Modified Hochschild and Periodic Cyclic Homology

Nicolae TELEMAN

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Décembre 2006

IHES/M/06/59

Modified Hochschild and Periodic Cyclic Homology.

Nicolae Teleman

December 4, 2006

1 Abstract

The Hochschild and (periodic) cyclic homology of the algebra of continuous functions on a smooth manifold are trivial, see Connes [1], [2]. In this paper we create an analogue of the Hochschild and periodic cyclic homology which gives the right result (i.e. Alexander-Spanier co-homology) when applied onto the algebra of continuous functions on smooth manifolds. This will be realized by replacing the Connes periodic bi-complex (b, B), see Connes [1], [2] and Loday [5], by the bi-complex $(\tilde{b}, d)$, where the operator $\tilde{b}$ is obtained by blending the Hochschild boundary b with the Alexander-Spanier boundary d; the operator $\tilde{b}$ anti-commutes with the operator d.

In order to reach this objective, as in the classical case, one has to consider the Alexander-Spanier complex of *germs* of functions. As the notion of germ is a locality notion, our procedure will apply to topological algebras.

More precisely, we show that the modified periodic cyclic homology of the algebra of continuous functions on a smooth manifold is the ordinary Z_2-graded de Rham co-homology of the manifold.

The problem of producing a tool able to extract the correct homology from the algebra of continuous functions was addressed before by Puschnigg [8].

2 Introduction

The Hochschild complex has the critical limitation that in the case of the C^*-algebra of continuous functions the Hochschild homology vanishes in positive degrees. The purpose of the present paper is to correct this deficiency.

The main idea of our procedure consists of replacing the Hochschild boundary operators b by operators of the form $\tilde{b} = bUb$. Such operators will be called modified Hochschild operators and the corresponding homology will be called modified Hochschild homology. The effect of such a replacement is that there are more $\tilde{b}-$ cycles and less $\tilde{b}-$boundaries than those in the Hochschild complex and hence the modified Hochschild homology will be bigger than the Hochschild homology.

In addition, whilst the Hochschild boundary does not behave well with respect to the Alexander-Spanier boundary operator, the new operator $\tilde{b}$ will anti-commute with it. This crucial commutativity relation will allow us to correct the above mentioned limitation of the original Hochschild homology and to replace the (b, B) Connes bi-complex by the $(\tilde{b}, d)$ bi-complex, called modified periodic cyclic bi-complex. The total homology of the modified periodic cyclic bi-complex will be called modified periodic cyclic homology.

The boundary operator $\tilde{b}$ will be realized by blending the Hochschild boundary b with the Alexander-Spanier boundary. In order to reach this objective we will have to restrict the Alexander-Spanier complex to *germs* of functions. As the notion of germ is a locality notion, it will apply on topological algebras. In the sequel we will be careful to perform only operations which are compatible with the *locality* phenomena, to encompass at least the case of scalar functions, sections in vector bundles and quasi-local operators. We stress on the observation that the Hochschild homology is local in nature, see Connes [1] and Teleman [10].

The construction of the operator $\tilde{b}$ will be based on the operator σ defined by the formula

$$db + bd = 1 - \sigma, \tag{2.1}$$

discussed in Sect.5-6.

Although the next considerations are general, to the end, in order to get interesting results, we will have to involve *locality.* As long as we perform algebraic operations only with the Alexander-Spanier and Hochschild operators, this objective is not obstructed.

In particular, it follows that the modified Hochschild homology of the algebra of continuous functions is not trivial. This result should provide the correct definition of non commutative differential forms for the algebra of continuous, or more singular, functions.

We stress that all our considerations do not make any kind of commutativity assumption on the associative algebra or the ground ring of the algebra.

In a subsequent note we intend to show that the modified periodic cyclic homology allows one to extract the Chern character from continuous direct connections (for direct connections see Teleman [11], [12], Kubarski-Teleman

[4]) in continuous vector bundles. For applications of linear direct connections, used as a tool, see Connes-Moscovici [3] and Mishchenko-Teleman [7].

The author thanks J.-M. Lescure for valuable remarks.

3 Alexander-Spanier Complex

Let $\mathcal{A}$ be an arbitrary associative algebra with unit 1 over an arbitrary ring K. Any commutativity assumption is made neither on the algebra $\mathcal{A}$ nor on the ground ring K. We assume of course that $\mathcal{A}$ is a K-bimodule.

In what follows we require only that the unit 1 commutes with all elements of the ring K and we assume that the tensor products are *circular* over K, i. e.

$$f_0 \otimes_K f_1 \otimes_K ... \otimes_K f_k.\alpha = \alpha.f_0 \otimes_K f_1 \otimes_K ... \otimes_K f_k \tag{3.2}$$

for any $\alpha \in K$. If K is a field, any tensor product over K is automatically circular.

