Reduction Theorems for characteristic functors on finite p-groups and applications to p-nilpotence criteria

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Abstract

We formalize various properties of characteristic functors on p–groups, and discuss relationships between them. Applications to the Thompson subgroup and certain of its analogues are then given.

1 Introduction

In a now classical paper ([8]), John Thompson introduced, for p a prime number and S a p–group, the subgroup $J_R(S)$ (there denoted by $J(S)$) generated by the abelian subgroups of S of maximal rank :

$$J_R(S) \equiv_{def.} < A \in ab(S) | \ m(A) = \max_{B \in ab(S)} m(B) > , \tag{1.1}$$

where $ab(S)$ denotes the set of all abelian subgroups of S, and, for C an abelian group, $m(C)$ denotes the minimal cardinality of a generating system of C.

Later on, in [2], Glauberman modified that definition to :

$$J(S) \equiv_{def.} < A \in ab(S) | \ |A| = \max_{B \in ab(S)} |B| > \ . \tag{1.2}$$

Thompson had formulated a p–nilpotence criterion using J_R ; this work was later built upon by Glauberman ([2]) with his ZJ–Theorem, and by Thompson himself ([9]). For the prime $p = 2$, it is often more convenient to work with the subgroup $J_e(S)$, defined using *elementary* abelian subgroups instead of abelian ones :

$$J_e(S) \equiv_{def.} < A \in ab_e(S) \ | \ |A| = \max_{B \in ab_e(S)} |B| > \tag{1.3}$$

where $ab_e(S)$ denotes the set of *elementary* abelian subgroups of S.

The functors J_e, J_R and J are *excellently abelian–generated characteristic p–functors* in the sense of §3 below. In §4, we shall establish various reduction results concerning such objects ; most notably, in certain cases, the normality of $W(S)$ in G (for $S \in \mathcal{S}yl_p(G)$ and W a characteristic p–functor) can be inferred from the (apparently much weaker) property of *control of p–nilpotence* by W (see Theorem 4.1(2)). In the fifth paragraph, we shall specialize our results to the prime $p = 2$ and the functors J_e and $\hat{J}$ (for the definition of the last one of which see [3]), and shall henceforth refine, in a very particular case, Thompson's Factorization Theorem ([9],Theorem 1(c)), thus recovering the results of [6].

In the course of the proof some reduction lemmas of independent interest, concerning normality of p–subgroups, and control of p–nilpotence, will be established.

Our notations are standard : for G a (finite) group and p a prime number, $O_p(G)$ will denote the largest normal p–subgroup of G, $O_{p'}(G)$ the largest normal subgroup of G with order prime to p, and $Z(G)$ the center of G. We set $o(G) = |G|$, $r_e(G) = m(G)$ if G is an elementary abelian p–group for some prime p, and $r_e(G) = 0$ else; for $(x, y) \in G^2$:

$$y^x := x^{-1}yx \ ,$$

and, for $A \subseteq G$ and $x \in G$:

$$A^x := \{y^x | y \in A\}.$$

As usual, by a slight abuse of language, G will be said to have p–length one if $G = O_{p',p,p'}(G)$. By a *class* of groups, we shall mean a family of groups containing every subgroup and every homomorphic image of each of its elements. $\mathcal{A}b$ will denote the class of finite abelian groups, $\mathcal{S}olv$ the class of finite solvable groups, and, for p a prime, $\mathcal{A}b_p$ the class of finite abelian p–groups. For H a finite group, $\mathcal{C}^{'}(H)$ will denote the class of finite groups, no section of which is isomorphic to H. For p a prime and $n \in \mathbf{N}$, $\mathcal{C}_p^n$ will denote the class of finite groups, one (*i.e.* all) of whose Sylow p–subgroups has (resp. have) nilpotency class at most n. By $ab(G)$ we shall denote the set of abelian subgroups of a group G. Finally, Σ_n will denote the symmetric group of degree n.

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2 A preliminary Lemma

The following result was first stated by Hayashi ([5],Lemma 3.9,p.101), though with an incomplete proof ; our own attempt at a proof ([6],Lemme) was not conclusive either (the sentence "Q, agissant sans point fixe sur le 2–groupe abélien élémentaire X, est donc cyclique" is ambiguous, as in order to thus establish the cyclicity of Q, we need to know that **each nonidentity element of Q** acts on X without fixed point, which is not obvious). Here, we shall take the opportunity to clarify the matter once and for all ; during the course of the proof, we shall feel free to use some ideas from [5] and [6].

Lemma 2.1 *Let G be a (solvable) $\{2,3\}$–group ; then the following statements are equivalent :*

(1) G is Σ_4–free , and :

(2) $G = O_{3,2,3}(G)$.

