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Classification of low energy sign-changing solutions of an almost critical problem *

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Abstract. In this paper we make the analysis of the blow-up of low energy sign-changing solutions of a semi-linear elliptic problem involving nearly critical exponent. Our results allow to classify these solutions according to the concentration speeds of the positive and negative part and, in high dimensions, lead to complete classification of them. Additional qualitative results, such as symmetry or location of the concentration points are obtained when the domain is a ball.

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1 Introduction

In this paper we consider the following semi-linear elliptic problem with subcritical nonlinearity:

$$(1) \qquad \begin{cases} -\Delta u = |u|^{2^*-2-\varepsilon}u & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, ε is a positive real parameter and $2^* = 2n/(n-2)$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$.

Problem (1) is related to the limiting problem (when $\varepsilon = 0$) which exhibits a lack of compactness and gives rise to solutions of (1) which blow up as $\varepsilon \to 0$.

In the last decades there have been many works devoted to the study of positive solutions of problem (1). In sharp contrast to this, very little study has been made concerning the sign-changing solutions. For details one can see [5] and the references therein.

The existence of sign-changing solutions of (1) for any $\varepsilon \in (0, p-1)$ has been proved in [4], [6] and [11]. On the other hand, when $\varepsilon = 0$, problem (1) becomes delicate. Pohozaev showed in [16] that if Ω is starshaped, problem (1) (with $\varepsilon = 0$) has no solutions whereas Clapp and

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Weth proved in [12] that problem (1) (with $\varepsilon = 0$) has a solution on domains with small holes and on some contractible domains with an involution symmetry.

In view of this qualitative change in the situation when $\varepsilon = 0$, it is interesting to study the following question: what happens to the solutions of (1) as $\varepsilon \to 0$? In this paper, we are mainly interested in the study of the behavior of low energy sign-changing solutions of (1). The first part of this paper is devoted to analyze the asymptotic behavior, as $\varepsilon \to 0$, of sign-changing solutions of (1) whose energy converges to the value $2S^{n/2}$, S being the Sobolev constant for the embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$. We prove that these solutions blow up at two points which may coincide and which are the limit of the concentration points $a_{\varepsilon,1}$ and $a_{\varepsilon,2}$ of the positive and negative part of the solutions. Moreover, we make a precise study of the location of these blow up points. More precisely, we have

Theorem 1.1 *Let $n \geq 3$ and let (u_ε) be a family of sign-changing solutions of (1) which satisfies*

$$||u_\varepsilon||^2 := \int_\Omega |\nabla u_\varepsilon|^2 \to 2S^{n/2} \qquad \text{as } \varepsilon \to 0. \tag{1.1}$$

Then, the set $\Omega \setminus \{x \in \Omega \,|\, u_\varepsilon(x) = 0\}$ has exactly two connected components.
Furthermore, there exist two points $a_{\varepsilon,1}$, $a_{\varepsilon,2}$ in Ω (one of them can be chosen to be the global maximum point of $|u_\varepsilon|$) and there exist two positive reals $\mu_{\varepsilon,1}$, $\mu_{\varepsilon,2}$ such that

$$||u_\varepsilon - P\delta_{(a_{\varepsilon,1},\mu_{\varepsilon,1})} + P\delta_{(a_{\varepsilon,2},\mu_{\varepsilon,2})}|| \to 0, \qquad \text{as } \varepsilon \to 0, \tag{1.2}$$

$$\mu_{\varepsilon,i} d(a_{\varepsilon,i}, \partial\Omega) \to +\infty, \quad |\mu_{\varepsilon,i}|^\varepsilon \to 1 \qquad \text{as } \varepsilon \to 0, \text{ for } i \in \{1,2\}, \tag{1.3}$$

where $P\delta_{(a,\mu)}$ denotes the projection of $\delta_{(a,\mu)}$ on $H_0^1(\Omega)$, that is,

$$\Delta P\delta_{(a,\mu)} = \Delta\delta_{(a,\mu)} \text{ in } \Omega,\ P\delta_{(a,\mu)} = 0 \text{ on } \partial\Omega, \text{ and } \delta_{(a,\mu)}(x) = \frac{\beta_n \mu^{(n-2)/2}}{(1+\mu^2|x-a|^2)^{(n-2)/2}}.$$

Here β_n is a constant chosen so that $-\Delta\delta_{(a,\mu)} = \delta_{(a,\mu)}^{(n+2)/(n-2)}$, ($\beta_n = (n(n-2))^{(n-2)/4}$).

Note that, for the supercritical case (that is for $\varepsilon < 0$), a recent result [9] shows that there is no sign-changing solution u_ε with low energy which satisfies (1.2) and (1.3).

Now, our aim is to give a complete classification of the solutions satisfying (1.1). To this aim, we divide this kind of solutions into two categories: the ones for which the positive and negative part blow up with the same rate (hypothesis (1.5) below) and the ones for which these rates are note comparable (hypothesis (1.9) below). In the first case we are able to prove that the concentration points of the positive and negative part of a solution of this type are distinct and away from the boundary and we characterize their limits in terms of the Green's function and of its regular part. Moreover, when the domain is a ball we prove that the limit concentration points are antipodal with respect to the center of the ball, the solution is axially symmetric with respect to the line joining these points and the nodal surface intersects the boundary. In the second case, i.e. when (1.9) holds, we are able to prove that, if $n \geq 4$, the positive and negative part of the solution concentrate at the same point (i.e. we have "bubble tower solutions") and we get a precise estimate of the blow up rates, in terms of the distance of the concentration points from the boundary of the domain. Moreover, if $n \geq 6$ we prove that the unique limit of the concentration points is away from the boundary and it is a critical point of the Robin's function. As far as we know this is the first time that the phenomenon of different concentration points converging to the same point is analyzed for critical exponent problems. Indeed this never happens in the case of positive solutions, see [14].

Note that solutions of both type exist, at least in symmetric domains. Indeed for the first type of solutions it is enough to take a positive solution in a symmetric cap of the domain and reflect it by antisymmetry. In the second case "bubble tower" solutions have been recently found by Pistoia and Weth [15].

To describe more precisely our results, we introduce some notations. We denote by G the Green's function of the Laplace operator defined by : $\forall x \in \Omega$

$$-\Delta G(x,.) = c_n \delta_x \quad \text{in} \quad \Omega, \quad G(x,.) = 0 \quad \text{on} \quad \partial\Omega,$$

where δ_x is the Dirac mass at x and $c_n = (n-2)\omega_n$, with ω_n denoting the area of the unit sphere of $\mathbb{R}^n$. We denote by H the regular part of G, that is,

$$H(x_1, x_2) = |x_1 - x_2|^{2-n} - G(x_1, x_2) \quad \text{for} \quad (x_1, x_2) \in \Omega^2.$$

For $x = (x_1, x_2) \in \Omega^2 \setminus \Gamma$, with $\Gamma = \{(y, y)/y \in \Omega\}$, we denote by $M(x)$ the matrix defined by

$$M(x) = (m_{ij})_{1\leq i,j\leq 2}, \quad \text{where} \quad m_{ii} = H(x_i, x_i),\ m_{12} = m_{21} = G(x_1, x_2). \tag{1.4}$$

Then we have

Theorem 1.2 *Let $n \geq 3$ and let (u_ε) be a family of sign-changing solutions of* (1) *satisfying* (1.1) *and let $a_{\varepsilon,1}$, $a_{\varepsilon,2}$ be the concentration points defined in Theorem 1.1. Assume that there exists a positive constant η such that*

$$\eta \leq -\max u_\varepsilon / \min u_\varepsilon \leq \eta^{-1}. \tag{1.5}$$

Then, $a_{\varepsilon,1}$ and $a_{\varepsilon,2}$ are two global extremum points of u_ε and we have

$$\mu_{\varepsilon,i} = |u_\varepsilon(a_{\varepsilon,i})|^{2/(n-2)-\varepsilon/2} / \beta_n^{2/(n-2)} \quad \textit{for} \quad i = 1, 2. \tag{1.6}$$

In addition, there exists a positive constant γ such that, for ε small,

$$|a_{\varepsilon,1} - a_{\varepsilon,2}| \geq \gamma, \quad d(a_{\varepsilon,i}, \partial\Omega) \geq \gamma \textit{ for } i = 1,\, 2. \tag{1.7}$$

More precisely, we have

$$(c_2\varepsilon/c_1)^{1/2} |u_\varepsilon(a_{\varepsilon,i})| \to \beta_n / \overline{\Lambda}_i, \quad a_{\varepsilon,i} \to \overline{a}_i \in \Omega \textit{ with } \overline{a}_1 \neq \overline{a}_2,\ \textit{for } i = 1, 2, \tag{1.8}$$

where $\overline{\Lambda}_i$ is a positive constant,

$$c_1 = \beta_n \int_{\mathbb{R}^n} \delta_{(0,1)}^{(n+2)/(n-2)} \quad \textit{and} \quad c_2 = \frac{n-2}{2} \beta_n^{2n/(n-2)} \int_{\mathbb{R}^n} Log(1 + |x|^2) \frac{(|x|^2 - 1)dx}{(1 + |x|^2)^{n+1}}.$$

Furthermore, $(\overline{a}_1, \overline{a}_2, \overline{\Lambda}_1, \overline{\Lambda}_2)$ is a critical point of the function

$$\Psi : \Omega^2 \setminus \Gamma \times (0, \infty)^2 \to \mathbb{R}; \quad (a, \Lambda) := (a_1, a_2, \Lambda_1, \Lambda_2) \rightarrowtail \frac{1}{2} {}^t\Lambda M(a) \Lambda - \log(\Lambda_1 \Lambda_2),$$

where $M(a)$ is the matrix defined by (1.4).

Remark 1.3 **1**- *The assumption* (1.5) *allows us to prove that the distance between the concentration points is bounded from below by a positive constant.*
2- *The low energy positive solutions of* (1) *have to blow up at a critical point of the Robin's function* $\varphi(x) = H(x,x)$*, see [13] and [18].*
3- *A similar matrix to that involved in the above function* Ψ *plays also a crucial role in the characterization of the concentration points for the positive solutions of* (1)*, see [3].*

The next result describes the asymptotic behavior, as $\varepsilon \to 0$, of low energy sign-changing solutions of (1) satisfying (1.5) outside the limit concentration points.

Theorem 1.4 *Let* $n \geq 3$ *and let* (u_ε) *be a family of sign-changing solutions of* (1) *satisfying* (1.1) *and* (1.5)*. Then the limit concentration points* $\overline{a}_1$ *and* $\overline{a}_2$*, defined in Theorem 1.2, are two isolated simple blow-up points of* (u_ε) *(see [14] for definition) and there exist positive constants* m_1 *and* m_2 *such that*

$$\varepsilon^{-1/2}u_\varepsilon \to \overline{u} := m_1 G(\overline{a}_1, .) - m_2 G(\overline{a}_2, .) \text{ in } C^2_{loc}(\overline{\Omega} \setminus \{\overline{a}_1, \overline{a}_2\}) \quad \text{as} \quad \varepsilon \to 0.$$

The second part of this paper is devoted to the study of symmetry properties of low energy sign-changing solutions of (1) of the first type when Ω is a ball. We shall prove the following results.

Theorem 1.5 *Assume that* $n \geq 3$*. Let* Ω *be the unit ball and let* (u_ε) *be a family of sign-changing solutions of* (1) *satisfying* (1.1) *and* (1.5)*. Then, up to a rotation of* Ω*, the limit concentration points* $\overline{a}_i$*'s and the reals* $\overline{\Lambda}_i$*'s, defined in Theorem 1.2, satisfy*

$$\overline{a}_1 = -\overline{a}_2 = a_* := (0, ..., 0, t_*),$$
$$\overline{\Lambda}_1 = \overline{\Lambda}_2 = \left(\frac{1}{H(a_*, a_*) + G(a_*, -a_*)}\right)^{1/2},$$

where t_* *is the unique solution of*

$$g(t) = \frac{t}{(1-t^2)^{n-1}} - \frac{1}{(2t)^{n-1}} + \frac{t}{(1+t^2)^{n-1}} = 0, \quad \text{for } t \in (0,1).$$

The characterization of the points $\overline{a}_i$'s and the reals $\overline{\Lambda}_i$'s allows us to improve the result of Theorem 1.4 and therefore we can prove that the nodal surface intersects the boundary. In fact, we have

Theorem 1.6 *Under the assumptions of Theorem 1.5, we have*
(a) The constants m_1 *and* m_2 *defined in Theorem 1.4 are equal.*
(b) The nodal surface of u_ε *intersects the boundary* $\partial\Omega$*.*

Observe that the limit function $G(\overline{a}_1, .) - G(\overline{a}_2, .)$, defined in Theorem 1.4 with $m_1 = m_2$, is symmetric with respect to any hyperplane passing through the points $\overline{a}_1$ and $\overline{a}_2$. Furthermore, it is antisymmetric with respect to the hyperplane passing through the origin and which is orthogonal to the line passing through the points $\overline{a}_1$ and $\overline{a}_2$. Moreover, it changes sign once.

