

Noncommutative Geometry Approach to Principal and Associated Bundles

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Abstract: We recast basic topological concepts underlying differential geometry using the language and tools of noncommutative geometry. This way we characterize principal (free and proper) actions by a density condition in (multiplier) C^* -algebras. We introduce the concept of piecewise triviality to adapt the standard notion of local triviality to fibre products of C^* -algebras. In the context of principal actions, we study in detail an example of a non-proper free action with continuous translation map, and examples of compact principal bundles which are piecewise trivial but not locally trivial, and neither piecewise trivial nor locally trivial, respectively. We show that the module of continuous sections of a vector bundle associated to a compact principal bundle is a cotensor product of the algebra of functions defined on the total space (that are continuous along the base and polynomial along the fibres) with the vector space of the representation. On the algebraic side, we review the formalism of connections for the universal differential algebras. In the differential geometry framework, we consider smooth connections on principal bundles as equivariant splittings of the cotangent bundle, as 1-form-valued derivations of the algebra of smooth functions on the structure group, and as axiomatically given covariant differentiations of functions defined on the total space. Finally, we use the Dirac monopole connection to compute the pairing of the line bundles associated to the Hopf fibration with the cyclic cocycle of integration over S^2 .

Contents

1	Topological aspects of principal and associated bundles	3
1.1	Proper maps	4
1.2	C*-characterisation of injectivity	7
1.3	Free and proper actions	8
1.4	Associated bundles	13
1.5	Local triviality and piecewise triviality	15
2	Elements of general algebra	20
2.1	Modules and comodules	21
2.2	The universal differential calculus and algebra	22
2.3	Projectivity and connections	24
3	Differential geometry of principal and vector bundles	26
3.1	Associated vector bundles	26
3.2	Connections and gauge transformations	28
3.3	Covariant derivatives and the Chern character	32
3.4	Hopf fibration and Dirac monopole	35

Introduction

This paper is mostly a survey. New results were derived to bridge gaps between the standard and noncommutative geometry approach to principal and associated bundles. It is intended to serve as an introduction to the noncommutative geometry of principal extensions of algebras. Since they play the role of quantum principal bundles, herein we analyse in detail the topology of classical principal actions. Following Henri Cartan, we define them as actions that are free and proper. Next, we adopt the point of view that principal bundles are monoidal functors from the category of finite-dimensional group representations to the category of modules that are finitely generated projective over the algebra of functions defined on the base space. The functor is given by associating a vector bundle to a principal bundle via a group representation, and then taking the module of global sections of the associated vector bundle. This leads us to an algebraic formulation of the module of continuous or smooth sections of a vector bundle. We focus in this context on compact groups and spaces, so as to take an advantage of Peter-Weyl theory and the Serre-Swan Theorem, and thus arrive at the aforementioned general algebraic formalism without losing track of the topological or smooth nature of vector bundles.

Our exposition of connections on principal and vector bundles is done in a way that provides a passage to the noncommutative setting. By standard Chern-Weil theory, connections are then used to define the Chern character. As an example of this general theory, we study the Dirac monopole

connection on the Hopf fibration, and prove that the integrated Chern classes of the associated line bundles coincide with minus the winding numbers of representations that define them. We interpret these topological invariants as the result of the pairing of K -theory and cyclic cohomology [9].

To make the paper reasonably self-contained, we enclose selected elements of general noncommutative algebra. In the special commutative case, they form an algebraic backbone of the differential-geometric section. On the other hand, we refrain from discussing the metric aspects of differential geometry. These are embodied in the theory of spectral triples on the noncommutative side, and that is beyond the scope of this paper.

1 Topological aspects of principal and associated bundles

The category of locally compact Hausdorff spaces with all continuous maps as morphisms is equivalent to the (appropriately defined) opposite category of commutative C^* -algebras [48]. The goal of this section is to describe principal (free and proper) actions of locally compact groups on locally compact Hausdorff spaces purely in terms of their C^* -algebras. This description will be the starting point for defining principal actions of locally compact quantum groups on general C^* -algebras. Throughout the paper, by a (locally) compact group we mean a topological group which, as a set, is (locally) compact Hausdorff.

A key technical concept appearing while considering principal actions and fibre bundles is that of a fibred product. For any three sets X, Y, Z , and any two maps $f : X \rightarrow Z, g : Y \rightarrow Z$, we can define the *fibred product* of X and Y over Z as the equaliser of the maps $F(x, y) = (x, f(x), y), G(x, y) = (x, g(y), y)$, i.e.

$$\{(x, y) \in X \times Y \mid f(x) = g(y)\} =: X \times_Z Y \xrightarrow{\subseteq} X \times Y \begin{array}{c} \xrightarrow{F} \\ \xrightarrow{G} \end{array} X \times Z \times Y \quad (1.0.1)$$

In the noncommutative setting, the fibred product will appear reincarnated as the cotensor product over a coalgebra and the tensor product over an algebra.

Recall that a continuous map $F \in C(X, Y)$ is proper iff the map

$$F \times \text{id} : X \times Z \longrightarrow Y \times Z \quad (1.0.2)$$

is closed for any topological space Z (see [5, Definition 1, p. 97]). A continuous group action $X \times G \rightarrow X$ is called *proper* iff the (principal) map

$$F^G : X \times G \ni (x, g) \longmapsto (x, xg) \in X \times X \quad (1.0.3)$$

is proper. Now, if X is Hausdorff and Y is locally compact Hausdorff, then a continuous map $F \in C(X, Y)$ is proper if and only if $F^{-1}(K)$ is compact for any compact subset K [5, Proposition 7, p.104]. In particular, if G is a locally compact group acting on a locally compact Hausdorff space X , we can say that this action is proper iff $(F^G)^{-1}(K)$ is compact for any compact subset K . (By a locally compact group we mean a topological group which is locally compact and Hausdorff.) A beautiful discussion and comparison study of different definitions of proper actions can be found in [4].

An action is called *free* iff F^G is injective, and *principal* iff this map is both injective and proper. In other words, principal means free and proper. It follows from Lemma 1.9 that, if the action is principal, F^G is a closed injection, so that it yields a homeomorphism F_X^G from $X \times G$ onto the fibred

product $X \times_{X/G} X$, which is the image of F^G . (Note here that X/G is again a locally compact Hausdorff space [5, Proposition 3, p. 253, Proposition 9, p. 257].) However, the opposite implication is not true: even when F_X^G is a homeomorphism the action of G on X might still fail to be proper (see Example 1.14). Notice also that an action can be proper without being free (e.g., the trivial action of a non-trivial compact group) and free without being proper (e.g., an ergodic action of \mathbb{R} on a torus). Principal actions lead to the concept of a topological principal bundle, and free but not proper actions give rise to foliations.

If the action is free, then there exists a (not necessarily continuous) map

$$\tilde{\tau} : X \times_{X/G} X \xrightarrow{(F_X^G)^{-1}} X \times G \xrightarrow{\pi_G} G \quad (1.0.4)$$

where π_G is the canonical surjection onto G . Since it determines which element of G translates one point of X to the other, it is called the *translation map* [25, Chapter 4, Definition 2.1]. The following properties of $\tilde{\tau}$ are immediate from the definition:

$$\tilde{\tau}(x, xg) = g, \quad (1.0.5)$$

$$x\tilde{\tau}(x, y) = y, \quad (1.0.6)$$

$$\tilde{\tau}(xg, yh) = g^{-1}\tilde{\tau}(x, y)h, \quad (1.0.7)$$

$$\tilde{\tau}(x, y)\tilde{\tau}(y, z) = \tilde{\tau}(x, z), \quad (1.0.8)$$

$$\tilde{\tau}(y, x) = \tilde{\tau}(x, y)^{-1}. \quad (1.0.9)$$

Here $x, y, z \in X$, $g, h \in G$, and e is the neutral element of G . As will be demonstrated later, the continuity of $\tilde{\tau}$ is guaranteed by the properness of the action. The noncommutative counterpart of the translation map plays a fundamental role in the theory of Galois-type extensions and coextensions.

There is some confusion in the literature concerning the definition of a principal action (principal bundle). Some authors call principal an action that is free and induces continuous translation map. While there are important results for which these two properties suffice, they do not guarantee that the quotient space is Hausdorff (see Example 1.14). This is why we adopt the more restrictive definition that calls principal an action that is free and proper, and which implies that the space of orbits is Hausdorff. As we shall argue later on, our definition is equivalent with that of H. Cartan, who requires not only the continuity of the translation map, but also that $X \times_{X/G} X$ be closed in $X \times X$.

1.1 Proper maps

To begin with, let us recall the separation axioms of topology and other basic definitions. Let X be a topological space, $O(X)$ its topology (the family of open subsets) and $C(X, Y)$ the space of continuous functions from X to some topological space Y . We say that X is

$$T_0 \quad \text{iff } \forall x, y \in X, x \neq y \exists U \in O(X) : (x \in U \text{ and } y \notin U) \text{ or } (x \notin U \text{ and } y \in U);$$

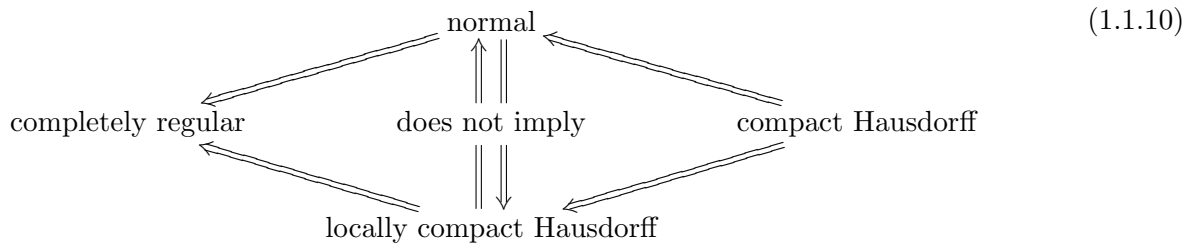
$$T_1 \quad \text{iff } \forall x, y \in X, x \neq y \exists U, V \in O(X) : (x \in U \text{ and } y \notin U) \text{ and } (x \notin V \text{ and } y \in V);$$

$$T_2 \quad \text{or Hausdorff iff } \forall x, y \in X, x \neq y \exists U, V \in O(X) : x \in U, y \in V \text{ and } U \cap V = \emptyset;$$

$$T_3 \quad \text{or regular iff it is } T_2 \text{ and } \forall x \in X, \text{ closed } C \subseteq X, x \notin C \exists U, V \in O(X) : \\ x \in U, C \subseteq V \text{ and } U \cap V = \emptyset;$$

- $T_{3\frac{1}{2}}$ or Tichonov or completely regular iff it is T_2 and
 $\forall x \in X$, closed $C \subseteq X$, $x \notin C \exists f \in C(X, [0, 1]) : f(x) = 0$ and $f(C) = \{1\}$;
- T_4 or normal iff it is T_2 and \forall closed $C, D \subseteq X$, $C \cap D = \emptyset \exists U, V \in O(X) :$
 $C \subseteq U$, $D \subseteq V$ and $U \cap V = \emptyset$.

A topological space is called *compact* if out of any of its open covers one can choose a finite subcover, and *locally compact* if any of its points is contained in an open set whose closure is compact. (In some books, e.g. [5, 15], (locally) compact means (locally) compact and Hausdorff.) Every compact Hausdorff space is normal [15, Theorem 3.1.9] and every locally compact Hausdorff space is completely regular [15, Theorem 3.3.1], though not necessarily normal [15, Example 3.3.14]. Also, since every metrizable space is normal [15, Corollary 4.1.13], the space of rational numbers is normal although one can directly check that it is not locally compact. Symbolically, we can write all this in the following way:



Having established the terminology, let us now review the relevant key facts.

LEMMA 1.1 (Urysohn Lemma). *Let X be a normal topological space and C, D disjoint closed subsets of X . Then there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(C) \subseteq \{0\}$ and $f(D) \subseteq \{1\}$.*

THEOREM 1.2 (Tietze-Urysohn Extension Theorem). *A topological space X is normal if and only if for any closed subset $C \subseteq X$ any \mathbb{C} -valued bounded continuous function on C can be extended to a bounded continuous function on X with the same sup norm.*

LEMMA 1.3. (cf. [15, Exercise 3.2.J]) *Let Y be a locally compact Hausdorff space and L a compact subset of Y . Then any continuous function $g : L \rightarrow \mathbb{C}$ can be extended to a continuous function on Y with compact support and the same sup norm.*

PROOF. Since L is compact and Y is locally compact Hausdorff, there exists an open set U containing L and whose closure \bar{U} is compact [15, Theorem 3.3.2]. Similarly, there exists open V such that $\bar{U} \subseteq V$ and \bar{V} is compact. Since Y is Hausdorff, \bar{V} and L are closed. The set L is also closed in \bar{V} , as $\bar{V} \setminus L = \bar{V} \cap (Y \setminus L)$. The set U is open and $U = U \cap \bar{V}$, so that it is open in \bar{V} . Thus $\bar{V} \setminus U$ is closed in \bar{V} . Next, we have $L \cap (\bar{V} \setminus U) = \emptyset$, so that we can define a continuous function $\tilde{g} : L \cup (\bar{V} \setminus U) \rightarrow \mathbb{C}$ by the formula

$$\tilde{g}(y) = \begin{cases} g(y) & \text{for } y \in L, \\ 0 & \text{for } y \in \bar{V} \setminus U. \end{cases} \quad (1.1.11)$$

(If the union of two closed subsets equals the space, then a function is continuous if and only if its restrictions to the closed subsets are continuous.) Clearly, $\|\tilde{g}\| = \|g\|$. Since \bar{V} is a compact Hausdorff

space, it is normal. Thus, by Theorem 1.1, there exists a continuous extension $\tilde{g} : \bar{V} \rightarrow \mathbb{C}$ of \tilde{g} such that $\|\tilde{g}\| = \|\tilde{g}\|$. Now define $g_Y : Y \rightarrow \mathbb{C}$ by the formula

$$g_Y(y) = \begin{cases} \tilde{g}(y) & \text{for } y \in \bar{V}, \\ 0 & \text{for } y \in Y \setminus U. \end{cases} \quad (1.1.12)$$

This function is well defined because $(Y \setminus U) \cap \bar{V} = \bar{V} \setminus U$ and $\tilde{g}(Y \setminus U) \subseteq \{0\}$. It is continuous because both \bar{V} and $Y \setminus U$ are closed and $\bar{V} \cup (Y \setminus U) = Y$. It extends g and has compact support by construction. Finally, its norm remains unchanged ($\|g_Y\| = \|g\|$). \square

A straightforward application of this lemma (or its simpler version), allows one to prove a C^* -characterisation of properness. To this end, recall first that a function $f : X \rightarrow \mathbb{C}$ is said to be *vanishing at infinity* iff

$$\forall \varepsilon > 0 \exists \text{compact } K \subseteq X \forall x \notin K : |f(x)| < \varepsilon. \quad (1.1.13)$$

The C^* -algebra of all continuous vanishing at infinity functions on a locally compact Hausdorff space is denoted by $C_0(X)$. Now we can state:

LEMMA 1.4. *Let X and Y be locally compact Hausdorff spaces. A continuous map $F : X \rightarrow Y$ is proper if and only if $F^*(C_0(Y)) \subseteq C_0(X)$.*

PROOF. Assume first that F is proper. We want to show that $f \circ F$ vanishes at infinity for any $f \in C_0(Y)$. Choose $\varepsilon > 0$. There exists compact $K_\varepsilon \subseteq Y$ such that $y \notin K_\varepsilon \Rightarrow |f(y)| < \varepsilon$. Since F is proper, $F^{-1}(K_\varepsilon)$ is compact. Moreover, we have $x \notin F^{-1}(K_\varepsilon) \Leftrightarrow F(x) \notin K_\varepsilon \Rightarrow |f(F(x))| < \varepsilon$. Hence $F^*(f) \in C_0(X)$, as needed.