Remark 2.2 According to Burnside's $p^a q^b$–Theorem, the solvability hypothesis is redundant.

Proof. The implication (2) $\Longrightarrow$ (1) is obvious, as the condition $G = O_{3,2,3}(G)$ is inherited by all sections of G, and $\Sigma_4 \neq O_{3,2,3}(\Sigma_4)$.

Let G denote a minimal counterexample to the statement that (1) $\Longrightarrow$ (2) ; it is clear that $O_3(G) = 1$, that G possesses a unique minimal non–trivial normal subgroup X, that X is a 2–group, and that $N_0 = O_{2,3}(G) \subset G$ is the unique maximal normal subgroup of G. It follows (as $O_3\left(\dfrac{G}{N_0}\right) = 1$) that $\dfrac{G}{N_0}$ has order 2; therefore one has $O^3(G) \not\subseteq N_0$, whence $G = O^3(G)$, thus

$$O^3(\frac{G}{X}) = \frac{O^3(G)X}{X} = \frac{G}{X} \ .$$

But, by the minimality of G, one may write

$$O_{3,2,3}(\frac{G}{X}) = \frac{G}{X} \ ,$$

whence

$$\frac{G}{X} = O_{3,2}(\frac{G}{X}) \ .$$

Take now $Q \in \mathcal{S}yl_3(G)$; we have just established that $QX \lhd G$, and the Frattini argument yields :

$$G = XN_G(Q) \ .$$

Let $L =_{def} N_G(Q)$; then $L \neq G$ and $G = LX$. Let us assume $L \subseteq H \subset G$; then

$$\begin{aligned} H &= H \cap G \\ &= H \cap LX \\ &= L(H \cap X) \ ; \end{aligned}$$

but $H \cap X \lhd < H, X >= G$, whence $H \cap X = 1$ or $H \cap X = X$. In the second case, $H = LX = G$, a contradiction ; therefore $H \cap X = 1$, and $H = L(H \cap X) = L$: L is a maximal subgroup of G. Taking now $H = L$ in the above argument yields :

$$L \cap X = 1 \ .$$

Let $C = C_L(X)$; then $C \lhd LX = G$, and $X \not\subseteq C$ (else one would have $G = LX = L$, a contradiction), therefore $C = 1$. As $X \lhd G$, $X \subseteq O_2(G)$, whence $X \lhd O_2(G)$ and $Y = X \cap Z(O_2(G)) \neq 1$; but $Y \lhd G$, therefore $Y = X$, *i.e.* $X \subseteq Z(O_2(G))$. It follows that

$$\begin{aligned} O_2(G) &\subseteq C_G(X) \\ &= C_G(X) \cap XL \\ &= XC_L(X) \\ &= X \ . \end{aligned}$$

Therefore $X = O_2(G)$. Let us set $\bar{G} = \dfrac{G}{X}$; then $O_2(\bar{G}) = \dfrac{O_2(G)}{X} = 1$, and (as $\bar{G}$ is solvable)

$$C_{\bar{G}}(O_3(\bar{G})) \subseteq O_3(\bar{G}).\ (*)$$

Let now $\bar{t} = tX$ denote an element of order 2 in $\bar{G} = \dfrac{G}{X}$; according to $(*)$, $\bar{t}$ does not centralize $O_3(\bar{G})$, therefore some $\bar{y} \in O_3(\bar{G})$ is not centralized by $\bar{t}$, thus $\bar{z} =_{def} [\bar{t}, \bar{y}] \neq 1$, $\bar{z} \in O_3(\bar{G})$, and

$$\begin{aligned} \bar{z}^{\bar{t}} &= \bar{t}^{-1}\bar{z}\bar{t} \\ &= \bar{t}^{-1}\bar{t}^{-1}\bar{y}^{-1}\bar{t}\bar{y}\bar{t} \\ &= \bar{y}^{-1}\bar{t}\bar{y}\bar{t} \\ &= (\bar{t}^{-1}\bar{y}^{-1}\bar{t}\bar{y})^{-1} \\ &= \bar{z}^{-1} \ . \end{aligned}$$

