Strongly homotopy Lie bialgebras and Lie quasi-bialgebras

Olga KRAVCHENKO

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

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OLGA KRAVCHENKO

This paper is dedicated to Jean-Louis Loday on the occasion of his 60th birthday with admiration and gratitude

Abstract

Structures of Lie algebras, Lie coalgebras, Lie bialgebras and Lie quasibialgebras are presented as solutions of Maurer-Cartan equations on corresponding governing differential graded Lie algebras. Cohomology theories of all these structures are described in a concise way using the big bracket construction of Kosmann-Schwarzbach. This approach provides a definition of an L_∞-(quasi)bialgebra (strong homotopy Lie (quasi)bialgebra). We recover an L_∞-algebra structure as a particular case of our construction. The formal geometry interpretation leads to a definition of an L_∞ (quasi)bialgebra structure on V as a differential operator Q on V, self-commuting with respect to the Poisson bracket. Finally, we establish an L_∞-version of a Manin (quasi) triple and get a correspondence theorem with L_∞-(quasi) bialgebras.

1. Introduction.

Algebraic structures are often defined as certain maps which must satisfy quadratic relations. One of the examples is a Lie algebra structure: a Lie bracket satisfies the Jacobi identity (indeed the Jacobi identity is a quadratic relation since the bracket appears twice in each summand). Other examples include an associative multiplication (associativity condition is quadratic), L_∞ and A_∞ algebras (also called strong homotopy Lie and strong homotopy associative algebras) and many others.

The subject of this article is a description of Lie (quasi) bialgebras and their L_∞-versions. The main philosophy is that the axioms of Lie (quasi)bialgebras (and their L_∞-versions) could be written in the form of a quadratic relation on a certain governing differential graded Lie algebra. We find the governing differential graded Lie algebras for Lie bialgebra and Lie quasi-bialgebra structures using Kosmann-Schwarzbach's big-bracket construction [9]. The L_∞ brackets are obtained by using the (higher) derived brackets of [12, 23, 1].

The quadratic relation on the structure can be expressed in the form of a Maurer-Cartan equation.

Classically [7], solutions of the Maurer-Cartan equation are considered only from the first graded component of the governing differential graded Lie algebra and they give the original algebraic structure. However solutions of the Maurer-Cartan equation in the whole governing differential graded Lie algebra provide a strong homotopy version of the original one.

Let us give here the definitions of Lie bialgebras, Lie quasibialgebras and Manin triples and pairs to start with.

A good reference on Lie (quasi)bialgebras is the book by Etingof and Schiffmann ([5], pages 32–34 and 150–152).

Definition 1.1. *A Lie bialgebra structure on a vector space V is the following data:*

- **a:** *a Lie bracket,* $\{\cdot,\cdot\} : V \wedge V \to V$;
- **b:** *a Lie cobracket, that is an element* $\delta : V \to V \wedge V$, *satisfying the coJacobi identity:*

$$Alt(\delta \otimes 1)\delta(x) = 0$$

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c: *a compatibility condition between* $\{\cdot,\cdot\}$ *and* δ*, meaning that* δ *is a 1-cocycle:* $\delta(\{x,y\}) = \{\delta(x), 1\otimes y + y\otimes 1\} + \{1\otimes x + x\otimes 1, \delta(y)\}$.

Where for x, y, z elements of V we denote $Alt(x\otimes y\otimes z) = x\otimes y\otimes z + y\otimes z\otimes x + z\otimes x\otimes y$.

Definition 1.2. *A Lie quasibialgebra structure on a vector space* V *is the following data:*

a: *a Lie bracket* $\{\cdot,\cdot\}$;

b: *an element* $\delta \in Hom(V, V\wedge V)$*, and an element* $\phi \in V\wedge V\wedge V$ *satisfying a modified coJacobi identity:*

$$\frac{1}{2}Alt(\delta\otimes 1)\delta(x) = [x\otimes 1\otimes 1 + 1\otimes x\otimes 1 + 1\otimes 1\otimes x, \phi]$$

and $(\delta\otimes 1\otimes 1 + 1\otimes\delta\otimes 1 + 1\otimes 1\otimes\delta)(\phi) = 0$.

c: δ *is a 1-cocycle with respect to the bracket* $\{\cdot,\cdot\}$.

Lie bialgebras are in one-to-one correspondence with Manin triples:

Definition 1.3. *A Manin triple* $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ *is a triple of finite dimensional Lie algebras, where* $\mathfrak{g}_+ \bigoplus \mathfrak{g}_- = \mathfrak{g}$ *as a vector space and* $\mathfrak{g}$ *is equipped with a nondegenerate symmetric invariant bilinear form* $< \cdot,\cdot >$ *such that* $\mathfrak{g}_+$ *and* $\mathfrak{g}_-$ *are Lagrangian subalgebras (that is maximal isotropic subspaces which are Lie subalgebras).*

We will formulate and prove an L_∞ (in other words strong homotopy) version of the Lie bialgebras - Manin triples correspondence.

Lie quasi-bialgebras turn out to be described by the notion of a Manin pair.

Definition 1.4. *A Manin pair is a pair* $(\mathfrak{g}, \mathfrak{g}_+)$ *where* $\mathfrak{g}$ *is a finite dimensional Lie algebra equipped with a non-degenerate symmetric invariant bilinear form* $< \cdot,\cdot >$ *and* $\mathfrak{g}_+$ *is a Lagrangian subalgebra.*

A Manin quasi-triple (also called a marked pair) is a pair $(\mathfrak{g}, \mathfrak{g}_+)$ *with a chosen Lagrangian complement of* $\mathfrak{g}_+$.

The main theorem here is by Drinfeld [4] which states that Manin quasi-triples are in one-to-one correspondence with Lie quasi-bialgebras.

We will also develop an L_∞ version of this correspondence.

It should be mentioned that an operad [17] or rather properade (or PROP) approach is not used in this paper. However, the definition of an L_∞-bialgebra coming from a minimal resolution of a Lie bialgebra PROP coincides with ours as shown explicitly in the work of Sergei Merkulov [20, Corollary 5.2] and also could be derived from works [18, 22] on Koszul PROPs, the Lie bialgebra PROP being one of them.

We use the Koszul sign convention: in a graded algebra whenever there is a change of places of two symbols there should be a corresponding sign. Throughout this paper, the summation convention is understood: indices $\alpha, \beta, \ldots$ once as superscript and once as subscript in a formula are to be summed over.

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2. Kosmann-Schwarzbach's big bracket

Treating exterior powers of a sum of a vector space with its dual as a super-Poisson algebra was pioneered in [13]. Then in 1991 Yvette Kosmann-Schwarzbach published an article [9] where the term big bracket was introduced in order to describe proto-bialgebras (a notion generalizing Lie bialgebras). This big bracket defines, in particular, the Lie structure of the governing dgLie algebra of Lie bialgebras (and, in fact, Lie algebras, Lie coalgebras and Lie quasibialgebras).

Here is the construction from [9] in a $\mathbb{Z}$-graded context.

On any $\mathbb{Z}$-graded space $X = \sum X^i$ there is an operation called de-suspension, mapping X to the space $X[1]$ so that $X[1]_i = X_{i+1}$. Let $\widetilde{V}$ be a finite dimensional k-vector space ($k = \mathbb{R}$ or $\mathbb{C}$), and $\widetilde{V}^*$ its dual. A non-graded space $\widetilde{V}^* \oplus \widetilde{V}$ could be considered as a graded one by assigning a 0-degree to each of its elements. Consider $V^* \oplus V = (\widetilde{V}^* \oplus \widetilde{V})[1]$, this means that "points" of $V^* \oplus V$ are in degree 1. The purpose of this shift of the degree is to see symmetric powers of the space $\widetilde{V}^* \oplus \widetilde{V}$ as exterior powers of the shifted space $V^* \oplus V$.

