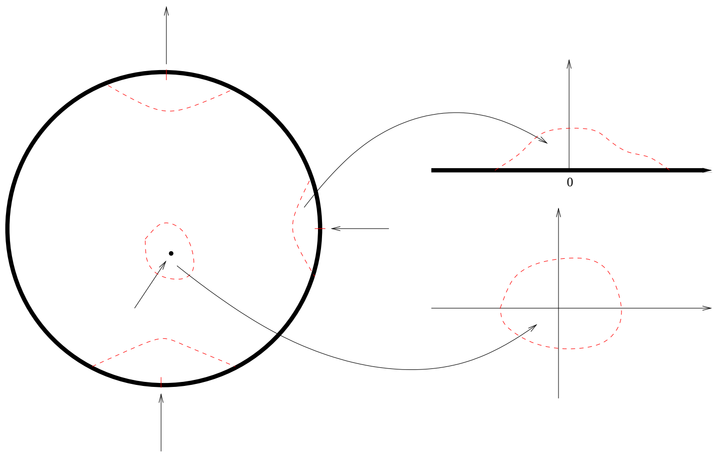
2.2 Swiss-cheese partial operad $\mathfrak{S}$

A *disk with strips and tubes of type* $(m_-, m_+; n_-, n_+)$ $(m_-, m_+, n_-, n_+ \in \mathbb{N})$ is a disk S with the following additional data:

1. $m_- + m_+$ distinct ordered punctures $p^B_{-m_-}, \ldots, p^B_{-1}, p^B_1, \ldots, p^B_{m_+}$ (called *boundary punctures*) on ∂S (the boundary of S), where $p^B_{-m_-}, \ldots, p^B_{-1}$ are negatively oriented and $p^B_1, \ldots, p^B_{m_+}$ are positively oriented, together with local coordinates:

$$(U^B_{-m_-}, \varphi^B_{-m_-}), \ldots, (U^B_{-1}, \varphi^B_{-1}); \quad (U^B_1, \varphi^B_1), \ldots, (U^B_{m_+}, \varphi^B_{m_+}),$$

where U^B_i is a neighborhood of p^B_i and $\varphi^B_i : U^B_i \to \tilde{\mathbb{H}}$ is an analytic map which vanishes at p^B_i and maps $U^B_i \cap \partial S$ to $\mathbb{R}$, for each $i = -m_-, \ldots, -1, 1, \ldots, m_+$.



2. $n_- + n_+$ distinct ordered points $p^I_{-n_-}, \ldots, p^I_{-1}, p^I_1, \ldots, p^I_{n_+}$ (called *interior punctures*) in the interior of S, where $p^I_{-n_-}, \ldots, p^I_{-1}$ are negatively oriented and $p^I_1, \ldots, p^I_{n_+}$ are positively oriented, together with local coordinates:

$$(U^I_{-n_-}, \varphi^I_{-n_-}), \ldots, (U^I_{-1}, \varphi^I_{-1}); \quad (U^I_1, \varphi^I_1), \ldots, (U^I_1, \varphi^I_{n_+}),$$

where U^I_j is a local neighborhood of p^I_j and $\varphi^I_j : U^I_j \to \mathbb{C}$ is an analytic map which vanishes at p^I_i for each $j = -n_-, \ldots, -1, 1, \ldots, n_+$.

Two disks with strips and tubes are conformal equivalent if there exists between them a biholomorphic map which maps punctures to punctures and preserves the order of punctures and the germs of local coordinate maps. We denote the moduli space of the conformal equivalence classes of disks with strips and tubes of type $(m_-, m_+; n_-, n_+)$ as $\mathbb{S}(m_-, m_+|n_-, n_+)$. The structure on $\mathbb{S}(m_-, m_+|n_-, n_+)$ will be discussed in [Ko2]. In this work, we are only interested in disks with strips and tubes of types $(1, m_+; 0, n_+)$ for $n, l \in \mathbb{N}$. For simplicity, we denote $\mathbb{S}(1, m_+|0, n_+)$ by $\Upsilon(m_+; n_+)$. For such disks, we label the only negatively oriented boundary puncture as the 0-th boundary puncture.

We can choose a canonical representative for each conformal equivalence class in $\Upsilon(m_+; n_+)$ just as we did for disks with strips [HKo1] and sphere with tubes [H4]. More precisely, for a disk with strips and tubes of type $(1, m_+; 0, n_+)$ where $m_+ > 0$, we first use a conformal map f to map the disk to $\hat{\mathbb{H}}$. Then we use an automorphism of $\hat{\mathbb{H}}$ to move the only negatively oriented puncture (the 0-th puncture) to ∞ and the smallest r_k to 0, and fix the local coordinate map f_0 at ∞ to be so that $\lim_{w\to\infty} wf_0(w) = -1$. As a consequence, the canonical representative of a generic conformal equivalent class of disk with strips and tubes $Q \in \Upsilon(m_+; n_+)$ is a disk $\hat{\mathbb{H}}$, together with a negatively oriented boundary puncture at $\infty \in \hat{\mathbb{H}}$ and local coordinate map given by

$$f^B_0(w) = -e^{\sum_{j=1}^{\infty} -B^{(0)}_j w^{-j+1}\frac{d}{dw}} \frac{1}{w},$$

where $B^{(0)}_j \in \mathbb{R}$, and positively oriented boundary punctures at $r_1, \ldots r_{m_+} \in \mathbb{R}_+ \cup \{0\}$ $(r_k = 0)$ and local coordinate maps given by

$$f^B_i(w) = e^{\sum_{j=1}^{\infty} B^{(i)}_j x^{j+1}\frac{d}{dx}} (b^{(i)}_0)^{x\frac{d}{dx}} x \bigg|_{x=w-r_i}, i = 1, \ldots, m_+,$$

where $B_j^{(i)} \in \mathbb{R}, b_0^{(i)} \in \mathbb{R}_+$, and positively oriented interior punctures at $z_1, \dots, z_{n_+} \in \mathbb{H}$ with local coordinate maps given by

$$f_i^I(w) = e^{\sum_{j=1}^{\infty} A_j^{(i)} x^{j+1} \frac{d}{dx}} (a_0^{(i)})^{x \frac{d}{dx}} x \Big|_{x=w-z_i}, i = 1, \dots, n_+.$$

where $A_j^{(i)} \in \mathbb{C}$ and $a_0^{(i)} \in \mathbb{C}^\times$.

Let $\Pi_\mathbb{R}$ ($\Pi_\mathbb{C}$) be the set of sequences of real numbers (complex numbers) $\{C_j\}_{j=1}^\infty$ such that $e^{\sum_{j>0} C_j x^{j+1} \frac{d}{dx}} x$ as a power series converges in some neighborhood of 0. We define

$$\tilde{\Lambda}^n := \{(r_1, \dots, r_n) | \ \exists \sigma \in S_n, \ r_{\sigma(1)} > \cdots > r_{\sigma(n)} = 0\}.$$

Using the data on the canonical representative of Q, we denote $Q \in \Upsilon(m_+; n_+)$ as follow

$$\begin{aligned}[r_1, \dots, r_{m_+-1}; B^{(0)}, (b_0^{(1)}, B^{(1)}), \dots, (b_0^{(m_+)}, B^{(m_+)})| \\ z_1, \dots, z_{n_+}; (a_0^{(1)}, A^{(1)}), \dots, (a_0^{(n_+)}, A^{(n_+)})],\end{aligned} \tag{2.9}$$

where $(r_1, \dots, r_{m_+}) \in \tilde{\Lambda}^{m_+}$ and $b_0^{(i)} \in \mathbb{R}_+, a_0^{(j)} \in \mathbb{C}^\times$, and $B^{(i)} = \{B_j^{(i)}\}_{j=1}^\infty \in \Pi_\mathbb{R}$ and $A^{(p)} = \{A_q^{(p)}\}_{q=1}^\infty \in \Pi_\mathbb{C}$ for all $i = 1, \dots, m_+, p = 1, \dots, n_+$.

Using notations (0.1) and (0.2), for $m_+ > 0$ and $n_+ \in \mathbb{N}$, we can express the moduli space of disks with strips and tubes of type $(1, m_+; 0, n_+)$ as follow:

$$\Upsilon(m_+; n_+) = \tilde{\Lambda}^{m_+-1} \times \Pi_\mathbb{R} \times (\mathbb{R}_+ \times \Pi_\mathbb{R})^{m_+} \times M_\mathbb{H}^{n_+} \times (\mathbb{C}^\times \times \Pi_\mathbb{C})^{n_+}.$$

For $m_+ = 0, n_+ \in \mathbb{N}$, we used automorphism of $\hat{\mathbb{H}}$ to fix $B_1^{(0)} = 0$. Hence, we have

$$\Upsilon(0; n_+) = \{B^{(0)} \in \Pi \mid B_1^{(0)} = 0\} \times (\mathbb{C}^\times \times \Pi_\mathbb{C})^{n_+}.$$

Note that $\Upsilon(m_+; 0)$ is nothing but $\Upsilon(m_+)$ introduced and studied in [HKo1]. $\Upsilon := \{\Upsilon(n)\}_{n \in \mathbb{N}}$ is a partial operad of disks with strips [HKo1]. The identity I_Υ is an element of $\Upsilon(1; 0)$. Also, for $m_+, n_+ \in \mathbb{N}$, S_{n_+} acts on $\Upsilon(m_+; n_+)$ in the obvious way. Let $\mathfrak{S}(m) = \cup_{n_+ \in \mathbb{N}} \Upsilon(m_+; n_+)$ for $m \in \mathbb{N}$, and $\mathfrak{S} = \cup_{m \in \mathbb{N}} \mathfrak{S}(m)$.

There are two kinds of sewing operations on $\mathfrak{S}$, The first kind is called *boundary sewing operation* which sews the a positively oriented boundary puncture in the first disk with a negatively oriented boundary puncture in the second disk. The second is called *interior sewing operation* which sews a positively oriented interior puncture in a disk with a negatively oriented puncture in a sphere with tubes. We describe these sewing operations more precisely. Let us consider $P \in \Upsilon(m_+; n_+)$ and $Q \in \Upsilon(p_+; q_+)$ ($Q \in K(p_+)$) for boundary sewing operations (for interior sewing operations). Let B^r ($\bar{B}^r$) denote the open (closed) ball in $\mathbb{C}$ center at 0 with radius r, φ_i the germs of local coordinate map at i-th boundary (interior) puncture p of P, and ψ_0 the germs of local coordinate map at 0-th puncture q of Q. Then we say that the i-th strip of P can be sewn with the 0-th strip (tube) of Q if there is a $r \in \mathbb{R}_+$ such that p and q are the only punctures in $\varphi_i^{-1}(\bar{B}^r)$ and $\psi_0^{-1}(\bar{B}^{1/r})$ respectively. A new disk with strips and tubes in $\Upsilon(m_+ + p_+ - 1, n_+ + q_+)$ ($\Upsilon(m_+, n_+ + p_+ - 1)$), denoted as $P_i \infty_0^B Q$ ($P_i \infty_0^I Q$), is obtained by cutting out $\varphi_i^{-1}(B^r)$ and $\psi_0^{-1}(B^{1/r})$ from P and Q respectively, and then identifying the boundary of $\varphi_i^{-1}(\bar{B}^r)$ and $\psi_0^{-1}(\bar{B}^{1/r})$ via the map

$$\psi^{-1} \circ J_{\hat{\mathbb{H}}} \circ \varphi_i$$

where $J_{\hat{\mathbb{H}}} : w \to -\frac{1}{w}$.

The boundary sewing operations and interior sewing operations induce the following partially defined substitution maps

$$\begin{aligned} &\Upsilon(k;l) \times \Upsilon(m_1;n_1) \times \ldots \times \Upsilon(m_k;n_k) \\ &\qquad\qquad \xrightarrow{\gamma^B} \Upsilon(m_1 + \ldots + m_k; l + n_1 + \ldots + n_k), \\ &\Upsilon(k;l) \times K(p_1) \times \cdots \times K(p_l) \xrightarrow{\gamma^I} \Upsilon(k; p_1 + \cdots + p_l), \end{aligned} \tag{2.10}$$

or equivalently

$$\begin{aligned} &\Upsilon(k;l) \times \Upsilon(m_1;n_1) \times \ldots \times \Upsilon(m_k;n_k) \times K(p_1) \times \cdots \times K(p_l) \\ &\qquad \xrightarrow{\gamma} \Upsilon(m_1 + \ldots + m_k; n_1 + \ldots + n_k + p_1 + \cdots + p_l). \end{aligned} \tag{2.11}$$

The following Proposition is clear.

Proposition 2.14. $(\mathfrak{S}|K, (\gamma^B, \gamma^I))$ is a $\mathbb{R}_+|\mathbb{C}^\times$-rescalable 2-colored partial operad.

This $\mathbb{R}_+|\mathbb{C}^\times$-rescalable 2-colored partial operad $(\mathfrak{S}|K, (\gamma^B, \gamma^I))$ is a generalization of Voronov's Swiss-cheese operad [V]. So we will call it *Swiss-cheese partial operad* and sometimes denote it by $\mathfrak{S}$ for simplicity. The relation between Swiss-cheese operad and Swiss-cheese partial operad is an analogue of that between little disk operad and sphere partial operad ([H4]).

2.3 Sewing equations and the doubling map δ

We are interested in finding the canonical representative of disk with strips and tubes obtained from sewing of two such disks or sewing of a disk and a sphere. In the case of sphere partial operad K, such canonical representatives were obtained by Huang [H1][H2][H4] by solving the so-called sewing equation. Similarly, the canonical representatives of disks with strips and tubes obtained by two types of sewing operations can also be determined by solving two types of sewing equations.

We start with the boundary sewing operations. For $Q \in \mathfrak{S}$, we denoted the canonical representative of Q as Σ_Q. Let $P \in \Upsilon(m;n)$ and $Q \in \Upsilon(p;q)$. Let g_0 be the local coordinate map at $\infty \in \Sigma_Q$ and f_i be that at $z_i \in \Sigma_P, 1 \le i \le m$. We assume that $P_i\infty_0^B Q$ exists. Then the canonical disk $\Sigma_{P_i\infty_0^B Q}$ can be obtained by solving the following sewing equation:

$$F^B_{(1)}(w) = F^B_{(2)}\left(g_0^{-1}\left(\frac{-1}{f_i(w)}\right)\right) \tag{2.12}$$

where $F^B_{(1)}$ is a conformal map from an open neighborhood of $\infty \in \Sigma_P$ to an open neighborhood of $\infty \in \Sigma_{P_i\infty_0^B Q}$, and $F^B_{(2)}$ is a conformal map from an open neighborhood of $0 \in \Sigma_Q$ to an open neighborhood of $0 \in \Sigma_{P_i \infty_0^B Q}$, with the following normalization conditions:

$$\begin{aligned} F^B_{(1)}(\infty) &= \infty, \\ F^B_{(2)}(0) &= 0, \\ \lim_{w\to\infty} \frac{F^B_{(1)}(w)}{w} &= 1. \end{aligned} \tag{2.13}$$

It is easy to see that the solution of (2.12) and (2.13) is unique. Notice that $F^B_{(1)}$, $F^B_{(2)}$, f_0 and f_i in (2.12) are all real analytic.

Similarly, let $P \in \Upsilon(m; n)$ and $Q \in K(p)$. We denote the canonical sphere with tubes of Q as Σ_Q. Let g_0 be the local coordinate map at $\infty \in Q$ and f_i be that at $z_i \in P, 1 \le i \le n$. We assume that $P_i\infty_0^I Q$ exists. Then $\Sigma_{P_i\infty_0^I Q}$ can be obtained by solving the following sewing equation:

$$F^I_{(1)}(w) = F^I_{(2)}\left(g_0^{-1}\left(\frac{-1}{f_i(w)}\right)\right) \tag{2.14}$$

where $F^I_{(1)}$ is a conformal map from an open neighborhood of $\hat{\mathbb{R}}$ in Σ_P to an open neighborhood of $\hat{\mathbb{R}}$ in $\Sigma_{P_i\infty_0^I Q}$, and $F^I_{(2)}$ is a conformal map from an open neighborhood of 0 in Σ_Q to an open subset of $\mathbb{H} \subset \Sigma_{P_i\infty_0^I Q}$, with the following normalization conditions:

$$\begin{aligned} F^I_{(1)}(\infty) &= \infty \\ F^I_{(1)}(0) &= 0 \\ \lim_{w\to\infty} \frac{F^I_{(1)}(w)}{w} &= 1. \end{aligned} \tag{2.15}$$

It is easy to see that the solution of (2.14) and (2.15) is unique as well.

Notice that $F^I_{(1)}$ is real analytic because it maps $\mathbb{R}$ to $\mathbb{R}$. Hence, $F^I_{(1)}$ is also the unique solution for the following equation

$$F^I_{(1)}(w) = \overline{F^I_{(2)}}\left(\overline{g_0^{-1}}\left(\frac{-1}{\overline{f_i}(w)}\right)\right) \tag{2.16}$$

with the same normalization condition (2.15).

In the sphere partial operad, we define a complex conjugation map $\mathrm{Conj} : K \to K$ as follow

$$\begin{aligned} \mathrm{Conj} : (z_1, \dots, z_{n-1}; A^{(0)}, (a_0^{(1)}, A^{(1)}), \dots, (a_0^{(n)}, A^{(n)})) \\ \longmapsto (\bar{z}_1, \dots, \bar{z}_{n-1}; \bar{A}^{(0)}, (\bar{a}_0^{(1)}, \bar{A}^{(1)}), \dots, (\bar{a}_0^{(n)}, \bar{A}^{(n)})). \end{aligned} \tag{2.17}$$

For simplicity, we denote $\mathrm{Conj}(Q)$ as $\bar{Q}$ for $Q \in K$.