Let $\omega(\bar{z}) = 3^m (m \geq 1)$, and $\bar{v} =_{def} \bar{z}^{3^{m-1}}$; then $\omega(\bar{v}) = 3$ and $\bar{v}^{\bar{t}} = \bar{v}^{-1}$, whence $< \bar{t}, \bar{v} > \simeq \Sigma_3$. Set now $V = X < t, t^v >$; then $\dfrac{V}{X} =< \bar{t}, \bar{v} > \simeq \Sigma_3$, and $O_3(V) \subseteq C_G(O_2(V)) \subseteq C_G(X) \subseteq X$, whence $O_3(V) = 1$. If $V \neq G$, then (by induction) $V = O_{3,2,3}(V)$, whence $V = O_{2,3}(V)$, $t \in O_2(V)$, $< t, t^v > \subseteq O_2(V)$, V is a 2–group, and hence also is $\bar{V}$, a contradiction. Therefore $V = G$ and $L \simeq \dfrac{G}{X} = \bar{V} \simeq \Sigma_3$. It follows that $G = LX = L \ltimes X$, X (as a minimal normal subgroup of G) being a nontrivial irreducible $\mathbf{F}_2 L \simeq \mathbf{F}_2\Sigma_3$–module. But then X has to be isomorphic to the canonical module ${\mathbf{F_2}}^2$ for $\Sigma_3 \simeq SL_2(\mathbf{F_2})$, and one obtains $G \simeq \Sigma_3 \ltimes {\mathbf{F_2}}^2 \simeq \Sigma_4$, a contradiction. □

3 Characteristic p–functors : generalities

For p a prime number, $\mathcal{G}_p$ will denote the category of finite p–groups (morphisms in $\mathcal{G}_p$ being the group isomorphisms in the usual sense).

Definition 3.1 ([2],p.1116) *By a characteristic p–functor we shall mean a functor $K : \mathcal{G}_p \to \mathcal{G}_p$ such that, for each $P \in \mathcal{G}_p$, $K(P) \subseteq P$ and $K(P) \neq 1$ if $P \neq 1$.*

Clearly, whenever K_1 and K_2 are characteristic p–functors, $K_1 \circ K_2$ (simply denoted by K_1K_2), defined by :

$$(K_1 \circ K_2)(P) \equiv_{def} K_1(K_2(P))$$

is one. Examples of characteristic p–functors include J_R, J, $\hat{J}$, J_e, Z, and Ω_n ($n \in \mathbf{N}$), the last one defined by :

$$\Omega_n(P) \equiv_{def} < x \in P | x^{p^n} = 1 > .$$

A general class of characteristic p–functors is obtained *via* :

Definition 3.2 *Let φ denote a mapping from $\mathcal{A}b_p$ to $\mathbf{N}$, invariant under isomorphisms, and such that*

$$A \neq 1 \implies \varphi(A) \geq 1 ;$$

then, for P a p–group, let

$$K_\varphi(P) \equiv_{def.} < A \text{ abelian subgroup of } P \quad | \ \varphi(A) = \max_{B \subseteq A; B \text{ abelian}} \varphi(B) > \quad .$$

It is easily seen that K_φ is a characteristic p–functor ; such characteristic p–functors will be termed *excellently abelian generated* . Clearly, J, J_R and J_e are such ; in fact , $J = K_o$, $J_R = K_m$ and $J_e = K_{r_e}$.

Definition 3.3 *The characteristic p–functor W is termed excellent if, whenever G is a finite group, $P \in \mathcal{S}yl_p(G)$, $x \in G$, and $W(P) \subseteq Q \subseteq P^x$, then $W(P) = W(Q) = W(P^x)(= W(P)^x)$. In particular, $W(P)$ is weakly closed in P, and characteristic in any p–subgroup of G that contains it.*

Lemma 3.4 *Any excellently abelian generated characteristic p–functor is excellent.*

Proof. For S a p–group, let

$$r_\varphi(S) =_{def} \max_{A \in ab(S)} \varphi(A) .$$

Let us assume that $K_\varphi(P) \subseteq Q \subseteq P^x$, and let $A_0 \in ab(P)$ such that

$$\varphi(A_0) = \max_{A \in ab(P)} \varphi(A) = r_\varphi(P) .$$

Obviously,

$$
\begin{aligned}
r_\varphi(Q) &\leq r_\varphi(P^x) \\
&= \max_{A \in ab(P^x)} \varphi(A) \\
&= \max_{C \in ab(P)} \varphi(C^x) \\
&= \max_{C \in ab(P)} \varphi(C) \\
&\quad \text{(as } \varphi \text{ is invariant under isomorphisms)} \\
&= r_\varphi(P) \\
&= \varphi(A_0) \\
&\leq r_\varphi(Q) (\text{as } A_0 \subseteq K_\varphi(P) \subseteq Q).
\end{aligned}
$$