Assume now that $F^*(C_0(Y)) \subseteq C_0(X)$. Since Y is Hausdorff, any compact subset K of Y is closed. Therefore, so is $F^{-1}(K)$ by the continuity of F . The case $K = \emptyset$ is trivial, so that, without loss of generality, we may assume $K \neq \emptyset$. Then, by Lemma 1.3, there exists a compactly supported continuous function f on Y such that $f(K) = \{1\}$. Due to our assumption $f \circ F$ vanishes at infinity. Hence, for $\varepsilon = \frac{1}{2}$, there exists a compact set $L \subseteq X$ such that $x \notin L \Rightarrow |f(F(x))| < \frac{1}{2}$. Consequently, $F^{-1}(K) \subseteq L$, and $F^{-1}(K)$ has to be compact as a closed subset of a compact set. This proves that F is proper. \square

For the sake of the general C^* -algebraic setting, it is useful to have the following refinement of the above lemma:

COROLLARY 1.5. *Let X and Y be locally compact Hausdorff spaces and $F : X \rightarrow Y$ a continuous map. Then F is proper if and only if $F^*(A) \subseteq C_0(X)$ for some norm dense subset A of $C_0(Y)$.*

PROOF. Due to Lemma 1.4, it suffices to show that $F^*(A) \subseteq C_0(X)$ for some norm dense subset A of $C_0(Y)$ if and only if $F^*(C_0(Y)) \subseteq C_0(X)$. To prove the left-to-right implication, recall that for any set S and any continuous function f , we have $f(\bar{S}) \subseteq \overline{f(S)}$. (Indeed, $S \subseteq f^{-1}(f(S)) \subseteq f^{-1}(\overline{f(S)})$. As the last set is closed by the continuity of f , we have $\bar{S} \subseteq f^{-1}(\overline{f(S)})$, whence $f(\bar{S}) \subseteq \overline{f(S)}$.) Therefore, since $C_0(X)$ is norm closed and F^* is continuous in the norm topology, we have:

$$F^*(C_0(Y)) = F^*(\bar{A}) \subseteq \overline{F^*(A)} \subseteq \overline{C_0(X)} = C_0(X). \quad (1.1.14)$$

The opposite implication is trivial. \square

1.2 C^* -characterisation of injectivity

We can pass now to the most demanding part of our topological introduction, which a C^* -characterisation of injectivity. Although such a result should have been proven decades ago, the only reference we are aware of is [14]. The main difficulty is to characterize injective but non-proper maps. (For instance, take $[0, 1) \ni \theta \mapsto e^{2\pi i\theta} \in \mathbb{C}$.) To this end, we need the concept of strict topology on multiplier C^* -algebras (see [31]). If A is a C^* -algebra and $M(A)$ is its multiplier C^* -algebra, then a subset $C \subseteq M(A)$ is *strictly dense* (dense in the strict topology) iff $\forall x \in M(A) \exists \{c_\alpha\}_{\alpha \in \text{directed set}} \subseteq C \forall a \in A$:

$$\lim_{\alpha} \|(c_\alpha - x)a\| + \|a(c_\alpha - x)\| = 0. \quad (1.2.15)$$

In the commutative case, we have $M(C_0(X)) = C_b(X)$ (the latter means the algebra of continuous bounded functions on X), and our C^* characterisation of injectivity takes the following form:

THEOREM 1.6 ([14]). *Let X and Y be locally compact Hausdorff spaces. A continuous map $F : X \rightarrow Y$ is injective if and only if $F^*(C_0(Y))$ is strictly dense in $C_b(X)$.*

PROOF. Suppose first that F is not injective. Let $x \neq y$ be such that $F(x) = F(y)$. Since X is Hausdorff, there exists a closed set containing y and not containing x . Since X is also locally compact, it is completely regular. Therefore, there exists a continuous function f such that $f(x) \neq f(y)$. On the other hand, as $\{x, y\}$ is compact, by Lemma 1.3, there exists $g \in C_0(X)$ such that $g(x) = 1 = g(y)$. If $F^*(C_0(Y))$ is strictly dense in $C_b(X)$, then $\lim_{\alpha} \|c_\alpha g - fg\| = 0$ for some generalised sequence $\{c_\alpha\}_{\alpha \in I} \subseteq F^*(C_0(Y))$. (Note that $c_\alpha(x) = c_\alpha(y)$.) Thus, to derive the desired contradiction, we can use the triangle inequality:

$$\begin{aligned} \|c_\alpha g - fg\| &\geq \max\{|(c_\alpha g)(x) - (fg)(x)|, |(c_\alpha g)(y) - (fg)(y)|\} \\ &\geq \frac{1}{2} (|(c_\alpha g)(y) - (fg)(x)| + |(c_\alpha g)(y) - (fg)(y)|) \\ &\geq \frac{|(fg)(x) - (fg)(y)|}{2} > 0. \end{aligned} \quad (1.2.16)$$

(We simply exploited the fact that the evaluation map is continuous in the strict topology.) Assume now that F is injective. Let $f \in C_b(X)$. By Lemma 1.3, to each compact subset K of X we can assign $c_K \in C_c(X)$ (compactly supported continuous function) such that $c_K(K) \subseteq \{1\}$. Define $f_K = fc_K$ and assume $K \neq \emptyset$. Since K is compact, F is injective and Y is Hausdorff, $F|_K : K \rightarrow F(K)$ is a homeomorphism (e.g., see [5, p.87]). Thus $f_K \circ (F|_K)^{-1}$ is continuous on the compact subset $F(K)$ of a locally compact Hausdorff space Y . Hence, by Lemma 1.3, it can be extended to $g_K \in C_c(Y)$ such that $\|g_K\| = \|f_K \circ (F|_K)^{-1}\|$. We want to show that $\lim_{\emptyset \neq K \subseteq X} F^*(g_K) = f$ (in the strict topology). This means that

$$\forall a \in C_0(X) : \lim_{\emptyset \neq K \subseteq X} \|(F^*(g_K) - f)a\| = 0. \quad (1.2.17)$$

If X is compact, we are done. Otherwise, choose $a \in C_0(X)$ and $\varepsilon > 0$. For any non-empty compact subset K of X , we have

$$\begin{aligned} \|F^*(g_K) - f\| &\leq \|g_K\| + \|f\| \\ &\leq \|f_K\| + \|f\| \\ &\leq \|f\| \|c_K\| + \|f\| \\ &= 2\|f\|. \end{aligned} \quad (1.2.18)$$

On the other hand, as $((F^*(g_K) - f)a)(K) = \{0\}$, we have

$$\begin{aligned}
\|(F^*(g_K) - f)a\| &= \sup_{x \notin K} |F^*(g_K) - f| |a| \\
&\leq \sup_{x \notin K} |F^*(g_K) - f| \sup_{x \notin K} |a| \\
&\leq \|F^*(g_K) - f\| \sup_{x \notin K} |a| \\
&\leq 2\|f\| \sup_{x \notin K} |a|.
\end{aligned} \tag{1.2.19}$$

Next, since a is a function vanishing at infinity, there exists a non-empty compact set K such that $2\|f\| \sup_{x \notin K} |a| < \varepsilon$. Therefore,

$$\forall a \in C_0(X), f \in C_b(X) : \lim_{\emptyset \neq K \subseteq X} \|(F^*(g_K) - f)a\| = 0, \tag{1.2.20}$$

i.e., $F^*(C_0(Y))$ is strictly dense in $C_b(X)$. \square

Again, we can refine the assertion of this theorem to suit our later purpose. To this end, let us first prove the following general lemma:

LEMMA 1.7. *Let $\varphi : S \rightarrow T$ be continuous and let $A \subseteq B \subseteq \overline{A} \subseteq S$. Assume also that T is equipped with an additional coarser topology (i.e., the closure in the coarser topology is bigger: $\overline{C} \subseteq \overline{C}^c \subseteq T$). Then $\overline{\varphi(A)}^c = \overline{\varphi(B)}^c$.*

PROOF. Since φ is continuous, we have $\varphi(\overline{A}) \subseteq \overline{\varphi(A)}$. Hence

$$\varphi(A) \subseteq \varphi(B) \subseteq \varphi(\overline{A}) \subseteq \overline{\varphi(A)} \subseteq \overline{\varphi(A)}^c. \tag{1.2.21}$$

Consequently, $\overline{\varphi(A)}^c \subseteq \overline{\varphi(B)}^c \subseteq \overline{\varphi(A)}^c$. \square

The strict topology is coarser than the norm topology. Therefore, taking $B = \overline{A} = C_0(Y)$ in the preceding lemma and combining it with Theorem 1.6, yields:

COROLLARY 1.8. *Let X and Y be locally compact Hausdorff spaces and $F : X \rightarrow Y$ a continuous map. If F is injective, then $F^*(A)$ is strictly dense in $C_b(X)$ for any norm dense subset A of $C_0(Y)$. If $F^*(A)$ is strictly dense in $C_b(X)$ for some norm dense subset A of $C_0(Y)$, then F is injective.*

1.3 Free and proper actions

We have just derived the C^* -characterisations of injectivity and properness as independent conditions. Applying it to the map (1.0.3), we obtain a C^* -definition of free actions and proper actions. Since we are interested here in actions that are both free and proper (principal), we would like to know what is a good C^* -characterisation of injectivity and properness treated simultaneously, as one condition. In other words, the question is what strict density of $\text{Im}F^*$ in $C_b(X)$ (injectivity of $F : X \rightarrow Y$) and the inclusion $\text{Im}F^* \subseteq C_0(X)$ (properness of F) add up to. The answer is: the surjectivity of F^* . To prove this assertion by direct topological arguments, let us first recall two more lemmas of general topology.

LEMMA 1.9. *Let X and Y be Hausdorff spaces, and let Y be locally compact. A continuous injection $F : X \rightarrow Y$ is proper if and only if it is closed.*

PROOF. Assume first that F is closed. Then $F(X)$ is closed in Y . Since Y is Hausdorff, any compact $K \subseteq Y$ is closed. Consequently, $F(X) \cap K$ is closed in Y and in K . Moreover, it is compact as a closed subset of a compact set. On the other hand, since F is injective and closed, the map $F^{-1} : F(X) \rightarrow X$ is continuous. Hence the set $F^{-1}(F(X) \cap K) = F^{-1}(K)$ is also compact. This shows that F is proper.

Suppose now that F is proper but not closed. Then there exists a closed $C \subseteq X$ such that $F(C)$ is not closed. This means that there exists $y \in \overline{F(C)} \setminus F(C)$. Since Y is locally compact, there exists a neighbourhood U of y such that \overline{U} is compact. The set $F^{-1}(\overline{U})$ is also compact due to the properness of F . As X is Hausdorff, $C \cap F^{-1}(\overline{U})$ is again compact by the same argument as in the first part of the proof. Therefore, $F(C \cap F^{-1}(\overline{U}))$ is compact, whence closed by the Hausdorffness of Y . Clearly, $F(C \cap F^{-1}(\overline{U})) \subseteq F(C)$, so that

$$y \notin F(C \cap F^{-1}(\overline{U})). \quad (1.3.22)$$

We want to show that $y \in \overline{F(C \cap F^{-1}(\overline{U}))} = F(C \cap F^{-1}(\overline{U}))$ and thus derive the desired contradiction. To this end, recall that

$$x \in \overline{A} \Leftrightarrow \forall \text{ open } U \ni x : A \cap U \neq \emptyset. \quad (1.3.23)$$

(Indeed, $x \notin \overline{A} \Leftrightarrow \exists \text{ closed } C \ni x, A \subseteq C \Leftrightarrow \exists \text{ closed } C, x \in X \setminus C, (X \setminus C) \cap A = \emptyset \Leftrightarrow \exists \text{ open } U \ni x, U \cap A = \emptyset$.) Now, take any neighbourhood V of y . Since $U \cap V$ is also a neighbourhood of $y \in \overline{F(C)}$, we have $U \cap V \cap F(C) \neq \emptyset$. Hence there exists $c \in C$ such that $F(c) \in U \cap V \subseteq \overline{U}$. Thus $c \in C \cap F^{-1}(\overline{U})$, whence $F(c) \in F(C \cap F^{-1}(\overline{U}))$. On the other hand, as $F(c) \in V$, we have $F(C \cap F^{-1}(\overline{U})) \cap V \neq \emptyset$. However, this means that $y \in \overline{F(C \cap F^{-1}(\overline{U}))} = F(C \cap F^{-1}(\overline{U}))$, which contradicts (1.3.22). \square

REMARK 1.10. In the second part of the proof we have not used the assumption that F is injective. Therefore, we can claim that any proper map from a Hausdorff space to a locally compact Hausdorff space is closed. \diamond

LEMMA 1.11. *Let Y be a locally compact Hausdorff space and X a closed subset of Y . Then any $f \in C_0(X)$ can be extended to an element of $C_0(Y)$.*

PROOF. Note first that any closed subset of a locally compact Hausdorff space is itself a locally compact Hausdorff space. Indeed, let $x \in X$ and U be a neighbourhood of x in Y such that \overline{U} is compact. Then $X \cap U$ is a neighbourhood of x in X and

$$\overline{X \cap U} = \bigcap_{C=\overline{C}, X \cap U \subseteq C \cap X} C \cap X \subseteq \bigcap_{C=\overline{C}, U \subseteq C} C \cap X = \left(\bigcap_{C=\overline{C}, U \subseteq C} C \right) \cap X = \overline{U} \cap X. \quad (1.3.24)$$