Algebraic functions on $\widetilde{V} \oplus \widetilde{V}^*$ are

(2.1)
$$B = \Lambda(V^* \oplus V) = \oplus_{j \geq -2} B^j, \quad \text{where for } B^j = \oplus_{p+q=j}(\Lambda^{p+1}V^* \otimes \Lambda^{q+1}V),\ j \geq -1, \quad \text{and } B^{-2} = k.$$

From this point of view B is an algebra of exterior powers of the odd space $V^* \oplus V$.

Grading in B is given by the sum $p+q$. That is, B^j consists of terms of the diagonal j, with $j = p+q$, for $p, q \geq -1$, so that the first few terms are as follows:

$$\begin{aligned}
B^{-2} &= k,\\
B^{-1} &= V \oplus V^*,\\
B^{0} &= V\wedge V \oplus V^*\otimes V \oplus V^*\wedge V^*,\\
B^{1} &= V\wedge V\wedge V \oplus V^*\otimes V\wedge V \oplus V^*\wedge V^*\otimes V \oplus V^*\wedge V^*\wedge V^*.
\end{aligned}$$

Let us write the elements of B in the table, using the bigrading on the number of V and V^* involved:

p / q	-1	0	1	2
	$\cdots$			
2	$V^*\wedge V^*\wedge V^*$	$\cdots$		
1	$V^*\wedge V^*$	$V^*\wedge V^* \otimes V$	$\cdots$	
0	V^*	$V^* \otimes V$	$V^* \otimes V\wedge V$	$\cdots$
-1	k	V	$V\wedge V$	$V\wedge V\wedge V$ $\cdots$

Let $\langle\cdot,\cdot\rangle$ be the natural pairing of $\widetilde{V}$ and $\widetilde{V}^*$. We extend it to a symmetric form on $\widetilde{V} \oplus \widetilde{V}^*$ as follows: for $x, y \in \widetilde{V}$ and $v, w \in \widetilde{V}^*$:

$$\langle x+v, y+w\rangle = \langle x, w\rangle + \langle v, y\rangle.$$

This symmetric form on $\widetilde{V} \oplus \widetilde{V}^*$ could be considered as an antisymmetric form on the de-suspended space $V \oplus V^*$. Moreover, on B it gives a Lie algebra structure by the following

Definition 2.1. *The big bracket is the graded Lie algebra structure on algebraic functions on $V \oplus V^*$ defined as follows.*

- *For $u, v \in B^{-2} \oplus B^{-1} = k \oplus V \oplus V^*$:*

$$[u,v] = \begin{cases} \langle u, v\rangle & \text{if} \quad u \in V \oplus V^* \text{ and } v \in V \oplus V^* \\ 0 & \text{if} \quad u \in k \quad \text{or} \quad v \in k \end{cases}$$

- *The bracket on other terms is defined by linearity and the graded Leibniz rule: for $u \in B^k,\ v \in B^l,\ w \in B^m$*

$$[u, v \wedge w] = [u,v] \wedge w + (-1)^{kl} v \wedge [u,w]$$

Remark 2.1. 1) $[u,v] = -(-1)^{kl}[v,u]; \quad u \in B^k,\ v \in B^l$, *that is the big bracket is skew-symmetric in the graded sense;*
2) $[\cdot,\cdot] : B^i \wedge B^j \to B^{i+j}$, *in other words: the bracket is of degree zero;*
3) *in particular,*

$$[\cdot,\cdot] : \ (\Lambda^k V^* \otimes \Lambda^l V) \ \wedge \ (\Lambda^m V^* \otimes \Lambda^n V) \longrightarrow \Lambda^{k+m-1} V^* \otimes \Lambda^{l+n-1} V;$$

4) B^0 *is a Lie subalgebra of* B;
5) B^{-2} *is the center of* $(B, [\cdot,\cdot])$.

3. Governing graded Lie algebras and Maurer-Cartan equations

Graded Lie algebra B has several graded Lie subalgebras. Verification that they are indeed Lie subalgebras is easy using Remark 2.1 (3). Some of these Lie subalgebras describe well known structures as it was described in [9]. We put them in the following table containing the $-1, 0$ and the first graded components of the governing Lie algebras:

$\mathfrak{g}$ - Lie subalgebra of B	$\mathfrak{g}^{-1}$	$\mathfrak{g}^0$	$\mathfrak{g}^1$	Solutions of Maurer-Cartan equation
$\mathcal{C}$: row $p = 0$	$u^* \in V^*$	$f \in V^* \otimes V$	$c \in V^* \otimes V \wedge V$	$[c,c] = 0$ Lie coalgebra structure on V
$\mathcal{L}$: column $q = 0$	$u \in V$	$f \in V^* \otimes V$	$l \in V^* \wedge V^* \otimes V$	$[l,l] = 0$ Lie algebra structure on V
$\mathcal{B}$: $p, q \neq -1$	—	$f \in V^* \otimes V$	$c \in V^* \otimes V \wedge V$ $l \in V^* \wedge V^* \otimes V$	$[c+l, c+l] = 0$ Lie bialgebra structure on V
$\mathcal{QB}$: $q \neq -1$	$u \in V$	$f \in V^* \otimes V$ $g \in V \wedge V$	$c \in V^* \otimes V \wedge V$ $l \in V^* \wedge V^* \otimes V$ $\phi \in V \wedge V \wedge V$	$[c+l+\phi, c+l+\phi] = 0$ Lie quasi-bialgebra structure on V

Let ΛV denote the sum $k \oplus V \oplus \Lambda^2 V \oplus \Lambda^3 V \oplus \cdots$. Then $\mathcal{C} = V^* \otimes \Lambda V \simeq Hom(V,\ \Lambda V)$, and $\mathcal{L} = \Lambda V^* \otimes V \simeq Hom(\Lambda V,\ V)$. In other words,

Proposition 3.1. *On a vector space V*

- **Coalgebras Lie:** $\mathcal{C} = V^* \otimes (\oplus_{l\geq 0}\Lambda^l V)$ *is the governing graded Lie algebra of Lie coalgebra structures on V. In particular, an element c of degree 1 such that $[c,c]=0$ defines a Lie coalgebra structure $\delta \in Hom(V, V\wedge V)$ as follows: $\delta(x) = [c,x]$.*
- **Algebras Lie:** $\mathcal{L} = \oplus_{k\geq 0}\Lambda^k V^* \otimes V$ *is the governing graded Lie algebra of Lie algebra structures on V. In particular, an element of degree $1, l$ such that $[l,l]=0$ defines a Lie algebra structure $\{\cdot,\cdot\} \in Hom(V\wedge V, V)$ as follows: $\{x,y\} = [[l,x],y]$.*
- **Lie bialgebras:** $\mathcal{B} = \oplus_{k\geq 1, l\geq 1}\Lambda^k V^* \otimes \Lambda^l V$ *is the governing graded Lie algebra of Lie bialgebra structures on V. In particular, elements of degree 1, $\theta \in V^* \otimes V\wedge V$ and $l \in V^*\wedge V^* \otimes V$ such that $[c+l, c+l] = 0$ define a Lie bialgebra structure with the cobracket $\delta(x) = [c,x]$ and the bracket $\{x,y\} = [[l,x],y]$.*
- **Lie quasi-bialgebras:** $\mathcal{Q}B = \oplus_{k\geq 0, l\geq 0}\Lambda^k V^* \otimes \Lambda^l V$ *is the governing graded Lie algebra of Lie quasibialgebra structures on V. In particular, elements of degree 1, $c \in V^* \otimes V\wedge V$, $l \in V^*\wedge V^* \otimes V$ and $\phi \in V\wedge V\wedge V$ such that*

$$[c+l+\phi, c+l+\phi] = 0 \tag{3.1}$$

define a Lie quasibialgebra structure with the 1-cocycle $\delta(x) = [c,x]$, the Lie bracket $\{x,y\} = [[l,x],y]$ and the 3-tensor ϕ.