For any non negative integer r define

$$C_k(\mathcal{A}) := \otimes_K^{k+1} \mathcal{A}; \tag{3.3}$$

its elements are called non commutative chains of degree k over $\mathcal{A}$. In the sequel the tensor product $\otimes$ is understood to mean $\otimes_K$. The formula

$$\alpha(a_0 \otimes a_1 \otimes ... \otimes a_k)\beta := (\alpha a_0) \otimes a_1 \otimes ... \otimes (a_k \beta) \tag{3.4}$$

defines an $\mathcal{A}$ bi-module structure on $C_k(\mathcal{A})$.

We define the Alexander-Spanier co-boundary face map $d_i : C_k(\mathcal{A}) \to C_{k+1}(\mathcal{A})$ by the formulas

$$d_i(a_0 \otimes a_1 \otimes ... \otimes a_k) := a_0 \otimes .... \otimes 1 \otimes a_i \otimes ... \otimes a_k, \quad for \quad 0 \leq i \leq k, \tag{3.5}$$

and

$$d_{k+1}(a_0 \otimes a_1 \otimes ... \otimes a_k) := a_0 \otimes a_1 \otimes ... \otimes a_k \otimes 1, \quad for \quad i = k+1. \tag{3.6}$$

The Alexander-Spanier boundary is defined by

$$d := \sum_{i=0}^{i=k+1} (-1)^i d_i; \tag{3.7}$$

it agrees with the classical Alexander-Spanier co-boundary operator, see e.g. Spanier [9]. It satisfies $d^2 = 0$.

In particular, for any $a \in \mathcal{A}$ one has

$$da = 1 \otimes a - a \otimes 1 \quad and \quad d1 = 0. \tag{3.8}$$

If $\alpha \in K$ and $a \in \mathcal{A}$, and as $\alpha.1 = 1.\alpha$ then

$$d(\alpha.a) = 1 \otimes (\alpha.a) - (\alpha.a) \otimes 1 = 1.\alpha \otimes a - (\alpha a) \otimes 1 = \alpha.da \tag{3.9}$$

The product

$$\times : C_r(\mathcal{A}) \otimes C_s(\mathcal{A}) \to C_{r+s}(\mathcal{A}) \tag{3.10}$$

of chains over A is defined precisely as in the case of the classical Alexander-Spanier co-chains

$$(a_0 \otimes a_1 \otimes ... \otimes a_r) \times (b_0 \otimes b_1 \otimes ... \otimes b_s) := a_0 \otimes a_1 \otimes ... \otimes (a_r b_0) \otimes b_1 \otimes ... \otimes b_s. \tag{3.11}$$

The complex $\mathcal{C}_*(\mathcal{A}) := \{\sum_{r=0}^{\infty} C_r(\mathcal{A}), d\}$ is a graded differential complex: for any $\omega \in C_r(\mathcal{A})$ and $\sigma \in C_s(\mathcal{A})$ one has

$$d(\omega \times \sigma) = (d\omega) \times \sigma + (-1)^r \omega \times (d\sigma). \tag{3.12}$$

If $\rho : \mathcal{A} \to K$ is a $K-$homorphism and $\rho(1_{\mathcal{A}}) = 1_K$, then $h : C_r(\mathcal{A}) \to C_{r-1}(\mathcal{A})$, defined by the formula

$$h(a_0 \otimes a_1 \otimes ... \otimes a_r) = \rho(a_0) a_1 \otimes ... \otimes a_r \tag{3.13}$$

satisfies the identity

$$dh + hd = 1 \tag{3.14}$$

and hence the complex $\{\mathcal{C}_*(\mathcal{A}), d\}$ is acyclic. In the case of the classical Alexander-Spanier complex the homomorphism ρ is given by the valuation of functions at one point.

If the algebra $\mathcal{A}$ has a locally convex topology, it is natural and customary (see Connes [1]) to replace the algebraic tensor product $\mathcal{C}_i(\mathcal{A})$ by a topological tensor product completion $\hat{\mathcal{C}}_i(\mathcal{A})$. The elements of $\hat{\mathcal{C}}_r(\mathcal{A})$ are called continuous Alexander-Spanier co-chains.

In the particular case of the algebra $\mathcal{A} = C^\infty(M)$, endowed with the *Fréchèt* topology, where M is a smooth manifold, for the projective tensor product completion, the continuous Alexander-Spanier co-chains consist of all smooth functions on various powers of M. The homology of this complex is acyclic, as explained above. If, however, the complex of continuous Alexander-Spanier chains is replaced by the complex of germs of such functions about the diagonals, the classical Alexander-Spanier theorem, see Spanier [9], states that its homology is canonically isomorphic to the de Rham cohomology.

It is very important to recall that the Alexander-Spanier theorem holds if smooth functions are replaced by arbitrary functions, or by special classes of functions (like measurable, Lipschitz, etc.); such generalizations hold if M is merely a CW-complex.

The main objective of this paper is to create an analogue of the Hochschild and periodic cyclic homology which does not give trivial results on algebras of functions as the algebra of continuous functions. This will be realized by blending the Hochschild boundary b with the Alexander-Spanier boundary d.