Following the idea of [10], we can prove, for ε small, that the functions u_ε satisfy also the symmetry property. More precisely, we have

Theorem 1.7 *Let $n \geq 3$ and let Ω be a ball and (u_ε) be a family of sign-changing solutions of* (1) *satisfying* (1.1) *and* (1.5)*. Then, for ε sufficiently small, the concentration points $a_{\varepsilon,1}$ and $a_{\varepsilon,2}$ of u_ε, given by Theorem 1.1, are far away from the origin and they lay on the same line passing through the origin and u_ε is axially symmetry with respect to this line.*
Moreover the points $a_{\varepsilon,1}$ and $a_{\varepsilon,2}$ lay on different sides with respect to T and all the critical points of u_ε belong to the symmetry axis and

$$\frac{\partial u_\varepsilon}{\partial \nu_T}(x) > 0 \qquad \forall x \in T \cap \Omega,$$

where T is any hyperplane passing through the origin but not containing $a_{\varepsilon,1}$ and where ν_T is the normal to T, oriented towards the half space containing $a_{\varepsilon,1}$.

Concerning the antisymmetric property, it is easy to construct a family of solutions (u_ε) which are antisymmetric by reflecting the positive minimizing solution on the half ball and hence interesting questions arise: Let (u_ε) be a family of solutions satisfying the assumptions of our Theorems, are the solutions (u_ε) antisymmetric? What about the uniqueness? A further investigation in this direction is in progress.

The last part of this paper is devoted to the study of the case where the assumption (1.5) is removed, that means the case when we can have sign-changing bubble tower solutions. We recall that Pistoia and Weth [15] have constructed such solutions in symmetric domains. Without loss of generality, we can assume, in the case where the assumption (1.5) is removed, that the following holds:

$$\frac{\max u_\varepsilon}{\min u_\varepsilon} \to -\infty \quad \text{as} \quad \varepsilon \to 0. \tag{1.9}$$

Our main result reads:

Theorem 1.8 *Let $n \geq 4$ and let (u_ε) be a family of sign-changing solutions of* (1) *satisfying* (1.1) *and* (1.9)*. Let $a_{\varepsilon,1}$, $a_{\varepsilon,2}$ be the concentration points and $\mu_{\varepsilon,1}$, $\mu_{\varepsilon,2}$ the speeds of the concentration points defined in Theorem 1.1. Then,*
(a) there exists a positive constant c such that, for ε small,

$$\frac{1}{c}\mu_{\varepsilon,1} \leq \mu_{\varepsilon,2}\left(\mu_{\varepsilon,2}d_{\varepsilon,2}\right)^2 \leq c\mu_{\varepsilon,1}, \quad \frac{1}{c}\varepsilon \leq \frac{1}{(\mu_{\varepsilon,2}d_{\varepsilon,2})^{n-2}} \leq c\varepsilon, \quad \mu_{\varepsilon,1}\mu_{\varepsilon,2}|a_{\varepsilon,1}-a_{\varepsilon,2}|^2 \leq c,$$
$$\mu_{\varepsilon,2}|a_{\varepsilon,1}-a_{\varepsilon,2}| \to 0, \quad \frac{|a_{\varepsilon,1}-a_{\varepsilon,2}|}{d_{\varepsilon,2}} \to 0, \quad \frac{d_{\varepsilon,1}}{d_{\varepsilon,2}} \to 1 \quad \text{as} \quad \varepsilon \to 0,$$

where $d_{\varepsilon,i} = d(a_{\varepsilon,i}, \partial\Omega)$ for $i = 1, 2$.
(b) the nodal surface of u_ε does not intersect the boundary $\partial\Omega$.
Furthermore, if $n \geq 6$, we have

$$\mu_{\varepsilon,1}\mu_{\varepsilon,2}|a_{\varepsilon,1}-a_{\varepsilon,2}|^2 \to 0\,, \quad d_{\varepsilon,i} \not\to 0 \quad \text{and} \quad a_{\varepsilon,i} \to \bar{a} \in \Omega \quad \text{for } i = 1, 2, \tag{1.10}$$

where $\bar{a}$ is a critical point of the Robin's function $\varphi(x) = H(x, x)$.

Let us mention that we are not able to extend the results of Theorem 1.8 to the dimension 3 because of serious technical difficulties. Also the restriction to $n \geq 6$ to deduce that the unique limit concentration point is away from the boundary is due to some technical difficulties but we think that the same result should be true also in lower dimensions, at least for $n = 4, 5$. Moreover in the case of the ball since the only critical point of the Robin's function is the center, from Theorem 1.8 we deduce, for $n \geq 6$, that the bubble tower concentration point is the center, with negative and positive part blowing up with a prescribed rate. This makes us to conjecture that this solution should be radial and hence should be the only solution of this type.

The outline of the paper is the following. Section 2 is devoted to the proof of Theorems 1.1 and 1.2. We prove Theorems 1.4, 1.5, 1.6 and 1.7 in Section 3. Finally, Section 4 is devoted to the proof of Theorem 1.8.

2 Proof of Theorems 1.1 and 1.2

First, we deal with the proof of Theorem 1.1. Regarding the connected components of $\Omega \setminus \{x \in \Omega \,|\, u_\varepsilon(x) = 0\}$, let Ω_1 be one of them. Multiplying (1) by u_ε and integrating on Ω_1, we derive that

$$\int_{\Omega_1} |\nabla u_\varepsilon|^2 \geq S^{n/2}(1 + o(1)), \tag{2.1}$$

where we have used Holder's inequality and the Sobolev embedding.
Since $\|u_\varepsilon\|^2 \to 2S^{n/2}$ as ε goes to 0, we deduce that there are only two connected components.

The following lemma shows that the energy of the solution u_ε converges to $S^{n/2}$ in each connected component. In fact we have

Lemma 2.1 *Let* (u_ε) *be a family of sign-changing solutions of* (1) *satisfying* (1.1)*. Then*

$$\begin{aligned}
&(i) \quad \int_\Omega |\nabla u_\varepsilon^+|^2 \to S^{n/2}, \ \int_\Omega |\nabla u_\varepsilon^-|^2 \to S^{n/2} \quad \text{as } \varepsilon \to 0,\\
&(ii) \quad \int_\Omega (u_\varepsilon^+)^{\frac{2n}{n-2}} \to S^{n/2}, \ \int_\Omega (u_\varepsilon^-)^{\frac{2n}{n-2}} \to S^{n/2} \quad \text{as } \varepsilon \to 0,\\
&(iii) \quad u_\varepsilon \rightharpoonup 0 \quad \text{as } \varepsilon \to 0,\\
&(iv) \quad M_{\varepsilon,+} := \max_\Omega u_\varepsilon^+ \to +\infty, \quad M_{\varepsilon,-} := \max_\Omega u_\varepsilon^- \to +\infty \qquad \text{as } \varepsilon \to 0,
\end{aligned}$$

where $u_\varepsilon^+ = \max(u_\varepsilon, 0)$ *and* $u_\varepsilon^- = \max(0, -u_\varepsilon)$*.*

Proof. The proof is the same as that of Lemma 2.1 of [8], so we omit it. □

Now, we are going to prove Theorem 1.1.
Proof of Theorem 1.1 Arguing as in the proof of Theorem 1.1 of [8], we obtain that there exist $a_{\varepsilon,1}$, $a_{\varepsilon,2}$, $\mu_{\varepsilon,1}$ and $\mu_{\varepsilon,2}$ such that, as $\varepsilon \to 0$,

$$||u_\varepsilon^+ - P\delta_{(a_{\varepsilon,1},\mu_{\varepsilon,1})}|| \to 0, \quad ||u_\varepsilon^- - P\delta_{(a_{\varepsilon,2},\mu_{\varepsilon,2})}|| \to 0, \quad \mu_{\varepsilon,i} d(a_{\varepsilon,i}, \partial\Omega) \to +\infty, \text{ for } i = 1, 2. \tag{2.2}$$

Next, we will prove that one of the points is the global maximum point of $|u_\varepsilon|$. Arguing as in the proof of Lemma 2.2 of [8], we obtain

$$M_\varepsilon^{2/(n-2)-\varepsilon/2} d(a_\varepsilon, \partial\Omega) \nrightarrow 0 \text{ as } \varepsilon \to 0, \tag{2.3}$$

where a_ε satisfies $|u_\varepsilon(a_\varepsilon)| = |u_\varepsilon|_\infty := M_\varepsilon$. Without loss of generality, we can assume that $u_\varepsilon(a_\varepsilon) > 0$. Now, let us define

$$\tilde{u}_\varepsilon(X) := M_\varepsilon^{-1} u_\varepsilon(a_\varepsilon + X/M_\varepsilon^{2/(n-2)-\varepsilon/2}), \text{ for } X \in \widetilde{\Omega}_\varepsilon := M_\varepsilon^{2/(n-2)-\varepsilon/2}(\Omega - a_\varepsilon).$$

By Lemma 2.1, the limit domain of $\widetilde{\Omega}_\varepsilon$, denoted by Π, has to be the whole space $\mathbb{R}^n$ or a half space and by (2.3), it contains the origin. Since $\tilde{u}_\varepsilon$ is bounded in $\widetilde{\Omega}_\varepsilon$, using the standard elliptic theory, it converges in $C^2_{loc}(\Pi)$ to a function u satisfying

$$-\Delta u = |u|^{2^*-2}u \text{ in } \Pi, \quad u(0) = 1, \quad u = 0 \text{ on } \partial\Pi, \quad \text{and} \quad \int_\Pi |\nabla u|^2 \leq 2S^{n/2}. \tag{2.4}$$

Observe that, any sign-changing solution w of (2.4) satisfies $\|w\|^2 > 2S^{n/2}$. Thus we derive that u is positive. It follows that Π has to be $\mathbb{R}^n$ and $u = \delta_{(0,\beta_n^{2/(2-n)})}$. We deduce that $\tilde{u}_\varepsilon > 0$ in any compact subset of $\widetilde{\Omega}_\varepsilon$. But we have $u_\varepsilon = 0$ on $\partial\Omega_+$, where $\partial\Omega_+ := \{x \in \Omega : u_\varepsilon(x) > 0\}$. Hence

$$\|u_\varepsilon^+ - M_\varepsilon^{\varepsilon(n-2)/2} P\delta_{(a_\varepsilon,\mu_\varepsilon)}\| \to 0 \text{ and } M_\varepsilon^{2/(n-2)-\varepsilon/2} d(a_\varepsilon, \partial\Omega_+) \to \infty \text{ as } \varepsilon \to 0, \tag{2.5}$$

where $\mu_\varepsilon := M_\varepsilon^{2/(n-2)-\varepsilon/2}/\beta_n^{2/(n-2)}$. Now, we observe that

$$\begin{aligned} M_\varepsilon^{\varepsilon(n-2)/2} S^{n/4}(1 + o(1)) &= M_\varepsilon^{\varepsilon(n-2)/2} \|P\delta_{(a_\varepsilon,\mu_\varepsilon)}\| \\ &\leq \|u_\varepsilon^+ - M_\varepsilon^{\varepsilon(n-2)/2} P\delta_{(a_\varepsilon,\mu_\varepsilon)}\| + \|u_\varepsilon^+\| \\ &= S^{n/4}(1 + o(1)). \end{aligned} \tag{2.6}$$

Thus, Lemma 2.1 and (2.6) imply that $M_\varepsilon^\varepsilon$ goes to 1 as $\varepsilon \to 0$.
The proof of our theorem is thereby completed. □

The goal of the sequel of this section is to prove Theorem 1.2. We start by the following proposition which gives a parametrization of the function u_ε. It follows from the corresponding statements in [2].