Proof. Let us write out in coordinates the condition for $\delta \in Hom(V, V\wedge V)$ to be a Lie cobracket. Let $\{e_i\}$ constitute a basis in V, and $\{e^i\}$ the dual basis in V^*. Then a Lie cobracket with structure coefficients c_k^{ij} :

$$\delta(e_k) = c_k^{ij} e_i e_j = [c_l^{ij} e_i e_j e^l, e_k]$$

(we omit the signs of $\wedge$ and $\otimes$). The condition $[c,c]=0$ thus becomes a condition on the structure coefficients c_k^{ij} of a Lie coalgebra:

$$< c_k^{ij} e_i e_j e^k, c_k^{ij} e_i e_j e^k >= (c_m^{ij} c_l^{mk} + c_m^{jk} c_l^{mi} + c_m^{ki} c_l^{mj}) e_i e_i e_k e^l = 0,$$

which is exactly the co-Jacobi identity on $\delta = ad_c$, where $c = c_l^{ij} e_i e_j e^l$.

Similarly, in the case of a Lie algebra structure, the condition $[l,l]=0$ is exactly the Jacobi identity.

The commutator $[l+c, l+c]$ lies in $\Lambda^3 V^* \otimes V \oplus V^* \otimes \Lambda^3 V \oplus$ and $\Lambda^2 V^* \otimes \Lambda^2 V \subset B$. Then $l+c$ defines a Lie bialgebra structure if the commutator is 0, that is all three components of it in B^2 must be equal to 0. This leads to three axioms of a Lie bialgebra:

- **a:** Jacobi identity follows from $[l,l]=0$, for the derived bracket given by $\{x,y\} = [[l,x]y]$.
- **b:** Co-Jacobi identity follows from $[c,c]=0$.
- **c:** The cocycle condition translates as: $[l,c]=0$.

In fact, a presentation of a Lie bialgebra structure as a square zero element in $\Lambda(V^*\oplus V)$ appeared first in [15] (see also [21] for the idea of a derived bracket involved).

In the same manner the equation 3.1 gives independent equations: $[l,l]=0$, $[c,c]+2[l,\phi]=0$, $[c,\phi]=0$ and $[c,l]=0$, which give the axioms of a Lie quasibialgebra.

One could look for complete proofs in [9]. □

In all the cases of Proposition 3.1 we get a graded Lie algebra with a differential d given by the adjoint action of an auto-commuting element in the first degree. Hence $\mathcal{C}, \mathcal{L}, \mathcal{B}$ and $\mathcal{QB}$ become dgLie algebras. On a dgLie algebra one can define a derived bracket. Here is a general fact:

Proposition 3.2. [10] *A differential graded Lie algebra $(\mathfrak{g}, [\cdot,\cdot], D)$ gives rise to a new bracket of degree $+1$, called a derived bracket: for $a, b \in \mathfrak{g}$, $\{a,b\} = [Da, b]$. This is a Loday-Leibniz algebra* *bracket in the sense of* [16]*. If W is an Abelian subalgebra of $\mathfrak{g}$, such that $[DW,W]$ is in W, then $\{\cdot,\cdot\}$ is a Lie bracket on W. Moreover, $D:\ (W,\{,\})\to\ (\mathfrak{g},[,])$ is a Lie algebra morphism.*

In our case, the algebra $\mathfrak{g}$ is one of $\mathcal{C},\mathcal{L},\mathcal{B}$ and $\mathcal{QB}$, while the subalgebra Lie W in all cases is the same $B^{-1}=V^*\oplus V$. The derived bracket with corresponding differentials on $\mathcal{C},\mathcal{L},\mathcal{B}$ will define the Lie bracket on $V^*\oplus V$, leading to Manin triples (we could see a Lie structure on V as a particular case of a Manin triple with a zero cobracket, and a Lie coalgebra structure - a Manin triple with zero Lie bracket). In the same way, we get a marked Manin pair from $\mathcal{QB}$.

However, we need a certain refinement of Proposition 3.2, since the Lie subalgebra B^{-1} is not Abelian.

Proposition 3.3. *Differential graded Lie algebra $(B,\ [\cdot,\cdot],\ d)$, gives rise to a derived bracket of degree $1:\ \{a,b\}=[da,b]$.*

This derived bracket restricted to B^{-1} is a Lie bracket and $d:\ (B^{-1},\{\cdot,\cdot\})\to\ (B^0,[\cdot,\cdot])$ is a Lie algebra morphism.

Proof. Notice that $\{B^{-1},B^{-1}\}\subset B^{-1}$. The subspace $B^{-1}=V^*\oplus V$ is not an Abelian Lie subalgebra of $(B,[\cdot,\cdot])$, and we cannot use Proposition 3.2. However, since the bracket $[\cdot,\cdot]$ on B^{-1} takes values in the center of B (namely, $B^{-2}=k$), the Jacobi identity on B gives us the skew-symmetry of the derived bracket $\{\cdot,\cdot\}$. Hence the derived bracket defines a Lie algebra structure (not just Loday's) on $V^*\oplus V$.

The Lie algebra morphism part is immediate: $d\{a,b\}=d[da,b]=[da,db]$. □

Depending on the terms involved in d we get several known cohomology theories:

Proposition 3.4. *Let us take the Lie algebra $B=\oplus_{p\geq-1,q\geq-1}\ \Lambda^{p+1}V^*\otimes\Lambda^{q+1}V$ with a differential*

$$d=ad_{l+c+\phi+\psi},\quad d\circ d=0,\quad l\in\Lambda^2V^*\otimes V,\ c\in V^*\otimes\Lambda^2V,\ \phi\in\Lambda^3V,\ \psi\in\Lambda^3V^*.$$

Cohomology of the differential graded Lie algebra $(B,\ [\cdot,\cdot],\ d)$ gives rise to

- *the Lie algebra cohomology defined with the differential $d=ad_l$, on its Lie subalgebra*
$$\mathcal{L}=\oplus_{p\geq0}\ \Lambda^{p+1}V^*\otimes V$$
($c=0,\ \phi=0,\ \psi=0$),
- *the Lie coalgebra cohomology for $d=ad_c$, on the Lie subalgebra*
$$\mathcal{C}=\oplus_{q\geq0}\ V^*\otimes\Lambda^{q+1}V$$
($l=0,\ \phi=0,\ \psi=0$),
- *the Lie bialgebra cohomology for $d=ad_{l+c}$ on the Lie subalgebra*
$$\mathcal{B}=\oplus_{p+q\geq0}\ \Lambda^{p+1}V^*\otimes\Lambda^{q+1}V$$
($\phi=0,\ \psi=0$),
- *the Lie quasi-bialgebra cohomology for $d=ad_{l+c+\phi}$ on the Lie subalgebra*
$$\mathcal{Q}B=\oplus_{p\geq-1,q\geq0}\ \Lambda^{p+1}V^*\otimes\Lambda^{q+1}V$$
($\psi=0$).