Proposition 2.15. Conj *is an partial operad automorphism of* K*.*

Proof. It is clear that $\mathrm{Conj}((\mathbf{0},(1,\mathbf{0}))) = (\mathbf{0},(1,\mathbf{0}))$ and Conj is equivariant with respect to the action of permutation group. Moreover, Conj is obviously bijective. It only remains to show that, for $1 \leq i \leq m$,

$$\overline{P_i\infty_0 Q} = \bar{P}_i\infty_0\bar{Q} \tag{2.18}$$

for any pair $P \in K(m), Q \in K(n)$ such that $P_i\infty_0 Q$ exists.

Let f_i be the local coordinate map at i-th puncture in P and g_0 be that at ∞ in Q. Then the local coordinate map at i-th puncture in $\bar{P}$ is $\bar{f}_i$ and that at ∞ in $\bar{Q}$ is $\bar{g}_0$. Also notice that the sewing equation and normalization equation for sphere partial operad ([H4]) are the same as the equation (2.12) and (2.13). Let $F^{(1)}_{sphere}, F^{(2)}_{sphere}$ be the solution of (2.12) and (2.13) for the sphere with tubes , then $\bar{F}^{(1)}_{sphere}, \bar{F}^{(2)}_{sphere}$ also satisfy the same normalization condition and the following sewing equation

$$\bar{F}^{(1)}_{sphere}(w) = \bar{F}^{(2)}_{sphere}\left(\overline{g_0^{-1}}\left(\frac{-1}{\bar{f}_i(w)}\right)\right),$$

which is the sewing equation for $\bar{P}_i\infty_0\bar{Q}$. Using the explicit formula ((A.6.1)-(A.6.5) in [H4]) of the moduli $\bar{P}_i\infty_0\bar{Q}$ in terms of $\bar{F}^{(1)}_{sphere}, \bar{F}^{(2)}_{sphere}, \bar{f}_i, \bar{g}_0$, one can easily see that (2.18) is true. ∎

There is a canonical doubling map $\delta : \mathfrak{S} \to K$ defined as follow. Let $Q \in \Upsilon(n;l)$ with form

$$\begin{aligned} Q \;=\; & [\, r_1,\ldots,r_{n-1}; B^{(0)}, (b_0^{(1)}, B^{(1)}),\ldots,(b_0^{(n)}, B^{(n)})| \\ & \quad z_1,\ldots,z_l; (a_0^{(1)}, A^{(1)}),\ldots,(a_0^{(l)}, A^{(l)})\,]. \end{aligned}$$

Then

$$\begin{aligned} \delta(Q) = (& z_1,\ldots,z_l,\bar{z}_1,\ldots,\bar{z}_l, r_1,\ldots,r_{n-1}; (a_0^{(1)}, A^{(1)}),\ldots,(a_0^{(l)}, A^{(l)}), \\ & (\overline{a_0^{(1)}}, \overline{A^{(1)}}),\ldots,(\overline{a_0^{(l)}}, \overline{A^{(l)}}); B^{(0)}, (b_0^{(1)}, B^{(1)}),\ldots,(b_0^{(n)}, B^{(n)})). \end{aligned} \tag{2.19}$$

Proposition 2.16. *Let $P \in \Upsilon(m;n)$ and $Q \in \Upsilon(p;q)$. Assume that $P_i\infty_0^B Q$ exists. We have*

$$\delta(P_i\infty_0^B Q) = \delta(P)_{2n+i}\infty_0\delta(Q). \tag{2.20}$$

Proof. Since $F^B_{(1)}$, $F^B_{(2)}$, g_0^{-1} and f_i in (2.12) are all real analytic, every solution of (2.12) and (2.13) for disk with strips and tubes can be extended to a solution of the same sewing equation for sphere with tubes by Schwarz's reflection principle. Then the Proposition follows immediately from this fact. ∎

Given a canonical disk with strips and tubes Σ_Q corresponding to moduli $Q \in \mathfrak{S}$, we consider its complex conjugation, denoted as $\overline{\Sigma_Q}$. $\overline{\Sigma_Q}$ is the lower half plane $\hat{\overline{\mathbb{H}}}$ together with the same boundary punctures and local coordinate maps as those in Σ_Q, and interior punctures which are the complex conjugation of the interior punctures in Σ_Q with local coordinate maps being the complex conjugation of those in Σ_Q. We can denote it as

$$\begin{aligned} & [r_1,\ldots,r_{m_+-1}; B^{(0)}, (b_0^{(1)}, B^{(1)}),\ldots,(b_0^{(m_+)}, B^{(m_+)})| \\ & \quad \bar{z}_1,\ldots,\bar{z}_{n_+}; (\overline{a_0^{(1)}}, \overline{A^{(1)}}),\ldots,(\overline{a_0^{(n_+)}}, \overline{A^{(n_+)}})] \end{aligned} \tag{2.21}$$

where $z_1,\ldots,z_{n_+} \in \mathbb{H}$.

Lemma 2.17. *Let $P \in \Upsilon(m;n)$ and $Q \in K(p)$. Assume that $P_i\infty_0^I Q$ exists for $1 \leq i \leq m$. Then $\overline{\Sigma_P}{}_i\infty_0^I \bar{Q}$ also exists and*

$$\overline{\Sigma_{P_i\infty_0^I Q}} \cong \overline{\Sigma_P}\,{}_i\infty_0^I\,\bar{Q}. \tag{2.22}$$

Proof. We can choose a canonical representative of $\overline{\Sigma_P}\,{}_i\infty_0^I\,\bar{Q}$ as a lower half plane $\hat{\overline{\mathbb{H}}}$ with two punctures at ∞ and 0 and the local coordinate map g_0 at ∞ being so that $\lim_{w\to\infty} wg_0(w) = -1$. We denote such representative of $\overline{\Sigma_P}\,{}_i\infty_0^I\,\bar{Q}$ (a lower half plane) as Σ_1. Σ_1 can be obtained by solving the sewing equation

$$G_{(1)}^I(w) = G_{(2)}^I\left(\overline{g_0^{-1}}\left(\frac{-1}{\overline{f_i}(w)}\right)\right), \tag{2.23}$$

where $G_{(1)}^I$ is a conformal map from a neighborhood of $\hat{\mathbb{R}} \subset \overline{\Sigma}_P$ to a neighborhood of $\hat{\mathbb{R}} \subset \Sigma_1$, and $G_{(2)}^I$ is a conformal map from a neighborhood of $0 \in \overline{\Sigma_Q}$ to an open subset of $\overline{\mathbb{H}} \subset \Sigma_1$, with the normalization equation (2.15). By comparing (2.23) with (2.16), we see that the unique solution $F_{(1)}^I$ (real analytic) and $\overline{F_{(2)}^I}$ of (2.16) and (2.15) exactly gives the unique solution $G_{(1)}^I$ and $G_{(2)}^I$of (2.23) and (2.15). Hence we have $\Sigma_1 = \overline{\Sigma_{P_i\infty_0^I Q}}$. ∎

Proposition 2.18. *Let $P \in \Upsilon(n;l), Q \in K(m)$ and $P_i\infty_0^I Q$ exists for $1 \leq i \leq l$. Then*

$$\delta(P_i\infty_0^I Q) = (\delta(P)_i\infty_0 Q)_{l+m-1+i}\infty_0\bar{Q} \tag{2.24}$$

Proof. Now we first consider the right hand side of (2.24). We denote the canonical representative of any $R \in K(l)$ as Σ_R. Then $\Sigma_{\delta(P)}$ can be viewed as a union of the closure of upper half plane, denoted as U_+, and the closure of lower half plane, denoted as U_-. Let Σ_+ be the Riemann surface obtained by sewing U_+ with Q, and Σ_- the Riemann surface obtained by sewing U_- with $\bar{Q}$. By identifying the real line in $U_+ \subset \Sigma_+$ with the real line in $U_- \subset \Sigma_-$ using identity map, we obtain the surface $\Sigma_+\#\Sigma_-$ which is isomorphic to the canonical sphere with tubes $\Sigma_{(\delta(P)_i\infty_0 Q)_{l+m-1+i}\infty_0\bar{Q}}$.

Both Σ_+ and Σ_- are disks with strips and tubes. Since $U_+ = \Sigma_P$, there is a unique biholomorphic map f from Σ_+ to the canonical disk with strips and tubes $\Sigma_{Q_i\infty_0^I P}$. Similarily, because $U_- = \overline{\Sigma_P}$, by Lemma 2.17, there is a unique biholomorphic map g from Σ_- to the canonical disk with strips and tubes $\overline{\Sigma_{P_i\infty_0^I Q}}$. The restriction of f on the neighborhood of $\hat{\mathbb{R}} = \partial U_+$ is nothing but the unique real analytic map $F_{(1)}^I$ satisfying (2.14) and (2.15). Meanwhile, the restriction of g on a neighborhood of $\hat{\mathbb{R}} = \partial U_-$ is nothing but the same $F_{(1)}^I$ satisfying (2.16) and (2.15). So we must have $f|_{\hat{\mathbb{R}}} = g|_{\hat{\mathbb{R}}}$, which further implies that $f^{-1}|_{\hat{\mathbb{R}}} = g^{-1}|_{\hat{\mathbb{R}}}$. Hence, f^{-1} can be extended to a biholomorphic map from $\Sigma_{\delta(P_i\infty_0^I Q)}$ to $\Sigma_+\#\Sigma_-$.

Therefore $\Sigma_{\delta(P_i\infty_0^I Q)}$ must be biholomorphic to $\Sigma_{(\delta(P)_i\infty_0 Q)_{l+m-1+i}\infty_0\bar{Q}}$. Since they are both canonical representatives of sphere with tubes, we must have the equality (2.24). ∎

Remark 2.19. Proposition 2.18 gives a geometric interpretation of the doubling trick used in physics literature [A][C1]. It also implies that the bulk theories must contain both chiral parts and anti-chiral parts.

Corollary 2.20. *For $P \in \delta(\Upsilon(n,l)) \subset K(n+2l)$ as in (2.19) and $Q \in K(m)$, the sewing operation, for $1 \le i \le l$,*

$$(P_i\infty_0 Q)_{l+m-1+i}\infty_0 \bar{Q} \in \delta(\Upsilon(n, l+m-1))$$

and it defines an action of the diagonal $\{(P,\bar{P}) \in K \times K\}$ of $K \times K$ on $\delta(\mathfrak{S})$.

From now on, we will identify $\mathfrak{S}$ with $\delta(\mathfrak{S})$ without referring to the doubling map δ unless we want to emphasis the difference.

2.4 The $\mathbb{C}$-extensions of $\mathfrak{S}$

In order to study open-closed conformal field theories with nontrivial central charges, we need study the $\mathbb{C}$-extensions of the Swiss-cheese partial operad $\mathfrak{S}$.

For $c \in \mathbb{C}$, let $\tilde{K}^c$ be the $\frac{c}{2}$-th power of determinant line bundle over K [H4]. We denote the pullback line bundle over $\mathfrak{S}$ through the doubling map δ as $\tilde{\mathfrak{S}}^c$. δ can certainly be extended to a map on $\tilde{\mathfrak{S}}^c$. We still denote it as δ. For any $n \in \mathbb{N}$, the restrictions of the sections ψ_{n+2l} of $\tilde{K}^c(n+2l)$ for $l \in \mathbb{N}$ to $\Upsilon(n;l)$ gives a section of $\tilde{\mathfrak{S}}^c(n)$ and we shall use $\psi_n^{\mathfrak{S}}$ to denote this section. It is clear that $\tilde{\Upsilon}^c$, the $\mathbb{C}$-extension of the partial operad of disks with strips, is the pullback bundle of the inclusion map $\Upsilon \hookrightarrow \mathfrak{S}$.

The boundary sewing operations in $\tilde{\mathfrak{S}}^c$ are naturally induced from the sewing operations of $\tilde{K}^c$. We denote the boundary sewing operations in $\tilde{\mathfrak{S}}^c$ as ${}_i\widetilde{\infty}_0^B$.[6] More explicitly, let $P \in \Upsilon(n;l)$ and $Q \in \Upsilon(m;k)$ be so that $P_i\infty_0^B Q$ exists. Let $\tilde{P}, \tilde{Q}$ be elements in the fiber over P and Q respectively. We define

$$\tilde{P}_i\widetilde{\infty}_0^B\tilde{Q} := \delta^{-1}(\delta(\tilde{Q})_{2l+i}\widetilde{\infty}_0\tilde{Q}), \tag{2.25}$$

where δ^{-1} is defined on the image of δ.

We would also like to lift an interior sewing operation ${}_i\infty_0^I$ to a sewing operation between an element in $\tilde{\mathfrak{S}}^c$ and an element in $\tilde{K}^c \otimes \overline{\tilde{K}^{\bar{c}}}$. We still call it interior sewing operation and denote it as ${}_i\widetilde{\infty}_0^I$. Let $P \in \Upsilon(n;l), Q \in K(m)$ such that $P_i\infty_0^I Q$ exists. Let $\tilde{P}, \tilde{Q}$ be elements in the fibers over P and Q respectively. Let $\psi_m \otimes \bar{\psi}_m$ be the canonical section on $\tilde{K}^c \otimes \overline{\tilde{K}^{\bar{c}}}(m)$. Then we have $\tilde{Q} = \lambda\psi_m \otimes \bar{\psi}_m(Q)$ for some $\lambda \in \mathbb{C}$. Then we define $\tilde{P}_i\widetilde{\infty}_0^I\tilde{Q}$ by

$$\tilde{P}_i\widetilde{\infty}_0^I\tilde{Q} := \delta^{-1}((\delta(\tilde{P})_i\widetilde{\infty}_0\lambda\psi_m(Q))_{l+m-1+i}\widetilde{\infty}_0\psi_m(\bar{Q})) \tag{2.26}$$

The following Lemma shows that the interior sewing operations are associative.

Lemma 2.21. *Let $P \in \Upsilon(n;l)$, $Q_1 \in K(m_1), Q_2 \in K(m_2)$ and $\tilde{P}, \tilde{Q}_1, \tilde{Q}_2$ be elements in fibers of line bundles $\tilde{\mathfrak{S}}^c$ and $\tilde{K}^c \otimes \overline{\tilde{K}^{\bar{c}}}$ over the base points P, Q_1, Q_2 respectively. Let $1 \le i \le l$ and $1 \le j \le m_1$. Then we have*

$$(\tilde{P}_i\widetilde{\infty}_0^I\tilde{Q}_1)_{i+j-1}\widetilde{\infty}_0^I\tilde{Q}_2 = \tilde{P}_i\widetilde{\infty}_0^I(\tilde{Q}_1{}_j\widetilde{\infty}_0\tilde{Q}_2) \tag{2.27}$$

assuming that the sewing operations appeared in (2.27) are all well-defined.

[6] Since we always work with a fixed $c \in \mathbb{C}$, it is convenient to make the dependence on c implicit in some notations.

Proof. Let $\lambda_i \in \mathbb{C}, i = 1,2$ be such that $\tilde{Q}_i = \lambda_i \psi_{m_i} \otimes \bar{\psi}_{m_i}(Q_i), i = 1,2$. Let $(a_0^{(i)}, A^{(i)})$ be the local coordinate map at j-th puncture of Q_1 and Let $B^{(0)}$ be the local coordinate map at ∞ in Q_2. By (2.26), the δ image of the left hand side of (2.27) equals to

$$
\begin{aligned}
(((\delta(\tilde{P})_i\widetilde{\infty}_0 \lambda_1 \psi_{m_1}(Q_1))_{l+m_1-1+i}\widetilde{\infty}_0 \psi_{m_1}(\bar{Q}_1)) \\
{}_{i+j-1}\widetilde{\infty}_0 \lambda_2 \psi_{m_2}(Q_2))_{l+m_1+m_2+i+j-3}\widetilde{\infty}_0 \psi_{m_2}(\bar{Q}_2)
\end{aligned}
$$

By the associativity of the partial operad $\tilde{K}^c$, the above formula equals to

$$
\begin{aligned}
&(\delta(\tilde{P})_i\widetilde{\infty}_0(\lambda_1\psi_{m_1}(Q_1)_j\widetilde{\infty}_0\lambda_2\psi_{m_2}(Q_2))) \\
&\qquad {}_{l+m_1+m_2-2+i}\widetilde{\infty}_0(\psi_{m_1}(\bar{Q}_1)_j\widetilde{\infty}_0\psi_{m_2}(\bar{Q}_2)) \\
&= (\delta(\tilde{P})_i\widetilde{\infty}_0(\lambda_1\lambda_2 e^{\Gamma(A^{(i)},B^{(0)},a_0^{(i)})c}\psi_{m_1+m_2-1}(Q_1\,{}_j\infty_0 Q_2))) \\
&\qquad {}_{l+m_1+m_2-2+i}\widetilde{\infty}_0 e^{\Gamma(\overline{A^{(i)}},\overline{B^{(0)}},\overline{a_0^{(i)}})c}\psi_{m_1+m_2-1}(\overline{Q_1\,{}_j\infty_0 Q_2}) \\
&= \delta\big(\tilde{P}_i\widetilde{\infty}_0^I(\lambda_1\lambda_2 e^{\Gamma(A^{(i)},B^{(0)},a_0^{(i)})c} e^{\Gamma(\overline{A^{(i)}},\overline{B^{(0)}},\overline{a_0^{(i)}})c} \\
&\qquad \psi_{m_1+m_2-1}\otimes\bar{\psi}_{m_1+m_2-1}(Q_1\,{}_j\infty_0 Q_2))\big) \\
&= \delta\big(\tilde{P}_i\widetilde{\infty}_0^I(\lambda_1\psi_{m_1}\otimes\bar{\psi}_{m_1}(Q_1)_j\widetilde{\infty}_0\lambda_2\psi_{m_2}\otimes\bar{\psi}_{m_2}(Q_2))\big) \\
&= \delta(\tilde{P}_i\widetilde{\infty}_0^I(\tilde{Q}_1\,{}_i\widetilde{\infty}_0\tilde{Q}_2)),
\end{aligned}
$$

which is nothing but the right hand side of (2.27). ∎

Boundary sewing operations and interior sewing operations induce the following partially defined substitution maps $\tilde{\gamma}$:

$$
\begin{aligned}
\tilde{\Upsilon}^c(n;l)\times\tilde{\Upsilon}^c(m_1;k_1)\times\cdots\times\tilde{\Upsilon}^c(m_n;k_n)\times\tilde{K}^c(n_1)\times\cdots\times\tilde{K}^c(n_l) \\
\xrightarrow{\tilde{\gamma}} \tilde{\Upsilon}^c(m_1+\cdots+m_n;k_1+\cdots+k_n+n_1+\cdots+n_l)
\end{aligned}
$$

The following Proposition is clear.