Proposition 2.2 *Let $n \geq 3$ and let (u_ε) be a family of sign-changing solutions of* (1) *satisfying* (1.1). *Then the following minimization problem*

$$min\{|| \, u_\varepsilon - \alpha_1 P\delta_{(a_1,\lambda_1)} + \alpha_2 P\delta_{(a_2,\lambda_2)} \, ||, \, \alpha_i > 0, \, \lambda_i > 0, \, a_i \in \Omega\}$$

has a unique solution $(\alpha_1, \alpha_2, a_1, a_2, \lambda_1, \lambda_2)$ (up to permutation). In particular, we can write u_ε as follows

$$u_\varepsilon = \alpha_1 P\delta_{(a_1,\lambda_1)} - \alpha_2 P\delta_{(a_2,\lambda_2)} + v,$$

where $v \in H_0^1(\Omega)$ such that $||v|| \to 0$ as $\varepsilon \to 0$ and

$$(V_0) : \langle v, \varphi \rangle = 0 \text{ for } \varphi \in \{P\delta_{(a_i,\lambda_i)}, \, \partial P\delta_{(a_i,\lambda_i)}/\partial\lambda_i, \, \partial P\delta_{(a_i,\lambda_i)}/\partial a_i^j, \, i = 1, 2, \, 1 \leq j \leq n\}, \tag{2.7}$$

where a_i^j denotes the j-th component of a_i.

Remark 2.3 *For each $i = 1, 2$, the point a_i is close to $a_{\varepsilon,i}$ and each parameter λ_i satisfies $\lambda_i/\mu_{\varepsilon,i}$ is close to 1, where $a_{\varepsilon,i}$, $\mu_{\varepsilon,i}$ are defined in Theorem 1.1 and a_i, λ_i are defined in Proposition 2.2.*

As usual in this type of problems, we first deal with the v-part of u_ε, in order to show that it is negligible with respect to the concentration phenomena. Using the following estimate (see Subsection 3.1 of [19])

$$\delta_i^{-\varepsilon}(x) = \beta_n^{-\varepsilon}\lambda_i^{-\varepsilon(n-2)/2} + O\left(\varepsilon \log(1+\lambda_i^2|x-a_i|^2)\right) = 1 + o(1), \tag{2.8}$$

(since $\lambda_i^\varepsilon \to 1$ as $\varepsilon \to 0$) and arguing as in Lemma 3.3 of [8] we derive the following lemma.

Lemma 2.4 *The function v defined in Proposition 2.2 satisfies the following estimate*

$$||v|| \le c\varepsilon + c\begin{cases} \sum_i \frac{1}{(\lambda_i d_i)^{n-2}} + \varepsilon_{12}(\log\varepsilon_{12}^{-1})^{(n-2)/n} & if \quad n < 6 \\ \sum_i \frac{1}{(\lambda_i d_i)^{(n+2)/2-\varepsilon(n-2)}} + \varepsilon_{12}^{(n+2)/2(n-2)}(\log\varepsilon_{12}^{-1})^{(n+2)/2n} & if \quad n \ge 6, \end{cases}$$

where

$$\varepsilon_{12} = \left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + \lambda_1\lambda_2|a_1 - a_2|^2\right)^{(2-n)/2}.$$

Now, arguing as in the proof of Propositions 3.4, 3.5 and 3.6 of [8] and using (2.8) we obtain the following results.

Proposition 2.5 *Assume that $n \ge 3$ and let α_i, a_i and λ_i be the variables defined in Proposition 2.2. We then have*

$$1 - \frac{\alpha_i^{4/(n-2)-\varepsilon}}{\beta_n^\varepsilon\lambda_i^{\varepsilon(n-2)/2}} = O\Big(\varepsilon + \frac{1}{(\lambda_i d_i)^{n-2}} + \varepsilon_{12} + \|v\|\Big), \tag{2.9}$$

$$\alpha_i c_1 \frac{n-2}{2}\frac{H(a_i,a_i)}{\lambda_i^{n-2}} - \alpha_j c_1\Big(\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} + \frac{n-2}{2}\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{(n-2)/2}}\Big) - \alpha_i\frac{n-2}{2}c_2\varepsilon$$
$$\le c\varepsilon^2 + c\begin{cases} \sum_{k=1,2}\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \varepsilon_{12}^{\frac{n}{n-2}}\log\varepsilon_{12}^{-1} & (\ if\ n \ge 4), \\ \sum_{k=1,2}\frac{1}{(\lambda_k d_k)^2} + \varepsilon_{12}^2(\log\varepsilon_{12}^{-1})^{2/3} & (\ if\ n = 3), \end{cases} \tag{2.10}$$

where i, $j \in \{1,2\}$ with $i \ne j$, and c_1, c_2 are defined in Theorem 1.2.

Proposition 2.6 *Let α_i, a_i and λ_i be the variables defined in Proposition 2.2.*
(a) For $n \ge 4$, we have

$$\alpha_i\frac{1}{\lambda_i^{n-1}}\frac{\partial H(a_i,a_i)}{\partial a_i} + 2\frac{\alpha_j}{\lambda_i}\Big(\frac{\partial\varepsilon_{12}}{\partial a_i} - \frac{\partial H}{\partial a_i}(a_1,a_2)\frac{1}{(\lambda_1\lambda_2)^{(n-2)/2}}\Big)$$
$$= O\Big(\sum_{k=1,2}\frac{1}{(\lambda_k d_k)^n} + \varepsilon_{12}^{\frac{n}{n-2}}\log\varepsilon_{12}^{-1} + \varepsilon\varepsilon_{12}(\log\varepsilon_{12}^{-1})^{\frac{n-2}{n}} + \frac{\varepsilon}{(\lambda_i d_i)^{n-1}}\Big), \tag{2.11}$$

where i, $j \in \{1,2\}$ with $i \neq j$.
(b) For $n=3$, we assume that $a_1 \neq a_2$ then we have for $i \in \{1,2\}$

$$\frac{1}{\lambda_i^2}\frac{\partial H(a_i,a_i)}{\partial a_i} + 2\frac{1}{\lambda_i}\Big(\frac{\partial \varepsilon_{12}}{\partial a_i} - \frac{\partial H}{\partial a_i}(a_1,a_2)\frac{1}{(\lambda_1\lambda_2)^{1/2}}\Big) = o\Big(\sum_{k=1,2}\frac{1}{(\lambda_k\eta_k)^2}\Big), \tag{2.12}$$

where η_1, η_2 are two positive parameters chosen such that

$$B(a_1,\eta_1)\cap B(a_2,\eta_2)=\emptyset \quad \textit{and} \quad B(a_i,\eta_i)\subset\Omega,\ \textit{for } i=1,2,.$$

Note that the proof of Propositions 2.5 and 2.6 are based on some integral estimates proved in [1] and [17].
To deal with dimension 3, we prove the following lemma.

Lemma 2.7 *Assume that $n=3$ and assume further* (1.5) *holds. Then there exists a positive constant $\overline{c}_0$ such that*

$$|a_1-a_2| \geq \overline{c}_0 \max(d_1,d_2),$$

where the points a_i's are defined in Proposition 2.2 and $d_i = d(a_i,\partial\Omega)$.

Proof. Arguing by contradiction, we assume that $|a_1-a_2| = o(\max(d_1,d_2))$. This implies that $d_1/d_2 \to 1$ as $\varepsilon \to 0$. Now, choosing $\eta_1=\eta_2=|a_1-a_2|/4$, we see that

$$B(a_1,\eta_1)\cap B(a_2,\eta_2)=\emptyset \quad \text{and} \quad B(a_i,\eta_i)\subset\Omega,\ \text{for } i=1,2,$$

and

$$\begin{aligned}
&\frac{1}{(\lambda_i\eta_i)^2} \leq \frac{c}{\lambda_1\lambda_2|a_1-a_2|^2} \leq c\varepsilon_{12}^2,\\
&\frac{1}{\lambda_i^2}\Big|\frac{\partial H(a_i,a_i)}{\partial a_i}\Big| \leq \frac{c}{(\lambda_i d_i)^2} = o\left(\frac{1}{\lambda_1\lambda_2|a_1-a_2|^2}\right) = o\left(\varepsilon_{12}^2\right),\\
&\frac{1}{\lambda_i}\frac{1}{(\lambda_1\lambda_2)^{1/2}}\Big|\frac{\partial H(a_1,a_2)}{\partial a_i}\Big| \leq \frac{c}{(\lambda_i d_i)^2} = o\left(\varepsilon_{12}^2\right),\\
&\frac{1}{\lambda_i}\frac{1}{(\lambda_1\lambda_2)^{1/2}}\Big|\frac{\partial}{\partial a_i}\left(\frac{1}{|a_1-a_2|}\right)\Big| = \frac{1}{\lambda_i(\lambda_1\lambda_2)^{1/2}|a_1-a_2|^2} \geq c\varepsilon_{12}^2,
\end{aligned}$$

where we have used the fact that λ_1 and λ_2 are of the same order. Applying (2.12) and the above estimates, we derive a contradiction and therefore our lemma follows. □

Next we prove the following crucial lemmas.

Lemma 2.8 *Under the assumptions of Theorem 1.2, there exists a positive constant $c_0>0$ such that the variable a_i, defined in Proposition 2.2, satisfy*

$$(i) \quad c_0 \leq \frac{d_1}{d_2} \leq c_0^{-1}; \quad (ii) \quad c_0 \leq \frac{|a_1-a_2|}{d_i} \leq c_0^{-1}, \quad \textit{for } i=1,2,$$

where $d_i = d(a_i,\partial\Omega)$.

Proof. On one hand, using (2.10), we have

$$\varepsilon = O\Big(\sum \frac{1}{(\lambda_j d_j)^{n-2}} + \varepsilon_{12}\Big). \tag{2.13}$$

On the other hand, using (1.5) an easy computation shows that

$$\varepsilon_{12} = \frac{1}{(\lambda_1\lambda_2|a_1-a_2|^2)^{(n-2)/2}} + O(\varepsilon_{12}^{n/(n-2)}), \tag{2.14}$$

$$\lambda_i \frac{\partial \varepsilon_{12}}{\partial \lambda_i} := -\frac{n-2}{2}\varepsilon_{12}\left(1 - 2\frac{\lambda_j}{\lambda_i}\varepsilon_{12}^{\frac{2}{n-2}}\right) = -\frac{n-2}{2}\frac{1}{(\lambda_1\lambda_2|a_1-a_2|^2)^{(n-2)/2}} + o(\varepsilon_{12}). \tag{2.15}$$

Thus, using (2.9), (2.13) and (2.15), the estimate (2.10) becomes

$$\frac{H(a_i,a_i)}{\lambda_i^{n-2}} + \frac{G(a_1,a_2)}{(\lambda_1\lambda_2)^{(n-2)/2}} - \frac{c_2}{c_1}\varepsilon = o\Big(\varepsilon_{12} + \sum \frac{1}{(\lambda_j d_j)^{n-2}}\Big). \tag{2.16}$$

Now we claim that

$$\frac{1}{\lambda_i^{n-1}}\frac{\partial H(a_i,a_i)}{\partial a_i} + \frac{2}{\lambda_i}\frac{\partial G}{\partial a_i}(a_1,a_2)\frac{1}{(\lambda_1\lambda_2)^{(n-2)/2}} = o\Big(\varepsilon_{12}^{(n-1)/(n-2)} + \sum \frac{1}{(\lambda_j d_j)^{n-1}}\Big). \tag{2.17}$$

For $n \geq 4$, (2.17) follows immediately from (2.9), (2.11) and (2.13).
For $n = 3$, choosing $\eta_i = \min(\overline{c}_0, 1)d_i/4$ in (2.12) where $\overline{c}_0$ is the positive constant defined in Lemma 2.7, Claim (2.17) follows from (2.12).
Now we are going to prove Claim (i) . Arguing by contradiction, we assume, for example, that $d_2 = o(d_1)$. Using (2.16) for $i = 1$ and $i = 2$, we get

$$\frac{1}{(\lambda_2 d_2)^{n-2}} = o(\varepsilon_{12}). \tag{2.18}$$

Using (2.14), (2.18) and the fact that $|\partial H(a_1,a_2)/\partial a_k| \leq c d_k^{-1}(d_1d_2)^{(2-n)/2}$, it is easy to obtain

$$\frac{1}{(\lambda_1\lambda_2)^{(n-2)/2}}\left|\frac{1}{\lambda_k}\frac{\partial G}{\partial a_k}(a_1,a_2)\right| \geq c\varepsilon_{12}^{(n-1)/(n-2)} \quad \text{for} \quad k = 1, 2. \tag{2.19}$$

Clearly, (2.17), (2.18) and (2.19) give a contradiction. Thus, we derive that d_1 and d_2 are of the same order. Hence Claim (i) is proved. Regarding Claim (ii), arguing by contradiction, we assume that $d_1 = o(|a_1 - a_2|)$. In this case, it is easy to obtain

$$\begin{aligned}\frac{1}{(\lambda_1\lambda_2)^{(n-2)/2}}\left|\frac{\partial G}{\partial a_1}(a_1,a_2)\right| &\leq \frac{1}{(\lambda_1\lambda_2)^{(n-2)/2}}\left(\frac{c}{|a_1-a_2|^{n-1}} + \frac{c}{d_1|a_1-a_2|^{n-2}}\right)\\ &= o\Big(\frac{1}{d_1(\lambda_1 d_1)^{n-2}}\Big).\end{aligned} \tag{2.20}$$

Thus, (2.17) and (2.20) give again a contradiction and we derive that $d_1/|a_1 - a_2|$ is bounded below. Hence, by Lemma 2.7, the proof is completed for $n = 3$.