Although the suiving considerations are general, in order to get interesting results, we will have to involve *locality.*

4 Recall of Hochschild and Periodic Cyclic Homology

In this section we recall some basic notions and results due to A. Connes [1], [2] which lay to the foundations of non commutative geometry.

We keep the hypotheses and notations from the previous section.

Let $b_r : C_k(\mathcal{A}) \to C_{k-1}(\mathcal{A})$, be the Hochschild boundary face operator defined on generators by

$$b_r(f_0 \otimes f_1 \otimes .... f_{k-1} \otimes f_k) = f_0 \otimes f_1 \otimes .... \otimes (f_r.f_{r+1}) \otimes ... \otimes f_k, \quad for \quad 0 \leq r \leq k-1 \tag{4.15}$$

and

$$b_k(f_0 \otimes f_1 \otimes .... f_{k-1} \otimes f_k) = (f_k f_0) \otimes f_1 \otimes ... \otimes f_{k-1}, \quad for \quad r = k. \tag{4.16}$$

Two boundary operators, b' and $b : C_k(\mathcal{A}) \to C_{k-1}(\mathcal{A})$ are introduced by the formulas

$$b' = \sum_{r=0}^{r=k-1} (-1)^r b_r \tag{4.17}$$

and

$$b = b' + (-1)^k b_k. \tag{4.18}$$

It is true that $(b')^2 = b^2 = 0$.

The complex $\{C_*(\mathcal{A}), b'\}$ is the bar complex; if the algebra $\mathcal{A}$ is unitary, as assumed, the bar complex is acyclic; it provides the so called bar resolution of the algebra $\mathcal{A}$, see [6]. The acyclicity of the bar resolution is provided by the homotopy

$$\chi(f_0 \otimes f_1 \otimes ....f_{k-1} \otimes f_k) = 1 \otimes (f_0 \otimes f_1 \otimes ....f_{k-1} \otimes f_k). \tag{4.19}$$

The complex $\{C_*(\mathcal{A}), b\}$ is the Hochschild complex of the algebra $\mathcal{A}$; its homology, denoted $H_*(\mathcal{A})$, is the Hochschild homology of the algebra.

If $\mathcal{A}$ is a topological real or complex algebra, the homology of the complex $\{\hat{C}_*(\mathcal{A}), b\}$ is the continuous Hochschild homology of the algebra $\mathcal{A}$, see A. Connes [1], [2].

The following theorem, proven by A. Connes [1] on compact manifolds was extended by N. Teleman [10] to paracompact manifolds.

Theorem 4.1 *For any smooth paracompact manifold*

$$H_k(C^\infty(M)) \approx \Omega_k(M), \tag{4.20}$$

where $\Omega_k(M)$ denotes the space of k forms.

5 The Operator σ.

Definition 5.1 *Let σ be the operator given by the formula*

$$db + bd := 1 - \sigma. \tag{5.21}$$

A general remark shows that σ commutes both with d and b, that is σ is a chain homomorphism both in the Alexander-Spanier and in the Hochschild complex. Consequently, the range of the operator σ, and its (fixed) powers, are subcomplexes both in the Alexander-Spanier and Hochschild complexes. Additionally, as σ is homotopic to the identity, the inclusions of these subcomplexes into the Alexander-Spanier, resp. Hochschild, complexes induce isomorphisms between their respective homologies.

Lemma 5.2
-i) b' anticommutes with d

$$b'd + db' = 0 \tag{5.22}$$

-ii) one has the identity

$$(bd+db)(f_0 \otimes f_1 \otimes ... \otimes f_k) = f_0 \otimes f_1 \otimes ... \otimes f_k - (-1)^{k+1}((df_k)f_0) \otimes f_1 \otimes ... \otimes f_{k-1}. \tag{5.23}$$

-ii') and therefore

$$\sigma(f_0 \otimes f_1 \otimes ... \otimes f_k) = (-1)^{k+1}((df_k)f_0) \otimes f_1 \otimes ... \otimes f_{k-1} \tag{5.24}$$

-iii) the k^{th} power of σ has the explicite expression

$$\sigma^k(f_0 \otimes f_1 \otimes ... \otimes f_r) = df_1.df_2...df_r.f_0. \tag{5.25}$$

Proof of Lemma 5.2 -i) The Alexander-Spanier and Hochschild boundary face operators satisfy the following relations on $C_k(\mathcal{A})$

$$d_i b_j = b_{j+1} d_i \quad for \quad 0 \leq i \leq j \leq k-1 \tag{5.26}$$

$$d_i b_j = b_j d_{i+1} \quad for \quad 0 \leq j < i \leq k \tag{5.27}$$

$$d_i b_i = b_{i-1} d_i = Id \quad for \quad 0 \leq i \leq k \tag{5.28}$$

$$b_0 d_0 = b_k d_{k+1} = Id \tag{5.29}$$

In virtue of the relations (5.26), (5.27) one has

$$db' = \sum_{0 \leq i \leq j \leq k-1} (-1)^{i+j} d_i b_j + \sum_{0 \leq j < i \leq k} (-1)^{i+j} d_i b_j =$$

$$\sum_{0 \leq i \leq j \leq k-1} (-1)^{i+j} b_{j+1} d_i + \sum_{0 \leq j < i \leq k} (-1)^{i+j} b_j d_{i+1}. \tag{5.30}$$