Proposition 2.22. $(\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}},\tilde{\gamma})$ *is a* $\mathbb{R}_+|\mathbb{C}^\times$*-rescalable* 2*-colored partial operad.*

We will call $(\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}},\tilde{\gamma})$ *Swiss-cheese partial operad with central charge c.* Note that $\tilde{\mathfrak{S}}^c$ restricted on Υ is just $\tilde{\Upsilon}^c$ which was introduced in [HKo1].

2.5 Smooth $\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}$-algebras

Let $V^O = \oplus_{n\in\mathbb{R}}V^O_{(n)}$, where $V^O_{(n)}$ has a structure of irreducible $\mathbb{R}_+$-module given by $r\mapsto r^n\mathrm{id}_{V^O_{(n)}}$ for $r\in\mathbb{R}_+$. Let $V^C = \oplus_{(m,n)\in\mathbb{R}\times\mathbb{R}}V^C_{(m,n)}$, where $V^C_{(m,n)}$ has a structure of irreducible $\mathbb{C}^\times$-module given by $z\mapsto z^m\bar{z}^n\,\mathrm{id}_{V^C_{(m,n)}}$ for $z\in\mathbb{C}^\times$ (recall (1.16) and (1.17)) for all m,n. This is also implies that $V^C_{(m,n)}=0$ for all $m-n\notin\mathbb{Z}$. Let

$$(V^O|V^C,\nu_{\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}},\nu_{\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}})$$

be a $\mathbb{R}_+|\mathbb{C}^\times$-rescalable $\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}$-algebra.

Definition 2.23. The $\mathbb{R}_+|\mathbb{C}^\times$-rescalable $\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}$-algebra $(V^O|V^C, \nu_{\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}, \nu_{\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}})$ is called *smooth* if it satisfies the following two conditions:

1. $\dim V^O_{(s)} < \infty$ for $s \in \mathbb{R}$, $V^O_{(n)} = 0$ for $n << 0$ and $\dim V^C_{(m,n)} < \infty$ for $m, n \in \mathbb{Z}$, $V^C_{(m,n)}$ for $m << 0$ or $n << 0$.
2. For $w \in (V^O)'$, $v_1, \dots, v_m \in V^O$, $u_1, \dots, u_n \in V^C$, and $\tilde{P} \in \tilde{\Upsilon}^c(m;n)$, the following map
$$\tilde{P} \mapsto \langle w, \nu_{\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}(\tilde{P})(v_1 \otimes \cdots \otimes v_m \otimes u_1 \otimes \cdots \otimes u_n)\rangle$$
is linear on fiber and smooth on the base space $\Upsilon(m;n)$.

By [H4], a vertex operator algebra U canonically gives a $\tilde{K}^c$-algebra. Using the doubling map δ, this $\tilde{K}^c$-algebra naturally gives a smooth $\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}$-algebra, in which $V^O = U$ and $V^C = U\otimes U$. It is also easy to see that this $\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}$-algebra canonically gives an analytic open-closed field algebra which is nothing but $(U\otimes U, U, \mathrm{id}_U, \mathrm{id}_U)$ or simply $(U\otimes U, U)$ discussed in Section 1. We still denote this smooth $\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}$-algebra as $(U\otimes U, U)$. A smooth $\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}$-algebra containing $(U\otimes U, U)$ as a subalgebra is called a *smooth* $\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}$*-algebra over* U.

Let $(V_{cl}, V_{op}, m_{cl-op})$ be an open-closed field algebra over V. By definition, V_{cl} is a conformal full field algebra over $V\otimes V$. By the results in [Ko1], V_{cl} has a structure of smooth $\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}$-algebra structure, we denote it as $(V_{cl}, \nu_{\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}})$.

Let $Q \in \Upsilon(n;l)$ of form (2.9) such that $r_1 > \dots > r_{n-1} > r_n = 0$. We define a map $\nu_{\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}} : \tilde{\mathfrak{S}}^c \to E^{\mathbb{R}_+|\mathbb{C}^\times}_{V_{op}|V_{cl}}$ as follow:

$$\begin{aligned}
&\nu_{\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}(\lambda\psi^{\mathfrak{S}}_n(Q))(v_1\otimes\cdots\otimes v_n\otimes u_1\otimes\cdots\otimes u_l)\\
&:= \lambda e^{-L_-(B^{(0)})} m^{(l;n)}_{cl-op}(e^{-L_+(A^{(1)})}(a^{(1)}_0)^{-L(0)} \otimes e^{-L_+(\overline{A^{(1)}})}(\overline{a^{(1)}_0})^{-L(0)}u_1,\\
&\qquad \dots e^{-L_+(A^{(l)})}(a^{(l)}_0)^{-L(0)} \otimes e^{-L_+(\overline{A^{(l)}})}(\overline{a^{(l)}_0})^{-L(0)}u_l;\\
&\qquad e^{-L_+(B^{(1)})}(b^{(1)}_0)^{-L(0)}v_1, \dots, e^{-L_+(B^{(n)})}(b^{(n)}_0)^{-L(0)}v_n;\\
&\qquad\qquad z_1, \bar{z}_1, \dots, z_l, \bar{z}_l; r_1, \dots, r_n),
\end{aligned} \tag{2.28}$$

where $L_\pm(A) = \sum_{j=1}^\infty L(\pm j)A_j$ for any $A = \{A_1, A_2, \dots\}, A_j \in \mathbb{C}$, for $u_1, \dots, u_l \in V_{cl}, v_1, \dots, v_n \in V_{op}$. Let e_l be the identity element of S_l. $\forall \sigma \in S_n$, we define

$$\nu_{\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}\big((\sigma, e_l)(\lambda\psi^{\mathfrak{S}}_n(Q))\big) = \nu_{\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}\big(\lambda\psi^{\mathfrak{S}}_n(Q)\big). \tag{2.29}$$

We have finished the definition of $\nu_{\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}$ in all cases. By results in [HKo1], the restriction of $\nu_{\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}$ on $\tilde{\Upsilon}^c$ clearly gives a morphism of partial nonassociative operad from $\tilde{\Upsilon}^c$ to $E^{\mathbb{R}_+}_{V_{op}}$.

Theorem 2.24. *$(V_{op}|V_{cl}, \nu_{\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}, \nu_{\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}})$ is a smooth $\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}$-algebra.*

Proof. By the permutation property of open-closed field algebra and (2.29), it is clear that $\nu_{\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}$is equivariant with the actions of permutation groups.

The conditions in Definition 2.23 are automatically satisfied. It remains to show that

$$\nu_{\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}} \circ \tilde{\gamma} = \Gamma \circ (\nu_{\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}, \dots, \nu_{\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}, \nu_{\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}, \dots, \nu_{\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}) \qquad (2.30)$$

as a map

$$\begin{aligned} \tilde{\Upsilon}^c(n;l) \times \tilde{\Upsilon}^c(m_1;k_1) \times \cdots \times \tilde{\Upsilon}^c(m_n;k_n) \times \tilde{K}^c(n_1) \times \cdots \times \tilde{K}^c(n_l) \\ \to \mathrm{Hom}(V_{cl}^{\otimes k_1+\cdots+k_n+n_1+\cdots+n_l} \otimes V_{op}^{\otimes n+m_1+\cdots+m_n}, \overline{V_{op}}). \end{aligned} \qquad (2.31)$$

Thanks to the doubling map δ and (2.27), $\tilde{\mathfrak{S}}^c$ can be viewed as a partial suboperad of $\tilde{K}^c$ with single-sewing operations for the punctures in $\mathbb{R}_+$ and double-sewing operations for mirror pairs of punctures in upper and lower half planes. Then using the V-invariant boundary condition, the chirality splitting property and the convergence and extension properties of any products and iterates of intertwining operators proved by Huang for any V satisfying the condition in Theorem 0.2, we can easily generalize the proof of Huang's fundamental result Proposition 5.4.1 in [H4] and results in [H5][H6] to shows that (2.30) holds.

Because the convergence properties of open-closed field algebra is quite strong, the proof can be shorten along the following lines.

It is enough to show that (2.30) is true when both sides of (2.30) act on the following two types of elements.

$$\begin{aligned} &(P, I_{\tilde{\Upsilon}^c}, \dots, I_{\tilde{\Upsilon}^c}, Q_1, I_{\tilde{\Upsilon}^c}, \dots, I_{\tilde{\Upsilon}^c}, I_{\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}, \dots, I_{\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}), \\ &(P, I_{\tilde{\Upsilon}^c}, \dots, I_{\tilde{\Upsilon}^c}, I_{\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}, \dots, I_{\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}, Q_2, I_{\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}, \dots, I_{\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}}), \end{aligned} \qquad (2.32)$$

where $P \in \tilde{\Upsilon}^c(l;n)$, $Q_1 \in \tilde{\Upsilon}^c(m;k)$ and $Q_2 \in \tilde{K}^c \otimes \overline{\tilde{K}^{\bar{c}}}(m)$. In other words, we only need consider the pairwise sewing operations of form $P_i\widetilde{\infty}_0^B Q_1$ and $P_j\widetilde{\infty}_0^I Q_2$ for some $i = 1, \dots, n$ and $j = 1, \dots, l$.

First, we consider the following three cases: 1. $P = \tilde{\Upsilon}^c(1;0)$, 2. $Q_1 = \tilde{\Upsilon}^c(1;0)$, 3. $Q_2 = \tilde{K}^c \otimes \overline{\tilde{K}^{\bar{c}}}(1)$, which will be all referred to as type I. In all three type I cases, only the relation between Virasoro algebra and $\mathbb{Y}_{cl-op}, Y_{op}$ is our concern. By the chiral splitting property and results in [HKo1], we know that both $\mathcal{Y}^{(i)}, i = 1, 2, 3, 4, Y_{op}$ are all intertwining operators with respect to V. This fact is enough to prove these three cases. The proof is exactly same as that of Proposition 5.4.1. in [H4].

Secondly, we consider the cases when two oppositely oriented punctures on P, Q_1 or P, Q_2, involved in the sewing operation both have standard local coordinate maps. We refer to these cases as type II cases. The convergence properties for type II cases are nothing but the convergence properties of open-closed field algebra.

Now we consider cases other than type I and type II. On the geometric partial operad side, for general P, Q_1, Q_2, the sewing operation between P and Q_1 (or Q_2) can always be decomposed into several sewing operations either of type I or type II. On the algebraic side, these sewing operations become multiple sums in a given iterate order. By our assumptions on V, it is easy to show that this multiple sums is absolutely convergent. As a consequence, we can freely switch the order of the iterate sums without changing the value of the sum. Then by choosing a proper order of the iterate sum, these general cases can be reduced to several cases of type I and type II.

That $(V \otimes V, V)$ is a subalgebra is obvious. ∎

Remark 2.25. We are not able to give an isomorphism theorem here because it is unclear to the author whether the axioms of smooth $\tilde{\mathfrak{S}}^c|\tilde{K}^c \otimes \overline{\tilde{K}^{\bar{c}}}$-algebra over V imply the chirality splitting property in the definition of open-closed field algebra over V.

3 Categorical formulation and constructions

In this section, we study open-closed field algebras over V from a tensor-categorical point of view. We first recall some basic ingredients of the vertex tensor categories. Then we reformulate the notion of open-closed field algebra over V categorically as open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra. In the end, we briefly discuss some categorical constructions.

3.1 Vertex tensor categories

The theory of tensor products for modules over a vertex operator algebra was developed by Huang and Lepowsky [HL2]-[HL6][H3]. By Theorem 0.2 and our assumption on V, the category of V-modules, denoted as $\mathcal{C}_V$, have a structure of vertex tensor category [HL2][HL6], In particular, it has a structure of semisimple braided tensor category.

We review some of the ingredients of vertex tensor category $\mathcal{C}_V$ and set our notations along the way.

There is a tensor product bifunctor $\boxtimes_{P(z)} : \mathcal{C}_V \times \mathcal{C}_V \to \mathcal{C}_V$ for each $P(z), z \in \mathbb{C}^\times$ in sphere partial operad K, where $P(z)$ is the conformal equivalent class of sphere with three punctures $0, z, \infty$ and standard local coordinates [H4]. We denote $\boxtimes_{P(1)}$ simply as $\boxtimes$. For any pair of V-modules W_1, W_2, the module $W_1 \boxtimes_{P(z)} W_2$ is spanned by the homogeneous components of $w_1 \boxtimes_{P(z_1)} w_2 \in \overline{W_1 \otimes W_2}, \forall w_1 \in W_1, w_2 \in W_2$.

For each V-module W, there is a left unit isomorphism $l_W : V \boxtimes W \to W$ defined by

$$\overline{l_W}(v \boxtimes w) = Y_W(v, 1)w, \qquad \forall v \in V, w \in W, \tag{3.1}$$

where $\overline{l_W}$ is the unique extension of l_W on $\overline{V \boxtimes W}$ and Y_W is the vertex operator which defines the module structure on W, and a right unit isomorphism $r_W : W \boxtimes V \to W$ defined by

$$\overline{r_W}(w \boxtimes v) = e^{L(-1)} Y_W(v, -1)w, \qquad \forall v \in V, w \in W. \tag{3.2}$$

Remark 3.1. We have used "overline" for the extensions of maps, algebraic completions of graded vector spaces and complex conjugations of complex variables. One shall not confuse them because they acts on different things.

Let W_1 and W_2 be V-modules. For a given path $\gamma \in \mathbb{C}^\times$ from a point z_1 to z_2, there is a parallel isomorphism associated to this path

$$\mathcal{T}_\gamma : W_1 \boxtimes_{P(z_1)} W_2 \longrightarrow W_1 \boxtimes_{P(z_2)} W_2.$$

Let $\mathcal{Y}$ be the intertwining operator corresponding to the intertwining map $\boxtimes_{P(z_2)}$ and $l(z_1)$ the value of the logarithm of z_1 determined by $\log z_2$ and analytic continuation along the path γ. For $w_1 \in W_1, w_2 \in W_2$, $\mathcal{T}_\gamma$ is defined by

$$\overline{\mathcal{T}_\gamma}(w_1 \boxtimes_{P(z_1)} w_2) = \mathcal{Y}(w_1, e^{l(z_1)})w_2,$$

where $\overline{\mathcal{T}_\gamma}$ is the natural extension of $\mathcal{T}_\gamma$. Moreover, the parallel isomorphism depends only on the homotopy class of γ.

For $z_1 > z_2 > z_1 - z_2 > 0$ and each triple of V-modules W_1, W_2, W_3, there is an associativity isomorphism:

$$\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \to (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3,$$

which is characterized by

$$\overline{\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}}(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) = (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)} \quad (3.3)$$

for $w_{(i)} \in W_i, i = 1, 2, 3$. The associativity isomorphism $\mathcal{A}$ of the braided tensor category is characterized by the following commutative diagram:

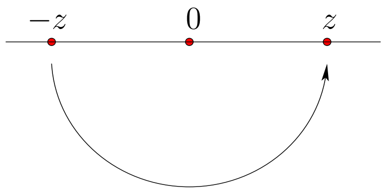
$$\begin{array}{ccc} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) & \xrightarrow{\mathcal{T}_{\gamma_1}\circ(\mathrm{id}_{W_1}\boxtimes_{P(z_1)}\mathcal{T}_{\gamma_2})} & W_1 \boxtimes (W_2 \boxtimes W_3) \\ \Big\downarrow \mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)} & & \Big\downarrow \mathcal{A} \\ (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 & \xrightarrow{\mathcal{T}_{\gamma_2}\circ(\mathcal{T}_{\gamma_3}\boxtimes_{P(z_2)}\mathrm{id}_{W_3})} & (W_1 \boxtimes W_2) \boxtimes W_3\,, \end{array} \quad (3.4)$$

where $\gamma_1, \gamma_2, \gamma_3$ are paths in $\mathbb{R}_+$ from $z_1, z_2, z_1 - z_2$ to 1, respectively.