It remains to prove that $d_1/|a_1-a_2|$ is bounded above for $n\geq 4$. To this aim, we argue by contradiction and we assume that $|a_1-a_2|=o(d_1)$. Therefore,

$$\frac{1}{\lambda_1^{n-1}}\left|\frac{\partial H(a_1,a_1)}{\partial a_1}\right|+\frac{1}{\lambda_1(\lambda_1\lambda_2)^{(n-2)/2}}\left|\frac{\partial H(a_1,a_2)}{\partial a_1}\right| \leq \frac{c}{(\lambda_1 d_1)^{n-1}}=o\left(\frac{1}{(\lambda_1\lambda_2|a_1-a_2|^2)^{(n-1)/2}}\right). \tag{2.21}$$

We now observe that

$$\left|\frac{1}{\lambda_1}\frac{\partial\varepsilon_{12}}{\partial a_1}\right|\geq\frac{c}{\lambda_1(\lambda_1\lambda_2)^{(n-2)/2}}\frac{1}{|a_1-a_2|^{n-1}}\geq\frac{c}{(\lambda_1\lambda_2|a_1-a_2|^2)^{(n-1)/2}}. \tag{2.22}$$

Hence, (2.17), (2.21) and (2.22) give a contradiction, and therefore $|a_1-a_2|/d_1$ is bounded below. Finally, using Claim (i), the proof of Claim (ii) is completed. □

Now, we will prove that the concentration points are in a compact set of Ω and they are far away of each other.

Lemma 2.9 *There exists a positive constant d_0 such that*

$$|a_1-a_2|\geq d_0 \quad ; \quad d_i\geq d_0 \quad \textit{for } i=1,2.$$

Proof. The proof is the same as that of Lemma 3.8 of [8], so we omit it. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 Without loss of generality we can assume that

$$M_\varepsilon:=\max u_\varepsilon\geq M_{\varepsilon,-}:=-\min u_\varepsilon.$$

Hence (2.5) holds. Now, let b_ε be such that $M_{\varepsilon,-}:=-u_\varepsilon(b_\varepsilon)$. Using (1.5) and arguing as in the proof of (2.5) we can prove that

$$\|u_\varepsilon^- - M_{\varepsilon,-}^{\varepsilon(n-2)/2}P\delta_{(b_\varepsilon,\mu_{\varepsilon,-})}\|\to 0 \text{ and } M_{\varepsilon,-}^{2/(n-2)-\varepsilon/2}d(b_\varepsilon,\partial\Omega_-)\to\infty \text{ as } \varepsilon\to 0, \tag{2.23}$$

where $\mu_{\varepsilon,-}:=M_{\varepsilon,-}^{2/(n-2)-\varepsilon/2}/\beta_n^{2/(n-2)}$ and $\Omega_-:=\{x\in\Omega : u_\varepsilon(x)<0\}$. Hence $a_{\varepsilon,1}$ and $a_{\varepsilon,2}$ can be chosen as a_ε and b_ε which are two global extremum points of u_ε.
Regarding, Claim (1.7), it follows from Lemma 2.9. Therefore each a_i converges to $\overline{a}_i\in\Omega$ with $\overline{a}_1\neq\overline{a}_2$.
Now, let us introduce the following change of variable

$$\frac{1}{\lambda_i^{(n-2)/2}}=\Lambda_i\left(\frac{c_2\varepsilon}{c_1}\right)^{1/2}.$$

Note that, (2.16) and (2.17) imply, for i, $j=1,2$ with $j\neq i$,

$$H(a_i,a_i)\Lambda_i+G(a_1,a_2)\Lambda_j-\frac{1}{\Lambda_i}=o(\Lambda_i), \tag{2.24}$$

$$\frac{\partial H(a_i,a_i)}{\partial a_i}\Lambda_i^2+2\frac{\partial G(a_1,a_2)}{\partial a_i}\Lambda_1\Lambda_2=o\left(\Lambda_i^2\right). \tag{2.25}$$

Since each a_i converges to $\overline{a}_i \in \Omega$ with $\overline{a}_1 \neq \overline{a}_2$, thus the functions H, G and its derivatives are bounded. Therefore, from (2.24) and (1.5), it is easy to see that for each $i = 1, 2$, Λ_i is bounded above and below. Hence, each Λ_i converges to $\overline{\Lambda}_i > 0$ (up to a sequence) which implies (1.8) (see (1.6) and Remark 2.3). Passing to the limit in (2.24) and (2.25), we get

$$H(\overline{a}_i, \overline{a}_i)\overline{\Lambda}_i + G(\overline{a}_1, \overline{a}_2)\overline{\Lambda}_j - \overline{\Lambda}_i^{-1} = 0, \tag{2.26}$$

$$\frac{\partial H(\overline{a}_i, \overline{a}_i)}{\partial \overline{a}_i}\overline{\Lambda}_i^2 + 2\frac{\partial G(\overline{a}_1, \overline{a}_2)}{\partial \overline{a}_i}\overline{\Lambda}_1\overline{\Lambda}_2 = 0, \tag{2.27}$$

where i, $j = 1, 2$ with $j \neq i$.
Equations (2.26) and (2.27) imply that $\nabla\Psi(\overline{a}_1, \overline{a}_2, \overline{\Lambda}_1, \overline{\Lambda}_2) = 0$. Hence $(\overline{a}_1, \overline{a}_2, \overline{\Lambda}_1, \overline{\Lambda}_2)$ is a critical point of Ψ. □

3 Proof of Theorems 1.4, 1.5, 1.6 and 1.7

Regarding Theorem 1.5, it follows immediately from Theorem 1.2 and the following lemma.

Lemma 3.1 *Let Ω be a ball and assume that $n \geq 3$. Then, up to a rotation of Ω, the function Ψ, defined in Theorem 1.2, has only one critical point $\overline{X} := (a, b, x, y)$. It satisfies*

$$a = -b = (0, ..., 0, t_*) \ \textit{with}\ t_* > 0,$$

$$x = y = \left(\frac{1}{H(a, a) + G(a, -a)}\right)^{1/2},$$

where t_* is the unique solution of

$$g(t) = \frac{t}{(1-t^2)^{n-1}} - \frac{1}{(2t)^{n-1}} + \frac{t}{(1+t^2)^{n-1}} = 0, \qquad \textit{for } t \in (0, 1).$$

Proof. Let (a_1, a_2, x_1, x_2) be a critical point of Ψ. Then, for i, $j = 1, 2$ with $j \neq i$, we derive

$$H(a_i, a_i)x_i + G(a_1, a_2)x_j = \frac{1}{x_i} \tag{3.1}$$

$$x_i \frac{\partial H(a, a)}{\partial a}_{|a=a_i} + 2x_j \frac{\partial G(a, a_j)}{\partial a}_{|a=a_i} = 0. \tag{3.2}$$

Multiplying (3.1) by x_i, we get

$$H(a_1, a_1)x_1^2 = H(a_2, a_2)x_2^2. \tag{3.3}$$

Recall that when Ω is the unit ball, we have

$$G(a, b) = \frac{1}{|a-b|^{n-2}} - \frac{1}{(|a|^2|b|^2 + 1 - 2\langle a, b\rangle)^{(n-2)/2}} \tag{3.4}$$

$$H(a, a) = \frac{1}{(1-|a|^2)^{n-2}}. \tag{3.5}$$

Thus,

$$\frac{\partial H(a,a)}{\partial a} = \frac{2(n-2)a}{(1-|a|^2)^{n-1}} \tag{3.6}$$

$$\frac{\partial G}{\partial a}(a,b) = \frac{(n-2)(b-a)}{|a-b|^n} - \frac{(n-2)(b-|b|^2a)}{(|a|^2|b|^2+1-2\langle a,b\rangle)^{n/2}}. \tag{3.7}$$

First, using (3.2), (3.6) and (3.7), it is easy to prove that $a_i \neq 0$ for $i = 1, 2$.
Without loss of generality, we can assume that $a_1 = (0, ..., 0, \gamma_1)$, where γ_1 is a constant. Taken the j-th component (for $j = 1, ..., n-1$) of the vector defined by (3.2), with $i = 1$, it follows that $a_2 = (0, ..., 0, \gamma_2)$, where γ_2 is a constant. Hence a_1 and a_2 lay in the same line passing through the origin. It remains to prove that $\gamma_1 = -\gamma_2$.
Using (3.2), (3.3), (3.6) and (3.7), we get

$$\frac{\gamma_i}{(1-\gamma_i^2)}\big(H(a_1,a_1)H(a_2,a_2)\big)^{1/2} + \frac{\gamma_j - \gamma_i}{|\gamma_2-\gamma_1|^n} - \frac{\gamma_j}{(1-\gamma_2\gamma_1)^{n-1}} = 0, \tag{3.8}$$

for i, $j = 1, 2$ with $j \neq i$.
Adding (3.8) for $i = 1$ and $i = 2$, we derive

$$\frac{(\gamma_1+\gamma_2)(1-\gamma_1\gamma_2)}{(1-\gamma_1^2)^{n/2}(1-\gamma_2^2)^{n/2}} = \frac{\gamma_1+\gamma_2}{(1-\gamma_1\gamma_2)^{n-1}}. \tag{3.9}$$

Thus, if $\gamma_1+\gamma_2 \neq 0$, (3.9) implies that $(1-\gamma_1\gamma_2)^2 = (1-\gamma_1^2)(1-\gamma_2^2)$ which implies that $\gamma_1 = \gamma_2$ and therefore $a_1 = a_2$ which is a contradiction. Thus $\gamma_1+\gamma_2 = 0$, that means $a_1 = -a_2 = (0, ..., 0, t_*)$, with t_* is the unique solution of

$$g(t) = \frac{t}{(1-t^2)^{n-1}} - \frac{1}{(2t)^{n-1}} + \frac{t}{(1+t^2)^{n-1}} = 0, \qquad \text{for } t \in (0,1),$$

where we have used (3.8).
Now using (3.3), (3.5) and the fact that the reals x_i's are positive, it is easy to obtain that $x_1 = x_2$. Using again (3.1) we derive that

$$x_1 = x_2 = \left(\frac{1}{H(a_1,a_1)+G(a_1,a_2)}\right)^{1/2} \tag{3.10}$$

which completes the proof of our lemma. □

Next we are going to prove Theorem 1.4.
Proof of Theorem 1.4 Observe that, by Theorems 1.1 and 1.2, we know that u_ε can be written as $P\delta_{(a_{\varepsilon,1},\mu_{\varepsilon,1})} - P\delta_{(a_{\varepsilon,2},\mu_{\varepsilon,2})} + v$ with $\|v\| \to 0$, $u_\varepsilon(a_{\varepsilon,1}) = \max u_\varepsilon$, $u_\varepsilon(a_{\varepsilon,2}) = \min u_\varepsilon$ and $\mu_{\varepsilon,i}|a_{\varepsilon,1} - a_{\varepsilon,2}| \to \infty$, for $i = 1, 2$. Furthermore, the concentration speeds satisfy (1.8).
Set $h_\varepsilon := \max d(x, \mathcal{S})^{(n-2)/2}|u_\varepsilon(x)|$ where $\mathcal{S} = \{a_{\varepsilon,1}, a_{\varepsilon,2}\}$. It is easy to prove that h_ε is bounded (if not, we can construct another blow-up point and therefore the energy of u_ε becomes bigger than $3S^{n/2}$ which gives a contradiction).
Let $d_{\varepsilon,1} = d(a_{\varepsilon,1}, \partial\Omega_+)$ and $d_{\varepsilon,2} = d(a_{\varepsilon,2}, \partial\Omega_-)$, where $\Omega_+ = \{x \in \Omega : u_\varepsilon(x) > 0\}$ and $\Omega_- =$