Therefore $r_\varphi(P) = r_\varphi(Q)$, whence

$$
\begin{aligned}
K_\varphi(Q) &= < A \in ab(Q) | \varphi(A) = r_\varphi(Q) > \\
&= < A \in ab(Q) | \varphi(A) = r_\varphi(P) > \\
&= < A \in ab(P) | \varphi(A) = r_\varphi(P) > \\
&= K_\varphi(P)
\end{aligned}
$$

(because $A \in ab(P)$ and $\varphi(A) = r_\varphi(P)$ yield $A \subseteq K_\varphi(P) \subseteq Q$)

Incidentally we have shown that $r_\varphi(Q) = r_\varphi(P^x)$, whence $K_\varphi(Q) \subseteq K_\varphi(P^x)$ and $K_\varphi(P) = K_\varphi(Q) \subseteq K_\varphi(P^x) = (K_\varphi(P))^x$, and equality all along follows. $\square$

4 A Reduction Theorem

Let p, W and $\mathcal{C}$ denote respectively a prime number, a characteristic p–functor, and a class of groups ; the following properties of the triple $(W, \mathcal{C}, p)$ will be considered (S denoting a Sylow p–subgroup of the group G) :

(P1) For each $G \in \mathcal{C}$, one has

$$G = N_G(W(S))O_{p'}(G) \ .$$

(P2) For each p–solvable $G \in \mathcal{C}$, one has

$$G = N_G(W(S))O_{p'}(G) \ .$$

(P3) For each solvable $G \in \mathcal{C}$, one has

$$G = N_G(W(S))O_{p'}(G) \ .$$

(P4) For each solvable $G \in \mathcal{C}$, all of whose Sylow q–subgroups for all primes $q \neq p$ are abelian, one has

$$G = N_G(W(S))O_{p'}(G) \ .$$

(P5) *W controls p–length 1 in $\mathcal{C}$, i.e.* for each *p–solvable* $G \in \mathcal{C}$, if $N_G(W(S))$ has p–length one then G has p–length one.

(P6) *W controls p–nilpotence in $\mathcal{C}$, i.e.* for each $G \in \mathcal{C}$, if $N_G(W(S))$ is p–nilpotent then G is p–nilpotent.

Stellmacher's result ([7]) asserts the existence of a (non–explicit) characteristic 2–functor W such that (P1)(and hence (P2),...,(P6)) hold for $(W, \mathcal{C}^{'}(\Sigma_4), 2)$.

Theorem 4.1 *(1) One has (P1) ⇒ (P2) ⇒ (P3) ⇒ (P4) ⇒ (P6), and (P3) ⇒ (P5) ⇒ (P6).*

(2) If $p = 2$, $W(S) \subseteq \Omega_1(S)$ for all S, and either

(i) $\mathcal{C} \subseteq \mathcal{C}_{2,2}$ and W is excellent ,

or

(ii) W is excellently abelian generated,

then (P6)⇒(P2), and hence properties (P2),...,(P6) are equivalent.

Proof. (1) The implications (P1) ⇒ (P2) ⇒ (P3) ⇒ (P4) are trivial.

In order to establish that (P3) ⇒ (P5), let us assume (P3), let G denote a counterexample to (P5) with minimal order. We shall use arguments similar to Bauman's in [1],pp.388–389. If $O_{p'}(G) \neq 1$, let $\bar{G} =_{def} \dfrac{G}{O_{p'}(G)}$; then one has :

$$\begin{aligned} N_{\bar{G}}(W(\bar{S})) &= N_{\bar{G}}\left(\frac{W(S)O_{p'}(G)}{O_{p'}(G)}\right) \\ &= \frac{N_G(W(S))O_{p'}(G)}{O_{p'}(G)} \\ &\text{(by the Frattini argument)} \\ &\simeq \frac{N_G(W(S))}{N_G(W(S)) \cap O_{p'}(G)} . \end{aligned}$$

Therefore $N_{\bar{G}}(W(\bar{S}))$ has p–length one, whence, by induction (as $\bar{G} \in \mathcal{C}$ and $\bar{G}$ is p–solvable), $\bar{G}$ has p–length one, hence so has G, a contradiction. Thus $O_{p'}(G) = 1$, whence (as G is p–solvable) $C_G(O_p(G)) \subseteq O_p(G)$; in particular, $O_p(G) \neq \{1\}$. Let now $\bar{G} = \dfrac{G}{O_p(G)}$, and let $\bar{H} = N_{\bar{G}}(W(\bar{S}))$; if $H = G$, then $W(\bar{S}) \lhd \bar{G}$, thus $W(\bar{S}) \subseteq O_p(\bar{G}) = 1$, $W(\bar{S}) = 1$, $\bar{S} = 1$, $S = O_p(G)$, $W(S) = W(O_p(G)) \lhd G$, and $G = N_G(W(S))$ has p–length one, a contradiction. Therefore $H \subset G$; as $N_H(W(S)) \subseteq N_G(W(S))$ has p–length one, so has H by induction, hence so has $\bar{H}$, hence so has $\bar{G}$, again by induction ($\bar{G}$ and H both belonging to $\mathcal{C}$). Let $\bar{K} = O_{p'}(\bar{G})$; it appears that $\bar{S}\bar{K} \lhd \bar{G}$, hence $SK \lhd G$; if $SK \neq G$, one finds by induction that SK has p–length 1;but $SK \lhd G$, whence $O_{p'}(SK) \lhd G$ and $O_{p'}(SK) \subseteq O_{p'}(G) = 1$. Therefore