Finally, to make explicit the connection to Manin pairs and triples we have the following

Proposition 3.5. *Let $l\in\Lambda^2V^*\otimes V,\ c\in V^*\otimes\Lambda^2V,\ \phi\in\Lambda^3V$. Then $(V\oplus V^*,\ \{a,b\}=[da,b])$, for*

- *$d=ad_{l+c}$, with a condition $[l+c,l+c]=0$ defines a Manin triple $(V\oplus V^*,V,V^*)$.*
- *$d=ad_{l+c+\phi}$, satisfying $[l+c+\phi,l+c+\phi]=0$ defines a Manin pair $(V\oplus V^*,V)$.*

4. L_∞ STRUCTURES

In the previous section we have seen that a Lie algebra structure on a space V is obtained as a derived bracket defined on the graded Lie algebra $\mathcal{L}$ with a differential given by an adjoint action of an element from $V^* \wedge V^* \otimes V \subset B^1$ whose bracket with itself is 0.

If we take an element from $\mathcal{L} = \oplus_{k=0}^{\infty} \Lambda^k V^* \otimes V$ (not just from $V^* \wedge V^* \otimes V$) whose bracket with itself is 0 we could define an L_∞ structure on V using higher derived brackets of Th. Voronov and Akman-Ionescu ([24, 1]).

The same idea works for other Lie subalgebras of B : an autocommuting element from $\mathcal{B}$ (or $\mathcal{QB}$) defines an L_∞ bialgebra (or L_∞ quasi-bialgebra) structure. Here we develop this theory.

A certain subtlety here is in defining what an element of degree 1 would mean in this context. For that we need to introduce a new notion of degree.

4.1. **Degree.** A starting point of any homotopy construction is a graded vector space $(W = \oplus_a W_a, \delta)$, with a differential of degree 1, is in other words a complex

$$\cdots \xrightarrow{\delta} W_a \xrightarrow{\delta} W_{a+1} \xrightarrow{\delta} \cdots . \tag{4.1}$$

Let the space V be graded. Then the space $B = \oplus_{p,q \geq -1}(\Lambda^{p+1} V^* \otimes \Lambda^{q+1} V)$, as well as the space of maps $\Lambda V \to \Lambda V$, inherit the grading from V in a consistent way. We will define a grading and a differential on these spaces as follows.

Lemma 4.1. *Let $(V = \oplus V_a, \delta)$ be a differential graded space over an ungraded field $\mathbb{K}$. Assume that each V_a is finite dimensional. Then on the dual space V^* there is a corresponding grading with the opposite sign and a differential (also denoted δ) of degree 1, $\delta : {V^*}_{-(a+1)} \to {V^*}_{-a}$:*

$$\cdots \xleftarrow{\delta} V^*_{-a} \xleftarrow{\delta} V^*_{-(a+1)} \xleftarrow{\delta} \cdots \tag{4.2}$$

Proof. If v is in V_a and $u = \delta v$ then their dual elements $v^*, u^* \in V^*$ are related as follows: $v^* = \delta u^*$. If u has a degree a then degree of $v = \delta u$ is $a+1$, while degree of v^* should be $-(a+1)$ and of u^* then is of $-a$. This relation is obtained from the condition:

$$[v^*, v] = 1 = [v^*, \delta u] = [\delta v^*, u] = [u^*, u]. \tag{4.3}$$

If $\delta : V \to V$ given by an adjoint action of a corresponding element $d \in V^* \otimes V$, $\delta = ad_d : V \to V$ or $ad_d : V^* \to V^*$ then the condition $[v^*, \delta u] = [\delta v^*, u]$ from 4.3 is just a consequence of the Jacobi identity and a fact that constants are in the center. □

This way $V^* \oplus V$ becomes a complex. We could extend the action of δ on $\Lambda^p V^* \otimes \Lambda^q V$ for any positive p, q by the Leibniz rule, so that the whole B becomes a complex.

Let us define a new grading on the complex B, taking into account the grading on V :

Definition 4.1. *Given a graded vector space $V = \oplus_a V_a$, consider elements from $\Lambda^p V^* \otimes \Lambda^q V$. The degree of elements from V^* is defined according to Lemma 4.1, and $\Lambda^0 V = \Lambda^0 V^* = \mathbb{K}$.*

Then the internal degree of $x_1^* x_2^* ... x_p^* y_1 y_2 ... y_q \in \Lambda^p V^* \otimes \Lambda^q V$ is $\sum \widetilde{x_i^*} + \sum \widetilde{y_i} - (p+q-2)$ where the degree of each element is denoted by $\widetilde{\ }$.

A map $\tau_{pq} : \Lambda^p V \to \Lambda^q V$ sending $v^1 v^2 ... v^p$ to $u^1 u^2 ... u^q$ has the internal degree n if $(\sum \widetilde{u_i}) - (\sum \widetilde{v_j}) = n - (p+q-2)$, which corresponds to the internal degree of $v_1^* v_2^* ... v_p^* u^1 u^2 ... u^q \in \Lambda^p V^* \otimes \Lambda^q V$, where v_i^* is defined by the following property: $[v_i^*, v^i] = 1$.

Let us define the solutions of the Maurer-Cartan equation of internal degree 1 in B. The correction of $(p+q-2)$ takes into account that elements entering the Maurer-Cartan equation are no longer necessarily from B^1 (that is when $p+q=3$) but from B^{p+q-2} for any p and q.

4.2. L_∞ **algebra structure from the multiple adjoint action.** Let us recall the definition of L_∞-algebras.

Definition 4.2. *Consider a map $\lambda_k \in Hom(\Lambda^k V, V)$. This map λ_k acts on $\Lambda^n V$ for any $n \in \mathbb{N}$ by coderivation of the unshuffle coproduct on the algebra of exterior powers of V :*

(4.4)
$$\lambda_k(v_1 \wedge \cdots \wedge v_n) = \begin{cases} 0, & \text{if } k > n \\ \sum_{\sigma \in Sh_n^k} (-1)^{sgn\sigma} \lambda_k(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)}) \wedge v_{\sigma(k+1)} \cdots \wedge v_{\sigma(n)}, & \text{otherwise.} \end{cases}$$

The set Sh_n^k is a set of all k-unshuffles in the permutation group of n elements, that is all permutations such that $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(n)$.

Definition 4.3. *An L_∞-structure on a graded space V is a set of maps of internal degree* 1,

$$\lambda_k : \Lambda^k V \to V[2-k], \; k \in \mathbb{N}$$

such that the following generalized form of the Jacobi identity is satisfied for any $n \geq 2$:

$$\sum_{k=1}^{n-1} [\lambda_k, \lambda_{n-k}] = 0. \tag{4.5}$$

The bracket $[\cdot,\cdot]$ here is the commutator of λ_k and λ_{n-k} considered as operators on ΛV. These equations are higher Jacobi identities and can be summarized in one equation:

$$\lambda^2 = 0, \; \text{where } \lambda = \sum_{k \geq 1} \lambda_k. \tag{4.6}$$

For a finite dimensional space V there is the following isomorphism $\mathrm{Hom}(\Lambda^k V, \Lambda^l V) \simeq \Lambda^k V^* \otimes \Lambda^l V$. Any operator $\tau_{kl} \in \mathrm{Hom}(\Lambda^k V, \Lambda^l V)$ is represented by $t_{kl} \in \Lambda^k V^* \otimes \Lambda^l V$ and a passage from t_{kl} to τ_{kl} could be made explicit by introducing the multiple adjoint action.