Since X is closed, $\overline{U} \cap X$ is compact. Hence it follows from (1.3.24) that $\overline{X \cap U}$ is compact. Consequently, X is a locally compact Hausdorff space. Let $X^+ = X \cup \{\infty\}$ and $Y^+ = Y \cup \{\infty\}$ denote the one-point compactifications of X and Y , respectively. (Recall that the open sets in X^+ are the open sets in X and the sets of the form $(X \setminus K) \cup \{\infty\}$, where K is compact.) Let us now show that any $f \in C_0(X)$ extends to $f^+ \in C(X^+)$ by the formula $f^+(\infty) = 0$. If $D \subseteq \mathbb{C}$ is an open disc not containing 0, then $\infty \notin (f^+)^{-1}(D)$, and we have $(f^+)^{-1}(D) = f^{-1}(D)$. By the continuity of f , the set $f^{-1}(D)$ is open in X , and consequently in X^+ . If $D_\varepsilon \subseteq \mathbb{C}$ is an open disc of radius ε centered at 0, then, as f is a function vanishing at infinity, there exists a compact subset K_ε of X such that $x \notin K_\varepsilon \Rightarrow f(x) \in D_\varepsilon$. Hence the set $(f^+)^{-1}(D_\varepsilon) = \{\infty\} \cup (X \setminus K_\varepsilon) \cup (f|_{K_\varepsilon})^{-1}(D_\varepsilon)$ is open in X^+ due to the continuity of f . Since any open set in \mathbb{C} can be decomposed into the union of open discs belonging to the above discussed family, we can conclude that f^+ is continuous on X^+ , as claimed. Next, since $Y^+ \setminus X^+ = Y \setminus X$ is open in Y , it is open in Y^+ . Hence X^+ is a closed subset of the compact Hausdorff space Y^+ . As the latter is normal, we can apply Theorem 1.2 to conclude that f^+ can be extended to $f_{Y^+}^+ \in C(Y^+)$. We need to prove now that the restriction of $f_{Y^+}^+$ to Y is an

element of $C_0(Y)$. First, note that $f_{Y^+}^+(\infty) = f^+(\infty) = 0$. Moreover, as $f_{Y^+}^+$ is continuous at ∞ , we have

$$\forall \varepsilon > 0 \exists \text{ compact } L_\varepsilon : y \in (Y \setminus L_\varepsilon) \cup \{\infty\} \Rightarrow |f_{Y^+}^+(y)| < \varepsilon. \quad (1.3.25)$$

Since L_ε is compact by construction, we have that $f_{Y^+}^+|_Y$ vanishes at infinity, as needed. Finally, $f_{Y^+}^+|_Y$ is an extension of f because $f_{Y^+}^+|_Y(x) = f_{Y^+}^+(x) = f^+(x) = f(x)$. \square

Combining Lemma 1.9 and Lemma 1.11 we obtain:

THEOREM 1.12. *Let X and Y be locally compact Hausdorff spaces. A continuous map $F : X \rightarrow Y$ is injective and proper if and only if $F^*(C_0(Y)) = C_0(X)$.*

Reasoning along the lines of (1.1.14) proves the following refinement of this theorem:

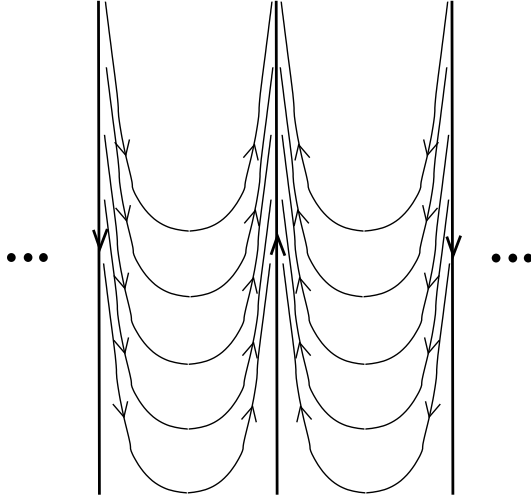
COROLLARY 1.13. *Let X and Y be locally compact Hausdorff spaces and $F : X \rightarrow Y$ a continuous map. Then F is injective and proper if and only if $\overline{F^*(A)} = C_0(X)$ for some norm dense subset A of $C_0(Y)$, and if and only if $\overline{F^*(A)} = C_0(X)$ for any norm dense subset A of $C_0(Y)$.*

Thus we are able to define a principal action of G on X entirely in terms of their C^* -algebras — it suffices to take $F = F^G$. However, as was already mentioned at the beginning of this section, for non-compact groups it is not sufficient to consider F_X^G . Indeed, with the help of Lemma 1.9, one can check that F^G is injective and proper if and only if F_X^G is a homeomorphism and its image is closed. On the other hand, F_X^G is a homeomorphism if and only if the translation map (1.0.4) is continuous. Therefore, the action is principal if and only if the translation map is continuous and $X \times_{X/G} X$ is closed in $X \times X$ [5, Proposition 6, p.255]. This way we have arrived at the definition of principal action provided by H. Cartan [8, condition (FP), p.6-05].

Recall that a continuous injection need not map homeomorphically its domain to its image, e.g., take $[0, 1) \ni \theta \mapsto e^{2\pi i\theta} \in \mathbb{C}$. The ergodic action of \mathbb{R} on T^2 gives an example of F_X^G which is a continuous bijection but not a homeomorphism (discontinuous translation map). Similarly, a continuous map that maps homeomorphically its domain to its image need not be proper, e.g., take $\mathbb{R} \ni x \mapsto \arctan(x) \in \mathbb{R}$. It is not proper because its image is not closed. This is precisely why it is not sufficient to assume that F_X^G is a homeomorphism. Let us consider the following example in which F_X^G is a homeomorphism and the action is not principal. (This example is essentially the same as in [37, p.298].)

EXAMPLE 1.14. Take $X = \mathbb{R}^2$ and $G = \mathbb{R}$. The idea is to define the action of \mathbb{R} on \mathbb{R}^2 by the flow of the (unital) smooth vector field on \mathbb{R}^2 given by the formula $v(x, y) = (\cos x, \sin x)$. This vector field is invariant with respect to the vertical translations. Its integral curves are vertical lines (special orbits) for $x = \frac{\pi}{2} + \mu\pi$, $\mu \in \mathbb{Z}$, and otherwise satisfy the equation $\gamma'(x) = \tan(x)$ (regular orbits). Hence $\gamma(x) = \log |\cos x|^{-1} + a$, $a \in \mathbb{R}$, or $x = \frac{\pi}{2} + \mu\pi$. Roughly speaking, what happens is that regular orbits converge to both of their neighbouring special orbits. Therefore, we can choose a sequence of pairs of points from the same regular orbit that converges to a pair of points belonging to two different orbits. This shows that the image of F^G is not closed. On the other hand, contrary to the Kronecker foliation case, if such a sequence of pairs converges to a pair of points on the same orbit, then the induced sequence of group elements determined by each pair also converges to the group element corresponding to the limit pair. This means that F_X^G is a homeomorphism despite F^G not being proper. Let us explicitly prove all of this.

First, we want to derive an explicit formula describing the action. On the special orbits, it is clear that the action is simply the vertical translation in one or the other direction. For a regular orbit, we have to determine the position of the moved point as a function of the initial point and the length of the curve joining the two points. (The latter is, by construction, the element of the group \mathbb{R} by



which the initial point has been moved.) Since all regular orbits can be obtained by a vertical shift of a chosen orbit, the first coordinate of the moved point depends only on the first coordinate x of the initial point and the group element t . Let us denote the value of the first coordinate of the moved point by $g(x, t)$. It is determined by the following equation

$$\int_x^{g(x,t)} \sqrt{1 + (\gamma'(s))^2} ds = t. \quad (1.3.26)$$

Solving this equation yields

$$g(x, t) = \begin{cases} \arcsin g_0(x, t) + 2\mu\pi & \text{for } x \in (-\frac{\pi}{2} + 2\mu\pi, \frac{\pi}{2} + 2\mu\pi), \mu \in \mathbb{Z}, \\ \pi - \arcsin g_0(x, t) + 2\mu\pi & \text{for } x \in (\frac{\pi}{2} + 2\mu\pi, \frac{3\pi}{2} + 2\mu\pi), \mu \in \mathbb{Z}, \end{cases} \quad (1.3.27)$$

$$g_0(x, t) := \frac{e^t(1 + \sin x) - e^{-t}(1 - \sin x)}{e^t(1 + \sin x) + e^{-t}(1 - \sin x)}.$$

Note that g remains well defined also at the special values of x , i.e., for $x = \frac{\pi}{2} + \mu\pi$, $\mu \in \mathbb{Z}$, it is continuous at all these points and independent of t . The latter agrees with the formula for the action on the special orbits. To find out the behaviour of the second coordinate under the group action, observe that the second coordinate of the moved point can be written as $y + \tilde{g}(x, t)$, where (x, y) stands for the initial point, t is the element of the group by which the initial point has been moved, and

$$\tilde{g}(x, t) := \gamma(g(x, t)) - \gamma(x) = \log \frac{e^t(1 + \sin x) + e^{-t}(1 - \sin x)}{2}. \quad (1.3.28)$$

Again, the vertical invariance makes it irrelevant which regular orbit γ we take. Thus we have arrived at the explicit formula for the continuous action:

$$\mathbb{R}^2 \times \mathbb{R} \ni ((x, y), t) \longmapsto (g(x, t), y + \tilde{g}(x, t)) \in \mathbb{R}^2. \quad (1.3.29)$$

It is entertaining to verify directly that this formula indeed gives an action, i.e., that $g(g(x, t), t') = g(x, t + t')$ and $\tilde{g}(x, t) + \tilde{g}(g(x, t), t') = \tilde{g}(x, t + t')$. Similarly, one can check that, for any $x \in \mathbb{R}$, if $g(x, t) = x$ and $\tilde{g}(x, t) = 0$, then $t = 0$. Hence the action is free, and F^G is a continuous injection. On the other hand, the sequence of pairs of points $((x_n, y_n), (x'_n, y'_n))$ given by the formulas

$$x_n = \frac{1}{n} - \frac{\pi}{2}, \quad y_n = 0, \quad x'_n = -\frac{1}{n} + \frac{\pi}{2}, \quad y'_n = 0, \quad (1.3.30)$$

is contained in $\mathbb{R}^2 \times_{\mathbb{R}^2/\mathbb{R}} \mathbb{R}^2$, but it converges to $((-\frac{\pi}{2}, 0), (\frac{\pi}{2}, 0)) \notin \mathbb{R}^2 \times_{\mathbb{R}^2/\mathbb{R}} \mathbb{R}^2$. Consequently, the image of F^G is not closed, so that, by Lemma 1.9, F^G is not a proper map.

Finally, to show that F_X^G is a homeomorphism, it suffices to prove that the translation map $\tilde{\tau}$ is continuous. Assume that (x, y) and (x', y') are two points on the same regular orbit. Solving the equation $g(x, t) = x'$ for t gives

$$t = \log \sqrt{\frac{1 - \sin x}{1 + \sin x} \frac{1 + \sin x'}{1 - \sin x'}} = \log \left(\left| \frac{\cos x}{\cos x'} \right| \frac{1 + \sin x'}{1 + \sin x} \right) = \log \left(\left| \frac{\cos x'}{\cos x} \right| \frac{1 - \sin x}{1 - \sin x'} \right). \quad (1.3.31)$$

Thus $\tilde{\tau}$ is evidently continuous on the (dense) subset of $\mathbb{R}^2 \times_{\mathbb{R}^2/\mathbb{R}} \mathbb{R}^2$ consisting of all pairs from regular orbits. Remembering the relationship $y' - y = \log \left| \frac{\cos x}{\cos x'} \right|$, we can re-write (1.3.31) in the form

$$t = y' - y + \log \frac{1 + \sin x'}{1 + \sin x} = y - y' + \log \frac{1 - \sin x}{1 - \sin x'}. \quad (1.3.32)$$

Now, let $(-\frac{\pi}{2} + 2\mu\pi, y_0), (-\frac{\pi}{2} + 2\mu\pi, y'_0), \mu \in \mathbb{Z}$, be a limit pair of some sequence

$$\{(x_n, y_n), (x'_n, y'_n)\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^2 \times_{\mathbb{R}^2/\mathbb{R}} \mathbb{R}^2. \quad (1.3.33)$$

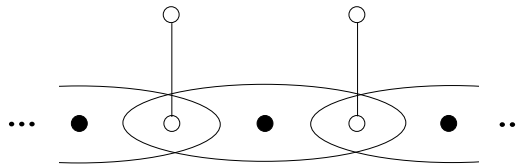
Then, for any sufficiently big n , both $\sin x_n$ and $\sin x'_n$ are different from 1. Therefore, due to (1.3.32), for any x_n and x'_n in a small neighbourhood of $-\frac{\pi}{2} + 2\mu\pi$, we have:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \tilde{\tau}((x_n, y_n), (x'_n, y'_n)) \\ &= \lim_{n \rightarrow \infty} \left(y_n - y'_n + \log \frac{1 - \sin x_n}{1 - \sin x'_n} \right) \\ &= y_0 - y'_0 \\ &= \tilde{\tau} \left(\left(-\frac{\pi}{2} + 2\mu\pi, y_0 \right), \left(-\frac{\pi}{2} + 2\mu\pi, y'_0 \right) \right). \end{aligned} \quad (1.3.34)$$

Much as above, if $(\frac{\pi}{2} + 2\mu\pi, y_0), (\frac{\pi}{2} + 2\mu\pi, y'_0), \mu \in \mathbb{Z}$, is a limit pair of some sequence from $\mathbb{R}^2 \times_{\mathbb{R}^2/\mathbb{R}} \mathbb{R}^2$, then

$$\begin{aligned} & \lim_{n \rightarrow \infty} \tilde{\tau}((x_n, y_n), (x'_n, y'_n)) \\ &= \lim_{n \rightarrow \infty} \left(y'_n - y_n + \log \frac{1 + \sin x'_n}{1 + \sin x_n} \right) \\ &= y'_0 - y_0 \\ &= \tilde{\tau} \left(\left(\frac{\pi}{2} + 2\mu\pi, y_0 \right), \left(\frac{\pi}{2} + 2\mu\pi, y'_0 \right) \right). \end{aligned} \quad (1.3.35)$$

Summarising, we have shown that F_X^G is a homeomorphism, whereas F^G is a continuous but not proper injection. The lack of properness manifests itself in the non-Hausdorffness of the quotient space \mathbb{R}^2/\mathbb{R} : the neighbouring special orbits cannot be separated by any open sets.



◇

1.4 Associated bundles

Let us first shortly recall some basic terminology related to topological bundles. In great generality, a *bundle* is a triple (E, π, M) , where E and M are topological spaces and $\pi : E \rightarrow M$ is a continuous surjective map. Here M is called the base space, E the total space, and π the projection of the bundle. For $p \in M$, the fibre over p is the topological space $\pi^{-1}(p)$. A *local section* of a bundle is a continuous map $s : U \rightarrow E$ with $\pi \circ s = \text{id}$, where U is an open subset of M . If each fibre of a bundle is endowed with a vector space structure such that the addition and scalar multiplication are continuous, we call it a *bundle of vector spaces*. (If in addition each fibre is finite dimensional, this coincides with the notion of a family of vector spaces [2, p.1].) A bundle (E, π, M) all of whose fibres are homeomorphic to a space F is called a *fibre bundle* with typical fibre F , and is denoted by (E, π, M, F) .