$S \lhd SK$, whence $S = O_p(SK) \lhd G$, and again $W(S) \lhd G$ and $G = N_G(W(S))$, a contradiction. Therefore $G = SK$, and $\bar{G} = \bar{S}\bar{K}$.

For $q \in \pi(\bar{K})$, let $\bar{Q}$ denote a Sylow q–subgroup of $\bar{K}$; the total number of Sylow q–subgroups of $\bar{K}$ is $|\bar{K} : N_{\bar{K}}(\bar{Q})| \not\equiv 0[p]$, therefore one of them, $\bar{K_q}$, is $\bar{S}$–invariant. If, for each $q \in \pi(\bar{K})$, one has $SK_q \neq G$, then, by induction, SK_q has p–length one; but $O_{p'}(SK_q) \subseteq C_G(O_p(SK_q)) \subseteq C_G(O_p(G)) \subseteq O_p(G)$, thus $O_{p'}(SK_q) = 1$ and $S \lhd SK_q$,thus $K_q \subseteq N_G(S)$, hence

$$\bar{K} =< \bar{K_q} | q \in \pi(\bar{K}) > \subseteq \overline{N_G(S)}$$

and $S \lhd SK = G$, a contradiction. Thus for some prime q one has $G = SK_q$, and it appears that G is solvable (in fact, a solvable $\{p,q\}$–group for some prime q). But now (P3) yields that $G = N_G(W(S))$, whence G has p–length one, a contradiction (in this proof, due to the hypotheses on $\mathcal{C}$, all the groups that appear *belong to* $\mathcal{C}$; such will be the case in all subsequent similar reasonings).

Assuming (P4), let G denote a counterexample to (P6), with minimal order ; then Thompson's arguments ([8],pp.43–44) yield that $O_{p'}(G) = 1$, $O_p(G) \neq 1$ and G is a $\{p,q\}$–group with (elementary) abelian Sylow subgroups for some prime $q \neq p$. But then (P4) yields that $G = N_G(W(S))$, whence G has p–length one, a contradiction. Therefore (P4) $\Rightarrow$ (P6) is established.

In order to establish that (P5) $\Rightarrow$ (P6), the same argument works ; here, we only need Thompson's reduction up to an earlier point, *viz.* $O_{p'}(G) = 1$ and G p–solvable.

(2) Let us assume all the conditions in (2), and let G denote a minimum counterexample to (P6) $\Rightarrow$ (P2) ; it is clear, as usual, that $O_{2'}(G) = 1$, and then (by the same reasoning as in (1)) that $O_{2'}(H) = 1$ for any subgroup H of G containing S, and therefore that $M :\equiv N_G(W(S))$ is the unique maximal subgroup of G containing S. Let $\bar{G} = \dfrac{G}{O_2(G)}$; then $\bar{G}$ is 2–solvable, and $\bar{M}$ is the unique maximal subgroup of G containing $\bar{S}$. By induction, one has

$$\begin{aligned} \bar{G} &= N_{\bar{G}}(W(\bar{S}))O_{2'}(\bar{G}) \\ &= N_{\bar{G}}(W(\bar{S}))(\bar{S}O_{2'}(\bar{G})) \; ; \end{aligned}$$

the two factors on the right–hand side of this equality contain $\bar{S}$, whence at least one is not contained in $\bar{M}$, *i.e.* either $N_{\bar{G}}(W(\bar{S})) = \bar{G}$ or $\bar{G} = \bar{S}O_{2'}(\bar{G})$. The first possibility leads to a contradiction as in the proof that (P3) $\Rightarrow$ (P5) ; therefore $\bar{G} = \bar{S}O_{2'}(\bar{G})$, *i.e.* G has 2–length one.