There is also a braiding isomorphism, for $z > 0$, $\mathcal{R}^{P(z)}_+ : W_1 \boxtimes_{P(z)} W_2 \to W_2 \boxtimes_{P(z)} W_1$ for each pair of V-modules W_1, W_2, defined as

$$\overline{\mathcal{R}^{P(z)}_+}(w_1 \boxtimes_{P(z)} w_2) = e^{L(-1)}\overline{\mathcal{T}}_{\gamma_+}(w_2 \boxtimes_{P(-z)} w_1), \quad (3.5)$$

where γ_+ is a path from $-z$ to z inside the lower half plane as shown in the following graph

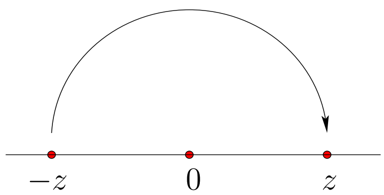


. (3.6)

The inverse of $\mathcal{R}^{P(z)}_+$ is denoted by $\mathcal{R}^{P(z)}_-$, which is characterized by

$$\overline{\mathcal{R}^{P(z)}_-}(w_2 \boxtimes_{P(z)} w_1) = e^{L(-1)}\overline{\mathcal{T}}_{\gamma_-}(w_1 \boxtimes_{P(-z)} w_2), \quad (3.7)$$

where γ_- is a path in the upper half plane as shown in the following graph



. (3.8)

We denote $\mathcal{R}^{P(1)}_\pm$ simply as $\mathcal{R}_\pm$.

Remark 3.2. Notice that our choice of braiding isomorphism is the opposite of that in [HKo1][H11][Ko1]. Accordingly, the twist θ must be chosen to be the inverse of the twist in [H11][Ko1] (see (3.11) and (3.14).

For $z_1, z_2 \in \mathbb{R}_+$, the naturalness of $\mathcal{T}$ implies the commutativity of the following diagram:

$$\begin{array}{ccc} W_1 \boxtimes_{P(z_1)} W_2 & \xrightarrow{\mathcal{T}_\gamma} & W_1 \boxtimes_{P(z_2)} W_2 \\ \mathcal{R}_\pm^{P(z_1)} \downarrow & & \downarrow \mathcal{R}_\pm^{P(z_2)} \\ W_1 \boxtimes_{P(z_1)} W_2 & \xrightarrow{\mathcal{T}_\gamma} & W_1 \boxtimes_{P(z_2)} W_2 \, , \end{array} \tag{3.9}$$

where γ is a path in $\mathbb{R}_+$ from z_1 to z_2.

Let $\mathcal{V}_{W_1 W_2}^{W_3}$ denotes the space of intertwining operators of type $\binom{W_3}{W_1 W_2}$. There is a canonical isomorphism between $\mathcal{V}_{W_1 W_2}^{W_3}$ and $\mathrm{Hom}_V(W_1 \boxtimes W_2, W_3)$ induced from the universal property of tensor product. Let $\mathcal{Y} \in \mathcal{V}_{W_1 W_2}^{W_3}$. We denote the corresponding morphism in $\mathrm{Hom}_V(W_1 \boxtimes W_2, W_3)$ as $m_{\mathcal{Y}}$. For $r \in \mathbb{Z}$, $\Omega_r : \mathcal{V}_{W_1 W_2}^{W_3} \to \mathcal{V}_{W_2 W_1}^{W_3}$ is an isomorphism defined as follow:

$$\Omega_r(\mathcal{Y})(w_2, x)w_1 = \mathcal{Y}(w_1, e^{(2r+1)\pi i}x)w_2$$

for all $w_1 \in W_1, w_2 \in W_2$. Then it is easy to see ([Ko1]) that

$$m_{\mathcal{Y}} = m_{\Omega_0(\mathcal{Y})} \circ \mathcal{R}_+ = m_{\Omega_{-1}(\mathcal{Y})} \circ \mathcal{R}_-. \tag{3.10}$$

We also define an isomorphism, called *twist*, $\theta_W \in \mathrm{Hom}_V(W, W)$ by

$$\theta_W = e^{-2\pi i L(0)} \tag{3.11}$$

for any V-module W.

The tensor product $V \otimes V$ is also a vertex operator algebra [FHL]. Moreover, it was shown in [HKo2] that $V \otimes V$ also satisfies the conditions in Theorem 0.2. Therefore, $\mathcal{C}_{V\otimes V}$, the category of $V \otimes V$-modules, also has a structure of semisimple braided tensor category. For $z, \zeta \in \mathbb{C}^\times$, let $\boxtimes_{P(z)P(\zeta)}$ be the tensor product bifunctor in $\mathcal{C}_{V\otimes V}$ defined by

$$(A \otimes B) \boxtimes_{P(z)P(\zeta)} (C \otimes D) = (A \boxtimes_{P(z)} B) \otimes (C \boxtimes_{P(\zeta)} D),$$

where A, B, C, D are V-modules. We denote the tensor product bifunctor $\boxtimes_{P(1)P(1)}$ simply as $\boxtimes$.

We showed in [Ko1] that there are a few different braiding structures on $\mathcal{C}_{V\otimes V}$. The one that is interesting to us is $\mathcal{R}_{+-}$, which is defined as follow. Consider two objects A, B in $\mathcal{C}_{V\otimes V}$ with the following decomposition:

$$A = \coprod_{i=1}^{m} A_i^L \otimes A_i^R, \quad B = \coprod_{j=1}^{n} B_j^L \otimes B_j^R. \tag{3.12}$$

Then we define $\mathcal{R}_{+-} : A \boxtimes B \to B \boxtimes A$ by the following commutative diagram:

$$\begin{array}{ccc} \coprod_{i=1}^M \coprod_{j=1}^N (A_i^L \boxtimes B_j^L) \otimes (A_i^R \boxtimes B_j^R) & \xrightarrow{\cong} & A \boxtimes B \\ \mathcal{R}_+ \otimes \mathcal{R}_- \downarrow & & \vdots \; \exists! \, \mathcal{R}_{+-} \\ \coprod_{i=1}^M \coprod_{j=1}^N (B_j^L \boxtimes A_i^L) \otimes (B_j^R \boxtimes A_i^R) & \xrightarrow{\cong} & B \boxtimes A, \end{array} \tag{3.13}$$

where the two horizontal maps are canonical morphisms due to the universal property of tensor product. Such defined $\mathcal{R}_{+-}$ is independent of the decomposition of A and B. From now on, we assume that $\mathcal{C}_{V\otimes V}$ is equipped with the braiding structure $\mathcal{R}_{+-}$.

For any object $A \in \mathcal{C}_{V\otimes V}$, we also define an isomorphism $\theta_A : A \to A$, called *twist*, as follow

$$\theta_A = e^{-2\pi i L^L(0)} \otimes e^{2\pi i L^R(0)}. \tag{3.14}$$

An object A is said to have a trivial twist if $\theta_A = \mathrm{id}_A$.

Now we recall the definition of (commutative) associative algebra in a braided tensor category $\mathcal{C}$ with tensor product $\otimes$, unit object $1_\mathcal{C}$, left unit isomorphism l_W and right unit isomorphism r_W for any object W, the associativity $\mathcal{A}$ and the braiding $\mathcal{R}$.

Definition 3.3. An associative algebra in $\mathcal{C}$ (or associative $\mathcal{C}$-algebra) is an object A in $\mathcal{C}$ along with $\mu_A : A \otimes A \to A$ and an monomorphism $\iota_A : 1_\mathcal{C} \to A$ such that the following conditions holds:

1. *Associativity*: $\mu_A \circ (\mu_A \otimes \mathrm{id}_A) = \mu_A \circ (\mathrm{id}_A \otimes \mu_A)$.
2. *Unit properties*: $\mu_A \circ (\iota_A \otimes \mathrm{id}_A) \circ l_A^{-1} = \mu_A \circ (\iota_A \otimes \mathrm{id}_A) \circ r_A^{-1} = \mathrm{id}_A$.

An associative $\mathcal{C}$-algebra is called *commutative* if an additional condition: $\mu_A = \mu_A \circ \mathcal{R}$ is satisfied.

The following Theorem is proved in [HKo1].

Theorem 3.4. *The category of open-string vertex operator algebras over V is isomorphism to the category of associative $\mathcal{C}_V$-algebras.*

The following Theorem is proved in [Ko1].

Theorem 3.5. *The category of conformal full field algebras over $V \otimes V$ is isomorphic to the category of commutative associative algebras in $\mathcal{C}_{V\otimes V}$ with a trivial twist.*

We are interested in studying the relation between above two algebras as ingredients of an open-closed field algebra over V. Notice that these two algebras live in different categories. So we will first discuss the relation between these two categories.

Recall a functor $T_{P(z)} : \mathcal{C}_{V\otimes V} \to \mathcal{C}_V$ [HL4]. In particular, for W_1, W_2 being V-module, $T_{P(z)}(W_1 \otimes W_2) = W_1 \boxtimes_{P(z)} W_2$. We will simply write $T_{P(1)}$ as T. Let $(\mathcal{C}_1, 1_{\mathcal{C}_1}, \otimes)$ and $(\mathcal{C}_2, 1_{\mathcal{C}_2}, \otimes)$ be any two monoidal categories. Recall that a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is called a monoidal functor (or tensor functor) [Ka] if there is an isomorphism morphism $\varphi_0 : \mathbf{1}_{\mathcal{C}_2} \to F(\mathbf{1}_{\mathcal{C}_2})$ and a natural isomorphism φ_2 between the following two functors

$$\begin{aligned} &\mathcal{C}_1 \times \mathcal{C}_1 \xrightarrow{F\times F} \mathcal{C}_2 \times \mathcal{C}_2 \xrightarrow{\otimes} \mathcal{C}_2, \\ &\mathcal{C}_1 \times \mathcal{C}_1 \xrightarrow{\otimes} \mathcal{C}_1 \xrightarrow{F} \mathcal{C}_2, \end{aligned} \tag{3.15}$$

satisfying three natural axioms expressed in terms of the following three commutative diagrams

$$\begin{array}{ccc} F(A)\otimes(F(B)\otimes F(C)) & \xrightarrow{\mathcal{A}} & (F(A)\otimes F(B))\otimes F(C) \\ \Big\downarrow{\scriptstyle \mathrm{id}_{F(A)}\otimes\varphi_2} & & \Big\downarrow{\scriptstyle \varphi_2\otimes\mathrm{id}_{F(C)}} \\ F(A)\otimes F(B\otimes C) & & F(A\otimes B)\otimes F(C) \\ \Big\downarrow{\scriptstyle \varphi_2} & & \Big\downarrow{\scriptstyle \varphi_2} \\ F(A\otimes(B\otimes C)) & \xrightarrow{F(\mathcal{A})} & F((A\otimes B)\otimes C) \end{array} \tag{3.16}$$

$$\begin{array}{ccc} \mathbf{1}_{\mathcal{C}_2}\otimes F(A) & \xrightarrow{l_{F(A)}} & F(A) \\ \Big\downarrow{\scriptstyle \varphi_0\otimes\mathrm{id}_{F(A)}} & & \Big\uparrow{\scriptstyle F(l_A)} \\ F(\mathbf{1}_{\mathcal{C}_1})\otimes F(A) & \xrightarrow{\varphi_2} & F(\mathbf{1}_{\mathcal{C}_1}\otimes A) \end{array} \tag{3.17}$$

$$\begin{array}{ccc} F(A)\otimes \mathbf{1}_{\mathcal{C}_2} & \xrightarrow{r_{F(A)}} & F(A) \\ \Big\downarrow{\scriptstyle \mathrm{id}_{F(A)}\otimes\varphi_0} & & \Big\uparrow{\scriptstyle F(r_A)} \\ F(A)\otimes F(\mathbf{1}_{\mathcal{C}_1}) & \xrightarrow{\varphi_2} & F(A\otimes \mathbf{1}_{\mathcal{C}_1}) \end{array} \tag{3.18}$$

In our case, T is a functor from $\mathcal{C}_{V\otimes V}$ to $\mathcal{C}_V$. Let

$$\varphi_0 := l_V^{-1} = r_V^{-1} : V \to V \boxtimes V = T(V\otimes V). \tag{3.19}$$

For each four V-modules $W_i^L, W_i^R, i = 1, 2$, notice that

$$\begin{aligned} T(W_1^L\otimes W_1^R)\boxtimes T(W_2^L\otimes W_2^R) &= (W_1^L\boxtimes W_1^R)\boxtimes(W_2^L\boxtimes W_2^R), && (3.20)\\ T((W_1^L\otimes W_2^R)\boxtimes(W_2^L\otimes W_2^R)) &= (W_1^L\boxtimes W_2^L)\boxtimes(W_1^R\boxtimes W_2^R). && (3.21)\end{aligned}$$

We define $\varphi_2 : (W_1^L\boxtimes W_1^R)\boxtimes(W_2^L\boxtimes W_2^R) \to (W_1^L\boxtimes W_2^L)\boxtimes(W_1^R\boxtimes W_2^R)$ by

$$\varphi_2 := \mathcal{A}\circ(\mathrm{id}_{W_1}\boxtimes\mathcal{A}^{-1})\circ(\mathrm{id}_{W_1}\boxtimes\mathcal{R}_-\boxtimes\mathrm{id}_{W_4})\circ(\mathrm{id}_{W_1}\boxtimes\mathcal{A})\circ\mathcal{A}^{-1}. \tag{3.22}$$

The above definition of φ_2 can be naturally extended to a morphism $T(A)\boxtimes T(B)\to T(A\boxtimes B)$ for each pair of objects A and B in $\mathcal{C}_{V\otimes V}$. We still denote the extended morphism as φ_2.

Proposition 3.6. *$T : \mathcal{C}_{V\otimes V} \to \mathcal{C}_V$ with φ_0 and φ_2 defined in (3.19) and (3.22) is a tensor functor.*

Proof. It is a direct check of axioms. We omit the detail. ∎

Proposition 3.7. *Let (A, μ_A, ι_A) be an associative algebra in tensor category $\mathcal{C}_1$ and $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a monoidal functor. If we define*

$$\begin{aligned} \mu_{T(A)} &:= F(\mu_A)\circ\varphi_2, && (3.23)\\ \iota_{T(A)} &:= F(\iota_A)\circ\varphi_0, && (3.24)\end{aligned}$$

then $(F(A), \mu_{F(A)}, \iota_{F(A)})$ is an associative algebra in $\mathcal{C}_2$.

Proof. This result is so standard that it must has been proven somewhere. But the author is not aware of where such proof is available. So we include the proof here.

We have the following commutative diagram:

$$
\begin{array}{ccccccc}
A & \xrightarrow{l_A^{-1}} & \mathbf{1}_{\mathcal{C}_1} \otimes A & \xrightarrow{\iota_A \otimes \mathrm{id}_A} & A \otimes A & \xrightarrow{\mu_A} & A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F(A) & \xrightarrow{F(l_A^{-1})} & F(\mathbf{1}_{\mathcal{C}_1} \otimes A) & \xrightarrow{F(\iota_A \otimes \mathrm{id}_A)} & F(A \otimes A) & \xrightarrow{F(\mu_A)} & F(A) \\
{\scriptstyle l_{F(A)}}\uparrow & & \uparrow{\scriptstyle \varphi_2} & & {\scriptstyle \varphi_2}\uparrow & & \\
\mathbf{1}_{\mathcal{C}_2} \otimes F(A) & \xrightarrow{\varphi_0 \otimes \mathrm{id}} & F(\mathbf{1}_{\mathcal{C}_1}) \otimes F(A) & \xrightarrow{F(\iota_A) \otimes \mathrm{id}} & F(A) \otimes F(A) & &
\end{array}
\tag{3.25}
$$

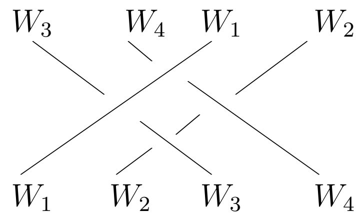
where the commutativity of the left lower subdiagram is due to (3.17) and that of right lower subdiagram is due to the naturalness of φ_2, the commutativity of the remaining subdiagrams follow from the functorial properties of F. The left unit property of associative algebra follows from the fact that two paths from $1_{\mathcal{C}_2} \otimes F(A)$ to $F(A)$ are commutative. The proof of the right unit property is similar.