Definition 4.4. *Consider an element $t_{kl} \in \Lambda^k V^* \otimes \Lambda^l V$.* Multiple adjoint action *of t_{kl} on $\Lambda^n V$ is defined as follows. For $n = k$ it defines a higher derived bracket as in* [1, 23]*:*

$$\tau_{kl}(v_1 \wedge \cdots \wedge v_k) = m\boldsymbol{ad}\,(t_{kl})(v_1 \wedge \cdots \wedge v_k) = [[\cdots[[t_{kl}, v_1], v_2]\cdots], v_k],$$

thus defining a map $\Lambda^k V^* \otimes \Lambda^l V \to Hom(\Lambda^k V, \Lambda^l V)$: $t_{kl} \mapsto \tau_{kl}$ More generally, we define an action of t_{kl} on any exterior power of V, with values in $\Lambda V^* \otimes \Lambda V$ as follows

(4.7)
$$m\boldsymbol{ad}\,(t_{kl}) \quad (v_1 \wedge \cdots \wedge v_n) = \begin{cases} \underbrace{[[\cdots[[}_{k-n \text{ times}} t_{kl}, v_1], v_2]\cdots], v_n], & \text{if } k \geq n \\ \sum_{\sigma \in Sh_n^k} (-1)^{sgn\sigma} \underbrace{[\cdots[}_{k \text{ times}} t_{kl}, v_{\sigma(1)}], v_{\sigma(2)}]\cdots], v_{\sigma(k)}] \wedge v_{\sigma(k+1)} \cdots \wedge v_{\sigma(n)}, & \text{otherwise.} \end{cases}$$

The set of permutations Sh_n^k is as in Definition 4.2. For the case $n = k$ one should keep in mind that $\Lambda^0 V^* = \mathbb{K}$.

It is shown already in [24] that L_∞-algebra can be obtained as a particular case of this multiple adjoint action.

We now state two theorems which are L_∞ analogues of results of Proposition 3.2.

Theorem 4.1. (see [24]) *Consider an element $L \in \mathcal{L}$, $L = \sum_{k=1}^{\infty} l_k$, $l_k \in \Lambda^k V^* \otimes V$ of internal degree* 1, *such that $[L, L] = 0$. The set of maps $\lambda_k = m\mathbf{ad}\; l_k \in Hom(\Lambda^k V, V)$ form an L_∞-algebra.*

In other words, $[L, L] = 0$ implies that $\lambda^2 = 0$ where $\lambda = m\boldsymbol{ad}\, L$.

The proof is a direct albeit tedious computation reducing the condition on the maps λ_k to $[L, L] = 0$ using the fact that ΛV is an Abelian Lie subalgebra of $\mathcal{L}$ and also that $[L, v] \in \Lambda V^*$ for any $v \in \Lambda V^*$.

We laso use the following statement analogous to the one about the Lie algebra morphism in Proposition 3.2.

Definition 4.5. *An L_∞-morphism from an L_∞-algebra $(V, \sum \lambda_k)$ to a Lie algebra $(W, [\,,\,])$ is a sequence of maps $\phi_l : \Lambda^l V \to W$ of internal degree 0 such that we have the following equality, for all $n \geq 2$,*

$$\sum_{k+l=n} \phi_l \lambda_k = \sum_{k+l=n+1} [\phi_k, \phi_l].$$

Theorem 4.2. *Consider a graded space V with an L_∞-structure given by the multiple adjoint action of $L = \sum l_k,\ l_k \in \Lambda^k V^* \otimes V,\ [L, L] = 0$. Consider the graded Lie algebra structure on $V^* \otimes V$ with the bracket $[\,,\,]$ given by the natural pairing.*

Then maps

$$m\mathbf{ad}\ (l_{k+1}) : \Lambda^k V \to V^* \otimes V$$

define an L_∞ morphism from $(V, \sum m\mathbf{ad}\ (l_k)) \to (V^* \otimes V,\ [\,,\,])$.

Checking this proposition amounts just to writing the higher Jacobi identities (4.5) on V.

4.3. L_∞**-coalgebras.**

Definition 4.6. *An L_∞ coalgebra structure on V is a sequence of maps*

$$\delta_{1l} : V \to \Lambda^l V\ [2 - l],\ \ l \geq 1$$

of internal degree 1 such that

$$\sum_{p \geq 1, q \geq 1} \delta_{1p} \delta_{1q} = 0.$$

In other words,

$$\delta^2 = 0,\ \ where\ \delta = \sum_{p \geq 1} \delta_{1p}.$$

Consider the Lie algebra governing the Lie coalgebra structure on V : $\mathcal{C} = \oplus_{q \geq 0} V^* \otimes \Lambda^{q+1} V$. The adjoint action of $c_{1l} \in V^* \otimes \Lambda^l V$ on $\Lambda^n V$ is a map from $\Lambda^n V$ to $\Lambda^{n+l-1} V$:

$$ad_{c_{1l}}(v_1 \wedge \cdots \wedge v_n) = \sum_{i=1}^{n} (-1)^{sgn} [c_{1l}, v_i] \wedge v_1 \wedge \cdots \check{v}_i \cdots \wedge v_n.$$

Theorem 4.3. *Let*

$$C = \sum_{l=1}^{\infty} c_{1l},\ \ c_{1l} \in V^* \otimes \Lambda^l V$$

be such that $[C, C] = 0$. Then the adjoint action of C defines an L_∞-coalgebra structure on V.

Moreover, the multiple adjoint action of C on ΛV^* defines an L_∞-algebra structure on V^*, the dual space to V.

Proof. The first statement follows from the definition of an L_∞ coalgebra, while an L_∞ structure on V^* is a consequence of Theorem 4.1. □

4.4. L_∞**-bialgebras.** Following the general philosophy of structures given by solutions of Maurer-Cartan equations on differential graded Lie algebras, we consider solutions of an equation $[Q, Q] = 0$ where Q belongs to Lie subalgebras of B : $\mathcal{C}, \mathcal{L}, \mathcal{B}$ and $\mathcal{QB}$ (3.4). They respectively define an L_∞-coalgebra, L_∞-algebra, L_∞-bialgebra and L_∞-quasibialgebra structures on V. In all cases we consider the multiple adjoint action 4.4 of corresponding Q on ΛV.

Let us give two new definitions: for a differential Lie bialgebra structure and a L_∞ bialgebra structure.

Definition 4.7. *An L_∞ bialgebra structure on V is a set of maps*

$$\tau_{kl} : \Lambda^k V \to \Lambda^l V[3 - (k + l)],\ \ k, l \geq 1$$

of internal degree 1 *such that each* $\tau_{kl} = m\boldsymbol{ad}\, t_{kl},\ t_{kl} \in \Lambda^k V^* \otimes \Lambda^l V$ *and*

$$\sum_{k+k'=p+1} \sum_{l+l'=q+1} [t_{kl}, t_{k'l'}] = 0, \tag{4.8}$$

for all $p \geq 2,\ q \geq 2$.

In other words, an element of $\mathcal{B}$ of internal degree 1 :

$$T = \sum_{k\geq 1,\ l\geq 1} t_{kl}, \qquad t_{kl} \in \Lambda^k V^* \otimes \Lambda^l V \text{ such that } [T, T] = 0. \tag{4.9}$$

defines an L_∞-bialgebra structure on V.