Let a topological group G act from the right on a topological space X . Then the triple $(X, \pi, X/G)$, where X/G is the orbit space (with the standard quotient topology) and π is the natural projection, is a bundle in the above sense. More generally, we call a bundle a *G-bundle* iff all the fibres are orbits of the G -action and its base space is homeomorphic to the orbit space. The projection of a G -bundle is necessarily an open map. Indeed, for any open $U \subseteq X$, we have $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} Ug$. Therefore, as G acts by homeomorphisms, $\pi^{-1}(\pi(U))$ is a union of open sets, whence itself open. Thus, by the definition of quotient topology, $\pi(U)$ is open, as claimed. Finally, if the action is principal, we arrive at the following fundamental definition:

DEFINITION 1.15. A *principal bundle* is a quadruple (X, π, M, G) such that

- (i) (X, π, M) is a bundle and G is a topological group acting continuously on X from the right,
- (ii) the action of G on X is principal (i.e., free and proper),
- (iii) $\pi(x) = \pi(y)$ if and only if $\exists g \in G : y = xg$ (the fibres are the orbits of G),
- (iv) the induced map $X/G \rightarrow M$ is a homeomorphism.

Observe that the induced map $X/G \rightarrow M$, existing by (iii), is always a continuous bijection but need not be a homeomorphism (e.g., if M has the indiscrete topology). However, it is an open map (and hence a homeomorphism) if and only if the bundle projection π is an open map. Next, note that if G is a topological group acting properly on a locally compact Hausdorff space X , then X/G is also locally compact and Hausdorff [5, Proposition 3, p.253; Proposition 9, p.257]. We say that a principal bundle is (locally) compact iff the involved spaces are (locally) compact Hausdorff.

Clearly, a principal action of G on X automatically makes the bundle $(X, \pi, X/G)$ a principal bundle. However, not every principal bundle has to be of this form. If we replace X/G by a homeomorphic space, not only we formally define a different bundle, but also it might happen that such a new bundle is not equivalent to $(X, \pi, X/G)$ [17, p.157]. Thus equivalent (equivariantly homeomorphic) G -spaces might lead to inequivalent principal bundles.

Since sometimes one can relax the proper-action assumption to the continuity-of-translation-map condition, we define a *quasi principal bundle* exactly as above except for point (ii) that is replaced by:

- (ii') the action of G on X is free and the translation map is continuous.

As argued below Corollary 1.13, a quasi principal bundle is a principal bundle if and only if the fibre product $X \times_M X$ is closed in $X \times X$. Note also that, due to the continuity of the translation map, each fibre of a quasi principal bundle is homeomorphic to G , so that it is always a fibre bundle. (See (1.5.45) and replace U by a one-point set.)

Let (X, π, M) be a G -bundle, and let G act on the left on another topological space F . Then $(x, v, g) \mapsto (xg, g^{-1}v)$ defines a right action of G on $X \times F$. The map $\pi_F : (X \times F)/G \rightarrow M$, $[(x, v)] \mapsto \pi(x)$, is well defined and continuous. Much as for a G -bundle, this projection is an open map. To show this, consider the commutative diagram of continuous surjections:

$$\begin{array}{ccc} X \times F & \xrightarrow{\pi_E} & (X \times F)/G \\ \text{pr}_1 \downarrow & & \downarrow \pi_F \\ X & \xrightarrow{\pi} & M \end{array} \quad (1.4.36)$$

(Here the left vertical and upper horizontal arrows are the obvious surjections.) We already know that the projections of G -bundles are open, and the canonical surjection pr_1 is open. Taking advantage of this and the surjectivity and continuity of π_E , we obtain that $\pi_F(U)$ is open for any open $U \subseteq (X \times F)/G$:

$$\pi_F(U) = \pi_F(\pi_E(\pi_E^{-1}(U))) = (\pi \circ \text{pr}_1)(\pi_E^{-1}(U)). \quad (1.4.37)$$

For a quasi principal bundle one can check that the assignment $v \mapsto [(x, v)]$ defines a homeomorphism from F to the fibre over $\pi(x)$. (Replace U by $\{\pi(x)\}$ in (1.5.50).) Thus, $((X \times F)/G, \pi_F, M, F)$ is a fibre bundle with typical fibre F . It is called an *associated fibre bundle*.

Global continuous sections of an associated fibre bundle can be identified with continuous equivariant maps defined on X with values in F . Let

$$\text{Hom}_G(X, F) := \{ f \in C(X, F) \mid f(xg) = g^{-1}f(x) \}, \quad (1.4.38)$$

and let $\Gamma(E)$ denote the space of continuous global sections of a bundle (E, π, M) . We have:

LEMMA 1.16. *Assume that (X, π, M, G) is a quasi principal bundle (G acts freely on X and the translation map is continuous). Let $E = (X \times F)/G$ be a fibre bundle associated to (X, π, M, G) . Then the formulas*

$$s_f : M \ni \pi(x) \longmapsto [(x, f(x))] \in E, \quad f_s : X \ni x \longmapsto v \in F, \quad s(\pi(x)) = [(x, v)],$$

define mutually inverse bijections between $\text{Hom}_G(X, F)$ and $\Gamma(E)$.

PROOF. One easily verifies that f_s and s_f are well defined. It is also clear that f_s is equivariant, $\pi \circ s_f = \text{id}$, and $f_{s_f} = f$, $s_{f_s} = s$. In order to see that the continuity of f entails the continuity of s_f , note that $\pi_E \circ (\text{id}, f) \circ \text{diagonal map} = s_f \circ \pi$, where π_E is the canonical quotient map $X \times F \rightarrow E$. The left hand side is evidently continuous, so that the continuity of s_f follows from the fact that π is an open surjection (see the second paragraph).

Indeed, for any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f_{AB}} & B \\ & \searrow f_{AC} & \downarrow f_{BC} \\ & & C \end{array} \quad (1.4.39)$$

where A, B, C are topological spaces, f_{AC} is continuous and f_{AB} is a surjection that maps open sets to open sets (not necessarily continuous), we can conclude that f_{BC} is continuous. To verify this, we employ the surjectivity of f_{AB} and compute:

$$f_{BC}^{-1}(U) = f_{AB}(f_{AB}^{-1}(f_{BC}^{-1}(U))) = f_{AB}(f_{AC}^{-1}(U)). \quad (1.4.40)$$

Hence, for any open $U \subseteq C$, the continuity of f_{AC} and the assumption that f_{AB} maps open sets to open sets entail that $f_{BC}^{-1}(U)$ is open. Thus f_{BC} is continuous, and substituting s_f for f_{BC} and π for f_{AB} yields the desired conclusion.

Conversely, assume that s is continuous. Note first that, since the map

$$\text{id} \times \pi_E : X \times X \times F \longrightarrow X \times E \tag{1.4.41}$$

is open (see above Definition 1.15) and $(\text{id} \times \pi_E)^{-1}(X \times_M E) = X \times_M X \times F$, by [5, Proposition 2a), p.51], the restriction of $\text{id} \times \pi_E$ to $X \times_M X \times F$ is an open map. Thus we obtain an open surjection $\overline{\pi}_E : X \times_M X \times F \rightarrow X \times_M E$. On the other hand, the continuity of the translation map implies that

$$T : X \times_M X \times F \ni (x, y, v) \longmapsto \tilde{\tau}(x, y)v \in F \tag{1.4.42}$$

is continuous. Furthermore, the property (1.0.7) of $\tilde{\tau}$ implies that T is well defined on the quotient $X \times_M E$. This yields the commutative diagram

$$\begin{array}{ccc} X \times_M X \times F & \xrightarrow{\overline{\pi}_E} & X \times_M E \\ & \searrow T & \downarrow \overline{T} \\ & & F \end{array} \tag{1.4.43}$$

Arguing as with (1.4.39), we infer that \overline{T} is continuous. Finally, since s is continuous by assumption and $f_s(x) = \overline{T}(x, s(\pi(x)))$, we can conclude that f_s is continuous, as desired. \square

If in addition F is equipped with a vector space structure compatible with its topology, and the action of G is given by a linear representation, the fibres of the associated fibre bundle

$$((X \times F)/G, \pi_F, M, F) \tag{1.4.44}$$

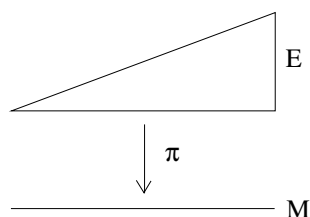
carry a natural vector space structure such that the homeomorphisms $v \mapsto [(x, v)]$ between the typical fibre and the fibres are linear. This way we obtain a bundle of vector spaces as an associated fibre bundle.

1.5 Local triviality and piecewise triviality

Local triviality is a notion commonly assumed in the classical setting of bundles — see standard definitions in [12, 16.12.1], [23, p.40], [34, p.13], [29, p.50].

DEFINITION 1.17. A bundle (E, π, M) is said to be *locally trivial*, if for every $p \in M$ there exists a neighbourhood U , a topological space F and a homeomorphism $\varphi : U \times F \rightarrow \pi^{-1}(U)$ that is fibre preserving, i.e. $\pi \circ \varphi : U \times F \rightarrow U$ is the canonical projection.

A very simple example of a bundle which is not locally trivial is exhibited by the following picture:



Among locally trivial bundles, particularly important are vector bundles — they are a starting point for K -theory.

DEFINITION 1.18. A bundle of vector spaces is called a *vector bundle* iff it is locally trivial, the trivialising maps are compatible with the linear structure, and each of the fibres is finite dimensional.

Note that a vector bundle need not be a fibre bundle. For instance, one might have 0-dimensional fibres over one connected component and 1-dimensional fibres over another connected component. However, a vector bundle is always a fibre bundle if its base space is connected [2, p.3].

Complex vector bundles (i.e. vector bundles whose fibres are complex vector spaces) are algebraically characterized as finitely generated projective modules over commutative C^* -algebras. This is one of the fundamentals of noncommutative geometry [39, pp.34–36].

THEOREM 1.19 (Serre [41], Swan [42]). *Let (E, π, M) be a complex vector bundle and M a compact Hausdorff space. Then the set $\Gamma(E)$ of all continuous global sections of E is a finitely generated projective module over the C^* -algebra $C(M)$ of continuous functions on M (with the natural module structure given by pointwise addition and multiplication). Conversely, any finitely generated projective module over $C(M)$ is isomorphic to $\Gamma(E)$ for some complex vector bundle (E, π, M) .*

There are generalisations of this theorem for general topological spaces using the notion of finite-type bundles of vector spaces. These are finite-dimensional bundles that are assumed to be locally trivial with respect to a finite covering of the base space admitting a partition of unity. (The partition-of-unity assumption allows one to go beyond paracompact spaces.) By [46, Theorem 1], the category of finite-type bundles of vector spaces over M is equivalent to the category of finitely generated projective modules over the algebra of all continuous functions on M . By [46, Theorem 2(3)] and the subsequent remark (see also [43]), the same is true for the algebra of bounded functions and the algebra of smooth functions, if M is a smooth manifold.

If E is a vector bundle, the trivialising map has to be compatible with its linear structure. For a G -bundle (X, π, M) , the trivialising map has to be G -equivariant. A traditional definition of a principal bundle assumes that it is locally trivial in this sense but does not require that the group action is proper [17, p.156]. For instance, the bundle studied in Example 1.14 is locally trivial (by Theorem 1.21 or direct inspection), so that it fits the aforementioned definition, but the group action is not proper. On the other hand, under the assumption of local triviality, one can remove from Definition 1.15 the condition that the quotient space is homeomorphic with the base space by the induced map.

Indeed, we already know that the induced map $f : X/G \rightarrow M$ is a continuous bijection. Thus it remains to see that it is an open map. Let $\{U_i\}_i$ be an open cover of M locally trivialising X , and let V be an open subset of X/G . Then each $V_i := V \cap f^{-1}(U_i)$ is an open set, and $V = \bigcup_i V_i$. Furthermore, $\pi^{-1}(f(V_i))$ is open because it is the preimage of V_i under the canonical quotient map. On the other hand, since $f(V_i) \subseteq U_i$, the preimage $\pi^{-1}(f(V_i))$ is mapped by a homeomorphism onto $f(V_i) \times G$. Therefore, $f(V_i) \times G$ is an open subset of $U_i \times G$. Consequently, $f(V_i)$ is an open subset of U_i (and hence M), because the projections from the Cartesian product of two sets onto its components are always open maps. Finally, it follows from the general fact $f(\bigcup_i A_i) = \bigcup_i f(A_i)$ (for any map and any family of sets) that $f(V) = \bigcup_i f(V_i)$. This is an open set, as desired.

PROPOSITION 1.20. *If (X, π, M, G) is a quasi principal bundle (G acts freely and the translation map is continuous), then local triviality is equivalent to the existence of local sections at each point of X .*

PROOF. If there exists a local section $\sigma : U \rightarrow X$, then we can construct a continuous map

$$\varphi_\sigma : U \times G \ni (u, g) \longmapsto \sigma(u)g \in \pi^{-1}(U). \quad (1.5.45)$$

Its inverse is given by $x \mapsto (\pi(x), \tilde{\tau}(\sigma(\pi(x)), x))$, so that φ_σ is evidently a trivialising map if $\tilde{\tau}$ is continuous. Conversely, if we have a trivialising map $\varphi : U \times G \rightarrow \pi^{-1}(U)$, then $u \mapsto \varphi(u, e)$ is clearly a local section. \square

The following theorem guarantees that, if $(X, \pi, M,)$ is a principal bundle and G is a Lie group, then it is a locally trivial bundle. The compact Lie group version of this result is attributed to A.M. Gleason [19], the general case is announced by J.-P. Serre [40, Théorème 1], and its proof can be found in [37, p.315].

THEOREM 1.21. *Let G be a Lie group acting freely on a completely regular space X . Then there exists a local section through each point of X if and only if the translation map is continuous.*

An immediate corollary of this theorem is that the quotient of a locally compact group by a closed Lie subgroup yields a locally trivial bundle. This is because in this situation the translation map is automatically continuous: $\tilde{\tau}(g, g') = g^{-1}g'$. Also, note that, since the translation map in Example 1.14 is continuous, it is an example of a locally trivial bundle over a non-Hausdorff space. On the other hand, the translation map for the Kronecker foliation (ergodic action of \mathbb{R} on T^2) is not continuous. Thus this is an example of a free action of a Lie group that does not allow local triviality.

EXAMPLE 1.22. In the light of Theorem 1.21, to construct an example of a principal bundle which is not locally trivial, we have to stay clear of Lie groups. What follows is a simple example of a compact principal bundle where we avoid the Lie group structure by constructing a group with the topology of the Cantor set. Let X be the infinite product of unitary groups $\prod_{\mathbb{N}} U(1)$, and G its subgroup that is the infinite product $\prod_{\mathbb{N}} (\mathbb{Z}/2\mathbb{Z})$ of the 2-element group $\mathbb{Z}/2\mathbb{Z}$ viewed as the subset $\{-1, 1\}$ of $U(1)$. Recall that the standard product topology is generated by the subsets $\prod_{i \in I} U_i \subseteq \prod_{i \in I} X_i$, where there exists a finite subset $J \subseteq I$ such that U_i is open in X_i for all $i \in J$ and $U_i = X_i$ for all $i \notin J$. With this topology, the group X is a compact Hausdorff space by the Tichonov Theorem, it is indeed a compact group (see [15, Problem 8.5.4.]) and G is a closed subgroup. Since the natural action of any subgroup on a group is free and any action of a compact group is proper, $(X, \pi, X/G, G)$ is automatically a principal G -bundle.