The associativity of $F(A)$ follows from the following commutative diagram:

$$
\begin{array}{ccccc}
F(A) \otimes (F(A) \otimes F(A)) & \xrightarrow{\mathrm{id} \otimes \varphi_2} & F(A) \otimes F(A \otimes A) & \xrightarrow{\mathrm{id} \otimes F(\mu_A)} & F(A) \otimes F(A) \\
\Big\downarrow & & \downarrow{\scriptstyle \varphi_2} & & {\scriptstyle \varphi_2}\downarrow \\
 & & F(A \otimes (A \otimes A)) & \xrightarrow{F(\mathrm{id} \otimes \mu_A)} & F(A \otimes A) \\
{\scriptstyle \mathcal{A}_2} & & \downarrow{\scriptstyle F(\mathcal{A})} & & {\scriptstyle F(\mu_A)}\downarrow \\
 & & & & F(A) \\
 & & & & \uparrow{\scriptstyle F(\mu_A)} \\
 & & F((A \otimes A) \otimes A) & \xrightarrow{F(\mu_A \otimes \mathrm{id})} & F(A \otimes A) \\
\Big\downarrow & & \uparrow{\scriptstyle \varphi_2} & & \uparrow{\scriptstyle \varphi_2} \\
(F(A) \otimes F(A)) \otimes F(A) & \xrightarrow{\varphi_2 \otimes \mathrm{id}} & F(A \otimes A) \otimes F(A) & \xrightarrow{F(\mu_A) \otimes \mathrm{id}} & F(A) \otimes F(A)
\end{array}
$$

in which the commutativity of the left subdiagram is due to (3.16), those of the right upper subdiagram and the right lower subdiagram are due to the naturalness of φ_2, and that of the right middle subdiagram is due to the functorial property of F and the associativity of A. ∎

For any four objects $W_i, i = 1, 2, 3, 4$ in $\mathcal{C}_V$, we define a morphism σ in $\mathrm{Hom}((W_1 \boxtimes W_2) \boxtimes (W_3 \boxtimes W_4), (W_3 \boxtimes W_4) \boxtimes (W_1 \boxtimes W_2))$ according to the following graph:



(3.26)

Clearly, for any $A \in \mathcal{C}_{V \otimes V}$, σ can be extended to an automorphism on $T(A) \boxtimes T(A)$, denoted as σ_A.

Proposition 3.8. *Let (A, μ_A, ι_A) be a commutative algebra in $\mathcal{C}_{V\otimes V}$. $(T(A), \mu_{T(A)}, \iota_{T(A)})$ is not only an associative algebra in $\mathcal{C}_V$ but also satisfies the following commutativity:*

$$\mu_{T(A)} = \mu_{T(A)} \circ \sigma_A. \tag{3.27}$$

Proof. Let us assume that A take the simple form $A = W_1 \otimes W_2$ with W_1, W_2 being V-modules (the proof for general A is the same). Consider the following diagram:

$$\begin{array}{ccccc} (W_1 \boxtimes W_2) \boxtimes (W_1 \boxtimes W_2) & \xrightarrow{\varphi_2} & (W_1 \boxtimes W_1) \boxtimes (W_2 \boxtimes W_2) & \xrightarrow{T(\mu_{cl})} & W_1 \boxtimes W_2 \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \mathcal{R}_+ \boxtimes \mathcal{R}_-} & & \downarrow{\scriptstyle \mathrm{id}} \\ (W_1 \boxtimes W_2) \boxtimes (W_1 \boxtimes W_2) & \xrightarrow{\varphi_2} & (W_1 \boxtimes W_1) \boxtimes (W_2 \boxtimes W_2) & \xrightarrow{T(\mu_{cl})} & W_1 \boxtimes W_2 \end{array} \tag{3.28}$$

The left subdiagram is commutative because two paths corresponding to the same braiding; the right subdiagram is commutative because of the commutativity of A. Hence the above diagram is commutative. (3.27) follows from the commutativity of (3.28). ∎

3.2 Open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebras

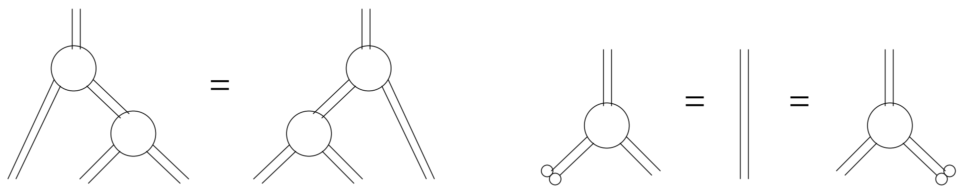
Let $((V_{cl}, m_{cl}, \iota_{cl}), (V_{op}, Y_{op}, \iota_{op}), \mathbb{Y}_{cl-op})$ be an open-closed field algebra over V throughout this subsection.

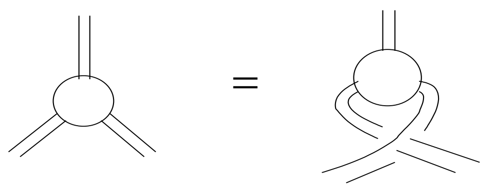
By Theorem 3.5, $(V_{cl}, m_{cl}, \iota_{cl})$, a conformal full field algebra over $V \otimes V$ is equivalent to a commutative associative algebra in $\mathcal{C}_{V\otimes V}$ equipped with braiding $\mathcal{R}_{+-}$, satisfying the condition $\theta_{V_{cl}} = \mathrm{id}_{V_{cl}}$. We denoted this algebra in $\mathcal{C}_{V\otimes V}$ as a triple

$$(V_{cl}, \mu_{cl}, \iota_{cl}),$$

where $\mu_{cl} : V_{cl} \boxtimes V_{cl} \to V_{cl}$ is the multiplication morphism induced from the formal intertwining operator $\mathbb{Y}_f$ (recall (1.31)), i.e. $\mu_{cl} = m_{\mathbb{Y}_f}$.

By Proposition 3.7 and 3.8, $(T(V_{cl}), \nu_{T(V_{cl})}, \iota_{T(V_{cl})})$ (recall the notation in Proposition 3.7) is an associative algebra in $\mathcal{C}_V$. The property of algebra $T(V_{cl})$ can be expressed in the following graphic equations:


, (3.29)

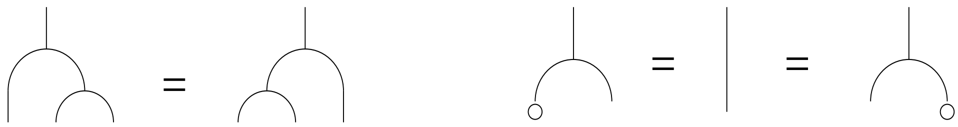

. (3.30)

By Theorem 3.4, $(V_{op}, Y_{op}, \iota_{op})$, an open-string vertex operator algebra over V is equivalent to an associative $\mathcal{C}_V$-algebra, denoted as a triple

$$(V_{op}, \mu_{op}, \iota_{op}),$$

where $\mu_{op} : V_{op} \boxtimes V_{op} \to V_{op}$ is the multiplication morphism canonically induced from the formal intertwining operator Y_{op}^f (recall (1.14) and (1.15)), i.e. $\mu_{op} = m_{Y_{op}^f}$.

The defining properties of $(V_{op}, \mu_{op}, \iota_{op})$ can be expressed in the following graphic equations:



(3.31)

Now we are ready to study the categorical formulation of the only remaining data $\mathbb{Y}_{cl-op}$. By the chirality splitting properties and the associativity of intertwining operator algebra [H7], there exist intertwining operators $\mathcal{Y}^{(5)}, \mathcal{Y}^{(6)}$ such that

$$\langle v', \mathbb{Y}_{cl-op}(u \otimes \bar{u}, z, \zeta) v \rangle = \langle v', \mathcal{Y}^{(5)}(\mathcal{Y}^{(6)}(u^L, z-\zeta)u^R, \zeta)v \rangle \tag{3.32}$$

for $v' \in V_{op}', v \in V_{op}, u^L \otimes u^R \in V_{cl}$ and $\zeta > z - \zeta > 0$. For $u^L \otimes u^R \in V_{cl}$, we have $u^L \boxtimes_{P(z-\zeta)} u^R \in \overline{T_{P(z-\zeta)}(V_{cl})}$. Using the universal property of tensor product [HL2]-[HL5], it is easy to see that there is a unique V-module map

$$\mu_{cl-op}^{P(z-\zeta)P(\zeta)} : T_{P(z-\zeta)}(V_{cl}) \boxtimes_{P(\zeta)} V_{op} \to V_{op} \tag{3.33}$$

for $z > \zeta > z - \zeta > 0$, such that, for $u^L \otimes u^R \in V_{cl}$ and $v \in V_{op}$,

$$\overline{\mu_{cl-op}^{P(z-\zeta)P(\zeta)}}((u^L \boxtimes_{P(z-\zeta)} u^R) \boxtimes_{P(\zeta)} v) = \mathcal{Y}^{(5)}(\mathcal{Y}^{(6)}(u^L, z-\zeta)u^R, \zeta)v, \tag{3.34}$$

where $\overline{\mu_{cl-op}^{P(z-\zeta)P(\zeta)}}$ is the unique extension of $\mu_{cl-op}^{P(z-\zeta)P(\zeta)}$ to the algebraic completion of $T_{P(z-\zeta)}(V_{cl}) \boxtimes_{P(\zeta)} V_{op}$. Let γ_1 be a path in $\mathbb{R}_+$ from 1 to $z - \zeta$ and γ_2 a path in $\mathbb{R}_+$ from 1 to ζ. Then we define a map $\mu_{cl-op} : T(V_{cl}) \boxtimes V_{op} \to V_{op}$ as follow:

$$\mu_{cl-op} := \mu_{cl-op}^{P(z-\zeta)P(\zeta)} \circ \mathcal{T}_{\gamma_2} \circ (\mathcal{T}_{\gamma_1} \boxtimes \mathrm{id}_{V_{op}}). \tag{3.35}$$

Since $\mathcal{T}_\gamma$ depends on path only homotopically, it is clear that above definition of μ_{cl-op} is independent of z, ζ in $\mathbb{R}_+$ and paths γ_1, γ_2 in $\mathbb{R}_+$. In particular, one can choose γ_1 and γ_2 to be the straight line between 1 and $z - \zeta, \zeta$ respectively.

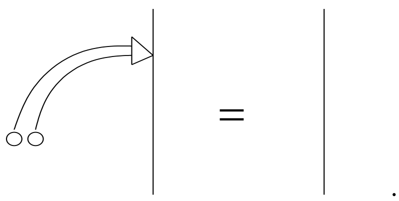
By Theorem 1.28, it is enough to study the categorical formulations of the unit property (1.82), the associativity I (1.52), the associativity II (1.56) and the commutativity I in Proposition 1.18.

We first consider the property (1.82).

Proposition 3.9. *The condition (1.82) is equivalent to the following condition:*

$$\mu_{cl-op} \circ ((T(\iota_{V_{cl}}) \circ \varphi_0) \boxtimes \mathrm{id}_{V_{op}}) \circ l_{V_{op}}^{-1} = \mathrm{id}_{V_{op}}, \tag{3.36}$$

which can also be expressed by the following graphic equation:



(3.37)

Proof. First, (1.82) is equivalent to the following condition:

$$\mathbb{Y}_{cl-op}(\mathbf{1}_{cl}; z, \zeta) = \mathrm{id}_{V_{op}}, \quad \text{for } z > \zeta > 0. \tag{3.38}$$

Recall that $\mathbf{1}_{cl} = \iota_{V_{cl}}(\mathbf{1} \otimes \mathbf{1})$. Replacing $u^L \otimes u^R$ by $\mathbf{1}_{cl}$ in (3.32), one can see that both sides of equation (3.32) are independent of z and ζ. Hence (3.38) holds for all $z, \zeta \in \mathbb{R}_+$. Using (3.32) and (3.34), we obtain

$$\overline{\mu_{cl-op}^{P(z-\zeta)P(\zeta)}}(T_{P(z-\zeta)}(\mathbf{1}_{cl}) \boxtimes_{P(\zeta)} v) = v \tag{3.39}$$

for $v \in V_{op}$ and $z, \zeta \in \mathbb{R}_+$. Therefore, we have, for $v \in V_{op}$,

$$\begin{aligned} &\mu_{cl-op}^{P(z-\zeta)P(\zeta)} \circ T_{P(z-\zeta)}(\iota_{V_{cl}}) \boxtimes_{P(\zeta)} \mathrm{id}_{V_{op}}((\mathbf{1} \boxtimes_{P(z-\zeta)} \mathbf{1}) \boxtimes_{P(\zeta)} v) \\ &\quad = \mu_{cl-op}^{P(z-\zeta)P(\zeta)}(T_{P(z-\zeta)}(\mathbf{1}_{cl}) \boxtimes_{P(\zeta)} v) \\ &\quad = v, \end{aligned} \tag{3.40}$$

which can be further expressed equivalently as

$$\begin{aligned} \mathrm{id}_{V_{op}} &= \mu_{cl-op}^{P(z-\zeta)P(\zeta)} \circ \big((T_{P(z-\zeta)}(\iota_{V_{cl}}) \circ \mathcal{T}_{\gamma_1}) \boxtimes_{P(\zeta)} \mathrm{id}_{V_{op}}\big) \circ \mathcal{T}_{\gamma_2} \circ (l_V^{-1} \boxtimes \mathrm{id}_{V_{op}}) \circ l_{V_{op}}^{-1} \\ &= \mu_{cl-op}^{P(z-\zeta)P(\zeta)} \circ (\mathcal{T}_{\gamma_1} \boxtimes_{P(\zeta)} \mathrm{id}_{V_{op}}) \circ \mathcal{T}_{\gamma_2} \circ \big((T(\iota_{V_{cl}}) \circ \varphi_0) \boxtimes \mathrm{id}_{V_{op}}\big) \circ l_{V_{op}}^{-1} \\ &= \mu_{cl-op}^{P(z-\zeta)P(\zeta)} \circ \mathcal{T}_{\gamma_2} \circ (\mathcal{T}_{\gamma_1} \boxtimes \mathrm{id}_{V_{op}}) \circ \big((T(\iota_{V_{cl}}) \circ \varphi_0) \boxtimes \mathrm{id}_{V_{op}}\big) \circ l_{V_{op}}^{-1} \\ &= \mu_{cl-op} \circ \big((T(\iota_{V_{cl}}) \circ \varphi_0) \boxtimes \mathrm{id}_{V_{op}}\big) \circ l_{V_{op}}^{-1} \end{aligned} \tag{3.41}$$

where γ_1 and γ_2 are paths in $\mathbb{R}_+$ from 1 to $z - \zeta$ and ζ respectively.

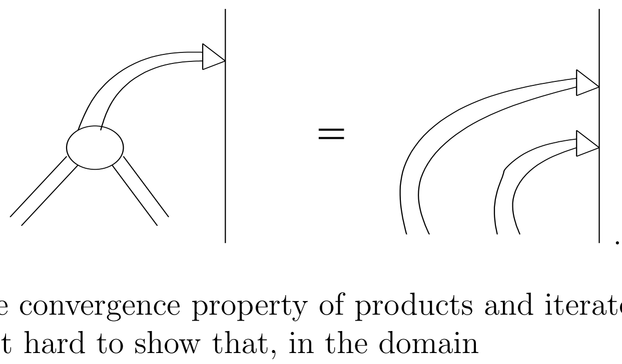
Conversely, from (3.39), (3.40) and (3.41), it is clear that (3.36) or (3.37) also implies (1.82). ∎

Now we consider the associativity II (recall Proposition 1.17).

Proposition 3.10. *The associativity II is equivalent to the following condition:*

$$\mu_{cl-op} \circ (\mathrm{id}_{T(V_{cl})} \boxtimes \mu_{cl-op}) = \mu_{cl-op} \circ \big((T(\mu_{cl}) \circ \varphi_2) \boxtimes \mathrm{id}_{V_{op}}\big) \circ \mathcal{A}, \tag{3.42}$$

which can also be expressed by the following graphic equation



. (3.43)

Proof. Using the convergence property of products and iterates of intertwining operators of V, it is not hard to show that, in the domain

$$\{(z_1, \zeta_1, z_2, \zeta_2) | z_1 > \zeta_1 > z_2 > \zeta_2 > 0, 2\zeta_2 > z_2, 2\zeta_1 > z_1 + z_2\}, \tag{3.44}$$

we have

$$
\begin{aligned}
&Y_{cl-op}(u_1^L \otimes u_1^R; z_1, \zeta_1) Y_{cl-op}(u_1^L \otimes u_1^R; z_1, \zeta_1) v \\
&\quad = \mathcal{Y}^{(5)}(\mathcal{Y}^{(6)}(u_1^L, z_1 - \zeta_1) u_1^R, \zeta_1) \mathcal{Y}^{(5)}(\mathcal{Y}^{(6)}(u_2^L, z_2 - \zeta_2) u_2^R, \zeta_2) v
\end{aligned} \tag{3.45}
$$

for $u_1^L \otimes u_1^R, u_2^L \otimes u_2^R \in V_{cl}$ and $v \in V_{op}$. By the universal property of tensor product, there is a unique morphism

$$
\mu_{cl-op}^{P(z_1-\zeta_1)P(\zeta_1)} \circ (\mathrm{id}_{T_{P(z_1-\zeta_1)}(V_{cl})} \boxtimes_{P(\zeta_1)} \mu_{cl-op}^{P(z_2-\zeta_2)P(\zeta_2)}) \tag{3.46}
$$

in $\mathrm{Hom}(T_{P(z_1-\zeta_1)}(V_{cl}) \boxtimes_{P(\zeta_1)} (T_{P(z_2-\zeta_2)}(V_{cl}) \boxtimes_{P(\zeta_2)} V_{op}), V_{op})$ such that

$$
\begin{aligned}
&Y_{cl-op}(u_1^L \otimes u_1^R; z_1, \zeta_1) Y_{cl-op}(u_1^L \otimes u_1^R; z_1, \zeta_1) v \\
&\quad = \overline{\mu_{cl-op}^{P(z_1-\zeta_1)P(\zeta_1)}} \circ (\mathrm{id}_{T_{P(z_1-\zeta_1)}(V_{cl})} \boxtimes_{P(\zeta_1)} \overline{\mu_{cl-op}^{P(z_2-\zeta_2)P(\zeta_2)}})\big(\\
&\qquad\quad (u_1^L \boxtimes_{P(z_1-\zeta_1)} u_1^R) \boxtimes_{P(\zeta_1)} ((u_2^L \boxtimes_{P(z_2-\zeta_2)} u_2^R) \boxtimes_{P(\zeta_2)} v)\big)
\end{aligned} \tag{3.47}
$$

for $(z_1, \zeta_1, z_2, \zeta_2)$ in the domain (3.44).