We see that T lives in the following bi-graded space:

k / l	1	2	3	4	
	$\cdots$				
4	$\Lambda^4 V^* \otimes V$	$\cdots$			
3	$\Lambda^3 V^* \otimes V$	$\Lambda^3 V^* \otimes \Lambda^2 V$	$\cdots$		
2	$\Lambda^2 V^* \otimes V$	$\Lambda^2 V^* \otimes \Lambda^2 V$	$\Lambda^2 V^* \otimes \Lambda^3 V$	$\cdots$	
1	$V^* \otimes V$	$V^* \otimes \Lambda^2 V$	$V^* \otimes \Lambda^3 V$	$V^* \otimes \Lambda^4 V$	$\cdots$

Let us look at the first few equations from (4.8). The first one is

$$[t_{11}, t_{11}] = 0, \tag{4.10}$$

providing the equation which defines a differential $d = \mathtt{ad}_{t_{11}}$ on V. The equations for $p = 1,\ q = 2$ and $p = 2,\ q = 1$ give respectively:

$$[t_{11}, t_{12}] = 0, \qquad [t_{11}, t_{21}] = 0, \tag{4.11}$$

which which gives the condition that d be a derivation of the cobracket and of the bracket. The equation for $p = 2,\ q = 2$:

$$[t_{11}, t_{22}] + [t_{12}, t_{21}] = 0 \tag{4.12}$$

shows that the bracket is a cocycle with respect to the cobracket up to homotopy. The homotopy is given by the element $t_{22} \in \Lambda^2 V^* \otimes \Lambda^2 V$.

The Jacobi identity for the bracket $m\mathbf{ad}\ (t_{21})$ also holds only up to homotopy; it is given by the equation for $p = 3, q = 1$:

$$[t_{11}, t_{31}] + \frac{1}{2}[t_{21}, t_{21}] = 0. \tag{4.13}$$

The co-Jacobi identity up to homotopy is the equation for $p = 1, q = 3$:

$$[t_{11}, t_{13}] + \frac{1}{2}[t_{12}, t_{12}] = 0. \tag{4.14}$$

Remark 4.1. 1) *The homology of V, $H^*(V, d = \mathtt{ad}\ t_{11})$ is a Lie bialgebra.*

2) *If V is not graded, that is all $V_n = \{0\}$ for all $n \neq 0$, then $V = V_0$ is an ordinary Lie bialgebra. Indeed, since t_{kl} is of internal degree 1, on an ungraded space only terms with k, l satisfying $1 - (k + l - 2) = 0$ survive. The result is that all t_{kl} except for t_{12} and t_{21} have to vanish. The Maurer-Cartan equation then gives the axioms of a Lie bialgebra from Definition 1.1: $\tau_{21} = m\mathbf{ad}\ t_{21}$ defines a bracket and $\tau_{12} = m\mathbf{ad}\ t_{12}$ a cobracket.*

3) *If all $t_{kl} = 0$ for all $k + l \geq 3$, then V is a differential graded Lie bialgebra (graded Lie bialgebra with a differential compatible with the bracket and the cobracket).*

4.5. **L_∞-quasi-bialgebra.** To have an L_∞-quasi-bialgebra structure we need to allow terms in $\Lambda^q V$. Hence the Maurer-Cartan equation of the previous subsection (4.9) becomes an equation on $S = \sum_{k\geq 0, l\geq 1} t_{kl},\ t_{kl} \in \Lambda^k V^* \otimes \Lambda^l V$ which differs from T because it contains elements $t_{0l} \in \Lambda^l V$. This S must satisfy a set of equations indexed by $p \geq 1, q \geq 2$:

$$[S, S] = \sum_{k+k'=p} \sum_{l+l'=q} [t_{kl}, t_{k'l'}] = 0,$$

	$\cdots$				
4	$\Lambda^4 V^* \otimes V$	$\cdots$			
3	$\Lambda^3 V^* \otimes V$	$\Lambda^3 V^* \otimes \Lambda^2 V$	$\cdots$		
2	$\Lambda^2 V^* \otimes V$	$\Lambda^2 V^* \otimes \Lambda^2 V$	$\Lambda^2 V^* \otimes \Lambda^3 V$	$\cdots$	
1	$V^* \otimes V$	$V^* \otimes \Lambda^2 V$	$V^* \otimes \Lambda^3 V$	$V^* \otimes \Lambda^4 V$	$\cdots$
0			$\Lambda^3 V$	$\Lambda^4 V$	$\cdots$
$(p+1)$ / $(q+1)$	1	2	3	4	

Terms t_{0l} do not act on ΛV, however they change the co-Jacobi condition and the other equations as well in comparison with the equations on T.

Moreover, terms $t_{01} \in V$ and $t_{02} \in V \wedge V$ change the nature of certain equations: for example, $[t_{11}, t_{11}] = 0$ is no longer true in the presence of t_{01}. Allowing non-zero terms t_{01} and t_{02} would lead us to a completely different setup of weak l_infty algebras and to avoid that we impose that these terms are 0.

Definition 4.8. *Consider an element of $\mathcal{QB}$ of internal degree* 1 :

$$S = \sum_{k\geq 1, l\geq 1} t_{kl} + \sum_{l\geq 3} t_{0l},\ t_{kl} \in \Lambda^k V^* \otimes \Lambda^l V, \tag{4.15}$$

such that

$$[S,S]=0. \tag{4.16}$$

An L_∞ quasi-bialgebra structure on V is the set of maps

$$\tau_{kl}:\Lambda^k V\to\Lambda^l V[3-(k+l)],\ k\geq 0,\ l\geq 1$$

of internal degree 1, where $\tau_{kl}=m\boldsymbol{ad}\, t_{kl}$.

Let us state properties of an L_∞-quasi-bialgebra similar to the properties stated in Remark 4.1:

Remark 4.2. 1) *The homology of $H^*V=H^*(V,\mathtt{ad}t_{11})$ is a Lie quasi-bialgebra.*

2) *If V is not graded then V is an ordinary Lie quasibialgebra. Indeed, since t_{kl} is of degree $2-(k+l-1)$ all t_{kl} except for t_{12},t_{21} and t_{03} have to vanish. Then $m\mathbf{ad}\ t_{12}$ defines a cobracket, $m\mathbf{ad}\ t_{21}$ a bracket and together with $t_{03}\in\Lambda^3V$ they satisfy the axioms of Lie quasibialgebras from Definition 1.2.*

3) *If all $t_{kl}=0$ for all $k+l\geq 3$ then V is a differential graded Lie quasi-bialgebra.*

Remark 4.3. *For L_∞-(quasi)bialgebras there is no analogue of Theorems 4.1, 4.3. The condition $[Q,Q]=0$ in general does not imply $(m\mathbf{ad}\ Q)^2=0$ for some $Q\in B$. Although it is true for Q either in $\mathcal{L}$ or in $\mathcal{C}$ in a more general case the equation $(m\mathbf{ad}\ Q)^2=0$ force higher order conditions on Q which are not necessarily satisfied even for an ordinary (not L_∞) Lie bialgebra or a Lie quasi-bialgebra.*

5. Formal geometry of L_∞-structures

We now move to a geometric definition of an L_∞-algebra structure on a finite dimensional space (see for example [2] or [19]). Consider a finite dimensional graded vector space $V=\oplus_k V_k$ with a chosen basis $\{e_\alpha\}$. Then a Lie algebra structure is defined by structure constants $c^\gamma_{\alpha,\beta}$:

$$[e_\alpha,e_\beta]=\sum c^\gamma_{\alpha,\beta}e_\gamma,$$

satisfying the structure equation which boils down to the Jacobi identity. We could also see the bracket acting from $V\wedge V$ to V as a differential operator $Q_{21}=c^\gamma_{\alpha,\beta}e_\gamma\otimes\dfrac{\partial}{\partial e_\alpha}\wedge\dfrac{\partial}{\partial e_\beta}$. We use the index 21 for this operator to underline that it is quadratic in $\dfrac{\partial}{\partial e}$'s and linear in e's.