Suppose now that this bundle is locally trivial. Then there exists an open subset U of $X/G \cong \prod_{\mathbb{N}} S^1$ over which we have a local section. On the other hand, U is a union of generating subsets defined above. Hence it must contain an open subset of the form $\prod_{i=0}^n U_i \times \prod_{i=n+1}^{\infty} S^1$, where each $U_i \subseteq S^1$ is open. However, this implies that the principal $\mathbb{Z}/2\mathbb{Z}$ -bundle $U(1) \rightarrow S^1$ has a global section. Thus we arrive at the desired contradiction proving that the principal bundle $(X, \pi, X/G, G)$ is not locally trivial.

Let us now prove that $\prod_{\mathbb{N}} (\mathbb{Z}/2\mathbb{Z})$ with the (product) topology inherited from $\prod_{\mathbb{N}} U(1)$ is homeomorphic to the Cantor set with the topology it receives as a subspace of $[0, 1]$. The Cantor set C is, by definition, obtained by removing all open “middle-third” intervals from $[0, 1]$. Equivalently, if we write an arbitrary $t \in [0, 1]$ as $t = \sum_{j=1}^{\infty} a_j 3^{-j}$, where $a_j \in \{0, 1, 2\}$, then C consists of all those $t \in [0, 1]$ such that all the a_j are either 0 or 2, i.e., 1 does not occur as an a_j . Since the coefficients a_j of $t \in C$ are uniquely determined, any such t can be viewed as a sequence of 0’s and 2’s. This establishes an evident bijection from the C to $\prod_{\mathbb{N}} (\mathbb{Z}/2\mathbb{Z})$ mapping a sequence of 0’s and 2’s to the sequence of -1 ’s and 1 ’s obtained by replacing each 0 by 1 and each 2 by -1 .

It now remains to prove that this bijection is a homeomorphism. Note that, since both C and $\prod_{\mathbb{N}} (\mathbb{Z}/2\mathbb{Z})$ are compact Hausdorff spaces, we need only prove continuity in one direction. Also, it is clear that it suffices to check that the preimages of the generating open sets are open. If U is a generating open set in $\prod_{\mathbb{N}} (\mathbb{Z}/2\mathbb{Z})$, then its preimage consists of all those $t \in C \subseteq [0, 1]$ satisfying conditions on finitely many a_j ’s. Thus, the preimage of U is the union of subsets of C defined by fixing

the first n a_j 's and taking all possible a_j 's for $j > n$. Since such sets are of the form (open interval) $\cap \mathbb{C}$, they are open. Hence the preimage of U is open, as needed. \diamond

In the setting of C^* -algebras, the concept of fibre products [38] replaces the notion of gluing of topological spaces. In general, a gluing of Hausdorff spaces need not be Hausdorff. For instance, the gluing of two closed unit intervals along the open unit interval is not a Hausdorff space. On the other hand, a gluing of Hausdorff spaces along closed subsets is always a Hausdorff space ([5, p.135, Exercise 8]). Since the disjoint union of finitely many compact spaces is compact and the quotient map is always continuous, the gluing of compact spaces is always compact. Thus we can conclude that the gluing of compact Hausdorff spaces over a closed subset is always compact Hausdorff.

Spaces obtained by gluing compact Hausdorff spaces along closed subsets can be viewed as covered by the spaces from which they were glued. Such coverings, however, are coverings by finitely many closed subsets. They are different from the usual open covers. We now discuss the relationship of these two concepts in the context of local triviality of principal bundles.

DEFINITION 1.23. A principal bundle (X, π, M, G) is said to be *piecewise trivial*, if there exists a covering of M by finitely many closed sets W_1, \dots, W_n , and fibre-preserving G -equivariant homeomorphisms $\varphi_i : \pi^{-1}(W_i) \rightarrow W_i \times G$, $\forall i \in \{1, \dots, n\}$.

The assumption of piecewise triviality implies that the induced map $f : X/G \rightarrow M$ is a homeomorphism. Since it is a continuous bijection, it suffices to show that it is a closed map. This can be shown reasoning much as under the assumption of local triviality. Instead of the openness of the projections onto components, we use the fact that a non-empty subset of a Cartesian product is closed if and only if each of the factors is closed [5, p.48]. Then, in the final argument, we take advantage of the fact that a finite union of closed sets is closed. Also, the analogue of Proposition 1.20 is true in this setting.

Let M be a paracompact space. Then the existence of a partition of unity implies that every open covering has a subordinate covering by closed sets. As a consequence, every locally trivial bundle over a compact Hausdorff space is piecewise trivial with respect to a finite closed cover. However, the converse is not true.

EXAMPLE 1.24. Let $C \subseteq [0, 1]$ be the Cantor set. As in Example 1.22, it can be given the structure of a compact abelian group. Define a base space M to be the gluing along C of two copies I_1, I_2 , of $[0, 1]$. Denote by $x \mapsto [x]$ be the quotient map from the disjoint union of I_1 and I_2 to M . Then M is a compact Hausdorff space with a closed covering given by the two sets $[I_1], [I_2]$. Define now a total space X as the gluing of $I_1 \times C$ and $I_2 \times C$ by the identifying homeomorphism

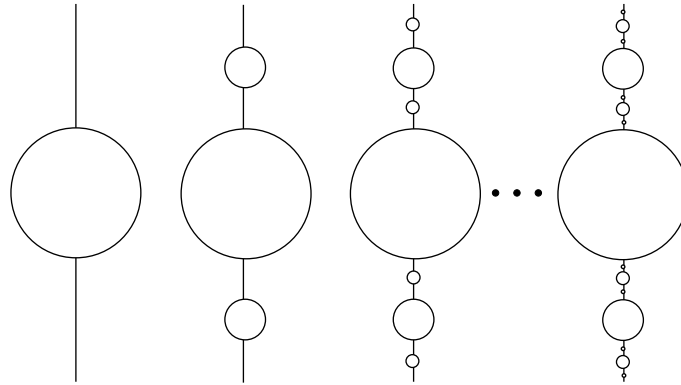
$$I_1 \times C \supseteq C \times C \ni (x, y) \longmapsto (x, xy) \in C \times C \subseteq I_2 \times C. \quad (1.5.46)$$

Since $C \times C$ is a closed subset of $[0, 1] \times C$, the total space X is also compact Hausdorff. It is straightforward to verify that the obvious right action of C on the disjoint union of $I_1 \times C$ and $I_2 \times C$ descends to a continuous free action on X . The action is obviously proper because C is a compact group. The obvious homeomorphism from the orbit space X/C to M defines a continuous surjection $\pi : X \rightarrow M$. Thus (X, π, M, C) is a piecewise trivial compact principal bundle.

Suppose now that X admits a global section. Then there exist continuous functions $f_1 : I_1 \rightarrow C$ and $f_2 : I_2 \rightarrow C$ satisfying the compatibility condition $f_2(x) = xf_1(x)$ for all $x \in C$. Since I_1 and I_2 are connected and the Cantor set is totally disconnected, both f_1 and f_2 must be constant maps. Therefore, the compatibility condition evidently cannot be satisfied, so that X is a non-trivial principal bundle. To prove that it is also not locally trivial, suppose that it is trivial over an open subset U

of M containing an element of C . Then there exists a local section over U , and the same argument as before applies to prove that this is impossible. (Any open set containing an element of C contains a copy of M .) Summarising, (X, π, M, C) is a piecewise trivial compact principal that is not locally trivial.

To end with, let us show that the space M is homeomorphic to a subset of the plane that can be visualised as follows (the bubble space):



It is created the same way one creates the Cantor set, only now instead of removing middle third intervals we replace them by circles whose diameters are equal to the lengths of replaced intervals. By cutting the bubble space vertically along the middle axis we obtain two homeomorphic subsets of a plane. They can be identified with the graph of a continuous function on $[0, 1]$. Indeed, the iterating procedure defining the bubble space can be easily translated into a sequence of uniformly convergent functions whose limit is the aforementioned function. Next, let $F_1 : I_1 \rightarrow \mathbb{C}$ and $F_2 : I_2 \rightarrow \mathbb{C}$ denote two copies of this function mapping onto the left and right “halves” of the bubble space, respectively. Since F_1 and F_2 agree only on the Cantor subsets of I_1 and I_2 , they define a continuous bijection from M onto the bubble space. This continuous bijection is a homeomorphism because M is compact and the bubble space is Hausdorff. \diamond

To complete the picture, we show that the principal bundle of Example 1.22 not only is not locally trivial, but also it is not piecewise trivial. Note first that for any finite cover $\{W_i\}_i$ of M there exists a set W_j such that $\bigcup_{i \neq j} W_i \neq M$. If the cover is also closed, then $M \setminus \bigcup_{i \neq j} W_i$ is a non-empty open set contained in W_j . Thus every finite closed cover has an element with a non-empty interior.¹ In the case of Example 1.22, local triviality over a closed set with non-empty interior would imply the existence of a local section over a non-empty open set. This, however, is explicitly proven in Example 1.22 to be impossible.

Finally, let us comment on the hierarchy of examples we have considered. Our first example was the action of \mathbb{R} on the 2-torus by translation under an irrational angle (leading to the Kronecker foliation). This is a free action which is not proper and has no continuous translation map. In Example 1.14, we have a free action, which is not proper, but has a continuous translation map. This yields a quasi principal bundle. (Notice that by removing the vertical lines where the orbits accumulate we get a trivial principal bundle, and that the whole bundle is locally trivial.) Example 1.22 provides an action that is principal, but not piecewise trivial or locally trivial. (The base space in this example is compact Hausdorff, so that the fact that it is not piecewise trivial implies that it is also not locally trivial.) Finally, Example 1.24 shows a principal bundle that is piecewise trivial but not locally trivial.

¹We owe this simple argument to S. L. Woronowicz. One can also argue using [5, Proposition 5, p.24].

If (E, π, M, F) is a locally trivial fibre bundle and U_1 and U_2 are trivialising neighbourhoods with homeomorphisms $\varphi_1 : U_1 \times F \rightarrow \pi^{-1}(U_1)$ and $\varphi_2 : U_2 \times F \rightarrow \pi^{-1}(U_2)$, then $\varphi_{21} := \varphi_2^{-1} \circ \varphi_1$ is defined as a map $(U_1 \cap U_2) \times F \rightarrow (U_1 \cap U_2) \times F$. This map is a homeomorphism of the form $\varphi_{21}(p, v) = (p, \vartheta_{21}(p, v))$, with continuous $\vartheta_{21} : (U_1 \cap U_2) \times F \rightarrow F$. For three trivialising neighbourhoods, the restrictions to $(U_1 \cap U_2 \cap U_3) \times F$ satisfy $\varphi_{31} = \varphi_{32} \circ \varphi_{21}$. In particular, if $\{U_i\}_{i \in I}$ is an open covering of M , where U_i are trivialising neighbourhoods with homeomorphisms $\varphi_i : U_i \times F \rightarrow \pi^{-1}(U_i)$, then there is an analogous equation for every triple of indices, i.e.

$$\forall i, j, k \in I : \varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}. \quad (1.5.47)$$

(This includes $\varphi_{ii} = \text{id}$ and $\varphi_{ij} = \varphi_{ji}^{-1}$.) Let us call the maps φ_{ij} the gluing functions of the bundle.

On the other hand, if one is given an open covering $\{U_i\}_{i \in I}$ of M and homeomorphisms $\varphi_{ij} : (U_i \cap U_j) \times F \rightarrow (U_i \cap U_j) \times F$ fulfilling the ‘‘cocycle condition’’ (1.5.47), then one can construct, by the standard gluing procedure, a locally trivial fibre bundle over M with typical fibre F . The bundle comes with canonical local trivialisations whose gluing functions coincide with the initial maps φ_{ij} (cf. [12, 16.13]).

Consider now a locally trivial G -bundle given by a free action inducing a continuous translation map. The locally trivialising maps φ_i are G -equivariant and thus of the form $\varphi_i(p, g) = \sigma_i(p)g$, where $\sigma_i : U_i \rightarrow \pi^{-1}(U_i)$ is a local section (see (1.5.45)). The corresponding gluing functions have the form

$$\varphi_{ij}(p, g) = \varphi_i^{-1}(\varphi_j(p, g)) = \varphi_i^{-1}(\sigma_j(p)g) = (p, \tilde{\tau}(\sigma_i(p), \sigma_j(p))g) =: (p, \theta_{ij}(p)g). \quad (1.5.48)$$

The continuous maps $\theta_{ij} := \tilde{\tau} \circ (\sigma_i \times \sigma_j) \circ \text{diagonal map} : U_i \cap U_j \rightarrow G$ are called the *transition functions* of the bundle. For any triple of indices i, j, k , they satisfy the cocycle condition (cf. [12, Exercice 1, 16.14])

$$\theta_{ij}(p) = \theta_{ik}(p)\theta_{kj}(p), \quad \forall p \in U_i \cap U_j \cap U_k. \quad (1.5.49)$$

Conversely, one can reconstruct the bundle from a given covering and transition functions θ_{ij} satisfying the cocycle condition (1.5.49) (again cf. [12, Exercice 1, 16.14]).

For a G -bundle (X, π, M) as above, the associated bundle $(X \times_G F, \pi_F, M, F)$ is also locally trivial. It has natural locally trivialising maps

$$\varphi_F : U \times F \longrightarrow \pi_F^{-1}(U), \quad \varphi_F(p, v) = [(\sigma(p), v)], \quad (1.5.50)$$

where $\sigma : U \rightarrow X$ is a local section. Indeed, φ_F is a continuous fibre-wise map and its inverse is given by

$$\varphi_F^{-1}([(x, v)]) = (\pi(x), \tilde{\tau}(\sigma(\pi(x)), x)v). \quad (1.5.51)$$

Hence, since $\tilde{\tau}$ is continuous, φ_F is a homeomorphism, and thus a locally trivialising map. The transition functions of the associated bundle have the form $(p, v) \mapsto (p, \theta_{ij}(p)v)$, where θ_{ij} are the transition functions of (X, π, M, G) .

2 Elements of general algebra

Introducing geometric structures on topological spaces needs some algebraic concepts. Differential objects such as differential forms, connections and sections of vector bundles have their algebraic backbones in the universal differential algebra, splittings of surjections on projective modules, and the cotensor product, respectively. In this section, we recall the aforementioned elements of algebra and explain in detail certain basic mechanisms. Throughout this part, we work with arbitrary associative unital algebras over a ground field k . We assume that all our vector spaces and unadorned tensor products are over k .

2.1 Modules and comodules

Herein, we gather a number of basic facts about actions and coactions. For more details, the interested reader is referred to textbooks, e.g., for modules and comodules in the context of Hopf algebras to [44, 1], and in the context of corings to [7].