On the other hand

$$b'd = \sum_{0 \leq i < j+1 \leq k} (-1)^{i+(j+1)} b_{j+1} d_i + \sum_{0 \leq j < i \leq k} (-1)^{(i+1)+j} b_j d_{i+1} + \tag{5.31}$$

$$\sum_{0 \leq j = i \leq k} (-1)^{(j+j)} b_j d_j + \sum_{0 \leq j \leq k, i=j+1} (-1)^{(j+j)} b_j d_{j+1} = \tag{5.32}$$

(relations (5.28), (5.29))

$$\sum_{0 \leq i < j+1 \leq k} (-1)^{i+(j+1)} b_{j+1} d_i + \sum_{0 \leq j < i \leq k} (-1)^{(i+1)+j} b_j d_{i+1} + \tag{5.33}$$

$$\sum_{0 \leq j = i \leq k} (-1)^{(j+j)} Id + \sum_{0 \leq j \leq k, i=j+1} (-1)^{j+(j+1)} Id. = \tag{5.34}$$

$$\sum_{0 \leq i < j+1 \leq k} (-1)^{i+(j+1)} b_{j+1} d_i + \sum_{0 \leq j < i \leq k} (-1)^{(i+1)+j} b_j d_{i+1} = -db' \tag{5.35}$$

in virtue of relations (5.31).

-ii) Where confusion might occur, the underscript (k) will specify that the corresponding operator is acting on chains of degree k. We have

$$(bd + db)_{(k)} = (b' + (-1)^{k+1} b_{k+1})d + d(b' + (-1)^k b_k) = \tag{5.36}$$

$$(-1)^{k+1}(b_{k+1}d - db_k). \tag{5.37}$$

A direct calculation shows that

$$(-1)^{k+1}(b_{k+1}d - db_k)(f_0 \otimes f_1 \otimes ... \otimes f_k) = \tag{5.38}$$

$$f_0 \otimes f_1 \otimes ... \otimes f_k + (-1)^k((df_k)f_0) \otimes f_1 \otimes ... \otimes f_{k-1}. \tag{5.39}$$

Therefore

$$(1 - \sigma)(f_0 \otimes f_1 \otimes ... \otimes f_k) = \tag{5.40}$$

$$f_0 \otimes f_1 \otimes ... \otimes f_k + (-1)^k((df_k)f_0) \otimes f_1 \otimes ... \otimes f_{k-1}. \tag{5.41}$$

We have proved that

$$\sigma(f_0 \otimes f_1 \otimes ... \otimes f_k) = (-1)^{k+1}((df_k).f_0) \otimes f_1 \otimes ... \otimes f_{k-1}, \tag{5.42}$$

which completes the proof of -ii) and -ii').

By iterating σ, one gets

$$\sigma^k(f_0 \otimes f_1 \otimes ... \otimes f_k) = df_1.df_2...df_k.f_0, \tag{5.43}$$

which completes the proof of the lemma.

Definition 5.3 *We introduce the operator*

$$\Pi_{(k)} := (1 - bd)\sigma^k_{(k)}. \tag{5.44}$$

Proposition 5.4
-i) The operator $\Pi_{(k)}$ has the explicite formula

$$\Pi_{(k)}(f_0 \otimes f_1 \otimes ... \otimes f_k) = f_0 \, . \, df_1 \, ... \, df_k \tag{5.45}$$

-ii) $\Pi_{(k)}$ *is a projector*

$$(\Pi_{(k)})^2 = \Pi_{(k)} \tag{5.46}$$

-iii) the operators d commute with the projectors $\Pi_{(k)}$ and hence they keep the ranges of $\Pi_{(k)}$ invariant

$$d\,\Pi_{(k)} = \Pi_{(k+1)}\,d \tag{5.47}$$

-iv) the operators b commute with the projectors $\Pi_{(k)}$ and hence they keep the ranges of $\Pi_{(k)}$ invariant

$$b\,\Pi_{(k)} = \Pi_{(k-1)}\,b. \tag{5.48}$$

Proof of Proposition 5.4

-i) From the Definition 5.3 and formula (5.43) we get

$$\Pi_{(k)}(f_0 \otimes f_1 \otimes ... \otimes f_k) = (1 - bd)\,df_1\,...\,df_k\,f_0 = \tag{5.49}$$

$$df_1\,...\,df_k\,f_0 - (-1)^k b\,(df_1\,...\,df_k\,df_0) = \tag{5.50}$$

$$df_1\,...\,df_k\,f_0 - (-1)^k b\,[df_1\,...\,df_k\,(1 \otimes f_0 - f_0 \otimes 1)] = \tag{5.51}$$

$$f_0\,df_1\,...\,df_k \tag{5.52}$$

because the Hochschild boundary faces b_i contract the factors $1 \otimes f_{i-1} - f_{i-1} \otimes 1$ into zero, for $0 \leq i \leq k-1$.