As $\bar{S}$ is contained into a unique maximal subgroup of $\bar{G}$ ($\bar{M}$), $O_{2'}(\bar{G})$ possesses a unique maximal $\bar{S}$–invariant proper subgroup : $O_{2'}(\bar{G}) \cap \bar{M}$. It follows, first, that $O_{2'}(\bar{G})$ is a q–group for some prime $q \neq 2$: $O_{2'}(\bar{G}) = \bar{Q}$ ($Q \in \mathcal{S}yl_q(G)$), and therefore $G = SQ$ is a solvable $\{2,q\}$–group, and secondly that $\bar{S}$ acts irreducibly on $\dfrac{\bar{Q}}{\Phi(\bar{Q})}$; in particular, $Z(\bar{S})$ is cyclic.

Let $N \equiv_{def} < W(S)^G > \lhd G$; then $O_{2'}(N) = 1$, and $S \cap N \in \mathcal{S}yl_2(N)$. If $N < G$, the minimality of G yields :

$$\begin{aligned} N &= N_N(W(S\cap N))O_{2'}(N) \\ &= N_N(W(S\cap N)) \ . \end{aligned}$$

But $W(S) \subseteq S\cap N \subseteq S$, whence $W(S) = W(S\cap N)$, as W is excellent (in case (i) by assumption, and in case (ii) by Lemma 3.4). The Frattini argument now yields that :

$$\begin{aligned} G &= NN_G(S\cap N) \\ &\subseteq NN_G(W(S\cap N)) \\ &\subseteq N_G(W(S\cap N)) \\ &\subseteq G \ , \end{aligned}$$

whence $G = N_G(W(S\cap N)) = N_G(W(S))$ is 2–nilpotent, a contradiction. Therefore $N = G$, *i.e.* $G =< W(S)^G >$; thence

$$\begin{aligned} \bar{G} &= < \overline{W(S)}^{\bar{G}} > \\ &= < \overline{W(S)}^{\bar{S}\bar{Q}} > \\ &= < \overline{W(S)}^{\bar{Q}} > \\ &\subseteq \overline{W(S)}\bar{Q} \text{ (as } \bar{Q} \lhd \bar{G}) \ , \end{aligned}$$

and $\bar{S} = \bar{S}\cap\overline{W(S)}\bar{Q} = \overline{W(S)}(\bar{S}\cap\bar{Q}) = \overline{W(S)}$, *i.e.* $S = W(S)O_2(G)$.

In case (ii), let $W = K_\varphi$; then $W(S) \not\subseteq O_2(G)$ (else one would have $S = W(S)O_2(G) = O_2(G) \lhd G$), whence there is an abelian subgroup A of S with $\varphi(A) = r_\varphi(P)$ and $A \not\subseteq O_2(G)$. Let $N =< A^G >\lhd G$; if $N \neq G$, then, by induction, it follows as above that $W(S\cap N) \lhd N$ whence $W(S\cap N) \subseteq O_2(N) \subseteq O_2(G)$. But

$$\varphi(A) \leq r_\varphi(S\cap N) \leq r_\varphi(S) = \varphi(A)$$

whence $\varphi(A) = r_\varphi(S\cap N)$ and $A \subseteq K_\varphi(S\cap N) = W(S\cap N) \subseteq O_2(N) \subseteq O_2(G)$, a contradiction. Therefore $G =< A^G >$, whence

$$\begin{aligned} \bar{G} &= < \bar{A}^{\bar{G}} > \\ &= < \bar{A}^{\bar{S}\bar{Q}} > \\ &= < \bar{A}^{\bar{S}} > \bar{Q} (\text{as } \bar{Q} \lhd \bar{G}) \ ; \end{aligned}$$

therefore

$$\begin{aligned} \bar{S} &= \bar{S}\cap\bar{G} \\ &= \bar{S}\cap < \bar{A}^{\bar{S}} > \bar{Q} \\ &= < \bar{A}^{\bar{S}} > (\bar{S}\cap\bar{Q}) \\ &= < \bar{A}^{\bar{S}} > \ . \end{aligned}$$

By a well–known property of p–groups, it follows that $\bar{S} = \bar{A}$; in particular, $\bar{S}$ is abelian.

In case (i), $\mathcal{C} \subseteq \mathcal{C}_{2,2}$, *i.e.* $cl(S) \leq 2$, whence

$$\begin{aligned}[S,S] &\subseteq Z(S) \\ &\subseteq C_G(O_2(G)) \\ &\subseteq O_2(G)\end{aligned}$$

(by the solvability of G and the Hall–Higman Lemma), whence, again, $\bar{S}$ is abelian. Therefore, $\bar{S}$ is abelian in both cases, (i) and (ii). Now, from the fact that $Z(\bar{S})$ is cyclic, follows that $\bar{S}$ itself is. But $\bar{S} = \overline{W(S)} \subseteq \overline{\Omega_1(S)} \subseteq \Omega_1(\bar{S})$ (by the hypothesis); therefore $\bar{S}$ has order 2.

