A generalization of residual finiteness

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Décembre 2006

IHES/M/06/60

A GENERALIZATION OF RESIDUAL FINITENESS

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Abstract

The concept of residual finiteness with respect to automorphic equivalence, a property generalizing residual finiteness and conjugacy separability is introduced. A sufficient condition for a group G to be residually finite with respect to automorphic equivalence is proven (Theorem). It is then used to give some examples of automorphic equivalent residually finite groups.

Résumé

Une généralisation de la finitude résiduelle. La notion de finitude résiduelle par rapport à une équivalence automorphique, une propriété qui généralise la finitude résiduelle et la finitude résiduelle par rapport à la conjugaison, est introduite. Une condition suffisante de la finitude résiduelle par rapport à une équivalence automorphique est prouvée (Théorème). Des exemples de groupes résiduellement finis par rapport à toute équivalence automorphique sont donnés.

Keywords

Residual properties, automorphic equivalence, residual finiteness which respect to automorphic equivalence, free products with amalgamation, HNN extensions.

Mots clés

Propriétés résiduelles, équivalence automorphique, finitude résiduelle par rapport à une équivalence automorphique, produits amalgamés, extensions HNN.

2000 Mathematics Subject Classification: 20E26, 20E06, 20E28, 20F05.

Version française abrégée

Soit G un groupe. Soit Φ un sous-groupe du groupe $\operatorname{Aut} G$ des automorphismes de G. Un sous-groupe distingué N de G est Φ-*invariant*, s'il est φ-invariant i.e $N\varphi = N$, pour tout automorphisme $\varphi \in \Phi$.

Deux éléments quelconques a et b de G sont dits Φ-*équivalents*, s'il existe un automorphisme $\varphi \in \Phi$ tel que $a = b\varphi$.

Un groupe G est dit *résiduellement fini par rapport à (ou relativement à) la* Φ-*équivalence* si pour toute paire d'éléments a et b de G qui ne sont pas Φ-équivalents, il existe un sous-groupe distingué N de G, qui est Φ-invariant et d'indice fini dans G tel que les éléments aN et bN du groupe quotient G/N ne soient pas $\overline{\Phi}$-équivalents, où $\overline{\Phi}$ est le sous-groupe du groupe $\operatorname{Aut}(G/N)$, constitué de tous les automorphismes de G/N, induits par les automorphismes de Φ.

La finitude résiduelle par rapport à la Φ-équivalence généralise la finitude résiduelle (si $\Phi = 1$) et la finitude résiduelle par rapport à la conjugaison (si $\Phi = \operatorname{Inn} G$, le groupe des automorphismes intérieurs de G).

La condition suffisante suivante de la finitude résiduelle par rapport à la Φ-équivalence est prouvée.

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Théorème. *Soit G un groupe. Soit Φ un sous-groupe de $Aut G$, le groupe des automorphismes de G, contenant le sous-groupe $Inn G$ des automorphismes intérieurs de G et tel que $Inn G$ soit d'indice fini dans Φ. Si G est finiment engendré et résiduellement fini par rapport à la conjugaison, alors G est résiduellement fini par rapport à la Φ-equivalence.*

Ce résultat nous permet de montrer que :

1. Si $k = \pm p^e$, où p est un nombre premier et $e \geq 1$, alors le groupe G_k de présentation $G_k = \langle a, b;\ a^{-1}ba = b^k \rangle$ est résiduellement fini par rapport à $\mathrm{Aut}\, G_k$-équivalence.

2. Tout groupe G_{mn} de présentation $G_{mn} = \langle a, b;\ [a^m, b^n] = 1 \rangle$ où $m, n > 1$ est résiduellement fini par rapport à $\mathrm{Aut}\, G_{mn}$-équivalence.

1. Introduction

A group G is *residually finite* if for any element $g \neq 1$ in G, there exists a normal subgroup N of finite index in G such that $gN \neq 1$ in the quotient group $\overline{G} = G/N$.

A well-known generalization of residual finiteness is conjugacy separability.

A group G is *conjugacy separable* if for any two non-conjugate elements f and g of G, there exists a normal subgroup N of finite index in G such that elements fN and gN are non-conjugate in the quotient group $\overline{G} = G/N$. It is evident that, if a group G is conjugacy separable then it is residually finite.

Some other residual properties generalizing residual finiteness are considered in [2,4].

In this paper, we introduce the concept of residual finiteness with respect to automorphic equivalence. We see that this concept generalizes residual finiteness and conjugacy separability. We prove a sufficient condition for a group to be residually finite with respect to some automorphic equivalence, namely:

Theorem. *Let subgroup Φ of $Aut G$, the group of the automorphisms of a given group G, contain group* $\mathrm{Inn}\, G$ *of the inner automorphisms of this group and let* $\mathrm{Inn}\, G$ *has a finite index in group Φ. If group G is finitely generated and conjugacy separable, then G is Φ-equivalent residually finite.*

We then use this theorem to give some examples of automorphic equivalent residually finite groups.