On the other hand of (1.56), it is proved in [HKo2] that $\mathbb{Y}$ can be expanded as follow:

$$
\mathbb{Y}_{an}(u_1^L \otimes u_1^R; z, \zeta) u_2^L \otimes u_2^R = \sum_{i=1}^{N} \mathcal{Y}_i^L(u_1^L, z) u_2^L \otimes \mathcal{Y}_i^R(u_1^R, \zeta) u_2^R \tag{3.48}
$$

for some $N \in \mathbb{Z}_+$. There is unique morphism

$$
\mu_{cl}^{P(z_1-z_2)P(\zeta_1-\zeta_2)} \in \mathrm{Hom}_{\mathcal{C}_{V\otimes V}}(V_{cl} \boxtimes_{P(z_1-z_2)P(\zeta_1-\zeta_2)} V_{cl}, V_{cl})
$$

such that, for $u, v \in V_{cl}$,

$$
\overline{\mu_{cl}^{P(z_1-z_2)P(\zeta_1-\zeta_2)}}(u \boxtimes_{P(z_1-z_2)P(\zeta_1-\zeta_2)} v) = \mathbb{Y}(u; z_1 - z_2, \zeta_1 - \zeta_2) v.
$$

By the convergence property of intertwining operator algebra, it not hard to see that, in the domain

$$
\{(z_1, \zeta_1, z_2, \zeta_2) | z_1 > z_2 > \zeta_1 > \zeta_2 > 0, 2\zeta_2 > z_1, 2z_2 > z_1 + \zeta_1\}, \tag{3.49}
$$

we have

$$
\begin{aligned}
&Y_{cl-op}(\mathbb{Y}(u_1^L \otimes u_1^R; z_1 - z_2, \zeta_1 - \zeta_2) u_2^L \otimes u_2^R; z_2, \zeta_2) v \\
&\quad = \sum_{i=1}^{N} Y_{cl-op}(\mathcal{Y}_i^L(u_1^L, z_1 - z_2) u_2^L \otimes \mathcal{Y}_i^R(u_1^R, \zeta_1 - \zeta_2) u_2^R; z_2, \zeta_2) v \\
&\quad = \sum_{i=1}^{N} \mathcal{Y}^{(5)}(\mathcal{Y}^{(6)}(\mathcal{Y}_i^L(u_1^L, z_1 - z_2) u_2^L, z_2 - \zeta_2) \mathcal{Y}_i^R(u_1^R, \zeta_1 - \zeta_2) u_2^R, \zeta_2) v.
\end{aligned} \tag{3.50}
$$

By the universal property of tensor product, there is a unique morphism

$$
\mu_{cl-op}^{P(z_2-\zeta_2)P(\zeta_2)} \circ (T_{P(z_2-\zeta_2)}(\mu_{cl}^{P(z_1-z_2)P(\zeta_1-\zeta_2)}) \boxtimes_{P(\zeta_2)} \mathrm{id}_{V_{op}}) \tag{3.51}
$$

in Hom$(T_{P(z_2-\zeta_2)}(V_{cl}(\boxtimes_{P(z_1-z_2)P(\zeta_1-\zeta_2)}V_{cl})\boxtimes_{P(\zeta_2)}V_{op}, V_{op})$ such that

$$\begin{aligned}
&\mathbb{Y}_{cl-op}(\mathbb{Y}(u_1^L\otimes u_1^R; z_1-z_2, \zeta_1-\zeta_2)u_2^L\otimes u_2^R; z_2,\zeta_2)v \\
&\quad = \overline{\mu_{cl-op}^{P(z_2-\zeta_2)P(\zeta_2)}} \circ \big(\overline{T_{P(z_2-\zeta_2)}(\mu_{cl}^{P(z_1-z_2)P(\zeta_1-\zeta_2)})} \boxtimes_{P(\zeta_2)} \mathrm{id}_{V_{op}}\big)\big(\\
&\qquad ((u_1^L \boxtimes_{P(z_1-z_2)} u_2^L) \boxtimes_{P(z_2-\zeta_2)} (u_1^R \boxtimes_{P(\zeta_1-\zeta_2)} u_2^R)) \boxtimes_{P(\zeta_2)} v\big)
\end{aligned} \tag{3.52}$$

for $(z_1,\zeta_1,z_2,\zeta_2)$ in the domain (3.49).

Notice that the domains (3.44) and (3.49) are disjoint. Now we fix a point $(z_1,\zeta_1,z_2,\zeta_2)$ in the domain

$$\{(z_1,\zeta_1,z_2,\zeta_2)|z_1>\zeta_1>z_2>\zeta_2>0, 2\zeta_2>z_1, 2\zeta_1>z_1+z_2, 2z_2>\zeta_1+\zeta_2\}, \tag{3.53}$$

which is a subdomain of (3.44). Let $\tilde{z}_2=\zeta_1, \tilde{\zeta}_1=z_2$. Then the quadruple $(z_1,\tilde{\zeta}_1,\tilde{z}_2,\zeta_2)$ is in the domain (3.49).

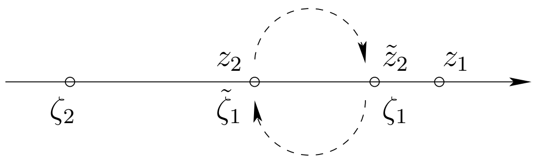
Both sides of associativity (1.56) can be extended to the domain $\mathbb{R}_+^4\cap M_{\mathbb{C}^\times}^4$. It is easy to see that

$$\mathbb{Y}_{cl-op}(u_1^L\otimes u_1^R; z_1,\zeta_1)\mathbb{Y}_{cl-op}(u_2^L\otimes u_2^R; z_2,\zeta_2)v,$$

and

$$\mathbb{Y}_{cl-op}(\mathbb{Y}(u_1^L\otimes u_1^R; z_1-\tilde{z}_2, \tilde{\zeta}_1-\zeta_2)u_2^L\otimes u_2^R; \tilde{z}_2,\zeta_2)v,$$

can be obtained from each other by analytic continuation along the following path



. (3.54)

Meanwhile, if we start from the element

$$(u_1^L \boxtimes_{P(z_1-\zeta_1)} u_1^R) \boxtimes_{P(\zeta_1)} ((u_2^L \boxtimes_{P(z_2-\zeta_2)} u_2^R) \boxtimes_{P(\zeta_2)} v)$$

in $\overline{T_{P(z_1-\zeta_1)}(V_{cl}) \boxtimes_{P(\zeta_1)} (T_{P(z_2-\zeta_2)}(V_{cl}) \boxtimes_{P(\zeta_2)} V_{op})}$ and apply associativity isomorphisms repeatedly and braiding isomorphism on it, we obtain

$$\begin{aligned}
&\xrightarrow{\overline{(\mathcal{A}_{P(z_1)P(\zeta_1)}^{P(z_1-\zeta_1)P(\zeta_1)})^{-1}}} u_1^L \boxtimes_{P(z_1)} (u_1^R \boxtimes_{P(\zeta_1)} ((u_2^L \boxtimes_{P(z_2-\zeta_2)} u_2^R) \boxtimes_{P(\zeta_2)} v)), \\
&\xrightarrow{\overline{\mathcal{A}_{P(\zeta_1)P(\zeta_2)}^{P(\zeta_1-\zeta_2)P(\zeta_2)}}} u_1^L \boxtimes_{P(z_1)} \big((u_1^R \boxtimes_{P(\zeta_1-\zeta_2)} (u_2^L \boxtimes_{P(z_2-\zeta_2)} u_2^R)) \boxtimes_{P(\zeta_2)} v\big), \\
&\xrightarrow{\overline{\mathcal{A}_{P(\zeta_1-\zeta_2)P(z_2-\zeta_2)}^{P(\zeta_1-z_2)P(z_2-\zeta_2)}}} u_1^L \boxtimes_{P(z_1)} \big(((u_1^R \boxtimes_{P(\zeta_1-z_2)} u_2^L) \boxtimes_{P(z_2-\zeta_2)} u_2^R) \boxtimes_{P(\zeta_2)} v\big), \\
&\xrightarrow{\overline{\mathcal{R}_-^{P(\zeta_1-z_2)}}} u_1^L \boxtimes_{P(z_1)} \big((e^{(\zeta_1-z_2)L(-1)} \cdot \\
&\qquad\qquad \overline{\mathcal{T}_{\gamma_-}}(u_2^L \boxtimes_{P(-\zeta_1+z_2)} u_1^R) \boxtimes_{P(z_2-\zeta_2)} u_2^R) \boxtimes_{P(\zeta_2)} v\big),
\end{aligned} \tag{3.55}$$

where we have ignored the obvious identity maps. Now we analytic continuate the last line of (3.55) along the path (3.54), we obtain

$$u_1^L \boxtimes_{P(z_1)} \big(((u_2^L \boxtimes_{P(\zeta_1-z_2)} u_1^R) \boxtimes_{P(z_2-\zeta_2)} u_2^R) \boxtimes_{P(\zeta_2)} v\big), \tag{3.56}$$

which is nothing but the third line of (3.55) except u_1^R and u_2^L being exchanged. Note that the last line of (3.55) and (3.56) are elements in the algebraic completion of the same V-module. We denote this V-module as W. Now we further apply associativity morphisms on (3.56) and use $z_2 = \tilde{\zeta}_1$ and $\zeta_1 = \tilde{z}_2$. We then obtain

$$\begin{aligned}
&\xrightarrow{\overline{(\mathcal{A}_{P(\tilde{z}_2-\zeta_2)}^{P(\tilde{z}_2-\tilde{\zeta}_2)P(\tilde{\zeta}_1-\zeta_2)})^{-1}}} u_1^L \boxtimes_{P(z_1)} \big((u_2^L \boxtimes_{P(\tilde{z}_2-\zeta_2)} (u_1^R \boxtimes_{P(\tilde{\zeta}_1-\zeta_2)} u_2^R) \boxtimes_{P(\zeta_2)} v\big),\\
&\xrightarrow{\overline{\mathcal{A}_{P(z_1)P(\zeta_2)}^{P(z_1-\zeta_2)P(\zeta_2)}}} \big(u_1^L \boxtimes_{P(z_1-\zeta_2)} (u_2^L \boxtimes_{P(\tilde{z}_2-\zeta_2)} (u_1^R \boxtimes_{P(\tilde{\zeta}_1-\zeta_2)} u_2^R))\big) \boxtimes_{P(\zeta_2)} v,\\
&\xrightarrow{\overline{\mathcal{A}_{P(z_1-\zeta_2)P(\tilde{z}_2-\zeta_2)}^{P(z_1-\tilde{z}_2)P(\tilde{z}_2-\zeta_2)}}} \big((u_1^L \boxtimes_{P(z_1-\tilde{z}_2)} u_2^L) \boxtimes_{P(\tilde{z}_2-\zeta_2)} v,\\
&\qquad\qquad\qquad (u_1^R \boxtimes_{P(\tilde{\zeta}_1-\zeta_2)} u_2^R)\big) \boxtimes_{P(\zeta_2)} v. \qquad (3.57)
\end{aligned}$$

Let m be the morphism $W \to V_{op}$ such that

$$\begin{aligned}
(3.46) \ = m \circ \mathcal{R}_-^{P(\zeta_1-z_2)} \circ \mathcal{A}_{P(\zeta_1-\zeta_2)P(z_2-\zeta_2)}^{P(\zeta_1-z_2)P(z_2-\zeta_2)} \circ\\
\mathcal{A}_{P(\zeta_1)P(\zeta_2)}^{P(\zeta_1-\zeta_2)P(\zeta_2)} \circ (\mathcal{A}_{P(z_1)P(\zeta_1)}^{P(z_1-\zeta_1)P(\zeta_1)})^{-1}. \qquad (3.58)
\end{aligned}$$

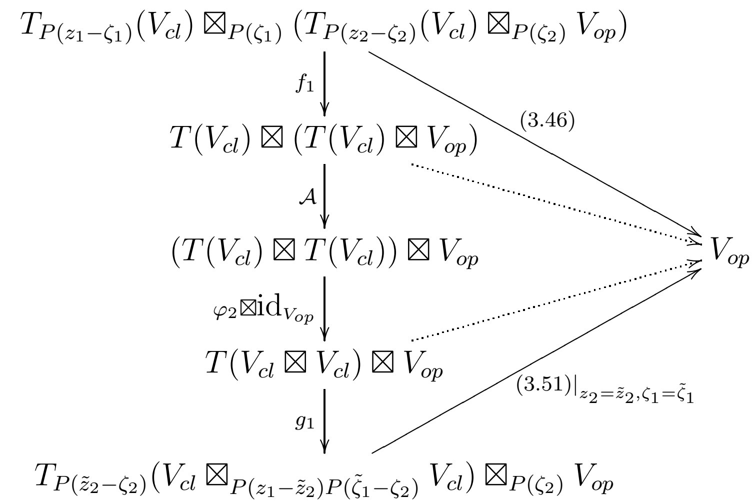
If we apply $\overline{m}$ on both the last line of (3.55) and (3.56), the two images of $\overline{m}$ are clearly the analytic continuation of each other along the path (3.54). On the other hand, combining this fact with (3.52) and (3.57), we obtain that

$$m = (3.51) \circ \mathcal{A}_{P(z_1-\zeta_2)P(\tilde{z}_2-\zeta_2)}^{P(z_1-\tilde{z}_2)P(\tilde{z}_2-\zeta_2)} \circ \mathcal{A}_{P(z_1)P(\zeta_2)}^{P(z_1-\zeta_2)P(\zeta_2)} \circ (\mathcal{A}_{P(\tilde{z}_2-\zeta_2)}^{P(\tilde{z}_2-\tilde{\zeta}_2)P(\tilde{\zeta}_1-\zeta_2)})^{-1} \qquad (3.59)$$

because the extensions of both sides of (3.59), applied on (3.56), give the same element in $\overline{V_{op}}$. Therefore, we further obtain from (3.58) and (3.59) the following identity:

$$\begin{aligned}
(3.46) \ = (3.51) \circ \mathcal{A}_{P(z_1-\zeta_2)P(\tilde{z}_2-\zeta_2)}^{P(z_1-\tilde{z}_2)P(\tilde{z}_2-\zeta_2)} \circ \mathcal{A}_{P(z_1)P(\zeta_2)}^{P(z_1-\zeta_2)P(\zeta_2)} \circ (\mathcal{A}_{P(\tilde{z}_2-\zeta_2)}^{P(\tilde{z}_2-\tilde{\zeta}_2)P(\tilde{\zeta}_1-\zeta_2)})^{-1}\\
\circ \mathcal{R}_-^{P(\zeta_1-z_2)} \circ \mathcal{A}_{P(\zeta_1-\zeta_2)P(z_2-\zeta_2)}^{P(\zeta_1-z_2)P(z_2-\zeta_2)} \circ \mathcal{A}_{P(\zeta_1)P(\zeta_2)}^{P(\zeta_1-\zeta_2)P(\zeta_2)} \circ (\mathcal{A}_{P(z_1)P(\zeta_1)}^{P(z_1-\zeta_1)P(\zeta_1)})^{-1}.\\
(3.60)
\end{aligned}$$

Using the commutative diagram (3.4)(3.9) and the definition of φ_2 (recall (3.22)), it is easy to see that (3.60) implies the commutativity of the following diagram:



(3.61)

where

$$f_1 = \mathrm{id}_{T(V_{cl})} \boxtimes (\mathcal{T}_{\gamma_4} \boxtimes \mathrm{id}_{H_{op}}) \circ \mathcal{T}_{\gamma_2} \boxtimes \mathcal{T}_{\gamma_3} \circ \mathcal{T}_{\gamma_1} \tag{3.62}$$
$$g_1 = T_{P(z_2-\zeta_2)}(\mathcal{T}_{\gamma_7} \otimes \mathcal{T}_{\gamma_8}) \boxtimes_{P(\zeta_2)} \mathrm{id}_{H_{op}} \circ (\mathcal{T}_{\gamma_6} \boxtimes_{P(\zeta_2)} \mathrm{id}_{V_{op}}) \circ \mathcal{T}_{\gamma_5} \tag{3.63}$$

in which $\gamma_i, i = 1, \dots, 4$ are paths in $\mathbb{R}_+$ from $\zeta_1, z_1 - \zeta_1, \zeta_2, z_2 - \zeta_2$ to 1 respectively and $\gamma_i, i = 5, \dots, 8$ are paths in $\mathbb{R}_+$ from 1 to $\zeta_2, z_2 - \zeta_2, z_1 - z_2, \zeta_1 - \zeta_2$ respectively.

Using (3.35), it is easy to see that

$$\begin{aligned} \mu_{cl-op} \circ (\mathrm{id}_{T(V_{cl})} \boxtimes \mu_{cl-op}) &= (3.46) \circ f_1^{-1}, \\ \mu_{cl-op} \circ (T(\mu_{cl}) \boxtimes \mathrm{id}_{V_{op}}) &= (3.51) \circ g_1. \end{aligned} \tag{3.64}$$

(3.64) together with the commutative diagram (3.61) implies (3.42), which is nothing but the commutativity of the subdiagram in the middle of (3.61).