A vector space M that, at the same time, is a left module of an algebra A and a right module of an algebra B with mutually commuting actions is called an (A, B) -bimodule. Recall that, given a right A -module M and a left A -module N , the *tensor product of A -modules* M and N is a vector space $M \otimes_A N$ defined by the following exact sequence

$$M \otimes A \otimes N \xrightarrow{\omega_A} M \otimes N \longrightarrow M \otimes_A N \longrightarrow 0, \quad (2.1.1)$$

where ω_A is given by $\omega_A : m \otimes a \otimes n \mapsto ma \otimes n - m \otimes an$. This means that, for all $a \in A$, $m \in M$ and $n \in N$, $ma \otimes_A n = m \otimes_A an$ (cf. (1.0.1)). The tensor product is a bifunctor from the categories of modules to the category of vector spaces. In particular, this means that, for any left A -module M and any right A -linear map $f : N \rightarrow \tilde{N}$, the map

$$f \otimes \text{id} : N \otimes_A M \longrightarrow \tilde{N} \otimes_A M, \quad n \otimes_A m \longmapsto f(n) \otimes_A m \quad (2.1.2)$$

is a homomorphism of vector spaces. A left A -module M is said to be *flat*, provided for any monomorphism (injection) f of right A -modules, the map $f \otimes_A \text{id}_M$ is also injective. This is equivalent to the statement that M is flat if and only if any (short) exact sequence of right A -linear maps remains exact after tensoring with M .

Recall that the dual concept to that of an algebra is the notion of a *coalgebra*. Thus, a coalgebra is a vector space C with a coassociative *coproduct* $\Delta : C \rightarrow C \otimes C$ and a *counit* $\varepsilon : C \rightarrow k$. Henceforth, to denote the action of Δ on elements we use the *Sweedler notation*

$$\Delta(c) = c_{(1)} \otimes c_{(2)}, \quad ((\Delta \otimes \text{id}) \circ \Delta)(c) = ((\text{id} \otimes \Delta) \circ \Delta)(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}, \quad (2.1.3)$$

(summation understood) etc. Given coalgebras C, D , a *coalgebra map* is a k -linear map $f : C \rightarrow D$ such that $\Delta \circ f = (f \otimes f) \circ \Delta$ and $\varepsilon \circ f = \varepsilon$.

A *left* (resp. *right*) C -comodule or *corepresentation* is a vector space M with a counital and coassociative coaction ${}_M\Delta : M \rightarrow C \otimes M$ (resp. $\Delta_M : M \rightarrow M \otimes C$). On elements the left C -coaction is denoted by ${}_M\Delta(x) = x_{(-1)} \otimes x_{(0)}$, and the right C -coaction is denoted by $\Delta_M(x) = x_{(0)} \otimes x_{(1)}$ (summation understood). A vector space M that is at the same time a left C -comodule and a right D -comodule with mutually commuting coactions is called a (C, D) -bicomodule.

For any pair of right (resp. left) C -comodules M, N , a linear map $f : M \rightarrow N$ is said to be *colinear* if it commutes with the coactions, i.e. $\Delta_N \circ f = (f \otimes \text{id}) \circ \Delta_M$ (resp. ${}_N\Delta \circ f = (\text{id} \otimes f) \circ {}_M\Delta$). The k -vector space of all right (resp. left) C -colinear maps $M \rightarrow N$ is denoted by $\text{Hom}^C(M, N)$ (resp. ${}^C\text{Hom}(M, N)$).

Next, let us recall the notion of a cotensor product. Given a coalgebra C , let M be a right C -comodule and N be a left C -comodule. The *cotensor product* $M \square_C N$ is defined by the exact sequence

$$0 \longrightarrow M \square_C N \longrightarrow M \otimes N \xrightarrow{\omega^C} M \otimes C \otimes N. \quad (2.1.4)$$

Here ω^C is the coaction equalising map $\Delta_M \otimes \text{id}_N - \text{id}_M \otimes {}_N\Delta$ (cf. (1.0.1)).

Since there are no unit elements in coalgebras, it is not clear how to define elements that are invariant under coalgebra coactions (cf. (2.1.7)). However, if a coalgebra C coacts (on the right) on

an algebra P , then, following M. Takeuchi, we can define the *subalgebra of invariants* $P^{coC} \subseteq P$ by

$$P^{coC} := \{b \in P \mid \forall p \in P : \Delta_P(bp) = b\Delta_P(p)\}. \quad (2.1.5)$$

The subalgebra of invariants of an algebra and a left C -comodule is defined in an analogous way.

This definition of invariants immediately implies that, for any left C -comodule V , the cotensor product $P \square_C V$ is a left P^{coC} -module with the action given by $b \sum_i p_i \otimes v_i = \sum_i bp_i \otimes v_i$. Similarly, for any right C -comodule W , the space $\text{Hom}^C(W, P)$ is a left P^{coC} -module with the action $(bf)(w) = bf(w)$. The relationship between these two modules is described in the following:

LEMMA 2.1. *Let C be a coalgebra and V a finite-dimensional left C -comodule. Let v_1, \dots, v_n be a basis of V and v^1, \dots, v^n the corresponding dual basis of $V^* := \text{Hom}(V, k)$. Then V^* is a right C -comodule by the formula*

$$\Delta_{V^*}(f) = \sum_i v^i \otimes v_{i(-1)} f(v_{i(0)}).$$

Moreover, for any algebra and right C -comodule P and for any finite-dimensional left C -comodule V , we have $P \square_C V \cong \text{Hom}^C(V^*, P)$ as left P^{coC} -modules.

PROOF. The coaction formula is a standard result in comodule theory (cf. [7, 3.11]). Next, one easily checks that the isomorphism of vector spaces

$$P \otimes V \longrightarrow \text{Hom}(V^*, P), \quad p \otimes v \longmapsto [f \mapsto pf(v)], \quad (2.1.6)$$

with the inverse $\varphi \mapsto \sum_i \varphi(v^i) \otimes v_i$, restricts to the isomorphism $P \square_C V \cong \text{Hom}^C(V^*, P)$ (cf. [7, 10.11]). The form of the map (2.1.6) immediately implies that this is an isomorphism of left P^{coC} -modules. \square

Having discussed basic facts concerning actions of algebras and coactions of coalgebras, let us pass to coactions of bialgebras. Recall first that an algebra H with a coalgebra structure such that the coproduct and counit are algebra maps is called a *bialgebra*. Given a bialgebra H , a *right H -comodule algebra* is an algebra and a right H -comodule P such that the coaction is an algebra map. In this case the subalgebra of invariants of P can be defined as

$$P^{coH} = \{b \in P \mid \Delta_P(b) = b \otimes 1_H\}. \quad (2.1.7)$$

One easily checks that this definition coincides with that of the subalgebra of invariants of P with respect to the coaction of *coalgebra* H in equation (2.1.5). A left H -comodule algebra and its subalgebra of invariants are defined in an analogous way.

2.2 The universal differential calculus and algebra

The *universal differential calculus* $\Omega^1 A$ is a bimodule defined by the exact sequence

$$0 \longrightarrow \Omega^1 A \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0, \quad (2.2.8)$$

i.e. as the kernel of the multiplication map. The differential is given by $da := 1 \otimes a - a \otimes 1$. We can identify $\Omega^1 A$ with $A \otimes A/k$ as left A -modules via the maps

$$\Omega^1 A \ni \sum_i a_i \otimes a'_i \mapsto \sum_i a_i \otimes \pi_A(a'_i) \in A \otimes A/k \ni x \otimes \pi_A(y) \mapsto xdy \in \Omega^1 A, \quad (2.2.9)$$

where $\pi_A : A \rightarrow A/k$ is the canonical surjection. Similarly, one can identify $\Omega^1 A$ with $A/k \otimes A$ as right A -modules ($\sum_i a_i \otimes a'_i \mapsto \sum_i \pi_A(a_i) \otimes a'_i$).

Note that the sequence (2.2.8) is split-exact as a sequence of right (or left) A -modules. The multiplication map has a right A -linear splitting $a \mapsto 1 \otimes a$ and a left A -linear splitting $a \mapsto a \otimes 1$. This implies that, for any left A -module N (no need for N to be flat), also the sequence

$$0 \longrightarrow \Omega^1 A \otimes_A N \longrightarrow A \otimes N \longrightarrow N \longrightarrow 0 \quad (2.2.10)$$

is exact. (The other-sided version is analogous.)

Note next that if B is a subalgebra of P , then we can write $(\Omega^1 B)P$ for the kernel of the multiplication map $B \otimes P \rightarrow P$. Indeed, $m((\Omega^1 B)P) = 0$, and if $\sum_i b_i \otimes p_i \in \text{Ker}(B \otimes P \xrightarrow{m} P)$, then

$$\sum_i b_i \otimes p_i = \sum_i (b_i \otimes p_i - 1 \otimes b_i p_i) = - \sum_i (db_i) p_i \in (\Omega^1 B)P. \quad (2.2.11)$$

Furthermore, it follows from the exactness of (2.2.10) that

$$\Omega^1 B \otimes_B P \cong (\Omega^1 B)P \quad (\text{cf. [21, p.251]}). \quad (2.2.12)$$

In order to define curvature and also for some problems in cyclic cohomology, it is helpful to introduce the concept of universal differential algebra. First, let us recall that the n -th order universal differential calculus is

$$\Omega^n A := \underbrace{\Omega^1 A \otimes_A \cdots \otimes_A \Omega^1 A}_{n\text{-times}} \quad (2.2.13)$$

We assume that $\Omega^0 A = A$ and define the *universal differential algebra* of A [27, 28] as

$$\Omega A := \bigoplus_{n \in \mathbb{N}} \Omega^n A. \quad (2.2.14)$$

It is understood as the tensor algebra of the A -bimodule $\Omega^1 A$, so that its product structure is given by $\alpha\beta = \alpha \otimes_A \beta$.

Now, to turn it into a differential algebra, we need to extend our differential $d : A \rightarrow \Omega^1 A$ to the higher order calculi. To this end, we first generalize the identification (2.2.9) to an arbitrary degree. One can easily check that the composite maps

$$\Omega^n A \xrightarrow{\subseteq} \underbrace{A^{\otimes 2} \otimes_A \cdots \otimes_A A^{\otimes 2}}_{n\text{-times}} \xrightarrow{\cong} A^{\otimes n+1} \longrightarrow A \otimes (A/k)^{\otimes n} \quad \text{and} \quad (2.2.15)$$

$$A \otimes (A/k)^{\otimes n} \xrightarrow{\text{id} \otimes \bar{d}^{\otimes n}} A \otimes (\Omega^1 A)^{\otimes n} \longrightarrow \underbrace{\Omega^1 A \otimes_A \cdots \otimes_A \Omega^1 A}_{n\text{-times}} \xrightarrow{\cong} \Omega^n A \quad (2.2.16)$$

are mutually inverse isomorphisms of left A -modules. Here $\bar{d} : A/k \ni [a] \mapsto 1 \otimes a - a \otimes 1 \in \Omega^1 A$, and other maps are constructed from the canonical surjections and the multiplication isomorphism $A \otimes_A M \cong M$, for any left A -module M .

As the next step, we define a map $A \otimes (A/k)^{\otimes n} \rightarrow A \otimes (A/k)^{\otimes n+1}$,

$$a_0 \otimes [a_1] \otimes \cdots \otimes [a_n] \mapsto 1 \otimes [a_0] \otimes [a_1] \otimes \cdots \otimes [a_n], \quad (2.2.17)$$

and transform it via the just described identification $\Omega^n A \cong A \otimes (A/k)^{\otimes n}$ into the desired differential $\Omega^n A \rightarrow \Omega^{n+1} A$. It is a standard fact that in this way we obtain a differential graded algebra, and that this algebra has an appropriate universality property in the category of differential graded algebras.

In the commutative setting, Kähler differential forms play the role of universal objects (see [33, 1.3.7., p.26]). For a commutative algebra A , the A -module of (first order) Kähler differentials is defined as $\Omega_{A|k}^1 := \Omega^1 A / (\Omega^1 A)^2$. Here we view $(\Omega^1 A)^2$ as an ideal in $A \otimes A$. If $C^\infty(M)$ is the algebra of smooth functions on a compact smooth manifold M , and $\Omega_{dR}^1(M)$ stands for the module of de Rham differential 1-forms, one obtains the following sequence of surjective $C^\infty(M)$ -bimodule maps:

$$\Omega^1 C^\infty(M) \longrightarrow \Omega_{C^\infty(M)|\mathbb{R}}^1 \longrightarrow \Omega_{dR}^1(M). \quad (2.2.18)$$

The second map is surjective but (in general) not injective. This lack of injectivity follows from the remarkable fact that for a transcendental function f , the expression $df - \frac{df}{dx} dx$ is non-zero as a Kähler form despite being obviously zero in the de Rham calculus, see [24, p.78] and [47, p.42].²

To prove the surjectivity, note first that every manifold M admits an atlas such that every coordinate function can be extended to all of M . Indeed, consider a chart (U, φ) with domain homeomorphic to an open unit ball in \mathbb{R}^n . There is a function g on the open unit ball whose support is contained in the ball of radius $\frac{3}{4}$, and which is the constant function 1 on the ball of radius $\frac{1}{2}$. Multiplying the coordinate functions with $\varphi^* g$, we end up with a new chart with smaller domain (homeomorphic to a ball with radius $\frac{1}{2}$). All coordinate functions of this chart can be extended to M by 0. Obviously, we can cover M with these charts, and in the compact case we can select finitely many of them.

Now, let $\{(U_i, \varphi_i)\}_{i \in \{1, \dots, n\}}$ be such an atlas, let φ_i^k be the k -th coordinate function given by φ_i , let $\widetilde{\varphi}_i^k$ be its extension to M , and let $\{\psi_i\}_{i \in \{1, \dots, n\}}$ be a smooth partition of unity subordinate to the covering $\{U_i\}_{i \in \{1, \dots, n\}}$, i.e., $\text{supp } \psi_i \subseteq U_i$, $\sum_{i=1}^n \psi_i = 1$. For any $\alpha \in \Omega_{dR}^1(M)$, its restriction $\alpha|_{U_i} = \sum_i f_i^k d\varphi_i^k$, for certain $f_i^k \in C^\infty(U_i)$. Let $\widetilde{\psi}_i f_i^k$ denote the extension by zero of $\psi_i|_{U_i} f_i^k$ to an element of $C^\infty(M)$. Then

$$\alpha = \sum_{i=1}^n \sum_{k=1}^{\dim M} \widetilde{\psi}_i f_i^k d\widetilde{\varphi}_i^k \quad (2.2.19)$$

is the desired presentation of α as a finite sum of elements of the form fdg . This proves the surjectivity of the second map in (2.2.18).