Now, as $\bar{S}$ acts irreducibly on the $\mathbf{F}_q$–module $M = \dfrac{\bar{Q}}{\Phi(\bar{Q})}$, the nontrivial element $\bar{t}$ of $\bar{S}$ either centralizes each element of M, or inverts each element of M ; now, irreducibility forces $|M| = q$, *i.e.* $\dfrac{\bar{Q}}{\Phi(\bar{Q})} = M$ is cyclic ; but then so are $\bar{Q}$, and $Q \simeq \bar{Q}$.

Let now $\bar{H} = \bar{S}\Phi(\bar{Q})$; then $H < G$ (in fact, $|G : H| = q$), and $S \subseteq H$. Therefore H is contained in $M = N_G(W(S))$, whence

$$\begin{aligned}[\bar{S}, \Phi(\bar{Q})] &= [\overline{W(S)}, \Phi(\bar{Q})] \\ &\subseteq [\overline{W(S)}, \bar{H}] \cap \Phi(\bar{Q}) \\ &\subseteq [\overline{W(S)}, \bar{M}] \cap \Phi(\bar{Q}) \\ &\subseteq \overline{W(S)} \cap \Phi(\bar{Q}) \\ &= 1 ,\end{aligned}$$

i.e. $\bar{S}$ centralizes $\Phi(\bar{Q})$. If $|\bar{Q}| \geq q^2$, then $\Omega_1(\bar{Q}) \subseteq \Phi(\bar{Q})$, whence $\bar{S}$ centralizes $\Omega_1(\bar{Q})$, and therefore $\bar{S}$ centralizes $\bar{Q}$, a contradiction. Thus $|\bar{Q}| = q$, and $\bar{G} = \bar{S}\bar{Q}$ is dihedral of order $2q$; it follows that $\bar{S}$ is a maximal subgroup of $\bar{G}$, *i.e.* S is a maximal subgroup of G. Therefore $S = M = N_G(W(S))$, and $N_G(W(S))$ is 2–nilpotent ; but now (P6) yields that G itself is 2–nilpotent, a contradiction. $\square$

5 Of J_e and $\hat{J}$

By a well–known variation([4],Theorem 1(c), and Remarks p.372) on Thompson's factorization([9]), any solvable Σ_3–free finite group G with Sylow 2–subgroup S satisfies :

$$G = N_G(J_e(S))C_G(Z(S))O_{2'}(G). \quad (5.1)$$

In [3] Glauberman introduced a new characteristic functor $\hat{J}$ having the property that, for each 2–group S, one has :

$$J_e(S) \subseteq \hat{J}(S) \subseteq S. \quad (5.2)$$

For this functor he was able to prove ([3], Theorem 7.4, p.48) that, for any 2–constrained Σ_4–free finite group G and each $S \in \mathcal{S}yl_2(G)$, one had :

$$G = N_G(\hat{J}(S))C_G(Z(S))O_{2'}(G) \text{ (5.3).}$$

By (5.2) one finds $J_e(S) = J_e(\hat{J}(S)) char \hat{J}(S)$ whence

$$N_G(\hat{J}(S)) \subseteq N_G(J_e(S)) \ ;$$

(5.3) is therefore stronger than (5.1).

In the particular case that S has nilpotence class at most two, we can state

Theorem 5.1 *Let G be a 2–constrained, Σ_4–free finite group with Sylow 2–subgroup S of nilpotence class at most two ; then one has :*

$$G = N_G(\hat{J}(S))O_{2'}(G).$$

By the above remark follows

Corollary 5.2 *In the situation of the Theorem,*

$$G = N_G(J_e(S))O_{2'}(G).$$

Thus one can assert

Corollary 5.3 *Let G be a finite solvable Σ_4–free group with Sylow 2–subgroup S of class at most two ; then :*

$$G = N_G(J_e(S))O_{2'}(G).$$

In other words, $(J_e, \mathcal{C}^{'}(\Sigma_4) \cap \mathcal{S}olv, 2)$ satisfies (P1), and hence (P2),...,(P6).

This Corollary was first proved by the author in [6].

Proof. of Theorem 5.1. Let G be a counterexample of minimal order.

(1) $\mathbf{O_{2'}(G) = 1}$.