$\{x \in \Omega : u_\varepsilon(x) < 0\}$. We need to prove that $d_{\varepsilon,i} \not\to 0$ as $\varepsilon \to 0$. Arguing by contradiction, assume that $d_{\varepsilon,1} \leq d_{\varepsilon,2}$ and $d_{\varepsilon,1} \to 0$. We define the following function

$$w_\varepsilon(X) := d_{\varepsilon,1}^{\alpha_\varepsilon} u_\varepsilon(a_{\varepsilon,1} + d_{\varepsilon,1}X) \quad \text{for } X \in \Omega'_{\varepsilon,+} := d_{\varepsilon,1}^{-1}(\Omega_+ - a_{\varepsilon,1}),$$

where $\alpha_\varepsilon = 2(n-2)/(4-\varepsilon(n-2))$. An easy computation shows that

$$-\Delta w_\varepsilon = w_\varepsilon^{p-\varepsilon}, \ w_\varepsilon > 0 \quad \text{in } \Omega'_{\varepsilon,+}, \quad w_\varepsilon = 0 \quad \text{on } \partial\Omega'_{\varepsilon,+}.$$

Observe that $B(0,1) \subset \Omega'_{\varepsilon,+}$ and $w_\varepsilon > 0$ in $\Omega'_{\varepsilon,+}$. Since h_ε is bounded and $d_{\varepsilon,1}^{(n-2)/2} u_\varepsilon(a_{\varepsilon,1}) \to \infty$ (see (2.5) and (2.23)), we derive that 0 is an isolated blow-up point of (w_ε). Thus, using [14], we deduce that 0 is isolated simple blow-up point of (w_ε). Hence, we have

$$w_\varepsilon(0) w_\varepsilon(y) \leq c|y|^{2-n}, \quad \text{for all } |y| \leq 1/2. \tag{3.11}$$

By standard elliptic theories, we derive that $w_\varepsilon(0)w_\varepsilon$ converges in $C^2_{loc}(\Pi)$ to a function w satisfying

$$-\Delta w = 0 \quad \text{in } \Pi \setminus \{0\}, \quad w = 0 \quad \text{on } \partial\Pi,$$

where Π is the limit domain of $\Omega'_{\varepsilon,+}$. Since 0 is an isolated simple blow-up point of (w_ε) we deduce that 0 is a nonremovable singularity and therefore $w = cG_\Pi$, where G_Π is the Green's function and c is a positive constant. Now, using Pohozaev identity in the form of Corollary 1.1 of [14] we obtain

$$\underline{c}\varepsilon(1+o(1)) \int_{B(0,\sigma)} w_\varepsilon^{p+1-\varepsilon} - \frac{\sigma}{p+1-\varepsilon} \int_{\partial B(0,\sigma)} w_\varepsilon^{p+1-\varepsilon} = \int_{\partial B(0,\sigma)} B(\sigma, x, w_\varepsilon, \nabla w_\varepsilon) dx, \tag{3.12}$$

where $\underline{c}$ is a positive constant and

$$B(\sigma, x, w_\varepsilon, \nabla w_\varepsilon) = \frac{n-2}{2} w_\varepsilon \frac{\partial w_\varepsilon}{\partial \nu} - \frac{\sigma}{2} |\nabla w_\varepsilon|^2 + \sigma \left(\frac{\partial w_\varepsilon}{\partial \nu} \right)^2.$$

Observe that, using (3.11), we obtain

$$w_\varepsilon^2(0) \sigma \int_{\partial B(0,\sigma)} w_\varepsilon^{p+1-\varepsilon} \leq c w_\varepsilon(0)^{1+\varepsilon-p} \sigma^{\varepsilon(n-2)-n} \to 0 \quad \text{as } \varepsilon \to 0, \tag{3.13}$$

$$\varepsilon(1+o(1)) w_\varepsilon^2(0) \int_{B(0,\sigma)} w_\varepsilon^{p+1-\varepsilon} \sim c d_{\varepsilon,1}^{2\alpha_\varepsilon} \varepsilon u_\varepsilon(a_{\varepsilon,1})^2 \to 0 \quad \text{as } \varepsilon \to 0, \tag{3.14}$$

where we have used (1.8) and the fact that $\alpha_\varepsilon \to (n-2)/2$ and $d_{\varepsilon,1} \to 0$ as $\varepsilon \to 0$.
For the last term in (3.12), an easy computation shows

$$\lim_{\varepsilon \to 0, \sigma \to 0} \int_{\partial B(0,\sigma)} B(\sigma, x, w_\varepsilon(0) w_\varepsilon, w_\varepsilon(0) \nabla w_\varepsilon) dx = c H_\Pi(0,0). \tag{3.15}$$

Clearly, (3.12),...,(3.15) and the fact that $\Pi \neq \mathbb{R}^n$ yield a contradiction. Hence $d_{\varepsilon,1} \not\to 0$ as $\varepsilon \to 0$ and therefore $\bar{a}_1$ is an isolated simple blow up point of (u_ε). The same holds for $\bar{a}_2$.
Now, arguing as in the proof of (4.10) of [7], the result follows. □

Now, we are going to prove Theorem 1.6.

Proof of Theorem 1.6 We start by proving Claim (a). By Theorem 1.4, the points $\overline{a}_i$'s are two isolated simple blow-up points of (u_ε). Thus, as in (3.12), we derive that

$$\underline{c}\varepsilon(1+o(1))\int_{B(a_{\varepsilon,i},\sigma)} u_\varepsilon^{p+1-\varepsilon} - \frac{\sigma}{p+1-\varepsilon}\int_{\partial B(a_{\varepsilon,i},\sigma)} u_\varepsilon^{p+1-\varepsilon} = \int_{\partial B(a_{\varepsilon,i},\sigma)} B(\sigma,x,u_\varepsilon,\nabla u_\varepsilon)dx, \tag{3.16}$$

for $i=1,2$, where $\underline{c}$ is a positive constant independent of i. As in (3.13), we have

$$\sigma u_\varepsilon^2(a_{\varepsilon,i})\int_{\partial B(a_{\varepsilon,i},\sigma)} u_\varepsilon^{p+1-\varepsilon} \le c u_\varepsilon(a_{\varepsilon,i})^{1+\varepsilon-p}\sigma^{\varepsilon(n-2)-n} \to 0 \quad \text{as } \varepsilon\to 0, \tag{3.17}$$

and using (1.8), we have

$$\varepsilon u_\varepsilon^2(a_{\varepsilon,i})\int_{B(a_{\varepsilon,i},\sigma)} u_\varepsilon^{p+1-\varepsilon} \sim S^{n/2}\varepsilon u_\varepsilon(a_{\varepsilon,1})^2 \to S^{n/2}\frac{c_1\beta_n^2}{c_2\overline{\Lambda}_i^2} \quad \text{as } \varepsilon\to 0. \tag{3.18}$$

It remains to study the right side integral of (3.16). Using again (1.8) and Theorem 1.4, we derive that

$$u_\varepsilon(a_{\varepsilon,i})u_\varepsilon \to m_i' G(\overline{a}_i,.) - m_j' G(\overline{a}_j,.) \tag{3.19}$$

where $j\neq i$ and m_1', m_2' are two positive constants satisfying $m_1'/m_2' = m_1/m_2$. Thus

$$\lim_{\varepsilon\to 0,\sigma\to 0}\int_{\partial B(a_{\varepsilon,i},\sigma)} B(\sigma,x,u_\varepsilon(a_{\varepsilon,i})u_\varepsilon, u_\varepsilon(a_{\varepsilon,i})\nabla u_\varepsilon)dx = c_1'\left(m_i' H(\overline{a}_i,\overline{a}_i) - m_j' G(\overline{a}_1,\overline{a}_2)\right), \tag{3.20}$$

for $i,j=1,2$ with $i\neq j$ and where c_1' is a positive constant independent of i and j.
Using (3.19) and (3.20) and the fact that $\overline{\Lambda}_1=\overline{\Lambda}_2$ (see Theorem 1.5), we deduce that

$$m_1' H(\overline{a}_1,\overline{a}_1) - m_2' G(\overline{a}_1,\overline{a}_2) = m_2' H(\overline{a}_2,\overline{a}_2) - m_1' G(\overline{a}_1,\overline{a}_2).$$

Hence since $\overline{a}_1 = -\overline{a}_2$, using (3.5), we derive that

$$(m_1'-m_2')\left(H(\overline{a}_1,\overline{a}_1) - G(\overline{a}_1,-\overline{a}_1)\right) = 0.$$

It is easy to verify that $H(\overline{a}_1,\overline{a}_1)\neq G(\overline{a}_1,-\overline{a}_1)$ and therefore we obtain that $m_1'=m_2'$ which implies that $m_1=m_2$. The proof of Claim (a) is thereby completed.

It remains to prove Claim (b). Arguing by contradiction and assuming that the set $\{x\in\Omega, u_\varepsilon(x)=0\}$ does not intersect the boundary $\partial\Omega$. Thus $\partial u_\varepsilon/\partial\nu$ does not change sign which implies that $\partial\overline{u}/\partial\nu$ does not change sign also, where $\overline{u}$ is defined in Theorem 1.4. Now, since $m_1=m_2$ (see Claim (a)), an easy computation shows that

$$\int_{\partial\Omega}\frac{\partial\overline{u}}{\partial\nu}(x)dx = m_1\int_{\partial\Omega}\frac{\partial}{\partial\nu}\left(G(\overline{a}_1,x)-G(\overline{a}_2,x)\right)dx = 0,$$

which implies a contradiction. Thus the result follows. □

Now, we are going to prove Theorem 1.7.

Proof of Theorem 1.7 According to Theorem 1.5, we know that, for ε close to 0, both points

$a_{\varepsilon,i}$ are far away from the origin and they lay on different sides with respect to T, where T is any hyperplane passing through the origin but not containing $a_{\varepsilon,1}$. Arguing now as in the proof of Lemma 4.1 of [8], we see that the points $a_{\varepsilon,i}$ lay on the same line passing through the origin. Lastly, the proof of the other statements of Theorem 1.7 is exactly the same as that of Theorem 1.5 of [8], so we omit it. This ends the proof of our result. □

4 Proof of Theorem 1.8

Throughout this section, c stands for a generic constant depending only on n and whose value may change in every step of the computations.