-ii) We recall that in the literature a chain of the form $f_0 \otimes f_1 \otimes ... \otimes f_k$ in which at least one of the factors $f_i = 1$, $1 \leq i \leq k$, is called degenerate. Then, it is easy to see that

$$\Delta(f_0 \otimes f_1 \otimes ... \otimes f_k) := (1 - \Pi_{(k)})(f_0 \otimes f_1 \otimes ... \otimes f_k) = \tag{5.53}$$

$$f_0 \otimes f_1 \otimes ... \otimes f_k - f_0\,df_1\,...\,df_k \tag{5.54}$$

is a finite sum of degenerate chains.

As $d\,1 = 0$, $\Pi_{(k)}$ carries any degenerate chain into zero. Therefore,

$$0 = \Pi_{(k)}\Delta = \Pi_{(k)}(1 - \Pi_{(k)}), \tag{5.55}$$

which proves the assertion.

It follows also that the range of the complementary projector $1 - \Pi_{(k)}$ consists precisely of degenerate chains.

-iii) In view of the identity (i) already proven, one has

$$d\,\Pi_{(k)}(f_0 \otimes f_1 \otimes ... \otimes f_k) = d(f_0\,.\,df_1\,...\,df_k) = \tag{5.56}$$

$$df_0\,.\,df_1\,...\,df_k = 1.df_0\,.\,df_1\,...\,df_k = \tag{5.57}$$

$$\Pi_{(k+1)}\, d_0\, (f_0 \otimes f_1 \otimes ... \otimes f_k) = \Pi_{(k+1)}\, d\, (f_0 \otimes f_1 \otimes ... \otimes f_k) \tag{5.58}$$

because $\Pi_{(k+1)}\, d_i\, (f_0 \otimes f_1 \otimes ... \otimes f_k) = 0$ for $1 \leq i \leq k+1$.
-iv)

$$b\, \Pi_{(k)} = b\,(1 - bd)\sigma_{(k)}^{k} = b\,(1 - bd)(1 - bd - db)\sigma_{(k)}^{k-1} = \tag{5.59}$$

$$b\,(1 - bd - db)\sigma_{(k)}^{k-1} = b\,(1 - db)\sigma_{(k)}^{k-1} = \tag{5.60}$$

$$(1 - bd)\, b\, \sigma_{(k)}^{k-1} = (1 - bd)\, \sigma_{(k-1)}^{k-1}\, b = \Pi_{(k-1)} b. \tag{5.61}$$

Definition 5.5 *Define the complex*

$$\tilde{C}_k(\mathcal{A}) \ := \ \Pi_{(k)}(C_k(\mathcal{A})) = \tag{5.62}$$

$$\{ \sum_{finite/series} f_0\, df_1\, df_2\, ...\, df_k,\ f_i \in \mathcal{A}\ \}. \tag{5.63}$$

Proposition 5.6
-i) One has the inclusion of complexes

$$\{\tilde{C}_*(\mathcal{A}), d\} \quad is \quad a \quad sub - complex \quad of \quad \{C_*(\mathcal{A}), d\} \tag{5.64}$$

-ii) one has the inclusion of complexes

$$\{\tilde{C}_*(\mathcal{A}), b\} \quad is \quad a \quad sub - complex \quad of \quad \{C_*(\mathcal{A}), b\} \tag{5.65}$$

-iii) on the sub-complex $\tilde{C}_k(\mathcal{A})$ one has the identity

$$(1 - bd)\sigma_{(k)}^{k} = 1 \tag{5.66}$$

-iv) the above sub-complex inclusions induce isomorphisms in the Alexander-Spanier, resp. Hochschild, homology.

In the literature the Hochschild complex modulo degenerate chains is known as the normalized Hochschild complex, see e.g. Loday's book [5]. It is also well known [5] that the Hochschild and the normalized Hochschild complexes have isomorphic homologies. Our considerations show also that the normalized Hochschild complex is precisely the sub-complex $\tilde{C}_k(\mathcal{A})$. The formula Proposition 5.6 -iii) is the formula (2.6.8.1-2) from Loday's book [5], pag. 85 formulated onto the normalized complex. Although the normalized complex and the sub-complex $\tilde{C}_k(\mathcal{A})$ coincide, we stress that from the very beginning we resisted the temptation to work into the normalized complex for the purpose of getting exact formulas in the non-normalized Hochschild and Alexander-Spanier complexes.

Proof of Proposition 5.6

The defining formula for σ shows that it (and any of its powers) is chain homotopic (both, with respect to d or b) to the identity. If $\omega \in C_k(\mathcal{A})$ and $d\omega = 0$, then $\sigma^k_{(k)}(\omega)$ is co-homologous to ω.