There is the following natural identification of exterior powers of V with symmetric powers of $V[1]$

$$Sym^n(V[1])\simeq\Lambda^n(V)[n]. \tag{5.1}$$

In particular, under this identification a Lie bracket $\Lambda^2V\to V$ is a map of degree 1: $\Lambda^2V[2]\to V[2]$ or equivalently $Sym^2(V[1])\to(Sym(V[1]))[1]$. On the other hand, the dual of the free cocommutative coalgebra $Sym(V[1])$ could be identified with the algebra of formal power series on the $\mathbb{Z}$-graded space $V[1]$. We say that the algebra of formal power series defines a formal manifold. This way a Lie bracket becomes a degree 1 quadratic vector field on it. The dual space to $V[1]$ is $V^*[-1]$. Elements of $Sym(V^*[-1])$ are coordinate functions on $V[1]$. Let us chose corresponding coordinates $\{x^\alpha\}$, dual to e_α, such that in the space $V^*[-1]$ x^α has degree shifted by -1 : $\tilde{x}^\alpha-1$.

A very natural generalization of this construction is the definition of an L_∞-algebra structure:

Proposition 5.1. *(see for example [8]) An L_∞-algebra structure on a graded space V is given by vector field Q of degree 1 on the $\mathbb{Z}$-graded formal manifold corresponding to $V[1]$ such that $[Q,Q]=0$.*

Definition 5.1. *(following [23]) A non-zero self-commuting operator (and in particular a vector field) on a formal manifold is called homological.*

In coordinates, an L_∞-algebra structure is given by a homological vector field:

$$Q = b^\beta_\alpha x^\alpha \frac{\partial}{\partial x^\beta} + c^\gamma_{\alpha\beta} x^\alpha x^\beta \frac{\partial}{\partial x^\gamma} + d^\delta_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma \frac{\partial}{\partial x^\delta} + \cdots,$$
$$Q = \sum_{k=1}^{\infty} Q_{k1}, \qquad Q_{k1} = f^\beta_{\alpha_1\cdots\alpha_k} x^{\alpha_1} \cdots x^{\alpha_k} \frac{\partial}{\partial x^\beta}.$$

In fact, each term in Q corresponds to a map of degree 1 on the dual space:

$$Q = \sum Q_{k1},\ Q_{k1} : V^*[-1] \to Sym^k(V^*[-1])[1]. \tag{5.2}$$

Counting degrees after the identification (5.1) we get $Q_{k1} : V^*[-1] \to \Lambda^k V^*[-k][1]$ which on the dual space becomes a map $\Lambda^k V \to V[2-k]$ as in Definition 4.3.

This way we are naturally brought to a geometric definition of an L_∞-bialgebra structure on V. In the sum (5.2) we should consider not only $Q_{k1} : V^* \to Sym^k V^*$, but any $Q_{kl} : Sym^l V^* \to Sym^k V^*$. So it becomes a differential operator on $V[1]$, acting on functions on $V[1]$, which are elements of $Sym(V^*[-1])$.

To keep track of degrees, we introduce a formal variable $\hbar$ of degree -2. We consider the ring of differential operators on $V[1]$, weighted with $\hbar$:

$$\mathbb{A}^\hbar(V) := Sym(V^*[-1])[\xi^1, \cdots, \xi^n, \hbar]$$

where $\xi^\alpha = \hbar \dfrac{\partial}{\partial x_\alpha}$. Notice that $\xi^\alpha \in V[1][-2] = V[-1]$.

Let $Op_\hbar$ be a map from $Sym(V^*[-1] \oplus V[1]) \to \mathbb{A}^\hbar(V)$ defined as follows:

$$\begin{gathered} Op_\hbar(v^1 \cdots v^p w^*_1 \cdots w^*_q) = \frac{1}{\hbar}(v^1 \cdots v^p \hbar \frac{\partial}{\partial w_1} \cdots \hbar \frac{\partial}{\partial w_p}) \\ \text{where } v^i \in V^*[-1],\ w^*_j \in V[1], \text{ and } \ < w^j, w^*_j >= 1. \end{gathered} \tag{5.3}$$

On $\mathbb{A}^\hbar(V)$ there is a Poisson bracket. For any two elements $E, F \in \mathbb{A}^\hbar(V)$ it could be defined in coordinates as follows:

$$\{E, F\} = \hbar \left(\frac{\partial E}{\partial x^\alpha} \frac{\partial F}{\partial \xi_\alpha} - (-1)^{\tilde{E}\tilde{F}} \frac{\partial F}{\partial x^\alpha} \frac{\partial E}{\partial \xi_\alpha} \right)$$

Proposition 5.2. *Any L_∞-bialgebra structure on V is given by an operator on the corresponding formal manifold, acting on $SymV^*[-1]$, such that*

- *the Poisson bracket of its image in $\mathbb{A}^\hbar(V)$ with itself is 0,*
- *its image in $\mathbb{A}^\hbar(V)$ is of degree 1.*

Proof. Consider an operator $Q = \sum Q_{pq}$, where $Q_{pq} = v^1 \cdots v^p w^*_1 \cdots w^*_q$. Its image in $\mathbb{A}^\hbar(V)$ being of degree 1 gives the following equality: $\sum(\tilde{v}_i + \tilde{\xi}_j) - \tilde{\hbar} = 1$. (the degree of $\dfrac{1}{\hbar}$ is $-\tilde{\hbar}$).

Such an operator acts as follows:

$$Q_{pq} : Sym^p(V^*[-1][2])[-2] \to Sym^q(V^*[-1])[1]$$

The image of Q_{pq} in $\mathbb{A}^\hbar(V)$ given by (5.3) has the degree $2 + \sum_{i=1}^p (\tilde{v}_i - 1) + \sum_{j=1}^q (\tilde{w}_j + 1 - 2) = \sum(\tilde{v}_i + \tilde{w}_j) + 2 - (p+q)$.

Thus, after the identification (5.1) we get that $Q_{pq} : \Lambda^p V^* \to \Lambda^q V^*[1][2-(p+q])$ which on the dual space gives $\Lambda^q V[p+q-3] \to \Lambda^p V$ leading to the right degrees.

The Poisson bracket of Q with itself being 0 corresponds to the self-commuting condition in (4.9). □

6. Manin L_∞-triple and L_∞-pair

We give a definition of a Manin L_∞-triple (which also could be called a strong homotopy Manin triple), a natural generalization of a Manin triple. For this we need a notion of an L_∞-subalgebra:

Definition 6.1. *Let (W, Q) be an L_∞-algebra. Then a subspace $V \subset W$ is an L_∞-subalgebra if V is Q-invariant.*

It means that V is an L_∞-subalgebra of W if the image of the operator Q acting on $V^*[-1]$ is in $SymV^*[-1][1]$. In other words it means that $Q = \sum_{k,l\geq 1} Q_{k1}$ restricted to V consists of maps $V^* \to \Lambda^k V^*[2-k]$ or, on the dual, it gives an L_∞ structure on V, $Q_{k1} : \Lambda^k V \to V[2-k]$.