2. Definition

Let G be an arbitrary group and let φ be an automorphism of G.

We remind that a normal subgroup N of group G is said to be *φ-invariant*, if $N\varphi = N$. If N is a φ-invariant normal subgroup of group G, then the mapping $\overline{\varphi}$ of the quotient group G/N onto itself defined by

$$(gN)\overline{\varphi} = (g\varphi)N \qquad (g \in G),$$

is an automorphism of group G/N. This automorphism is called the automorphism *induced* by φ.

Let Φ be a subgroup of group $\operatorname{Aut} G$ of all automorphisms of G.

A normal subgroup N of group G is said to be Φ-*invariant*, if it is φ-invariant for any automorphism $\varphi \in \Phi$.

If now N is a Φ-invariant normal subgroup of group G, then we denote by $\overline{\Phi}$ the subgroup of group $\operatorname{Aut}(G/N)$ of all the automorphisms of group G/N, induced by the automorphisms of Φ.

Let G be a group and $\Phi \leqslant \operatorname{Aut} G$. Elements a and b of group G are said to be Φ-*equivalent*, if there exists an automorphism $\varphi \in \Phi$ such that $a = b\varphi$.

Let's now formulate our main concept.

Let G be an arbitrary group. Let Φ be a subgroup of group $\operatorname{Aut} G$ of all the automorphisms of G.

Group G is said to be *residually finite with respect to* Φ-*equivalence* (or Φ-*equivalent residually finite*), if for any non Φ-equivalent elements a and b of group G, there exists a normal Φ-invariant subgroup N of finite index in G such that elements aN and bN of the quotient group G/N are not $\overline{\Phi}$-equivalent.

It is clear that, particular cases of this notion are residual finiteness (when subgroup Φ consists only of the identical automorphism) and conjugacy separability (when Φ coincides with $\operatorname{Inn} G$, the group of all the inner automorphisms of group G).

If $\Phi = \operatorname{Aut} G$, then Φ-equivalent residual finiteness is just $\operatorname{Aut} G$-equivalent residual finiteness i.e. residual finiteness with respect to any automorphic equivalence.

We now prove the sufficient condition of Φ-equivalent residual finiteness (Theorem).

3. Proof of Theorem

Let $\psi_1, \psi_2, \dots, \psi_r$ be a fixed representative system of cosets of subgroup $\operatorname{Inn} G$ in group Φ.

Let a and b be non-Φ-equivalent elements of group G. Assume for $i = 1, 2, \dots, r$, $b_i = b\psi_i$. Since subgroup Φ contains group $\operatorname{Inn} G$, element a can not be conjugate in group G to elements $b_1, b_2, \dots, b_r$. Further, since group G is conjugacy separable, there exists a normal subgroup M of finite index of G such that in the quotient group G/M element aM is not conjugate to elements $b_1M, b_2M, \dots, b_rM$.

It is well known that any arbitrary subgroup of finite index of a finitely generated group G contains some characteristic subgroup which has a finite index in G. Let N be the characteristic (and consequently the Φ-invariant) subgroup of finite index of our group G, contained in subgroup M. Then, in the quotient group G/N, element aN is not conjugate to elements $b_1N, b_2N, \dots, b_rN$.

We now assert that elements aN and bN of the quotient group G/N are not $\overline{\Phi}$-equivalent. In fact, let in the contrary that for some automorphism $\varphi \in \Phi$, the equality $aN = (bN)\overline{\varphi}$ takes place; i.e. $aN = (b\varphi)N$. Let's write the automorphism φ as $\varphi = \psi_i\gamma$ for some $i \in \{1, 2, \dots, r\}$ and some inner automorphism γ of group G. Then, in group G, element $b\varphi$ is conjugate to some element b_i, and consequently, in the quotient group

G/N, element $aN = (b\varphi)N$ is conjugate to some element b_iN. This contradicts the selection of subgroup N. So, the Theorem is demonstrated.

4. Examples

Now, using the above Theorem, we have:

Example 1. If $k = \pm p^e$, where p is a prime integer and $e \geq 1$, then the group G_k with presentation

$$G_k = \langle a, b;\ a^{-1}ba = b^k \rangle$$

is $\operatorname{Aut} G_k$-equivalent residually finite.

Indeed, group G_k is conjugacy separable [5]. From [1], group $\operatorname{Aut} G_k$ can be described as follows:

Proposition 1. *Let $G_k = \langle a, b;\ a^{-1}ba = b^k \rangle$, where $|k| \neq 1$. Let $k = \delta p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, where $\delta = \pm 1$, $p_1, p_2, \ldots, p_r$ are distinct primes and $e_i \geq 1$, $(i = 1, \ldots, r)$. Then the group $Aut\, G_k$ has presentation:*

$$\langle \varphi, \psi_1, \psi_2, \ldots, \psi_r, \tau;\ \psi_i^{-1} \varphi \psi = \varphi^{p_i},\ \psi_i \psi_j = \psi_j \psi_i,\\ \tau^2 = 1,\ \tau\psi_i = \psi_i \tau,\ \tau^{-1} \varphi \tau = \varphi^{-1}\ (i, j = 1, \ldots, r) \rangle.$$

In this presentation the automorphisms are defined by:

(a) $a\varphi = ab,\ b\varphi = b$;

(b) $a\psi_i = a,\ b\psi_i = b^{p_i}\ (i = 1, \ldots, r)$;

(c) $a\tau = a,\ b\tau = b^{-1}$.