2.3 Projectivity and connections

Recall that a left B -module P is said to be a *projective module* provided it is a direct summand of a free module. Equivalently, P is a projective left B -module if and only if the module structure map $B \otimes P \rightarrow P$ splits as a left B -module map. Note that this last description is possible since we restricted our considerations to algebras over a field, so that $B \otimes P$ is a free left B -module, and the above statement is equivalent to say that P is a direct summand of a free module.

For a module P , being a direct summand of a free module means that there exists an idempotent automorphism of a free module whose image is isomorphic with P . Indeed, if $P \xrightarrow{i} B \otimes V \xrightarrow{\pi} P$, $\pi \circ i = \text{id}$, then $(i \circ \pi)^2 = i \circ \pi$ and $(i \circ \pi)(B \otimes V) \cong P$ via appropriately restricted i and π . Similarly, if e is an idempotent automorphism and $f : P \rightarrow e(B \otimes V)$ is an isomorphism, then the composite maps

$$P \xrightarrow{f} e(B \otimes V) \xrightarrow{\subseteq} B \otimes V \quad \text{and} \quad B \otimes V \xrightarrow{e} e(B \otimes V) \xrightarrow{f^{-1}} P \quad (2.3.20)$$

realise P as a direct summand of $B \otimes V$.

²We are grateful to M. Wodzicki for a discussion clarifying this point.

REMARK 2.2. At this point let us recall the following simple property of idempotent operators, which will be used later on. If E, F are idempotent linear operators on a vector space V with the same kernel, then $EF = E$. Indeed, since $v - F(v) \in \text{Ker } F$, we have $v - F(v) \in \text{Ker } E$. Therefore, $E(v) = E(v - F(v) + F(v)) = E(F(v))$. \diamond

Every projective module can be equivalently characterised by the existence of a *connection* [10, Proposition 8.2], i.e. a linear map

$$\nabla : P \longrightarrow \Omega^1 B \otimes_B P \quad \text{such that} \quad \nabla(bp) = db \otimes_B p + b\nabla(p), \quad \forall b \in B, p \in P. \quad (2.3.21)$$

Indeed, the projectivity of P is tantamount to the existence of a left B -linear splitting s of the product map $m : B \otimes P \rightarrow P$. Taking such a splitting and viewing elements of $\Omega^1 B \otimes_B P$ in $B \otimes P$, we can define a connection by the formula:

$$\nabla(p) = 1 \otimes p - s(p). \quad (2.3.22)$$

Thus every projective module admits a connection. Vice versa, if P is a left B -module admitting a connection ∇ , then the formula

$$s(p) = 1 \otimes p - \nabla(p) \quad (2.3.23)$$

defines a left B -linear splitting of the multiplication map, so that P is projective. The formulas (2.3.22) and (2.3.23) provide mutually inverse bijections between the space of splittings and the space of connections.

This correspondence between splittings and connections can be applied to show that every connection is always obtained from the differential and an idempotent. Indeed, let ∇ be a connection and s_∇ the splitting of the multiplication map associated to ∇ via (2.3.23). Then

$$\nabla(p) = 1 \otimes p - s_\nabla(p) = ((\text{id} \otimes m) \circ (d \otimes \text{id}) \circ s_\nabla)(p), \quad \forall p \in P. \quad (2.3.24)$$

In general, if P is a direct summand of a free module $B \otimes V$ via the maps $P \xrightarrow{i} B \otimes V \xrightarrow{\pi} P$, then the formula

$$\nabla(p) = ((\text{id} \otimes \pi) \circ (d \otimes \text{id}) \circ i)(p), \quad \forall p \in P, \quad (2.3.25)$$

determines a connection. Such connections are called *Graßmann connections*.

The definition of connection can be easily extended from P to $\Omega B \otimes_B P$. Indeed, there exists a unique extension of ∇ to a degree-one map $\Omega P \otimes_B P \rightarrow \Omega P \otimes_B P$ satisfying the graded Leibniz rule:

$$\nabla(\alpha\xi) = d\alpha \otimes_B \xi + (-1)^n \alpha \nabla(\xi), \quad \forall \alpha \in \Omega^n B \otimes_B P, \xi \in \Omega B \otimes_B P. \quad (2.3.26)$$

One can immediately verify that ∇^2 is linear over ΩB . This is a crucial property allowing us to view ∇^2 as a matrix with entries in ΩB . We call ∇^2 the *curvature* of a connection ∇ . This concept is fundamental for the construction of the Chern character.

In noncommutative geometry projective modules that are also finitely generated are of particular interest, since they correspond to vector bundles. A left B -module P is finitely generated projective if and only if there exist a positive integer n , an $n \times n$ -matrix $e = (e_i^j)_{i,j=1}^n$ with $e_i^j \in B$ such that $e^2 = e$, and an isomorphism of left B -modules $P \cong B^n e$. Here B^n is understood as a space of row vectors (b_1, \dots, b_n) , and the multiplication represents the standard multiplication of a row vector by a matrix. Any two idempotent matrices e and \tilde{e} determine isomorphic finitely generated projective modules if and only if these two idempotent matrices are stably similar — i.e. by enlarging e and \tilde{e} by zeroes, there then exists an invertible matrix which conjugates one enlarged matrix to the other. (cf. [39, Lemma 1.2.1]).

Finitely generated projective modules are equivalently characterised by the existence of *finite dual bases*. Thus P is a finitely generated projective left B -module if and only if there exists a finite set $\{e_i \in P, e^i \in \text{Hom}_B(P, B)\}_i$ such that, for all $p \in P$, $p = \sum_i e^i(p)e_i$. The corresponding idempotent matrix e has entries $e_i^j := e^j(e_i)$, while the splitting $s : B \rightarrow B \otimes P$ of the product map reads $s(p) = \sum_i e^i(p) \otimes e_i$. Much as in (2.3.24), the connection ∇ associated to s via (2.3.22) can be expressed in terms of the dual basis $\{e_i \in P, e^i \in \text{Hom}_B(P, B)\}_i$ and the corresponding idempotent as

$$\nabla(e_i) = \sum_j \text{de}_i^j \otimes_B e_j. \quad (2.3.27)$$

Indeed, to check (2.3.27), note first that the definition of a dual bases and the corresponding idempotent imply that $e_i = \sum_j e_i^j e_j$. Hence equation (2.3.22) yields

$$\nabla(e_i) = 1 \otimes e_i - \sum_j e^j(e_i) \otimes e_j = \sum_j 1 \otimes e_i^j \otimes_B e_j - \sum_j e_i^j \otimes 1 \otimes_B e_j = \sum_j \text{de}_i^j \otimes_B e_j. \quad (2.3.28)$$

Finally, let us observe that a straightforward computation shows that

$$\nabla^2(e_i) = - \sum_{k,l} \text{de}_i^k \text{de}_k^l \otimes_B e_l. \quad (2.3.29)$$

Applying $\text{id} \otimes_B e^j$ to the right hand side of this equation gives the matrix of two-forms: $-(\text{de})(\text{de})e = -e(\text{de})(\text{de})$.

3 Differential geometry of principal and vector bundles

Thus far we have analysed purely topological aspects of principal and associated bundles and recalled some relevant general algebraic concepts. Herein, we study them in their traditional context of differential geometry. Throughout this section, spaces are assumed to be compact Hausdorff, which removes most of the subtleties discussed in Section 1. On the other hand, this assumption makes the geometry on these spaces accessible by the aforementioned general (i.e., not bound to commutative cases) algebraic tools. We use them to recast the standard differential-geometric concepts of associated vector bundles and connections on principal bundles into a language that not only lends itself to noncommutative generalisations, but also functions naturally therein. We end the paper by exemplifying our general considerations on the Dirac-monopole connection on the Hopf fibration $S^3 \rightarrow S^2$.

To be in line with the usual setting of noncommutative geometry, we always work with complex-valued functions and complexified versions of vector fields, differential forms and other geometric objects. Therefore, we adopt a shorthand notation, $C^\infty(M, \mathbb{C}) = C^\infty(M)$, $\Gamma^\infty(E, \mathbb{C}) = \Gamma^\infty(E)$, $H_{dR}(M, \mathbb{C}) = H_{dR}(M)$, etc.

3.1 Associated vector bundles

The goal of this subsection is to give a description of the module of continuous (or smooth) sections of an associated vector bundle as an appropriate cotensor product. We first identify sections of an associated vector bundle with equivariant maps (Lemma 1.16). Then we show that the latter are in a natural correspondence with elements of a certain cotensor product (Lemma 3.2).

Let G be a compact group acting principally on the right on a compact Hausdorff space X . Dualising this right action gives a $*$ -homomorphism $\Delta_R : C(X) \rightarrow C(X \times G) \cong C(X) \overline{\otimes} C(G)$. Here,

$\overline{\otimes}$ denotes the C^* -completed tensor product. (Recall that all C^* -tensor products are equivalent for commutative C^* -algebras, and correspond to the Cartesian product of underlying spaces.) Similarly, the group multiplication in G gives rise to a $*$ -homomorphism $\Delta : C(G) \rightarrow C(G \times G) \cong C(G) \overline{\otimes} C(G)$. By analogy with the algebraic situation, we say that Δ is a coproduct on $C(G)$ and Δ_R is a right coaction of $C(G)$ on $C(X)$ (cf. [45]).

REMARK 3.1. The principal map F^G (see (1.0.3)) has as its dual (pull-back) the canonical map

$$\text{can} : C(X) \overline{\otimes} C(X) \longrightarrow C(X) \overline{\otimes} C(G), \quad \text{can}(a \otimes a') = a \Delta_R(a'). \quad (3.1.1)$$

The C^* -algebra $C(X/G)$ can be identified with the subalgebra of $C(X)$ of functions that are constant along every fibre, i.e., the subalgebra of invariants:

$$C(X)^{\text{co}C(G)} = \{b \in C(X) \mid \Delta_R(b) = b \otimes 1\}. \quad (3.1.2)$$

Note that $\text{Ker}(\text{can})$ coincides with the closure of the linear span of elements of the form $ab \otimes a' - a \otimes ba'$, with $a, a' \in C(X)$ and $b \in C(X)^{\text{co}C(G)}$. This can be proved using the Stone-Weierstraß theorem. \diamond

By Peter-Weyl theory, the matrix elements of irreducible unitary representations of G generate a Hopf algebra $\mathcal{O}(G)$ which is a dense $*$ -subalgebra of the C^* -algebra $C(G)$, and any coaction ${}_V\Delta : V \rightarrow C(G) \otimes V$ on a finite-dimensional vector space V takes values in $\mathcal{O}(G) \otimes V$. The coproduct of the Hopf algebra $\mathcal{O}(G)$ is the restriction to $\mathcal{O}(G)$ of the pull-back of the multiplication map $G \times G \rightarrow G$. (Due to Woronowicz's generalisation of Peter-Weyl theory [49], [50], analogous statements are true for representations of compact quantum groups.)

For any left coaction of $C(G)$ on V , we can define the cotensor product:

$$C(X) \square_{C(G)} V := \{u \in C(X) \otimes V \mid (\Delta_R \otimes \text{id})(u) = (\text{id} \otimes {}_V\Delta)(u)\}. \quad (3.1.3)$$

If V is finite dimensional with a basis $\{v_1, \dots, v_n\}$ and $\sum_i p_i \otimes v_i \in C(X) \square_{C(G)} V$, then it follows that $\Delta_R(p_i) \in C(X) \otimes \mathcal{O}(G)$, i.e., the p_i 's are elements of

$$\mathcal{C}(X) := \{p \in C(X) \mid \Delta_R(p) \in C(X) \otimes \mathcal{O}(G)\}. \quad (3.1.4)$$

It is immediate that $\mathcal{C}(X)$ is a right $\mathcal{O}(G)$ -comodule algebra, and we have

$$C(X) \square_{C(G)} V = \mathcal{C}(X) \square_{\mathcal{O}(G)} V. \quad (3.1.5)$$

This is an equality of $C(X/G)$ -modules (with the module structure by multiplication in the left leg of the cotensor product). The algebra $\mathcal{C}(X)$ is fundamental in our approach to associated vector bundles, and should be thought of as the algebra of functions that are continuous along the base space and polynomial along the fibre.

Next, let

$$\text{Hom}_G(X, V) = \{f \in C(X, V) \mid f(xg) = \rho(g^{-1})f(x)\}. \quad (3.1.6)$$

This is a $C(X/G)$ -module by pointwise multiplication. We can now claim (cf. Lemma 2.1):

LEMMA 3.2. *Let $\rho : G \rightarrow GL(V)$ be a finite-dimensional representation, $\{v_i\}_{i=1}^n$ a basis of V with the dual basis $\{v^i\}_{i=1}^n$, and ${}_V\Delta : V \rightarrow \mathcal{O}(G) \otimes V$ the left coaction given by*

$${}_V\Delta(v_i) = \sum_{j=1}^n v^j (\rho(\cdot^{-1})(v_i)) \otimes v_j. \quad (3.1.7)$$

Then $\text{Hom}_G(X, V) \cong \mathcal{C}(X) \square_{\mathcal{O}(G)} V$ as $C(X/G)$ -modules.