If not, $\bar{G} = \dfrac{G}{O_{2'}(G)}$ is of smaller order than G and satisfies the hypothesis, whence

$$\bar{G} = N_{\bar{G}}(\hat{J}(\bar{S}))O_{2'}(\bar{G}) = N_{\bar{G}}(\hat{J}(\bar{S})).$$

But the canonical map $S \to \dfrac{SO_{2'}(G)}{O_{2'}(G)} = \bar{S}$ is an isomorphism, whence $\hat{J}(\bar{S}) = \dfrac{\hat{J}(S)O_{2'}(G)}{O_{2'}(G)}$ and

$$N_{\bar{G}}(\hat{J}(\bar{S})) = \frac{N_G(\hat{J}(S)O_{2'}(G))}{O_{2'}(G)} = \frac{N_G(\hat{J}(S))O_{2'}(G)}{O_{2'}(G)} \ ,$$

by the Frattini argument. Thus we get $G = N_G(\hat{J}(S))O_{2'}(G)$, a contradiction.

(2) $\mathbf{C_G(O_2(G)) \subseteq O_2(G)}$.

Obvious, because G is 2–constrained and $O_{2'}(G) = 1$.

(3) $\mathbf{M = N_G(\hat{J}(S))}$ **is the unique maximal subgroup of G that contains S**.

By hypothesis $M \subset G$. Let H be a proper subgroup of G containing S; one has $O_2(G) \subseteq S \subseteq H$, whence (as in the proof of Theorem 4.1(1))

$$O_2(G) \subseteq O_2(H)$$

and :

$$\begin{aligned} C_H(O_2(H)) &= H \cap C_G(O_2(H)) \\ &\subseteq H \cap C_G(O_2(G)) \\ &\subseteq H \cap O_2(G) \ \text{(by (2))} \\ &\subseteq O_2(H). \end{aligned}$$

Therefore $O_{2'}(H) = 1$ and H is 2–constrained with Sylow 2–subgroup S ; the minimality of G now yields :

$$\begin{aligned} H &= N_H(\hat{J}(S))O_{2'}(H) = N_H(\hat{J}(S)) \\ &\subseteq N_G(\hat{J}(S)) = M. \end{aligned}$$

Thus M is a proper subgroup of G that contains any proper subgroup of G containing S ; the result follows.

(4) $\mathbf{Z(S) \subseteq Z(G)}$.

By (5.3) one has

$$G = N_G(\hat{J}(S))C_G(Z(S))O_{2'}(G) = MC_G(Z(S)) \ ;$$

thus $S \subseteq C_G(Z(S)) \nsubseteq M$, whence $C_G(Z(S)) = G$ by (3).

(5) **G centralizes** $\mathbf{\dfrac{O_2(G)}{Z(G)}}$.

Let $C = C_G(\dfrac{O_2(G)}{Z(G)}) \lhd G$; then

$$[S, O_2(G)] \subseteq [S, S] \subseteq Z(S) \subseteq Z(G)$$

(by (4) and the hypothesis on S). It follows that $S \subseteq C$, whence

$$G = CN_G(S) \ ,$$

again by the Frattini argument. If C were different from G, one would have $C \subseteq M$ (because of (3)) and

$$G = CN_G(S) \subseteq MN_G(S) \subseteq M.M = M \ ,$$

a contradiction. Thus $C = G$.

(6) **The End.**

By (5) one has $[G, O_2(G)] \subseteq Z(G)$, i.e.

$$[G, O_2(G), G] = [O_2(G), G, G] = 1.$$

Philip Hall's Three Subgroups Lemma now yields

$$[G, G, O_2(G)] = 1 \ ,$$

that is :

$$G' \subseteq C_G(O_2(G)) \ ,$$

whence $G' \subseteq O_2(G)$ by (2). Therefore $H = \dfrac{G}{O_2(G)}$ is an abelian group with $O_2(H) = 1$, *i.e.* an abelian $2'$–group ; it appears that $S = O_2(G) \triangleleft G$, whence $\hat{J}(S) \triangleleft G$, thus $G = M$ and again a contradiction ensues. This concludes the proof. □

Remark 5.4 It seems difficult to generalize directly Corollary 5.2, and even Corollary 5.3, as the counter-examples to the ZJ–Theorem for $p = 2$ given by Glauberman in the last paragraph of [2] show. Such a counterexample G is solvable, with Sylow 2–subgroup S of nilpotence class 3 (this is not difficult to see), and S possesses a unique abelian subgroup of maximal order A, that is elementary abelian. Therefore $J_e(S)$, $J_R(S)$, $J(S)$ and $ZJ(S)$ all coïncide with A, and neither is normal in G.

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