Definition 6.2. *A finite dimensional Manin L_∞-triple is a triple of finite dimensional L_∞-algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ equipped with a nondegenerate bilinear form $<,>$ such that*

- *$\mathfrak{g}_+, \mathfrak{g}_-$ are L_∞-subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as a vector space;*
- *$\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic with respect to $<,>$;*
- *the n-brackets constituting the L_∞-algebra structure $\lambda_n : \Lambda^n(V^* \oplus V) \to V^* \oplus V$ are invariant with respect to the bilinear form $<,>$, that is*

$$< \lambda_n(v_1 \wedge \cdots \wedge v_n), v_0 >= (-1)^{\widetilde{v_n}\widetilde{v_0}} < \lambda_n(v_1 \wedge \cdots \wedge v_{n-1} \wedge v_0), v_n > \tag{6.1}$$

Notice that a Manin triple is an example of a Manin L_∞-triple.

The invariance is in fact cyclic, since λ_n (being a map on $\Lambda^n V$) is antisymmetric in all variables.

Theorem 6.1. *The notions of a (finite dimensional) Manin L_∞-triple and an L_∞-bialgebra are equivalent.*

Proof. Consider a Manin L_∞-triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$. Let us show that there is an L_∞-bialgebra structure on $\mathfrak{g}_+$. Let the L_∞-algebra structure on $\mathfrak{g}$ be given by maps $\lambda_k : \Lambda^k \mathfrak{g} \to \mathfrak{g}$, that is $\lambda_k : \Lambda^k(\mathfrak{g}_+ \oplus \mathfrak{g}_-) \to (\mathfrak{g}_+ \oplus \mathfrak{g}_-)$. These maps can be split as follows:

$$\lambda_{mn+} : \Lambda^m \mathfrak{g}_+ \otimes \Lambda^n \mathfrak{g}_- \to \mathfrak{g}_+ \text{ and } \lambda_{mn-} : \Lambda^m \mathfrak{g}_+ \otimes \Lambda^n \mathfrak{g}_- \to \mathfrak{g}_- \quad \text{with } m + n = k.$$

Using non-degeneracy and invariance (6.1) of the internal product $<,>$ after an identification of $\mathfrak{g}_-^*$ with $\mathfrak{g}_+$, we get the equivalences

$$\begin{aligned} &\mathrm{Hom}_{inv}(\Lambda^m \mathfrak{g}_+ \otimes \Lambda^n \mathfrak{g}_-, \mathfrak{g}_+) \simeq \mathrm{Hom}(\Lambda^m \mathfrak{g}_+, \Lambda^{n+1} \mathfrak{g}_+) \simeq \Lambda^m \mathfrak{g}_+^* \otimes \Lambda^{n+1} \mathfrak{g}_+ \quad \text{and} \\ &\mathrm{Hom}_{inv}(\Lambda^m \mathfrak{g}_+ \otimes \Lambda^n \mathfrak{g}_-, \mathfrak{g}_-) \simeq \mathrm{Hom}(\Lambda^{m+1} \mathfrak{g}_+, \Lambda^n \mathfrak{g}_+) \simeq \Lambda^{m+1} \mathfrak{g}_+^* \otimes \Lambda^n \mathfrak{g}_+. \end{aligned}$$

Let $l_{mn+} \in \Lambda^m \mathfrak{g}_+^* \otimes \Lambda^{n+1} \mathfrak{g}_+$ correspond to λ_{mn+}. This way, l_{mn+} would act by a multiple adjoint action:

$$m\mathbf{ad}\ (l_{mn+}) : \Lambda^m \mathfrak{g}_+ \to \Lambda^{n+1} \mathfrak{g}_+.$$

On the other hand $\lambda_{mn-} : \Lambda^m \mathfrak{g}_+ \otimes \Lambda^n \mathfrak{g}_- \to \mathfrak{g}_-$ again is given by a multiple adjoint action of an element $l_{mn-} \in \Lambda^{m+1} \mathfrak{g}_+^* \otimes \Lambda^n \mathfrak{g}_+$ and

$$m\mathbf{ad}\ (l_{mn-}) : \Lambda^{m+1} \mathfrak{g}_+ \to \Lambda^n \mathfrak{g}_+.$$

Now, $L = \sum_{m,n\geq 1}(l_{mn+} + l_{mn-})$ satisfies $[L, L] = 0$ from the condition on λ.

In the other direction it is actually easier. Let an L_∞-bialgebra structure on V be defined by the multiple adjoint action of

$$T = \sum_{k\geq 1, l\geq 1} t_{kl}, \quad \text{where } t_{kl} \in \Lambda^k V^* \otimes \Lambda^l V.$$

Then the corresponding Manin L_∞-triple is $(V \oplus V^*, V, V^*)$ with the nondegenerate form given by the natural pairing of V and V^*.

The L_∞ structure on $V^* \oplus V$ is given by $\tau = m\mathbf{ad}\ (T)$. In

$$m\mathbf{ad}\ (t_{kl}) : (V^* \oplus V) \to \Lambda^{k-1}V^* \otimes \Lambda^l V \oplus \Lambda^k V^* \otimes \Lambda^{l-1}V$$

we recognize the L_∞-coalgebra structure dual to an L_∞-algebra structure we are looking for.

To get the L_∞-subalgebra structure on V we consider the quadratic equations (4.8) for $q = 2$. We get:

$$\sum_{k+k'=p} t_{k1} t_{k'1} = 0,$$

the set of equations defining an L_∞-structure on V (by taking $t_{k1} = l_k$ in (4.1)).

The operator $\tau_V = \sum_{k\geq 1} \mathtt{ad}(t_{k1}) : V^* \to \Lambda V^*$ is the restriction of τ to V^* making V an L_∞-subalgebra of $V^* \oplus V$, since T_V is self-commuting.

In the same way, elements $t_{1l} \in V^* \otimes \Lambda^l V$ satisfy the following quadratic equation (equations (4.8) for $p = 2$):

$$\sum_{l+l'=q} t_{1l} t_{1l'} = 0$$

therefore they induce maps $m\mathbf{ad}\ (t_{1l}) : V \to \Lambda^l V$, on V thus defining an L_∞-algebra structure on V^*. □

Remark 6.1. *In* [3, Corollary 2] *it is noticed that an odd self-commuting element from B defines an L_∞ structure on ΛV (as well as on ΛV^*) by the multiple adjoint action.*

The notion which leads to an L_∞-quasi-bialgebra is a Manin L_∞-pair (strong homotopy Manin pair):

Definition 6.3. *A finite dimensional Manin L_∞-pair is a pair of finite dimensional L_∞-algebras $\mathfrak{g} \supset \mathfrak{g}_+$ equipped with a nondegenerate invariant (6.1) bilinear form $<,>$, such that $\mathfrak{g}_+$ is an isotropic L_∞-subalgebra of $\mathfrak{g}$.*

A Manin L_∞-quasi-triple is a Manin L_∞-pair $(\mathfrak{g}, \mathfrak{g}_+)$ with a chosen Lagrangian complement of $\mathfrak{g}_+$.

There is also a correspondence like in Theorem 6.1 with a similar proof:

Theorem 6.2. *The notions of a (finite dimensional) Manin L_∞-quasi-triple and an L_∞-quasi-bialgebra are equivalent.*

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Departement de mathématiques, Université Lyon 1, 69622 Villeurbanne Cedex, France
E-mail address: ok@alum.mit.edu