Moreover, $(\sigma_{(k)} + db)\sigma^k_{(k)}(\omega) = (1 - bd)\sigma^k_{(k)}(\omega)$ is co-homologous to $\sigma^k_{(k)}(\omega)$; hence any homology class in the Alexander-Spanier complex is co-homologous to its projection into the subcomplex $\{\tilde{C}_*(\mathcal{A}), d\}$. Therefore, the homology of the subcomplex is at least as big as the Alexander-Spanier homology.

The same argument works for the Hochschild complex, using the original definition of the projection $\Pi_{(k)}$.

On the other hand, suppose ω is a $d-$cycle in $\tilde{C}_*(\mathcal{A})$ and that $\omega = d\phi$, where $\phi \in C_{k-1}(\mathcal{A})$. Then, in virtue of Proposition 5.4 -iii)

$$\omega = \Pi_{(k)}(\omega) = \Pi_{(k)}(d\phi) = d\Pi_{(k-1)}(\phi), \tag{5.67}$$

which shows that ω is a boundary in the subcomplex $\tilde{C}_*(\mathcal{A})$.

The same argument works for the Hochschild complex.

Corollary 5.7 *-i) The Alexander-Spanier, resp. Hochschild, complex decomposes in a direct sum of complexes*

$$\{\tilde{C}_*(\mathcal{A}), d\} \oplus \{degenerate \quad chains, d\} \tag{5.68}$$

respectively,

$$\{\tilde{C}_*(\mathcal{A}), b\} \oplus \{degenerate \quad chains, b\} \tag{5.69}$$

-ii) the Alexander-Spanier, resp. Hochschild, sub-complex of degenerate chains is acyclic.

6 Modified Hochschild boundary.

From now on we will be working only on the sub-spaces $\{\tilde{C}_*(\mathcal{A})\}$, which are Alexander-Spanier and Hochschild sub-complexes.

In the previous section we introduced the operator σ defined by the formula

$$db + bd = 1 - \sigma. \tag{6.70}$$

We have shown that σ satisfies on $\{\tilde{C}_*(\mathcal{A})\}$ the identity

$$\sigma^k = 1 + bd\sigma^k. \tag{6.71}$$

Replacing

$$\sigma = 1 - (db + bd) \tag{6.72}$$

in the above identy one gets

$$1 = (1 - bd)\sigma^n = (1 - bd)[1 - (bd + db)]^n = \tag{6.73}$$

$$(1 - bd)[1 + \sum_{k=1}^{n} (-1)^k C_n^k (bd + db)^k] = \tag{6.74}$$

$$(1 - bd)\{1 + \sum_{k=1}^{n} (-1)^k C_n^k [(bd)^k + (db)^k]\} = \tag{6.75}$$

$$1 + \sum_{k=1}^{n} (-1)^k C_n^k [(bd)^k + (db)^k] - bd - \sum_{k=1}^{n} (-1)^k C_n^k (bd)^{k+1}, \tag{6.76}$$

and hence

$$0 = bd + \sum_{k=1}^{n} (-1)^{k-1} C_n^k [(bd)^k + (db)^k] + \sum_{k=1}^{n} (-1)^k C_n^k (bd)^{k+1} = \tag{6.77}$$

$$bd + \sum_{k=1}^{n} (-1)^{k-1} C_n^k (bd)^k + \sum_{k=1}^{n} (-1)^k C_n^k (bd)^{k+1} + \sum_{k=1}^{n} (-1)^{k-1} C_n^k (db)^k = \tag{6.78}$$

$$(1+n)bd + \sum_{k=2}^{n} (-1)^{k-1} C_n^k (bd)^k + \sum_{k=1}^{n} (-1)^k C_n^k (bd)^{k+1} + \sum_{k=1}^{n} (-1)^{k-1} C_n^k (db)^k = \tag{6.79}$$

$$(1+n)bd + \sum_{k=2}^{n} (-1)^{k-1} C_n^k (bd)^k + \sum_{k=1}^{n-1} (-1)^k C_n^k (bd)^{k+1} + (-1)^n C_n^n (bd)^{n+1} + \sum_{k=1}^{n} (-1)^{k-1} C_n^k (db)^k = \tag{6.80}$$

$$(1+n)bd + \sum_{k=2}^{n} (-1)^{k-1} C_n^k (bd)^k + \sum_{k=2}^{n} (-1)^{k-1} C_n^{k-1} (bd)^k + (-1)^n C_n^n (bd)^{n+1} + \sum_{k=1}^{n} (-1)^{k-1} C_n^k (db)^k = \tag{6.81}$$

$$(1+n)bd + \sum_{k=2}^{n} (-1)^{k-1} (C_n^k + C_n^{k-1}) (bd)^k + (-1)^n C_n^n (bd)^{n+1} + \sum_{k=1}^{n} (-1)^{k-1} C_n^k (db)^k = \tag{6.82}$$