To prove Theorem 1.8, we need a delicate analysis and careful estimates. First, for ε sufficiently small, Proposition 2.2 implies that u_ε can be uniquely written as

$$u_\varepsilon = \alpha_{1,\varepsilon} P\delta_{(a_{1,\varepsilon},\lambda_{1,\varepsilon})} - \alpha_{2,\varepsilon} P\delta_{(a_{2,\varepsilon},\lambda_{2,\varepsilon})} + v_\varepsilon, \tag{4.1}$$

where $v_\varepsilon \in H_0^1(\Omega)$, $\| v_\varepsilon \| \to 0$ as $\varepsilon \to 0$ and v_ε satisfies (2.7).
To simplify the notations, we write α_i, λ_i, $P\delta_i$ and v instead of $\alpha_{i,\varepsilon}$, $\lambda_{i,\varepsilon}$, $P\delta_{(a_{i,\varepsilon},\lambda_{i,\varepsilon})}$ and v_ε respectively. We will also use the following notations:

$$d_i = d(a_i, \partial\Omega) \quad \text{and} \quad \varepsilon_{12} = \left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + \lambda_1\lambda_2 \mid a_1 - a_2 \mid^2 \right)^{(2-n)/2}.$$

Now, we prove the following crucial lemmas:

Lemma 4.1 *Assume that $n \geq 4$ and*

$$\varepsilon_{12} = \left(\lambda_1\lambda_2 \mid a_1 - a_2 \mid^2\right)^{(2-n)/2} + o(\varepsilon_{12}) \quad as \quad \varepsilon \to 0. \tag{4.2}$$

Then, for ε small, there exists a positive constant c such that

$$(i)\ \frac{d_1}{c} \leq d_2 \leq c d_1 \quad and \quad (ii)\ \frac{\lambda_1}{c} \leq \lambda_2 \leq c\lambda_1.$$

Proof. On one hand, using (4.2), we derive that (2.15) holds. Thus, using (2.9), (2.10), (2.13) and (2.15), we see that (2.16) holds. On the other hand, (2.9), (2.11) and (2.13) imply that (2.17) holds. Now, arguing by contradiction we assume, for example, that $\lambda_2 d_2 = o(\lambda_1 d_1)$. Using (2.16) for $i = 1$ and $i = 2$, we obtain (2.18). Using (4.2) and (2.18), we get

$$\frac{1}{(\lambda_1\lambda_2)^{(n-2)/2}} \mid \frac{1}{\lambda_2}\frac{\partial G}{\partial a_2}(a_1, a_2) \mid \geq c\varepsilon_{12}^{(n-1)/(n-2)}. \tag{4.3}$$

Clearly, (2.16) for $i = 2$, (4.2), (2.18) and (4.3) give a contradiction. Thus, we derive that $\lambda_1 d_1$ and $\lambda_2 d_2$ are of the same order. Assume, for example, that $d_1 = o(d_2)$. In this case, it is easy to obtain that $\mid a_1 - a_2 \mid \geq d_2 - d_1 \geq d_2/2$. Thus (2.20) holds. Obviously, (2.16) and (2.20) give a contradiction and therefore Claims (i) and (ii) are proved. □

Lemma 4.2 *Let $n \geq 4$ and assume that* (1.9) *holds. Then, for ε small, we have*

$$\mid a_1 - a_2 \mid = o(d_2) \quad \text{and} \quad d_1 = d_2 + o(d_2).$$

Proof. Observe that Remark 2.3 implies that $\lambda_1/\lambda_2 \to +\infty$ as $\varepsilon \to 0$. Thus, using Lemma 4.1, we derive that there exists a positive constant c such that

$$\frac{\lambda_1}{\lambda_2} \geq c\lambda_1\lambda_2 \mid a_1 - a_2 \mid^2 .$$

Thus $\lambda_2 \mid a_1 - a_2 \mid \leq c$. Now, since $\lambda_2 d_2 \to +\infty$ as $\varepsilon \to 0$, we derive that $\mid a_1 - a_2 \mid = o(d_2)$ and therefore $d_1 = d_2 + o(d_2)$. □

Remark 4.3 *Notice that assumption $\lambda_1/\lambda_2 \to +\infty$ as $\varepsilon \to 0$ and Lemma 4.1 imply that*

$$\varepsilon_{12} \geq c\left(\frac{\lambda_2}{\lambda_1}\right)^{(n-2)/2} \qquad \textit{for } \varepsilon \textit{ small.}$$

Lemma 4.4 *Let $n \geq 4$ and assume that* (1.9) *holds. Then, there exists a positive constant c such that, for ε small,*

$$(i) \quad \frac{1}{c}\varepsilon_{12} \leq \frac{1}{(\lambda_2 d_2)^{n-2}} \leq c\varepsilon_{12}$$

$$(ii) \quad \frac{1}{c}\varepsilon \leq \varepsilon_{12} \leq c\varepsilon$$

$$(iii) \quad \frac{1}{c}\lambda_1 \leq \lambda_2(\lambda_2 d_2)^2 \leq c\lambda_1.$$

Proof. Using (2.10) for $i = 1$, we see that

$$c_1\varepsilon_{12} - c_2\varepsilon = o\left(\frac{1}{(\lambda_2 d_2)^{n-2}} + \varepsilon_{12} + \varepsilon\right).$$

Thus

$$\varepsilon = \frac{c_1}{c_2}\varepsilon_{12} + o\left(\frac{1}{(\lambda_2 d_2)^{n-2}} + \varepsilon_{12}\right). \tag{4.4}$$

Using (2.13), (2.15) and (2.10) for $i = 1$ and for $i = 2$, we derive that

$$c_1\frac{H(a_2,a_2)}{\lambda_2^{n-2}} - 2c_1\varepsilon_{12}\varepsilon_{12}^{\frac{2}{n-2}}\frac{\lambda_1}{\lambda_2}(1+o(1)) = o\left(\frac{H(a_2,a_2)}{\lambda_2^{n-2}} + \varepsilon_{12}\right). \tag{4.5}$$

But, by Remark 4.3, we know that there exists $c > 0$ such that

$$c \leq \frac{\lambda_1}{\lambda_2}\varepsilon_{12}^{2/n-2} \leq 1. \tag{4.6}$$

Clearly, (4.4),(4.5), (4.6) and the fact that $c \leq H(a_2, a_2)d_2^{n-2} \leq 1$ imply Claims (i) and (ii). Now, it follows from Claim (i) and Lemma 4.1 that the following holds:

$$c"\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{n-2}{2}} \leq c'\varepsilon_{12} \leq \frac{1}{(\lambda_2 d_2)^{n-2}} \leq c\varepsilon_{12} \leq c\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{n-2}{2}},$$

where c, c' and c" are positive constants.
Therefore Claim (iii) follows. □

Lemma 4.5 *Let $n \geq 4$ and assume that* (1.9) *holds. Then, there exists a positive constant c such that, for ε small,*

$$(i) \quad \lambda_1\lambda_2 \mid a_1 - a_2 \mid^2 \leq c$$
$$(ii) \quad \lambda_2 \mid a_1 - a_2 \mid \leq c\varepsilon_{12}^{1/n-2} \to 0 \quad as \quad \varepsilon \to 0.$$

Proof. Using (2.11) for $i = 2$ and Lemma 4.4, we obtain

$$\frac{1}{\lambda_2^{n-1}}\frac{\partial H}{\partial a_2}(a_2, a_2) + 2(n-2)\lambda_1(a_1 - a_2)\varepsilon_{12}^{\frac{n}{n-2}} = o\left(\frac{1}{(\lambda_2 d_2)^{n-1}}\right) = o\left(\varepsilon_{12}^{\frac{n-1}{n-2}}\right). \tag{4.7}$$

Now, from (4.6) we deduce that

$$c\sqrt{\lambda_1\lambda_2} \mid a_1 - a_2 \mid \varepsilon_{12}^{\frac{n-1}{n-2}} \leq \lambda_1 \mid a_1 - a_2 \mid \varepsilon_{12}^{n/(n-2)} \leq \sqrt{\lambda_1\lambda_2} \mid a_1 - a_2 \mid \varepsilon_{12}^{\frac{n-1}{n-2}}. \tag{4.8}$$

Arguing by contradiction, assume that $\lambda_1\lambda_2 \mid a_1 - a_2 \mid^2 \to +\infty$. Using (4.7), (4.8) and the fact that

$$\mid \frac{1}{\lambda_2^{n-1}}\frac{\partial H}{\partial a_2}(a_2, a_2) \mid \leq \frac{c}{(\lambda_2 d_2)^{n-1}} \leq c\varepsilon_{12}^{\frac{n-1}{n-2}},$$

we obtain a contradiction and therefore Claim (i) follows. Finally, Claim (ii) follows from Claim (i) and Remark 4.3. □

Now, to deal with the case of $n \geq 6$, we need the following crucial proposition which improves the estimate (2.11).

Proposition 4.6 *Let $n \geq 6$ and assume that* (1.9) *holds. Then we have*

$$\frac{1}{\lambda_1}\frac{\partial \varepsilon_{12}}{\partial a_1} = o\left(\varepsilon_{12}^{(n+1)/(n-2)}\right).$$

Proof. For sake of simplicity, we will use the following notations:

$$P\delta_i = P\delta_{(a_i,\lambda_i)}, \ \delta_i = \delta_{(a_i,\lambda_i)}, \ \theta_i = \delta_i - P\delta_i,$$
$$\bar{u}_\varepsilon = \alpha_1 P\delta_1 - \alpha_2 P\delta_2, \ \varphi_1 = \frac{1}{\lambda_1}\frac{\partial P\delta_1}{\partial a_1}, \ \psi_1 = \frac{1}{\lambda_1}\frac{\partial \delta_1}{\partial a_1}.$$

Multiplying (1) by φ_1 and integrating on Ω, we obtain

$$\alpha_1 \int_\Omega \delta_1^p \varphi_1 - \alpha_2 \int_\Omega \delta_2^p \varphi_1 = \int_\Omega |u_\varepsilon|^{p-1-\varepsilon} u_\varepsilon \varphi_1, \tag{4.9}$$

where $p=(n+2)/(n-2)$.
For the left-hand side of (4.9), it follows from [1] that

$$\int_\Omega \delta_1^p \frac{1}{\lambda_1}\frac{\partial P\delta_1}{\partial a_1} = -\frac{1}{2}\frac{c_1}{\lambda_1^{n-1}}\frac{\partial H(a_1,a_1)}{\partial a_1} + O\Big(\frac{1}{(\lambda_1 d_1)^n}\Big), \tag{4.10}$$

$$\begin{aligned}\int_\Omega \delta_2^p \frac{1}{\lambda_1}\frac{\partial P\delta_1}{\partial a_1} &= \int_{\mathbb{R}^n} \delta_2^p \frac{1}{\lambda_1}\frac{\partial \delta_1}{\partial a_1} + O\left(\int_{\Omega^c}\delta_2^p\delta_1 + \int_\Omega \delta_2^p \frac{1}{\lambda_1} \mid \frac{\partial \theta_1}{\partial a_1}\mid\right)\\ &= \frac{c}{\lambda_1}\frac{\partial \varepsilon_{12}}{\partial a_1} + O\left(\lambda_2 \mid a_1 - a_2 \mid \varepsilon_{12}^{\frac{n+1}{n-2}}\right) + o\left(\frac{1}{(\lambda_1 d_1)^{\frac{n-2}{2}}(\lambda_2 d_2)^{\frac{n}{2}}}\right)\\ &= \frac{c}{\lambda_1}\frac{\partial \varepsilon_{12}}{\partial a_1} + o\left(\varepsilon_{12}^{(n+1)/(n-2)}\right), \end{aligned} \tag{4.11}$$

where we have used Lemmas 4.2, 4.4 and 4.5. For the other integral in (4.9), an easy expansion implies that

$$\begin{aligned}\int_\Omega |u_\varepsilon|^{p-1-\varepsilon}u_\varepsilon\varphi_1 &= \int_\Omega |\bar u_\varepsilon|^{p-1-\varepsilon}\bar u_\varepsilon\varphi_1 + p\int_\Omega |\bar u_\varepsilon|^{p-1} v\varphi_1\\ &\quad + O\left(\|\, v\,\|^2 + \varepsilon \,\|\, v\,\| + \|\, v\,\|^{p-\varepsilon}\left(\|\,\theta_1\,\| + \varepsilon_{12}^{1/2}(\log \varepsilon_{12}^{-1})^{\frac{n-2}{2n}}\right)\right)\\ &= \int_\Omega |\bar u_\varepsilon|^{p-1-\varepsilon}\bar u_\varepsilon\varphi_1 + p\int_\Omega |\bar u_\varepsilon|^{p-1-\varepsilon} v\varphi_1 + o\left(\varepsilon_{12}^{(n+1)/(n-2)}\right), \end{aligned} \tag{4.12}$$

where we have used in the last inequality Lemmas 2.4 and 4.4 and the fact that $\|\theta_1\| \le c(\lambda_1 d_1)^{(2-n)/2}$. For the last integral in (4.12), arguing as in (323) – (326) in [8], we obtain

$$\int_\Omega |\bar u_\varepsilon|^{p-1} v\varphi_1 \le c \,\|\, v\,\| \left(\varepsilon_{12}^{\frac{n+2}{2(n-2)}}(\log \varepsilon_{12}^{-1})^{\frac{n+2}{2n}} + \frac{\log(\lambda_1 d_1)}{(\lambda_1 d_1)^{(n+2)/2}}\right) = o\left(\varepsilon_{12}^{\frac{n+1}{n-2}}\right), \tag{4.13}$$

where we have used in the last inequality Lemmas 2.4 and 4.4. It remains to study the first integral in the right-hand side of (4.12). Denoting $f=\alpha_1\delta_1-\alpha_2\delta_2$, we observe that

$$\begin{aligned}\int_\Omega |\bar u_\varepsilon|^{p-1-\varepsilon}\bar u_\varepsilon\varphi_1 &= \int_{\mathbb{R}^n} |f|^{p-1-\varepsilon} f\psi_1 + (p-\varepsilon)\int_\Omega |f|^{p-1-\varepsilon}(\alpha_1\theta_1 - \alpha_2\theta_2)\varphi_1\\ &\quad + O\left(\int_\Omega |f|^{p-\varepsilon}\frac{1}{\lambda_1}\mid \frac{\partial\theta_1}{\partial a_1}\mid + \int_{2(\theta_1+\theta_2)\le |f|} |f|^{p-2}(\theta_1^2+\theta_2^2)\mid \varphi_1\mid\right)\\ &\quad + O\left(\frac{1}{(\lambda_1 d_1)^{\frac{n-2}{2}}(\lambda_2 d_2)^{\frac{n+2}{2}}}\right) + O\left(\int (\theta_1^p + \theta_2^p)(\mid \psi_1\mid + \frac{1}{\lambda_1}\mid \frac{\partial \theta_1}{\partial a_1}\mid)\right). \end{aligned} \tag{4.14}$$