Now, the following proposition can be derived.

Proposition 2. *Subgroup $Inn\, G_k$ has a finite index in group $Aut\, G_k$ if and only if $k = \pm p^e$, where p is a prime integer and $e \geq 1$.*

Proof. Assume first that subgroup $\operatorname{Inn} G_k$ has a finite index in group $\operatorname{Aut} G_k$. Then any automorphism of G_k should have a finite order modulo $\operatorname{Inn} G_k$. In particular, for some integer $n > 0$, the automorphism ψ_1^n should be inner i.e. for some element $g \in G_k$, the equalities $g^{-1}ag = a$ and $g^{-1}bg = b^{p_1^n}$ should be satisfied. Since condition $|k| \neq 1$ implies that $Z_{G_k}(a) = \langle a \rangle$ ($Z_{G_k}(a)$ is the centralizer in group G_k of element a), we have $g = a^m$ for some integer m. So, the equality $g^{-1}bg = b^{p_1^n}$ has the form $a^{-m}ba^m = b^{p_1^n}$. But G_k is the HNN-extension with base group the infinite cycle $\langle b \rangle$ and stable letter a. Since the left hand side of the last equality is not reduced, $|k| \neq 1$ and $n > 0$, we should have $m > 0$. Consequently, using the defining relations of group G_k, this equality has the form $b^{k^m} = b^{p_1^n}$, which gives $k^m = p_1^n$. Hence, p_1 is the unique prime divisor of k, which is required.

Conversely, let $k = \delta p^e$, where $\delta = \pm 1$, p a prime number and $e > 0$. By Proposition 1, for this case, group $\operatorname{Inn} G_k$ is generated by the automorphisms φ, τ and ψ (where φ and τ are defined above and ψ_1 is defined by: $a\psi = a,\ b\psi = b^p$) and is defined by the relations

$$\psi_i^{-1} \varphi \psi = \varphi^p,\ \tau^2 = 1,\ \tau\psi = \psi\tau,\ \tau^{-1} \varphi \tau = \varphi^{-1}.$$

Since for any integer n $a\varphi^n = ab^n$ and from the defining relations of group G_k $bab^{-1} = ab^{k-1}$, it follows that automorphism φ^{k-1} is inner. But $k-1 \neq 0$, thus automorphism φ has a finite order modulo $\operatorname{Inn} G_k$.
Similarly, the equality $b\psi^n = b^{p^n}$ which is satisfied for any integer $n \geq 0$ shows that, if $\delta = 1$, then automorphism ψ^e is inner and if $\delta = -1$, automorphism ψ^{2e} is inner.
So let $f : \operatorname{Aut} G_k \longrightarrow \operatorname{Aut} G_k/\operatorname{Inn} G_k$ be the canonical homomorphism of group $\operatorname{Aut} G_k$ onto the quotient group $\operatorname{Aut} G_k/\operatorname{Inn} G_k$ and let X and Y be the image by f of the subgroups generated by φ and ψ respectively. We see that

$$1 \leq X \leq Y \leq \operatorname{Aut} G_k/\operatorname{Inn} G_k$$

is a subnormal sequence with finite cyclic factors. Thus the factor group $\operatorname{Aut} G_k/\operatorname{Inn} G_k$ is finite and proposition 2 is proved.

So, G_k is conjugacy separable and subgroup $\operatorname{Inn} G_k$ has a finite index in group $\operatorname{Aut} G_k$ if $k = \pm p^e$, where p is a prime number and $e \geq 1$. Thus, applying Theorem, G_k is $\operatorname{Aut} G_k$-equivalent residually finite.

Example 2 Any group with presentation

$$G_{mn} = \langle a, b;\ [a^m, b^n] = 1\rangle \quad \text{where} \quad m > 1 \text{ and } n > 1$$

is $\operatorname{Aut} G_{mn}$-equivalent residually finite.

Group G_{mn} is conjugacy separable [3] and $\operatorname{Inn} G_{mn}$ has a finite index in $\operatorname{Aut} G_{mn}$ [6]. So, by Theorem, group G_{mn} is $\operatorname{Aut} G_{mn}$-equivalent residually finite.

We mention here that groups of form G_k are the Baumslag-Solitar groups which are HNN extensions [1,5] whereas groups of form G_{mn} are free products with amalgamations [6].

Acknowledgements

The first author gratefully acknowledges the financial support from the "Institut des Hautes Etudes Scientifiques (IHES)" for his visit at IHES.

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