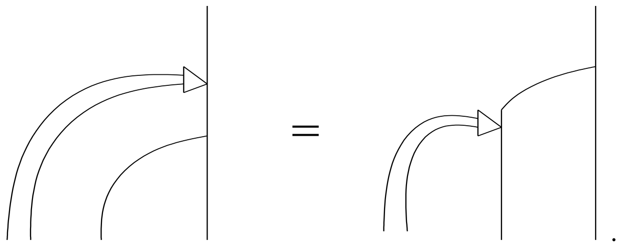
Conversely, (3.42) implies the commutativity of the diagram (3.61). It is easy to see that above arguments can be reserved. Therefore, we can also obtain the associativity II (1.56) from (3.42). ∎

We now study the categorical formulations of the rest conditions needed in Theorem 1.28. The proof of them are essentially same as that of Proposition 3.10. So we will only sketch the proofs below.

Proposition 3.11. *The associativity I (recall Proposition 1.16) is equivalent to the following condition:*

$$\mu_{cl-op}(\mathrm{id}_{T(V_{cl})} \boxtimes \mu_{op}) = \mu_{op}(\mu_{cl-op} \boxtimes \mathrm{id}_{V_{op}}) \circ \mathcal{A} \tag{3.65}$$

which can also be expressed in the following graph:



(3.66)

Proof. The left hand side of (1.52) gives arise to a morphism

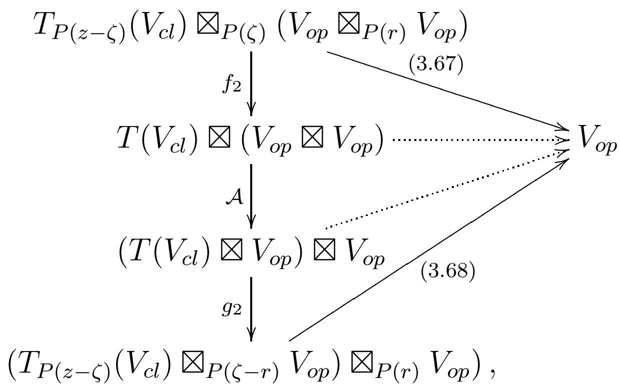
$$\mu_{cl-op}^{P(z-\zeta)P(\zeta)} \circ (\mathrm{id}_{T_{P(z-\zeta)}(V_{cl})} \boxtimes_{P(\zeta)} \mu_{op}^{P(r)}) \tag{3.67}$$

in $\mathrm{Hom}(T_{P(z-\zeta)}(V_{cl}) \boxtimes_{P(\zeta)} (V_{op} \boxtimes_{P(r)} V_{op}), V_{op})$. The right hand side of (1.52) gives arise to a morphism

$$\mu_{op}^{P(r)} \circ (\mu_{cl-op}^{P(z-\zeta)P(\zeta-r)} \boxtimes_{P(\zeta-r)} \mathrm{id}_{V_{op}}) \tag{3.68}$$

in $\mathrm{Hom}((T_{P(z-\zeta)}(V_{cl}) \boxtimes_{P(\zeta)} V_{op}) \boxtimes_{P(r)} V_{op}, V_{op})$.

For z, ζ, r in a proper subdomain of $z > \zeta > r > 0$, using similar arguments as in Proposition 3.10, we obtain that the associativity (1.52), for some $z, \zeta, r, \tilde{r} \in \mathbb{R}_+$, implies the following commutative diagram:



(3.69)

where

$$f_2 = (\mathcal{T}_{\gamma_2} \boxtimes \mathcal{T}_{\gamma_3}) \circ \mathcal{T}_{\gamma_1}, \tag{3.70}$$

$$g_2 = \mathcal{T}_{\gamma_6} \boxtimes_{P(\zeta)} \mathrm{id}_{V_{op}}) \boxtimes_{P(r)} \mathrm{id}_{V_{op}} \circ (\mathcal{T}_{\gamma_5} \boxtimes_{P(r)} \mathrm{id}_{V_{op}}) \circ \mathcal{T}_{\gamma_4}, \tag{3.71}$$

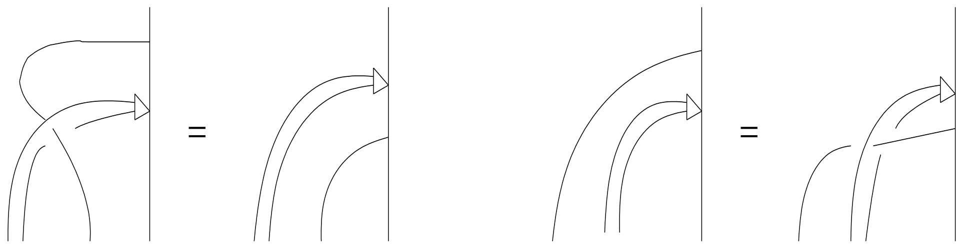
where $\gamma_i, i = 1, 2, 3$ are paths in $\mathbb{R}_+$ from $\zeta, z-\zeta$ and r to 1 respectively and $\gamma_i, i = 4, 5, 6$ are paths in $\mathbb{R}_+$ from 1 to $r, \zeta - r$ and $z - \zeta$ respectively. The commutativity of outside loop in (3.69) implies immediately the commutativity of the subdiagram in the middle of (3.69), which is nothing but the identity (3.65) or (3.66).

Conversely, using (3.69) and reversing above arguments, it is clear that (3.65) or (3.66) also implies the associativity (1.52). ∎

Proposition 3.12. *The commutativity I (recall Proposition 1.18) is equivalent to the following identity:*

$$\mu_{cl-op}(\mathrm{id}_{T(V_{cl})} \boxtimes \mu_{op}) = \mu_{op}(\mathrm{id}_{V_{op}} \boxtimes \mu_{cl-op}) \circ \mathcal{A}^{-1} \circ \tilde{\sigma}_1 \circ \mathcal{A}, \tag{3.72}$$

or the following graphic identities:



. (3.73)

Proof. There is a morphism

$$\mu_{cl-op}^{P(z-\zeta)P(\zeta)} \circ (\mathrm{id}_{T_{P(z-\zeta)}(V_{cl})} \boxtimes_{P(\zeta)} \mu_{op}^{P(r)}) \tag{3.74}$$

in $\mathrm{Hom}(T_{P(z-\zeta)}(V_{cl}) \boxtimes_{P(\zeta)} (V_{op} \boxtimes_{P(r)} V_{op}))$ associated to (1.58). There is another morphism

$$\mu_{op}^{P(\tilde{r})} \circ (\mathrm{id}_{V_{op}} \boxtimes_{P(\tilde{r})} \mu_{cl-op}^{P(z-\zeta)P(\zeta)}) \tag{3.75}$$

in $\mathrm{Hom}(V_{op} \boxtimes (T(V_{cl}) \boxtimes V_{op}))$ associated to (1.59). We introduce a map

$$\sigma_1 \in \mathrm{Hom}(T(V_{cl}) \boxtimes V_{op}, V_{op} \boxtimes T(V_{cl})).$$

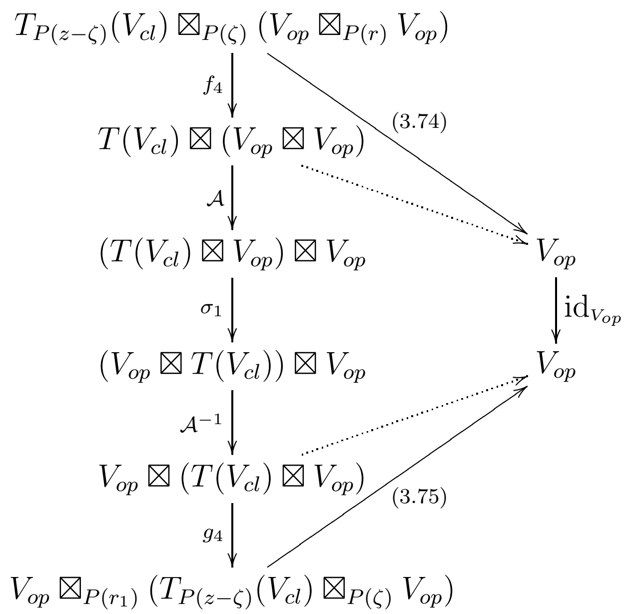
Let $W_i \otimes W_j$ be an direct summand in V_{cl}, then the restriction of σ_1 on $T(W_i \otimes W_j) \boxtimes V_{op}$ is defined as

$$\sigma_1 := (\mathcal{R}_+ \boxtimes \mathrm{id}_{W_j}) \circ \mathcal{A} \circ (\mathrm{id}_{W_i} \boxtimes \mathcal{R}_-) \circ \mathcal{A}^{-1}. \tag{3.76}$$

This completely determines σ_1.

For z, ζ, r, r_1 in a proper subdomain of $r_1 > z > \zeta > r > 0$, using similar arguments as in Proposition 3.10, we obtain that the commutativity I, implies the following commutative diagram:

(3.77)



where

$$f_4 = (\mathcal{T}_{\gamma_2} \boxtimes \mathcal{T}_{\gamma_3}) \circ \mathcal{T}_{\gamma_1}, \tag{3.78}$$
$$g_4 = (\mathrm{id}_{V_{op}} \boxtimes (\mathcal{T}_{\gamma_5} \boxtimes \mathcal{T}_{\gamma_6}) \circ \mathcal{T}_{\gamma_4}, \tag{3.79}$$

in which $\gamma_i, i = 1, 2, 3$ are paths in $\mathbb{R}_+$ from ζ, $z - \zeta$ and r respectively to 1 and $\gamma_i, i = 4, 5, 6$ are path in $\mathbb{R}_+$ from 1 to r_1, $z - \zeta$ and ζ respectively. Above commutative diagram immediately implies that the subdiagram in the middle of (3.77) is commutative. This is nothing but the commutativity (3.72) or the first formula in (3.73). Moreover, it also easy to see that the two formula in (3.73) are actually equivalent.

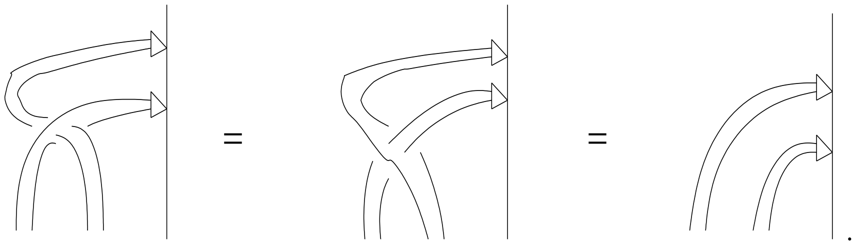
Conversely, using commutative diagram (3.77) and reversing above arguments, it is clear that (3.73) implies the commutativity of rational $\tilde{\mathfrak{S}}^c$. ∎

Commutativity II (recall Proposition 1.19) is not needed in Theorem 1.28 because it automatically follows from associativity II and skew symmetry of V_{cl}. It also has a very nice categorical formulation as given in the following proposition, which follows from (3.30) and (3.43) immediately.

Proposition 3.13. *For an open-closed field algebra over V, we have*

$$\mu_{cl-op} \circ \mathrm{id}_{T(V_{cl})} \boxtimes \mu_{cl-op} = \mu_{cl-op} \circ \mathrm{id}_{T(V_{cl})} \boxtimes \mu_{cl-op} \circ \mathcal{A}^{-1} \circ \sigma \circ \mathcal{A},$$

or equivalently



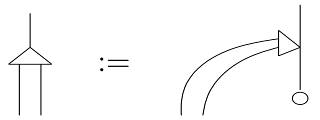
(3.80)

In summary, we have already completely reformulated the all the data and conditions in Theorem 1.28 in the language of tensor category as (3.35), (3.36), (3.42), (3.65) and (3.72) or equivalently as graphic identities (3.37), (3.43), (3.66) and (3.73).

We define a morphism $\iota_{cl-op} : T(V_{cl}) \to V_{op}$ as the composition of the following maps:

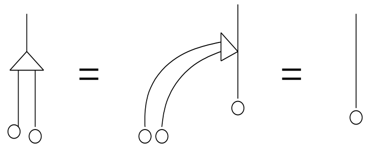
$$T(V_{cl}) \xrightarrow{r^{-1}_{T(V_{cl})}} T(V_{cl}) \boxtimes \mathbf{1}_{\mathcal{C}_V} \xrightarrow{\mathrm{id}\boxtimes \iota_{V_{op}}} T(V_{cl}) \boxtimes V_{op} \xrightarrow{\mu_{cl-op}} V_{op}. \tag{3.81}$$

or in the following graphic formula:

ι_{cl-op} =  . (3.82)

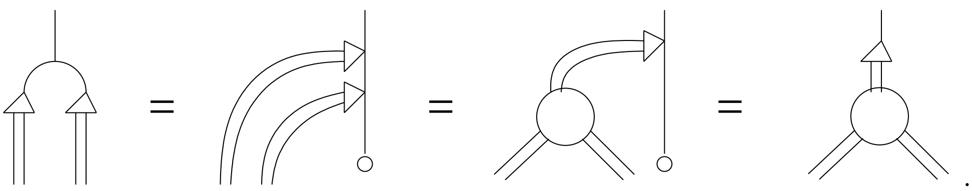
Lemma 3.14. *ι_{cl-op} is an algebra morphism from $T(V_{cl})$ to V_{op}.*

Proof. That ι_{cl-op} maps identity to identity is proved as follow:



(3.83)

The homomorphism property $\mu_{op} \circ (\iota_{cl-op} \boxtimes \iota_{cl-op}) = \iota_{cl-op} \circ \tilde{m}_{cl}$ is proven as follow:



(3.84)

∎

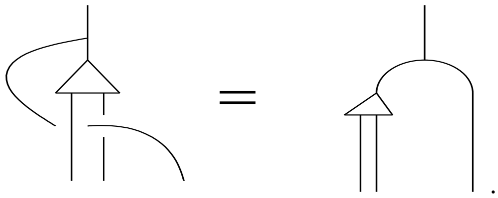
Definition 3.15. An *open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra*, denoted as

$$(\,(A_{op}, \mu_{op}, \iota_{op})|(A_{cl}, \mu_{cl}, \iota_{cl}), \iota_{cl-op}\,)$$

or simply $(A_{op}|A_{cl}, \iota_{cl-op})$, consists of an associative algebra $(A_{op}, \mu_{op}, \iota_{op})$ in $\mathcal{C}_V$, a commutative associative algebra with a trivial twist $(A_{cl}, \mu_{cl}, \iota_{cl})$ in $\mathcal{C}_{V\otimes V}$ and an associative algebra homomorphism $\iota_{cl-op} : T(A_{cl}) \to A_{op}$, satisfying the following commutativity:

$$\mu_{op} \circ (\iota_{cl-op} \boxtimes \mathrm{id}_{V_{op}}) = \mu_{op} \circ (\mathrm{id}_{V_{op}} \boxtimes \iota_{cl-op}) \circ \sigma_1, \tag{3.85}$$

or equivalently,



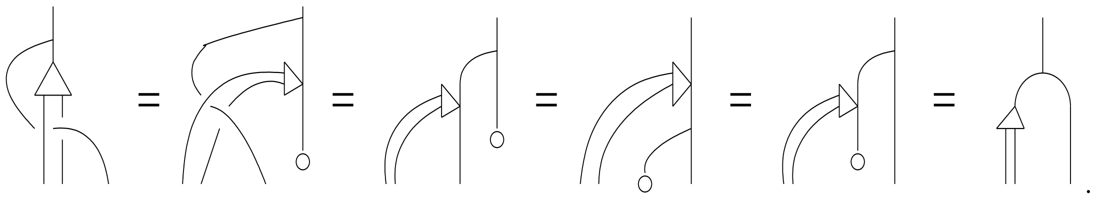
(3.86)

Theorem 3.16. *The following two notions*

1. *open-closed field algebra over* V*,*
2. *open-closed* $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$*-algebra,*

are equivalent in the sense that the categories of above two notions are isomorphic.

Proof. Given an open-closed field algebra over V, we have shown that it gives a triple $(V_{cl}, V_{op}, \mu_{cl-op})$, in which V_{cl} is a commutative associative algebra in $\mathcal{C}_{V\otimes V}$ with a trivial twist, and V_{op} is an algebra in $\mathcal{C}_V$, and μ_{cl-op} satisfies (3.37),(3.43), (3.66) and (3.73). Moreover, we have shown that ι_{cl-op} defined by (3.82) gives an morphism of associative algebra. Now we prove (3.86) as follow:

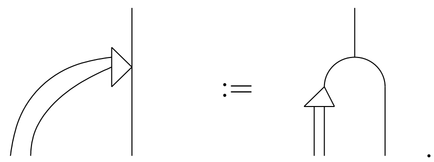


Hence $(V_{op}|V_{cl}, \iota_{cl-op})$ is an open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra.

Hence we have obtain a functor from the category of open-closed field algebras over V to that of open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebras.