Now, using Lemma 4.4, we can write

$$\int(\theta_1^p+\theta_2^p)\frac{1}{\lambda_1}\mid\frac{\partial\theta_1}{\partial a_1}\mid\leq\left[\frac{1}{(\lambda_1 d_1)^{\frac{n+2}{2}}}+\frac{1}{(\lambda_2 d_2)^{\frac{n+2}{2}}}\right]\frac{1}{(\lambda_1 d_1)^{\frac{n}{2}}}=o\left(\varepsilon_{12}^{\frac{n+1}{n-2}}\right);\tag{4.15}$$

$$\int\theta_1^p\mid\psi_1\mid\leq\int_{B(a_1,d_1)}\theta_1^{n/(n-2)}\delta_1^{n/(n-2)}+\int_{B(a_1,d_1)^c}\delta_1^{p+1}$$
$$\leq c\frac{\log(\lambda_1 d_1)}{(\lambda_1 d_1)^n}=o\left(\varepsilon_{12}^{\frac{n+1}{n-2}}\right);\tag{4.16}$$

$$\int\theta_2^p\mid\psi_1\mid\leq\int_{B(a_1,d_1)}\theta_2^{p/2}\delta_2^{p/2}\mid x-a_1\mid\delta_1^{n/(n-2)}+\int_{B(a_1,d_1)^c}\theta_2^p\delta_1$$
$$\leq\frac{c}{d_2^{\frac{n+2}{2}}}\frac{d_1}{\lambda_1^{\frac{n}{2}}}+\frac{c}{(\lambda_1 d_1)^{\frac{n-2}{2}}(\lambda_2 d_2)^{\frac{n+2}{2}}}=o\left(\varepsilon_{12}^{\frac{n+1}{n-2}}\right).\tag{4.17}$$

Note that, since $p-2\leq 0$, we get

$$\int_{2(\theta_1+\theta_2)\leq|f|}|f|^{p-2}(\theta_1^2+\theta_2^2)\mid\varphi_1\mid\leq\int(\theta_1^p+\theta_2^p)(\mid\psi_1\mid+\frac{1}{\lambda_1}\mid\frac{\partial\theta_1}{\partial a_1}\mid)=o\left(\varepsilon_{12}^{\frac{n+1}{n-2}}\right).\tag{4.18}$$

where we have used (4.15)-(4.17). Now, using Lemmas 4.2 and 4.4, we derive that

$$\int_\Omega|f|^{p-\varepsilon}\frac{1}{\lambda_1}\mid\frac{\partial\theta_1}{\partial a_1}\mid+\int_\Omega|f|^{p-1-\varepsilon}\theta_1\mid\varphi_1\mid\leq\frac{1}{(\lambda_1 d_1^2)^{\frac{n-2}{2}}}\int(\delta_1^p+\delta_2^p)$$
$$\leq\frac{c}{(\lambda_1 d_1)^{\frac{n-2}{2}}(\lambda_2 d_2)^{\frac{n-2}{2}}}=o\left(\varepsilon_{12}^{\frac{n+1}{n-2}}\right).\tag{4.19}$$

Using again Lemmas 4.2 and 4.4, we obtain

$$\int_\Omega|f|^{p-1-\varepsilon}\theta_2\mid\varphi_1\mid\leq\int\delta_1^p\theta_2+\int\delta_2^{p-1}\theta_2\mid x-a_1\mid\delta_1^{\frac{n}{n-2}}+\int_\Omega\delta_2^{p-1}\theta_2\frac{1}{\lambda_1}|\frac{\partial\theta_1}{\partial a_1}|$$
$$\leq\frac{1}{(\lambda_1 d_1)^{\frac{n-2}{2}}}\frac{1}{(\lambda_2 d_2)^{\frac{n-2}{2}}}+\frac{c}{(\lambda_1 d_1)^{\frac{n}{2}}}+\frac{c}{(\lambda_2 d_2)^{\frac{n-2}{2}}}\frac{1}{(\lambda_1 d_1)^{\frac{n}{2}}}=o\left(\varepsilon_{12}^{\frac{n+1}{n-2}}\right).\tag{4.20}$$

Now, we deal with the first integral in the right-hand side of (4.14). To this aim, we set $\tilde{\delta}_2:=\delta_{(a_1,\lambda_2)}$. We note that, using Lemma 4.5, we derive that

$$\delta_2=\tilde{\delta}_2+O\left(\lambda_2\mid a_2-a_1\mid\tilde{\delta}_2\right).\tag{4.21}$$

We now introduce the following sets:

$$A_1:=\{x\in\mathbb{R}^n:\delta_1\leq\varepsilon_{12}^{1/6}\tilde{\delta}_2\}=\{x\in\mathbb{R}^n:\mid x-a_1\mid\geq\frac{\beta}{\sqrt{\lambda_1\lambda_2}}\}$$
$$A_2:=\{x\in\mathbb{R}^n:\tilde{\delta}_2\leq\varepsilon_{12}^{1/6}\delta_1\}=\{x\in\mathbb{R}^n:\mid x-a_1\mid\leq\frac{1}{\beta\sqrt{\lambda_1\lambda_2}}\}$$
$$A_3:=\mathbb{R}^n\setminus(A_1\cup A_2),$$

where

$$\beta := \left(\lambda_1 - \varepsilon_{12}^{1/(3(n-2))}\lambda_2\right)^{1/2}\left(\lambda_1\varepsilon_{12}^{1/(3(n-2))} - \lambda_2\right)^{-1/2} \sim \varepsilon_{12}^{-1/(3(n-2))} \quad \text{as} \quad \varepsilon \to 0.$$

We will estimate the first integral in the right-hand side of (4.14) on each set A_i for $i = 1, 2, 3$. First, we write

$$\int_{A_1} \mid f \mid^{p-1-\varepsilon} f\psi_1 = -\int_{A_1} (\alpha_2\delta_2)^{p-\varepsilon}\psi_1 + O\left(\int_{A_1} \delta_2^{p-1}\delta_1^{\frac{2n-2}{n-2}} \mid x - a_1 \mid\right). \tag{4.22}$$

Recall that ε_{12} satisfies (4.6). Hence we get

$$\begin{aligned}\int_{A_1} \delta_2^{p-1}\delta_1^{\frac{2n-2}{n-2}} \mid x - a_1 \mid &\leq \varepsilon_{12}^{\frac{n-4}{6(n-2)}} \int_{A_1} \delta_2^{\frac{n}{n-2}}\delta_1^{\frac{n+2}{n-2}} \mid x - a_1 \mid \\ &\leq \varepsilon_{12}^{\frac{n-4}{6(n-2)}} \lambda_2^{\frac{n}{2}} \int_{A_1} \delta_1^{\frac{n+2}{n-2}} \mid x - a_1 \mid = o\left(\varepsilon_{12}^{\frac{n+1}{n-2}}\right),\end{aligned} \tag{4.23}$$

$$\begin{aligned}\int_{A_1} \delta_2^{p-\varepsilon}\psi_1 &= \int_{\mathbb{R}^n} \delta_2^{p-\varepsilon}\psi_1 - \int_{A_2\cup A_3} \delta_2^{p-\varepsilon}\psi_1 \\ &= \int_{\mathbb{R}^n} \delta_2^{p}\psi_1 + O\left(\varepsilon\varepsilon_{12}(\log(\varepsilon_{12}^{-1})^{\frac{n-2}{n}}\right) + O\left(\lambda_2 \mid a_1 - a_2 \mid \int_{A_2\cup A_3} \tilde{\delta}_2^{p} \mid \psi_1 \mid\right),\end{aligned} \tag{4.24}$$

where we have used the evenness of $\tilde{\delta}_2$, the oddness of ψ_1, (2.8) and (4.21). Note that, as in (4.21), we get

$$\int_{A_2\cup A_3} \tilde{\delta}_2^{p} \mid \psi_1 \mid \leq \varepsilon_{12}^{\frac{-1}{6(n-2)}} \int_{A_2\cup A_3} \tilde{\delta}_2^{\frac{n+1}{n-2}} \mid x - a_1 \mid \delta_1^{\frac{n+1}{n-2}} \leq \varepsilon_{12}^{\frac{-1}{6(n-2)}} \varepsilon_{12}^{\frac{n+1}{n-2}} \log(\varepsilon_{12}^{-1}). \tag{4.25}$$

Combining (4.22)-(4.25), (ii) of Lemma 4.5 and using the estimate $F11$ of [1], we obtain

$$\int_{A_1} \mid f \mid^{p-1-\varepsilon} f\psi_1 = \alpha_2^{p-\varepsilon}\frac{c}{\lambda_1}\frac{\partial\varepsilon_{12}}{\partial a_1} + o\left(\varepsilon_{12}^{\frac{n+1}{n-2}}\right). \tag{4.26}$$

Secondly, we write

$$\begin{aligned}\int_{A_2} \mid f \mid^{p-1-\varepsilon} f\psi_1 = &\int_{A_2} (\alpha_1\delta_1)^{p-\varepsilon}\psi_1 - (p-\varepsilon)\int_{A_2} (\alpha_1\delta_1)^{p-1-\varepsilon}\alpha_2\delta_2\psi_1 \\ &+ O\left(\int_{A_2} \delta_2^2\delta_1^{\frac{6}{n-2}} \mid x - a_1 \mid\right).\end{aligned} \tag{4.27}$$

Note that

$$\int_{A_2} \delta_2^2\delta_1^{\frac{6}{n-2}} \mid x - a_1 \mid \leq \varepsilon_{12}^{\frac{n-5}{6(n-2)}} \int_{A_2} \delta_2^{\frac{n+1}{n-2}}\delta_1^{\frac{n+1}{n-2}} \mid x - a_1 \mid = o\left(\varepsilon_{12}^{\frac{n+1}{n-2}}\right), \tag{4.28}$$

$$(p-\varepsilon)\int_{A_2} \delta_1^{p-1-\varepsilon}\delta_2\psi_1 = p\int_{\mathbb{R}^n} \delta_1^{p-1}\delta_2\psi_1 - p\int_{A_1\cup A_3} \delta_1^{p-1}\delta_2\psi_1 + O\left(\varepsilon\varepsilon_{12}(\log(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}}\right). \tag{4.29}$$

Using the evenness of $\tilde{\delta}_2$, the oddness of ψ_1 and (4.21), we get

$$\int_{A_1\cup A_3} \delta_1^{p-1}\delta_2\psi_1 = O\left(\lambda_2 \mid a_1 - a_2 \mid \int_{A_1\cup A_3} \delta_1^{p-1}\tilde{\delta}_2 \mid \psi_1 \mid\right). \tag{4.30}$$

Arguing as in (4.23), we obtain

$$\int_{A_1\cup A_3} \delta_1^{p-1}\delta_2\psi_1 = o\left(\varepsilon_{12}^{\frac{n+1}{n-2}}\right). \tag{4.31}$$