$(C_n^k + C_n^{k-1} = C_{n+1}^k)$

$$= (1+n)bd + \sum_{k=2}^{n} (-1)^{k-1} C_{n+1}^k (bd)^k + (-1)^n C_{n+1}^{n+1} (bd)^{n+1} + \sum_{k=1}^{n} (-1)^{k-1} C_n^k (db)^k \tag{6.83}$$

or

$$0 = \sum_{k=1}^{n+1} (-1)^{k-1} C_{n+1}^k (bd)^k + \sum_{k=1}^{n} (-1)^{k-1} C_n^k (db)^k. \tag{6.84}$$

The operator $\tilde{b}_n$, acting on n-forms, is defined by the formula

$$\tilde{b}_n := \sum_{k=1}^{n} (-1)^{k-1} C_n^k (bd)^{k-1} b. \tag{6.85}$$

The above relation (6.84) becomes

$$0 = \tilde{b}_{n+1} d_n + d_{n-1} \tilde{b}_n \tag{6.86}$$

and hence, the operators $\tilde{b}$ and d anti-commute.

As the operator $\tilde{b}_n$ is of the form $\tilde{b}_n = bTb$, it follows that $\tilde{b}\tilde{b} = 0$ and hence it a boundary operator. The complex obtained by replacing the operator b by $\tilde{b}$ will be called *modified Hochschild complex*, denoted $C_*(\tilde{b}, d)$. We intend to study its homology.

Given the above mentioned stucture of the operator $\tilde{b}_n$, it follows immediately that in the modified Hochschild complex the Hochschild cycles remain cycles and no new boundaries appear. Therefore, the homology of the modified Hochschild complex, called *modified Hochschild homology,* is bigger than the Hochschild homology (characteristic zero assumed).

7 Modified Periodic Cyclic Homology

In analogy with the periodic cyclic complex due to A. Connes [2], see also J.-L. Loday [5] Sect. 5.1.7., pag.159, we introduce the *modified periodic cyclic bi-complex* of the (topological) algebra $\mathcal{A}$, by

$$\tilde{C}^{\lambda,per}(\mathcal{A}) = \{\tilde{C}_{p,q}\}_{(p,q)\in\mathbf{Z}\times\mathbf{Z}}, \tag{7.87}$$

where

$$\tilde{C}_{p,q} = \tilde{C}_{q-p}(\mathcal{A}) \;\; for \;\; p \leq q, \;\; and \;\; \tilde{C}_{p,q} = 0 \;\; for \;\; q < p, \tag{7.88}$$

while the bi-complex operators $\tilde{b}$ and d are acting as such

$$\tilde{b} : \tilde{C}_{p,q} \rightarrow \tilde{C}_{p,q-1}$$

$$d : \tilde{C}_{p,q} \rightarrow \tilde{C}_{p,q+1};$$

as seen in the previous section, they anti-commute and hence they are legitimate bi-complex operators.

The *modified periodic cyclic homology* of the algebra $\mathcal{A}$, $\tilde{H}_*^{\lambda,per}(\mathcal{A})$, is by definition the homology of the total complex associated to the modified cyclic periodic complex $\tilde{C}^{\lambda,per}(\tilde{b}, d)$.

As applications of the above considerations, in the next subsections 7.1, 7.2 we are going to compute the modified periodic cyclic homology both in the case of the algebra of smooth functions and the algebra of continuous functions, on smooth manifolds. We recall the expression of the boundary operator $\tilde{b}$

$$\tilde{b}_r := b \sum_{k=1}^{r} (-1)^{k-1} C_n^k (db)^{k-1} \tag{7.89}$$

and the expression of the Alexander-Spanier boundary operator d

$$(df_r)(x_0, x_1, ..., x_r, x_{r+1}) = \sum_{i=0}^{i=r+1} (-1)^i f_r(x_0, ...\hat{x_i}, ..., x_{k+1}). \tag{7.90}$$

In both application we consider d to be the Alexander-Spanier co-boundary acting on germs of smooth, resp. continuous, functions defined on neighborhoods of the diagonal in the different powers of the base space.

7.1 The smooth case.

It is clear that the operator d is well defined on germs of functions about the diagonals. It is also important to notice that the Hochschild boundary is also well defined on germs and that the Hochschild homology depends only on the quotient complex consisting of germs, see Teleman [10].

We use the spectral sequence associated to the first filtration of the bicomplex (with respect to the $d-$degree of chains) to compute the homology of the total complex. We are going to prove that the corresponding terms $E^1_{p,q}$ and $E^2_{p,q}$ of the spectral sequence, in the case of the algebra $A := C^\infty(M)$, are

$$E^1_{p,q} = H_p(C_{*,q}, d) \cong H^{q-p}_{dR}(M)$$

and

$$E^2_{p,q} = H_q(E^1_{p,*}, \tilde{b}) \cong H^{q-p}_{dR}(M).$$

For the computation of the term E^1 we use the isomorphism of the Alexander-Spanier homology of the complex of smoth chains with the de Rham cohomology.