Conversely, given an open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra, $(V_{op}|V_{cl}, \iota_{cl-op})$, we define a morphism $\mu_{cl-op} \in \mathrm{Hom}(T(V_{cl}) \boxtimes V_{op}, V_{op})$ as

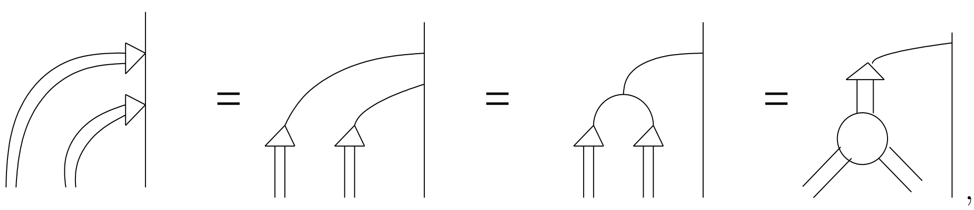
μ_{cl-op} =



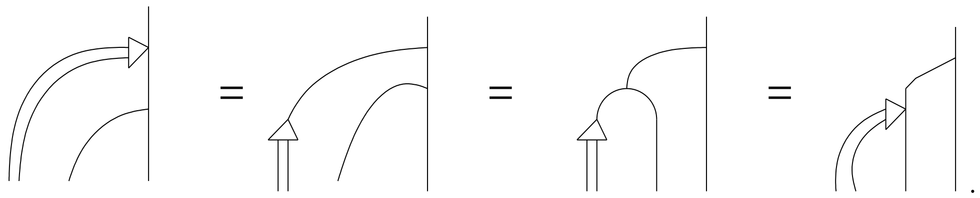
(3.87)

Since ι_{cl-op} is an algebra homomorphism, it maps unit to unit (recall (3.83)). Thus the identity property of μ_{cl-op} holds. Then the identity property of $\mathbb{Y}_{cl-op}$ follows.

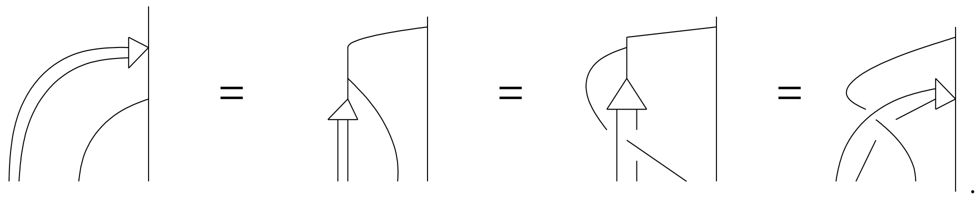
Furthermore, we have



which gives the associativity II (3.43). The associativity I (3.66) follows from



By using (3.2), we can prove the commutativity (3.73) as follow:

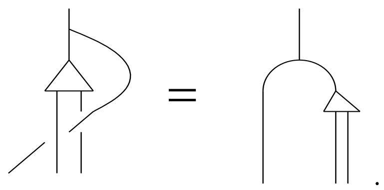


Note that the other half of (3.73) is equivalent to the first half. Notice also that the commutativity (3.80) simply follows from (3.27) and the fact that ι_{cl-op} is an algebra homomorphism. We don't need this fact.

Therefore we obtain a functor from the category of open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebras to that of open-closed field algebras over V.

The isomorphism of category follows from (3.82) and (3.87) easily. ∎

Remark 3.17. The condition (3.86) is equivalent to the following condition:



Remark 3.18. Comparing the categorical formulation of open-closed conformal field algebra over V with the result of 2-dimensional open-closed topological field theories [La][Mo1][Mo2][MS], one can see that genus-zero open-closed conformal field theories are still very similar to topological theories.

3.3 Categorical constructions

In this section, we discuss some simple categorical constructions of open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebras. We leave more thorough study of categorical constructions for the future.

An open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra is very easy to construct. For example, let A be an associative algebra in $\mathcal{C}_{V\otimes V}$ and $C_l(A)$ the left center of A [O][FFRS] [RFFS] and $\iota : C_l(A) \hookrightarrow A$ the natural embedding. Let A_0 be any subalgebra of $C_l(A)$. Then it is clear that $(A_0|T(A), T(\iota))$ gives an open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra, which further gives an open-closed field algebra over V and a smooth $\tilde{\mathfrak{S}}^c$-algebra over V. If we express the examples of such algebras given in [HKo1] in terms of open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebras, they are of the form $(A_{op}|V\otimes V, \iota_{A_{op}})$, where A_{op} is an associative algebra in $\mathcal{C}_V$. In general, $(A_0|T(A), T(\iota))$ is a nontrivial generalization of the construction given in [HKo1] in the sense that $A_0 \subset C_l(A)$ is a nontrivial extension of $V \otimes V$. The existence of such nontrivial extensions can be seen from the following fact. Let $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ be an abelian braided tensor category with braiding natural transformation $\mathcal{R}$. Let (A, μ_A, ι_A) be an associative algebra in $\mathcal{C}$ and (B, μ_B, ι_B) a commutative associative algebra in $\mathcal{C}$. Then $A \otimes B$ has a natural structure of associative algebra with $m_{A\otimes B}$ given by

$$(\mu_A \otimes \mu_B) \circ (\mathrm{id}_A \otimes \mathcal{R}_{AB} \otimes \mathrm{id}_B) \tag{3.88}$$

and $\iota_{A\otimes B} := \iota_A \otimes \iota_B$ [FFRS]. Then it is easy to see that $C_l(A) \otimes B$ is in the left center of the associative algebra $A \otimes B$ and is clearly nontrivial.

In above paragraph, we construct open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebras from associative algebras A in $\mathcal{C}_V$. We can also first start from a given commutative associative algebra in $\mathcal{C}_{V\otimes V}$ with a trivial twist, and ask which associative algebra in $\mathcal{C}_V$ can make it into an open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra. This dual point of view is associated to the so-called open-closed duality in string theory. Now we present such point of view of constructions below. The resulting open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebras are special cases of constructions discussed in the last paragraph. More interesting examples will be discussed in [Ko2].

Because ι_{cl-op} is an algebra map from $T(A_{cl})$ to A_{op}, A_{op} has a natural structure of $T(A_{cl})$-module. Moreover, A_{op} is a certain "algebraic" object in the category of $T(A_{cl})$-modules, although we should be careful since the category of $T(A_{cl})$-modules may not be monoidal. Due to the work of Kirillov and Ostrik ([KOs]), the category of modules for a commutative associative algebra in braided tensor category is itself monoidal. $T(A_{cl})$ is not commutative, but A_{cl} is. So we can first look at the category of A_{cl}-modules.

Let us recall some results from [KOs]. Let $\mathcal{C}$ be an abelian braided tensor category over $\mathbb{C}$ with tensor product bifunctor $\otimes$, unit object $1_{\mathcal{C}}$, left unit morphism l_W and right unit morphism r_W for any object W, associativity $\mathcal{A}$, braiding $\mathcal{R}_+$ and antibraiding $\mathcal{R}_-$. Let (A, μ_A, ι_A) be a commutative associative algebra in $\mathcal{C}$. A A-module is a pair (W, μ_W) where W is an object in $\mathcal{C}$ and $\mu_W \in \mathrm{Mor}(A \otimes W, W)$ satisfying natural axioms. We denote the category of A-modules as Rep$A$. A subcategory Rep^0A ([KOs][Pa]) of Rep$A$ consists of all objects W in Rep$A$ such that

$$\mu_W \circ \mathcal{R}_+^{WA} \circ \mathcal{R}_+^{AW} = \mu_W,$$

where the subscripts ^{WA} and ^{AW} on $\mathcal{R}$ indicate that the domains of these two braiding isomorphisms are $W \otimes A$ and $A \otimes W$ respectively. The following Theorem is due to Kirillov and Ostrik ([KOs]).

Theorem 3.19. *RepA is a monoidal category with unit object A and tensor product $\otimes_A$ defined as, for $W_1, W_2 \in RepA$,*

$$W_1 \otimes_A W_2 := Coker(\mu_1 - \mu_2), \tag{3.89}$$

where $\mu_1, \mu_2 : A \otimes W_1 \otimes W_2 \to W_1 \otimes W_2$ are defined by

$$\begin{aligned} \mu_1 &:= \mu_{W_1} \otimes \mathrm{id}_{W_2}, \\ \mu_2 &:= (\mathrm{id}_{W_1} \otimes \mu_{W_2}) \circ \mathcal{R}_+^{AW_1}. \end{aligned} \tag{3.90}$$

Rep^0A has a structure of braided monoidal category.

We briefly recall some basic structures in Rep$A$ and Rep^0A. Let us denote the natural projection $W_1 \otimes W_2 \to W_1 \otimes_A W_2$ as δ_{W_1,W_2}. The left unit morphism l^A_W and the right unit morphism r^A_W for a given A-module W is given in the following commutative diagram:

$$\begin{array}{ccccccc} A\otimes_A W & \xleftarrow{\delta_{A,W}} & A\otimes W & \xrightarrow{\mathcal{R}^{AW}_{+}} & W\otimes A & \xrightarrow{\delta_{W,A}} & W\otimes_A A\,, \\ & {\scriptstyle \exists!\, l^A_W}\searrow & \downarrow{\scriptstyle \mu_W} & & & \swarrow{\scriptstyle \exists!\, r^A_W} & \\ & & W & & & & \end{array} \tag{3.91}$$

where the existence and the uniqueness of l^A_W and r^A_W follow from the universal properties of cokernels. Similarly, the associativity isomorphisms $\mathcal{A}^A$ in Rep$A$ are induced from $\mathcal{A}$ canonically in the following commutative diagrams:

$$\begin{array}{ccc} W_1\otimes(W_2\otimes W_3) & \xrightarrow{\quad\mathcal{A}\quad} & (W_1\otimes W_2)\otimes W_3 \\ \downarrow{\scriptstyle \delta_{W_2,W_3}} & & \downarrow{\scriptstyle \delta_{W_1,W_2}} \\ W_1\otimes(W_2\otimes_A W_3) & & (W_1\otimes_A W_2)\otimes W_3 \\ \downarrow{\scriptstyle \delta_{W_1,W_2\otimes_A W_3}} & & \downarrow{\scriptstyle \delta_{W_1\otimes_A W_2,W_3}} \\ W_1\otimes_A(W_2\otimes_A W_3) & \xrightarrow{\quad\exists!\,\mathcal{A}^A\quad} & (W_1\otimes_A W_2)\otimes_A W_3 \end{array} \tag{3.92}$$

for any A-modules W_1, W_2, W_3. And the (anti)braiding isomorphisms $\mathcal{R}^A_\pm$ in A are induced from $\mathcal{R}_\pm$ canonically in the following commutative diagram:

$$\begin{array}{ccc} W_1\otimes W_2 & \xrightarrow{\quad\mathcal{R}_\pm\quad} & W_2\otimes W_1 \\ \downarrow{\scriptstyle \delta_{W_1,W_2}} & & \downarrow{\scriptstyle \delta_{W_2,W_1}} \\ W_1\otimes_A W_2 & \xrightarrow{\quad\exists!\,\mathcal{R}^A_\pm\quad} & W_2\otimes_A W_1 \end{array} \tag{3.93}$$

for any pair of A-modules W_1, W_2.

We need the following Lemma for our construction.

Lemma 3.20. *Let (B, μ^A_B, ι^A_B) be an associative algebra in Rep$^0\,A$ and two morphisms $\mu_B : B \otimes B \to B$ and $\iota_B : V \to B$ given by $\mu_B := \mu^A_B \circ \delta_{B,B}$ and $\iota_B := \iota^A_B \circ \iota_A$. The triple (B, μ_B, ι_B) is an associative $\mathcal{C}$-algebra and $\iota^A_B : A \to B$ is a $\mathcal{C}$-algebra homomorphism. Moreover, the following diagram:*

$$\begin{array}{ccccc} A\otimes B & \xrightarrow{\iota^A_B\otimes\,\mathrm{id}_B} & B\otimes B & \xrightarrow{\mu_B} & B \\ \downarrow{\scriptstyle \mathcal{R}_\pm} & & & & \downarrow{\scriptstyle \mathrm{id}_B} \\ B\otimes A & \xrightarrow{\mathrm{id}_B\otimes\,\iota^A_B} & B\otimes B & \xrightarrow{\mu_B} & B \end{array} \tag{3.94}$$

is commutative.

Proof. To prove the left unit property, we consider the following diagram:

$$\begin{array}{ccccccc} 1_{\mathcal{C}}\otimes B & \xrightarrow{\iota_A\otimes\mathrm{id}_B} & A\otimes B & \xrightarrow{\delta_{A,B}} & A\otimes_A B & \xrightarrow{l^A_B} & B \\ & & \downarrow{\scriptstyle \iota^A_B\otimes\mathrm{id}_B} & & \downarrow{\scriptstyle \iota^A_B\otimes_A\mathrm{id}_B} & \nearrow{\scriptstyle \mu^A_B} & \\ & & B\otimes B & \xrightarrow[\delta_{B,B}]{} & B\otimes_A B. & & \end{array} \tag{3.95}$$

The square in the middle of diagram (3.95) is exactly how $\iota_B^A \otimes_A \mathrm{id}_B$ is constructed from $\iota_B^A \otimes \mathrm{id}_B$ so that $\otimes_A$ becomes a bifunctor $\mathrm{Rep}^0 A \times \mathrm{Rep}^0 A \to \mathrm{Rep}^0 A$. In other words, the square is commutative. The triangle in the right part of diagram (3.95) is commutative because of the unit property of B as an associative algebra in $\mathrm{Rep}^0 A$. Therefore we have

$$\begin{aligned} \mu_B \circ (\iota_B \otimes \mathrm{id}_B) &= \mu_B^A \circ \delta_{B,B} \circ (\iota_B^A \otimes \mathrm{id}_B) \circ (\iota_A \otimes \mathrm{id}_B) \\ &= l_B^A \circ \delta_{A,B} \circ (\iota_A \otimes \mathrm{id}_B). \end{aligned} \tag{3.96}$$

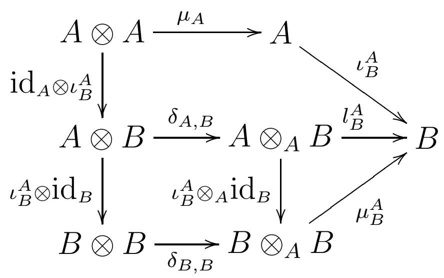
Using the construction of l_B^A in (3.91) and the unit property of B as A-module, we further obtain that

$$\mu_B \circ (\iota_B \otimes \mathrm{id}_B) = \mu_B \circ (\iota_A \otimes \mathrm{id}_B) = l_B,$$

which is nothing but the left unit property of (B, μ_B, ι_B) in $\mathcal{C}_{V\otimes V}$. The right unit property can be proved similarly.

The associativity of (B, μ_B, ι_B) follows immediately from that of (B, μ_B^A, ι_B^A) and the commutative diagram (3.92).

Hence (B, μ_B, ι_B) is an associative $\mathcal{C}$-algebra. That ι_B^A is a $\mathcal{C}$-algebra homomorphism follows immediately if the following diagram:



(3.97)

is commutative. It is clear that the lower two subdiagrams of (3.97) is commutative. Notice that $l_B^A \circ \delta_{A,B} = \mu_B$ by (3.91). The commutativity of the upper subdiagram of (3.97) simply follows from the fact that ι_B^A is a A-module map. Hence (3.97) is commutative.

The commutativity (3.94) also follows easily from the commutative diagram (3.93), the identity $l_B^A = r_B^A \circ \mathcal{R}_{\pm}^A$ [Ka] and the unit properties of (B, μ_B^A, ι_B^A). ∎

Proposition 3.21. *Let (A, μ_A, ι_A) be a commutative associative algebra in $\mathcal{C}_{V\otimes V}$ with a trivial twist and (B, μ_B^A, ι_B^A) an associative algebra in Rep^0 A. Let $\mu_B := \mu_B^A \circ \delta_{B,B}$ and $\iota_B := \iota_B^A \circ \iota_A$. Then*

$$((A, \mu_A, \iota_A)|(T(B), T(\mu_B) \circ \varphi_2, T(\iota_B) \circ \varphi_0), T(\iota_B^A))$$

is an open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra.

Proof. By Lemma 3.20 and Proposition 3.7, the triple

$$(T(B), T(\mu_B) \circ \varphi_2, T(\iota_B) \circ \varphi_0)$$

is an associative algebra in $\mathcal{C}_V$.

$T(\iota_B^A) : T(A) \to T(B)$ clearly maps unit to unit. Hence to show that $T(\iota_B^A)$ is an algebra homomorphism, we only need to prove the commutativity of the following diagram:

$$\begin{array}{ccccc} T(A) \boxtimes T(A) & \xrightarrow{\varphi_2} & T(A \boxtimes A) & \xrightarrow{T(\mu_A)} & T(A) \\ \downarrow{\scriptstyle T(\iota_B^A) \boxtimes T(\iota_B^A)} & & \downarrow{\scriptstyle T(\iota_B^A \boxtimes \iota_B^A)} & & \downarrow{\scriptstyle T(\iota_B^A)} \\ T(B) \boxtimes T(B) & \xrightarrow{\varphi_2} & T(B \boxtimes B) & \xrightarrow{T(\mu_B)} & T(B). \end{array} \tag{3.98}$$

The commutativity of the left subdiagram in (3.98) follows from the naturalness of φ_2 and that of the right subdiagram in (3.98) is obvious. Hence $T(\iota_B^A)$ is an algebra homomorphism.

The commutativity (3.86) follows from the commutativity of the diagram (3.94) and a similar diagram as (3.28) immediately. ∎

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