Thus, using the evenness of δ_1, the oddness of ψ_1, the estimate $F11$ of [1] and combining (4.27)-(4.31), we derive that

$$\int_{A_2} \mid f \mid^{p-1-\varepsilon} f\psi_1 = -\alpha_1^{p-1-\varepsilon}\alpha_2 \frac{c}{\lambda_1}\frac{\partial\varepsilon_{12}}{\partial a_1} + o\left(\varepsilon_{12}^{\frac{n+1}{n-2}}\right). \tag{4.32}$$

Lastly, denoting by $\tilde{f} = \alpha_1\delta_1 - \alpha_2\tilde{\delta}_2$, and using the evenness of $\tilde{f}$, we obtain

$$\int_{A_3} \mid f \mid^{p-1-\varepsilon} f\psi_1 = O\left(\int_{A_3} \mid \tilde{f} \mid^{p-1}\mid \delta_2 - \tilde{\delta}_2 \mid\mid \psi_1 \mid + \int_{2\mid\delta_2-\tilde{\delta}_2\mid\geq\mid\tilde{f}\mid} \mid \delta_2 - \tilde{\delta}_2 \mid^p\mid \psi_1 \mid\right). \tag{4.33}$$

Note that the set $\{x : 2 \mid \delta_2 - \tilde{\delta}_2 \mid\geq\mid \tilde{f} \mid\}$ implies that $\delta_1 = \tilde{\delta}_2(1 + o(1))$. Hence it is contained in $F := \{x :\mid x - a_1 \mid\geq c(\lambda_1\lambda_2)^{-1/2}\}$. Therefore, as in (4.23), we get

$$\int_{2\mid\delta_2-\tilde{\delta}_2\mid\geq\mid\tilde{f}\mid} \mid \delta_2 - \tilde{\delta}_2 \mid^p\mid \psi_1 \mid\leq c\left(\lambda_2 \mid a_1 - a_2 \mid\right)^{\frac{n+2}{n-2}} \int_F \tilde{\delta}_2^{\frac{n}{n-2}} \mid x - a_1 \mid \delta_1^p = o\left(\varepsilon_{12}^{\frac{n+1}{n-2}}\right). \tag{4.34}$$

Now, arguing as in (4.25), we find

$$\int_{A_3} \mid \tilde{f} \mid^{p-1}\mid \delta_2 - \tilde{\delta}_2 \mid\mid \psi_1 \mid\leq c\left(\lambda_2 \mid a_1 - a_2 \mid\right) \int_{A_3} (\delta_1^{p-1} + \tilde{\delta}_2^{p-1})\tilde{\delta}_2 \mid x - a_1 \mid \delta_1^{\frac{n}{n-2}} = o\left(\varepsilon_{12}^{\frac{n+1}{n-2}}\right). \tag{4.35}$$

Combining (4.33)-(4.35), we obtain

$$\int_{A_3} \mid f \mid^{p-1-\varepsilon} f\psi_1 = o\left(\varepsilon_{12}^{\frac{n+1}{n-2}}\right). \tag{4.36}$$

Clearly, our Proposition follows from (4.9)-(4.20), (4.26), (4.32) and (4.36). □

Now, we are ready to prove Theorem 1.8.

Proof of Theorem 1.8 According to Remark 2.3 and Lemmas 4.2, 4.4 and 4.5, Claim (a) follows. Now we will prove (1.10) in the case where $n \geq 6$. On one hand, observe that Proposition 4.6 and Remark 4.3 imply

$$\mid \frac{1}{\lambda_2}\frac{\partial\varepsilon_{12}}{\partial a_2} \mid=\mid \frac{\lambda_1}{\lambda_2}\frac{1}{\lambda_1}\frac{\partial\varepsilon_{12}}{\partial a_2} \mid= o\left(\varepsilon_{12}^{\frac{n-1}{n-2}}\right). \tag{4.37}$$

On the other hand, Remark 4.3 implies that

$$\mid \frac{1}{\lambda_2}\frac{\partial\varepsilon_{12}}{\partial a_2} \mid= c\lambda_1 \mid a_1 - a_2 \mid \varepsilon_{12}^{\frac{n}{n-2}} \geq c\sqrt{\lambda_1\lambda_2} \mid a_1 - a_2 \mid \varepsilon_{12}^{\frac{n-1}{n-2}}. \tag{4.38}$$

Clearly, from (4.37), (4.38) and Remark 2.3, we get the first claim of (1.10). Now, using (4.37), Lemma 4.4 and (2.11) for $i=2$, we derive that

$$\frac{1}{\lambda_2^{n-1}}\frac{\partial H}{\partial a_2}(a_2,a_2)=o\left(\frac{1}{(\lambda_2 d_2)^{n-1}}\right).$$

This implies that

$$d_2\not\to 0 \quad \text{and} \quad \frac{\partial H}{\partial a_2}(a_2,a_2)\to 0 \quad \text{as} \quad \varepsilon\to 0.$$

This completes the proof of (1.10).
It remains to prove Claim (b). Observe that, by Theorem 1.1 and assumption (1.9), we have

$$u_\varepsilon = P\delta_{(a_{\varepsilon,1},\mu_{\varepsilon,1})}-P\delta_{(a_{\varepsilon,2},\mu_{\varepsilon,2})}+v \;\text{ with }\; \|v\|\to 0 \quad \text{and} \quad u_\varepsilon(a_{\varepsilon,1})=\max|u_\varepsilon|. \tag{4.39}$$

First we claim that

$$\exists\, m>0 \quad s.t. \quad h_\varepsilon:=\max_{x\in\Omega}|x-a_{\varepsilon,1}|^{(n-2)/2}|u_\varepsilon(x)|\le m. \tag{4.40}$$

In fact, if $h_\varepsilon\to+\infty$, then, using the method of R. Schoen [14], we can construct a concentration point b_ε with a concentration speed $c|u_\varepsilon(b_\varepsilon)|^{2/(n-2)-\varepsilon/2}$ and the function u_ε becomes close to

$$P\delta_{(a_{\varepsilon,1},\mu_{\varepsilon,1})}-P\delta_{(b_\varepsilon,c|u_\varepsilon(b_\varepsilon)|^{2/(n-2)-\varepsilon/2})}.$$

Observe that, since $h_\varepsilon\to+\infty$, we derive that

$$c|u_\varepsilon(b_\varepsilon)|^{2/(n-2)-\varepsilon/2}|b_\varepsilon-a_{\varepsilon,1}|\to\infty,$$

which contradicts the conclusions of Theorem 1.8. Hence our claim is proved.

Now, let

$$\Omega_1:=\Omega\setminus B(a_{\varepsilon,2},\mu_{\varepsilon,2}^{\varepsilon(n-2)/4-1}),$$

and we introduce the following function

$$w_\varepsilon(X):=\frac{1}{\mu_{\varepsilon,2}^{(n-2)/2}}u_\varepsilon(a_{\varepsilon,2}+\mu_{\varepsilon,2}^{\varepsilon(n-2)/4-1}X) \;\text{ for }\; X\in\Omega'_\varepsilon:=\mu_{\varepsilon,2}^{1-\varepsilon(n-2)/4}(\Omega-a_{\varepsilon,2}).$$

The function w_ε satisfies

$$\begin{cases}-\Delta w_\varepsilon=|w_\varepsilon|^{2^*-2-\varepsilon}w_\varepsilon & \text{in } \Omega'_\varepsilon,\\ w_\varepsilon=0 & \text{on } \partial\Omega'_\varepsilon.\end{cases}$$

Observe that, using (4.40) we derive that

$$|w_\varepsilon(x)|\le m \quad \text{for each } x\in\widetilde{\Omega}_1:=\Omega'_\varepsilon\setminus B(0,1).$$

Hence w_ε converges in $C^2_{loc}(\mathbb{R}^n\setminus B(0,1))$ to a function w satisfying

$$-\Delta w=|w|^{2^*-2}w \text{ in } \mathbb{R}^n\setminus B(0,1).$$

But from (4.39), we deduce that w has to be $-\delta_{(0,\beta_n^{2/(n-2)})}$ (where β_n is defined in Theorem 1.1). Thus $w < -c < 0$ in $\partial B(0,2)$ which implies that $w_\varepsilon < 0$ in the same set. Hence $u_\varepsilon < 0$ in $\partial B_\varepsilon := \partial B(a_{\varepsilon,2}, 2\mu_{\varepsilon,2}^{\varepsilon(n-2)/4-1})$. Now since $\mu_{\varepsilon,2} d(a_{\varepsilon,2}, \partial\Omega) \to \infty$ (see Theorem 1.1) we derive that $\overline{B_\varepsilon}$ is contained in a compact set of Ω. Finally, using the fact that $u_\varepsilon(a_{\varepsilon,1}) > 0$, $a_{\varepsilon,1} \in B_\varepsilon$ (since $\mu_{\varepsilon,2}|a_{\varepsilon,2} - a_{\varepsilon,1}| \to 0$) and $\Omega \setminus \{x : u_\varepsilon(x) = 0\}$ has exactly two connected components, we deduce that the nodal surface does not intersect the boundary of Ω. Hence Claim (b) follows.
This completes the proof of our Theorem. □

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References

[1] A. Bahri, *Critical point at infinity in some variational problems,* Pitman Res. Notes Math, Ser **182**, Longman Sci. Tech. Harlow 1989.

[2] A. Bahri and J.M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of topology of the domain,* Comm. Pure Appl. Math. **41** (1988), 255-294.

[3] A. Bahri, YY. Li and O. Rey, *On a variational problem with lack of compactness: The topological effect of the critical points at infinity,* Calc. Var. Partial Differential Equations **3** (1995), 67-94.

[4] T. Bartsch, *Critical point theory on partially ordered Hilbert spaces*, J. Funct. Anal. **186** (2001), 117-152.

[5] T. Bartsch, A. M. Micheletti and A. Pistoia, *On the existence and the profile of nodal solutions of elliptic equations involving critical growth*, Calc. Var. Partial Differential Equations **26** (2006), 265-282.

[6] T. Bartsch and T. Weth, *A note on additionnal properties of sign-changing solutions to superlinear elliptic equations*, Top. Metho. Nonlin. Anal. **22** (2003), 1-14.

[7] M. Ben Ayed, K. El Mehdi and F. Pacella, *Blow-up and nonexistence of sign-changing solutions to the Brezis-Nirenberg problem in dimension three*, Ann. I.H.Poincaré, Anal. non linéaire **23** (2006), 567-589.

[8] M. Ben Ayed, K. El Mehdi and F. Pacella, *Blow-up and symmetry of sign-changing solutions to some critical elliptic equations*, Journal of Differential Equations, **230** (2006), 771-795.

[9] M. Ben Ayed and K. Ould Bouh, *Nonexistence results of sign-changing solutions to a supercritical nonlinear problem*, preprint 2006.

[10] D. Castorina, and F. Pacella, *Symmetry of positive solutions of an almost critical problem in an annulus*, Calc. Var. Partial Differential Equations **23** (2005), 125-138.

[11] A. Castro, J. Cossio and J.M. Neuberger, *A sign-changing solution for a superlinear Dirichlet problem*, Rocky Mountain J. Math. **27** (1997), 1041-1053.

[12] M. Clapp and T. Weth, *Minimal nodal solutions of the pure critical exponent problem on a symmetric domain*, Calc. Var. Partial Differential Equations **21** (2004), 1-14.

[13] Z. Han, *Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire **8** (1991), 159–174.

[14] Y. Y. Li, *Prescribing scalar curvature on S^n and related topics, Part I*, Journal of Differential Equations, **120** (1995), 319-410.

[15] A. Pistoia and T. Weth, *Sign-changing bubble tower solutions in a slightly subcritical semilinear Dirichlet problem*, Ann. I.H.Poincaré, Analyse non linéaire, to appear.

[16] P.S. Pohozaev, *On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Dokhl. Akad. Nauk SSSR **165** (1965), 36-39.

[17] O. Rey, *The role of Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent,* J. Funct. Anal. **89** (1990), 1-52.

[18] O. Rey, *Proof of two conjectures of H. Brezis and L. A. Peletier*, Manuscripta Math. **65** (1989), 19-37.

[19] O. Rey, *The topological impact of critical points at infinity in a variational problem with lack of compactness: the dimension* 3, Adv. Differential Equations **4** (1999), 581–616.