For the computation of the term $E^2_{p,q} = H_q(E^1_{p,*}, \tilde{b})$ we show that the differential $d^1 = \tilde{b}_*$ on the complex E^1 is equal to zero. Indeed, the operator $\tilde{b}$ being local, we may express it in local co-ordinates $(x^1, x^2, ..., x^n)$. Let $\gamma \in H^p_{dR}(M)$. We may represent γ, locally, by a smooth closed differential form

$$\gamma = \sum_{i_1,i_2,...,i_p} \omega_{i_1,i_2,...,i_p}(x) dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_p} =$$

or by the differentiable function expressed locally by

$$\sum_{i_1,i_2,...,i_p} \omega_{i_1,i_2,...,i_p}(x_0)(x_1^{i_1} - x_0^{i_1}) \wedge (x_2^{i_2} - x_0^{i_2}) \wedge .... \wedge (x_p^{i_p} - x_0^{i_p}),$$

where $\wedge$ signifies anti-symmetrization. The Hochschild boundary b of such a representative is clearly zero, and hence, a fortiori, $\tilde{b}\gamma = 0$; this proves the assertion.

From here it follows that

$$E^2_{p,q} = H^{q-p}_{dR}(M).$$

Therefore, we have proven the

Theorem 7.1 *In the case of the algebra of smooth functions $C^\infty(M)$ on the smooth manifold M, the modified periodic cyclic homology and the periodic cyclic homology defined by A. Connes [2] coincide*

$$H^{\lambda,per}_k = H_k(C^{per}(\tilde{b}, d)) = \oplus^{r \cong k(mod.2)} H^r_{dR}(M).$$

7.2 The continuous case.

In this subsection we compute the modified periodic cyclic homology of the algebra of continuous functions, $C(M)$, on the *smooth* manifold M.

For its computation we proceed along the same lines as in the case of smooth functions. To begin with, we observe that the modified periodic cyclic bicomplex of the algebra of smooth functions is a sub-bicomplex of the modified periodic cyclic bicomplex of the algebra of continuous functions. Given that the inclusion of the Alexander-Spanier complex of smooth functions into the Alexander-Spanier complex of continuous functions induces isomorphism in homology, one infers that the bicomplex inclusion induces isomorphisms between the corresponding terms E^1 and therefore E^2

$$E^1_{p,q} = H_p(C_{*,q}, d) \cong H^{q-p}_{dR}(M) \tag{7.91}$$

$$E^2_{p,q} = H_q(E^1_{p,*}, \tilde{b}) \cong H^{q-p}_{dR}(M). \tag{7.92}$$

This proves the

Theorem 7.2 *The modified periodic cyclic homology of the algebra of real valued continuous functions $C^0(M)$ on the smooth manifold M, coincides with the periodic cyclic homology of the algebra of smooth functions*

$$H^{\lambda,per}_k(C^0(M)) = \oplus^{r \cong k(mod.2)} H^r_{dR}(M) \cong \oplus^{r \cong k(mod.2)} H^r(M, \boldsymbol{R}). \tag{7.93}$$

We expect, of course, the same result to hold if M is merely a topological manifold, or a CW-complex.

Corollary 7.3 *The modified Hochschild homology of the algebra $(C^0(M))$ is not trivial.*

This result should be used as the correct definition of non commutative differential forms in the case of algebras of continuous, or more singular, functions.

References

[1] Connes A.: Noncommutative differential Geometry, Publ. Math. IHES 62 (1985), pp.257 - 360

[2] Connes A.: Noncommutative Geometry, Academic Press, (1994)

[3] Connes A., Moscovici H.: Cyclic Cohomology, The Novikov Conjecture and Hyperbolic Groups, Topology Vol. 29, pp. 345-388, 1990.

[4] Kubarski J., Teleman N. Linear Direct Connections, Proceedings 7th Conference on "Geometry and Topology of Manifolds - The Mathematical Legacy of Charles Ehresmann", Betlewo, May 2005.

[5] Loday J.-L.: Cyclic Homology, Grundlehren in mathematischen Wissenschaften 301, Springer Vedrlag, Berlin Heidelberg, 1992.

[6] McLane S.: Homology, Third Ed., Grundlehren der mathematischen Wissenschaften in Einzeldarstellung Band 114, Springer Verlag, Heidelberg, 1975.

[7] Mishchenko A.S., Teleman N.: Almost flat bundles and almost flat structures. To appear.

[8] Puschnigg M.: Diffeotopy Functors of Ind-Algebras and Local Cyclic Cohomology, Documenta Mathematica Vol. 8, 143-245, 2003.

[9] Spanier E. H.: Algebraic Topology, McGraw - Hill Series in Higher Mathematics, New York, 1966

[10] Teleman N.: Microlocalization de l'Homologie de Hochschild, Compt. Rend. Acad. Scie. Paris, Vol. 326, 1261-1264, 1998.

[11] Teleman N.: Distance Function, Linear quasi connections and Chern Character, IHES Prepublications M/04/27, June 2004.

[12] Teleman N.: Direct Connections and Chern Character. Proceedings of the International Conference in Honour of Jean-Paul Brasselet, Luminy, May 2005. Will appear in World Scientific.