PROOF. Observe first that the equivariance of $f \in \text{Hom}_G(X, V)$ entails

$$(v^j \circ f)(xg) = \sum_{k=1}^n v^j(\rho(g^{-1})(v_k))(v^k \circ f)(x). \quad (3.1.8)$$

Therefore, $\Delta_R(v^j \circ f)(x, g) = (\sum_{k=1}^n (v^k \circ f) \otimes v^j(\rho(\cdot^{-1})(v_k)))(x, g)$, so that $v^j \circ f \in \mathcal{C}(X)$. Furthermore, it is immediate that $\sum_{j=1}^n \Delta_R(v^j \circ f) \otimes v_j = \sum_{j=1}^n (v^j \circ f) \otimes v \Delta(v_j)$. Thus, we obtain a well-defined $C(X/G)$ -linear map $\text{Hom}_G(X, V) \ni f \mapsto \sum_{j=1}^n (v^j \circ f) \otimes v_j \in \mathcal{C}(X) \square_{\mathcal{O}(G)} V$. The other way around, if $\sum_{j=1}^n f^j \otimes v_j \in \mathcal{C}(X) \square_{\mathcal{O}(G)} V$, then

$$\sum_{j=1}^n f^j(xg)v_j = \sum_{j=1}^n (\Delta_R(f^j))(x, g)v_j = \sum_{j,k=1}^n f^j(x)v^k(\rho(g^{-1})(v_j))v_k = \rho(g^{-1}) \sum_{j=1}^n f^j(x)v_j. \quad (3.1.9)$$

Consequently, $\sum_{j=1}^n f^j v_j \in \text{Hom}_G(X, V)$, and $\sum_{j=1}^n f^j \otimes v_j \mapsto \sum_{j=1}^n f^j v_j$ yields the desired inverse homomorphism. The $C(X/G)$ -linearity is obvious. \square

Combining this with Lemma 1.16, we arrive at:

THEOREM 3.3. *Let a compact group G act principally on a compact Hausdorff space X , ρ be a representation of G in a finite dimensional vector space V , and E be the vector bundle associated to the principal bundle (X, π, M, G) via ρ . Then the space $\Gamma(E)$ of continuous sections of E is isomorphic as a $C(M)$ -module to the cotensor product $\mathcal{C}(X) \square_{\mathcal{O}(G)} V$ defined by the coactions given by the formulas $\Delta_R(f)(x, g) = f(xg)$ and $v \Delta(v) = v^j(\rho(\cdot^{-1})(v)) \otimes v_j$.*

The foregoing algebraic formulation of associated bundles is also possible in the smooth setting. To begin with, there is a well-known analogue of Lemma 1.16 (see, e.g., [12, 16.14, Exercise 8]). Then, it is straightforward to check that the above considerations are valid also when replacing the algebra of continuous functions on a compact Hausdorff space by the algebra of smooth functions on a compact manifold, and the compact group by a compact Lie group. One finally arrives at an isomorphism of the $C^\infty(M)$ -module $\Gamma^\infty(E)$ of smooth sections of a smooth associated vector bundle with the cotensor product $C^\infty(X) \square_{\mathcal{O}(G)} V$, where

$$C^\infty(X) := \{p \in C^\infty(X) \mid \Delta_R(p) \in C^\infty(X) \otimes \mathcal{O}(G)\}. \quad (3.1.10)$$

3.2 Connections and gauge transformations

In this subsection, we consider finite-dimensional compact smooth manifolds without boundary, and assume mappings to be smooth. We denote by T the tangent functor, and by $\Lambda^k(M)$ the space of differential k -forms on a manifold M . Let (X, π, M, G) be a principal bundle. The tangent spaces to the fibres form the canonical subbundle $\text{Ver} \subseteq TX$ coinciding with the kernel of $T\pi$. We shall denote by $R : X \times G \rightarrow X$ the right action, and by

$$R_g : X \ni x \mapsto xg \in X, \quad R_x : G \ni g \mapsto xg \in X, \quad R_{\check{x}} : \pi^{-1}(\pi(x)) \ni y \mapsto \check{\tau}(x, y) \in G, \quad (3.2.11)$$

the induced maps. (Recall that $\check{\tau}$ stands for the translation map (1.0.4).) The adjoint action of G on itself will be denoted by

$$\text{Ad}^G : g \longrightarrow \text{Aut}(G), \quad \text{Ad}_g^G(g') := gg'g^{-1}. \quad (3.2.12)$$

A connection on (X, π, M, G) is usually defined as a G -equivariant horizontal distribution Hor in TX that is complementary to the distribution Ver of vertical subspaces [29, p.63]. Here equivariance

means $TR_g(\text{Hor}_x) \subseteq \text{Hor}_{xg}$, where x and g are arbitrary elements of X and G , respectively. Equivalently, this can be rephrased by saying that a connection on a principal bundle is an equivariant horizontal lift of tangent vectors (cf. [13, 20.2.2]). The idea of a horizontal lift, combined with the existence of finite dual bases for the finitely generated projective modules of smooth sections of tangent and cotangent bundles, allows us to prove very useful identifications. We take it for granted that connections always exist for paracompact (in particular compact) manifolds.

The horizontal-distribution definition of a connection can be easily reformulated in a dual way, referring to differential forms instead of tangent vectors. First, recall that a horizontal form is, by definition, a form that vanishes if at least one of its arguments is a vertical vector (tangent to a fibre). We denote the space of horizontal 1-forms on X by $\Lambda_{\text{hor}}^1(X)$.

DEFINITION 3.4. A connection on a principal bundle is a $C^\infty(X)$ -linear idempotent homomorphism $\Pi : \Lambda^1(X) \rightarrow \Lambda^1(X)$ such that $\text{Ker } \Pi = \Lambda_{\text{hor}}^1(X)$ and $\Pi \circ R_g^* = R_g^* \circ \Pi$, for all $g \in G$.

Let us denote the space of connections described in the foregoing definition by $\mathcal{P}(X)$. Since connections are known to form an affine space, it is natural to ask what kind of a vector space is generated by differences of elements of $\mathcal{P}(X)$. To this end, let us define the following subspace of *nilpotent* endomorphisms:

$$\mathcal{N}(X) := \{ N \in \text{End}_{C^\infty(X)}(\Lambda^1(X)) \mid \text{Im } N \subseteq \Lambda_{\text{hor}}^1(X) \subseteq \text{Ker } N, R_g^* \circ N = N \circ R_g^*, \forall g \in G \}. \quad (3.2.13)$$

It is straightforward to verify that $\mathcal{N}(X)$ is a $C^\infty(M)$ -module. In particular, it is a vector space, and we have:

PROPOSITION 3.5. *The space of connections $\mathcal{P}(X)$ is an affine space over $\mathcal{N}(X)$.*

PROOF. Let $N := \Pi - \Pi'$ be the difference of two connections. Obviously, it is an equivariant $C^\infty(X)$ -module endomorphism with $\Lambda_{\text{hor}}^1(X) \subseteq \text{Ker } N$. If E is any idempotent linear operator on $\Lambda^1(X)$ with $\text{Ker } E = \Lambda_{\text{hor}}^1(X)$, then Remark 2.2 implies $EN = E\Pi - E\Pi' = E - E = 0$. Hence $\text{Im } N \subseteq \text{Ker } E = \Lambda_{\text{hor}}^1(X)$. On the other hand, if $\Pi \in \mathcal{P}(X)$ and $N \in \mathcal{N}(X)$, then again it is clear that $\Pi + N$ is an equivariant $C^\infty(X)$ -module endomorphism. Since $\Pi \circ (\Pi + N) = \Pi$, we obtain $\text{Ker } (\Pi + N) \subseteq \text{Ker } \Pi = \Lambda_{\text{hor}}^1(X)$. The reverse inclusion is immediate, so that $\text{Ker } (\Pi + N) = \Lambda_{\text{hor}}^1(X)$. Finally, as $\text{Im}(\text{id} - \Pi) = \text{Ker } \Pi = \Lambda_{\text{hor}}^1(X)$, we have $N \circ (\text{id} - \Pi) = 0$. Consequently, $(\Pi + N)^2 = \Pi + N$. \square

Traditionally, connection forms are defined as appropriate differential 1-forms on X with values in the Lie algebra of G . The following is a version of connection forms avoiding the use of tangent vectors and Lie algebras.

DEFINITION 3.6. A *connection form* is a linear map $\omega : C^\infty(G) \rightarrow \Lambda^1(X)$ with the properties:

- (i) $\omega(h_1 h_2) = \omega(h_1)h_2(e) + h_1(e)\omega(h_2)$, $\forall h_1, h_2 \in C^\infty(G)$, $e \in G$ the neutral element;
- (ii) $\text{ev}_e \circ R_x^* \circ \omega = \text{ev}_e \circ \text{d}$, $\forall x \in X$, $\text{ev}_e : \Gamma^\infty(T^*G) \ni \alpha \mapsto \text{ev}_e(\alpha) := \alpha_e \in T_e^*G$;
- (iii) $R_g^* \circ \omega = \omega \circ (\text{Ad}_{g^{-1}}^G)^*$, $\forall g \in G$.

It is straightforward to show:

PROPOSITION 3.7. *Let $\tilde{\omega}$ be a traditionally defined connection 1-form, Y a vector field on X , and h a smooth function on G . Then the formula $(\omega(h))(Y) = (\tilde{\omega}(Y))(h)$ implements a bijective correspondence between connection forms and traditionally defined connection forms.*

We shall denote the space of connection forms by $\mathcal{F}(X)$. There is an obvious affine structure on $\mathcal{F}(X)$. The difference of any two such connection forms satisfies the defining conditions (i) and (iii), and annihilates vertical vectors due to the defining condition (ii).

Let us now pass to the third description of the concept of a connection: covariant differentiation. This time there is no need to reformulate its traditional definition. What we need to show instead is that it is an equivalent formulation of the connection understood as an appropriate equivariant distribution in the cotangent bundle T^*X . Recall that the equivalence of connections and traditionally defined connection forms is considered completely standard and is proven in many books, e.g., in [29]. Therefore, since the traditional definition of connection forms is equivalent to ours, we will take for granted the equivalence of connections and connection forms, and focus on the equivalence of connections and covariant differentiations. The latter are axiomatically defined in the following manner:

DEFINITION 3.8. An (exterior) *covariant differentiation* is a linear map $D : C^\infty(X) \rightarrow \Lambda_{\text{hor}}^1(X)$ with the following properties:

- (i) $D(fg) = D(f)g + fD(g), \quad \forall f, g \in C^\infty(X);$
- (ii) $db = Db, \quad \forall b \in \pi^*(C^\infty(M));$
- (iii) $R_g^* \circ D = D \circ R_g^*, \quad \forall g \in G.$

To proceed further, we need the following technical result:

LEMMA 3.9. *The space $\Lambda_{\text{hor}}^1(X)$ of horizontal 1-forms coincides with $C^\infty(X)\pi^*(\Lambda^1(M))$.*

PROOF. The inclusion $C^\infty(X)\pi^*(\Lambda^1(M)) \subseteq \Lambda_{\text{hor}}^1(X)$ is immediate. To prove the reverse inclusion, let us choose a connection. It defines the horizontal projection, vertical projection and horizontal lift. We denote these maps by hor , ver and $\widehat{}$, respectively. Let $\{e_\mu, e^\nu\}_{\mu, \nu}$ be finite dual bases of the finitely generated projective $C^\infty(M)$ -modules $\Gamma^\infty(TM)$ and $\Gamma^\infty(T^*M) = \Lambda^1(M)$. Then, for any horizontal form $\alpha \in \Lambda_{\text{hor}}^1(X)$ and tangent vector $Y_x \in T_x X$, we obtain:

$$\begin{aligned}
\left(\sum_\mu \alpha(\widehat{e_\mu}) \pi^*(e^\mu)\right)(Y_x) &= \sum_\mu \alpha\left(\left(\widehat{(e_\mu)_{\pi(x)}}\right)_x\right) e^\mu(\pi_* Y_x) \\
&= \alpha\left(\left(\sum_\mu (e_\mu)_{\pi(x)} e^\mu(T\pi(Y_x))\right)_x\right) \\
&= \alpha\left(\widehat{T\pi(Y_x)}_x\right) \\
&= \alpha(\text{hor}(Y_x)) \\
&= \alpha(\text{hor}(Y_x) + \text{ver}(Y_x)) \\
&= \alpha(Y_x).
\end{aligned} \tag{3.2.14}$$

Hence $\alpha = \sum_\mu \alpha(\widehat{e_\mu}) \pi^*(e^\mu) \in C^\infty(X)\pi^*(\Lambda^1(M))$. □

We are now ready for:

PROPOSITION 3.10. (cf. [20, p.254–255]) *The formulas $j_{pd}(\Pi) = (\text{id} - \Pi) \circ d$ and $j_{dp}(D)(df) = (d - D)(f)$ define mutually inverse bijections between the space of connections and the space of covariant differentiations.*

PROOF. Let Π be a connection in the sense of Definition 3.4. Since it is an idempotent whose kernel is the space of horizontal 1-forms, we have $\text{Im}(\text{id} - \Pi) = \text{Ker } \Pi = \Lambda_{\text{hor}}^1(X)$. Therefore, $D_\Pi := (\text{id} - \Pi) \circ d$ is a linear map from $C^\infty(X)$ into $\Lambda_{\text{hor}}^1(X)$. Define $D_\Pi := (\text{id} - \Pi) \circ d$. The Leibniz rule for D_Π comes from the Leibniz rule for d and $C^\infty(X)$ -linearity of Π . The form $d(f \circ \pi)$ is evidently horizontal, so that $D(f \circ \pi) = d(f \circ \pi)$. Finally, the equivariance of D_Π is obvious from the equivariance properties of Π and d . Thus D_Π enjoys all the necessary properties.

Now, let D be a covariant differentiation, Y a vector field on X , and f a smooth function on X . Define $D_Y(f) := (D(f))(Y)$. The Leibniz rule for D implies that D_Y is a derivation of $C^\infty(X)$. Therefore, it is a vector field on X , and we can define $\Pi_D(\alpha) := \alpha(Y - D_Y)$, $\alpha \in \Lambda^1(X)$. The thus defined Π_D is evidently $C^\infty(X)$ -linear. Hence, as any $\alpha \in \Lambda^1(X)$ is a finite $C^\infty(X)$ -linear combination of exact 1-forms (see the end of Section 2.2), it is uniquely determined by its values on exact 1-forms, where it coincides with $j_{dp}(D)$. Thus $j_{dp}(D) = \Pi_D$, and the formula for $j_{dp}(D)$ defines a $C^\infty(X)$ -linear endomorphism of $\Lambda^1(X)$. It is immediate that Π_D is equivariant. Furthermore, if $\Pi_D(\alpha) = 0$, then $\alpha(Y) = \alpha(D_Y)$. Presenting α as a finite sum $\sum_k f_k dg_k$ yields $\alpha(Y) = \sum_k f_k (Dg_k)(Y)$. Since $\text{Im}(D) \subseteq \Lambda_{\text{hor}}^1(X)$, we obtain $\alpha(Y) = 0$ if Y is vertical. Hence $\text{Ker } \Pi_D \subseteq \Lambda_{\text{hor}}^1(X)$. The reverse inclusion follows from Lemma 3.9, which allows us to choose the functions g_k from $\pi^*(C^\infty(M))$. Finally, the just proven inclusion $\text{Ker } \Pi_D \supseteq \Lambda_{\text{hor}}^1(X)$ together with $\Lambda_{\text{hor}}^1(X) \supseteq \text{Im}(D)$ implies that $(\Pi_D)^2 = \Pi_D$.

To end the proof, we need to verify that $D_{\Pi_D} = D$ and $\Pi_{D_\Pi} = \Pi$. The former is immediate, and the latter is obvious when remembering that any 1-form is a finite sum $\sum_k f_k dg_k$. \square

Thus we have shown that connections on a principal bundle can be equivalently defined as exterior covariant differentiations. (Observe that the equation $D_\Pi = (\text{id} - \Pi) \circ d$ of the foregoing proposition is just the usual definition $D = d \circ \text{hor}$.) We denote the space of all exterior covariant differentiations by $\mathcal{D}(X)$. It has an obvious affine structure. The difference of any two exterior covariant differentiations is a linear map $C^\infty(X) \rightarrow \Lambda_{\text{hor}}^1(X)$ that is equivariant, satisfies the Leibniz rule and annihilates $\pi^*(C^\infty(M))$.

We end this section by considering the behaviour of connections under gauge transformations. A gauge transformation of a principal bundle (X, π, M, G) is a vertical automorphism of the bundle, i.e., a G -equivariant fibre-preserving diffeomorphism $\gamma : X \rightarrow X$. Gauge transformations form a group with group multiplication given by the composition of maps. We denote this group by $\mathcal{G}(X)$, and define its left action on the spaces $\mathcal{P}(X)$, $\mathcal{F}(X)$ and $\mathcal{D}(X)$ of connections, connection forms and covariant differentiations, respectively, by

$$\Pi^\gamma := (\gamma^{-1})^* \circ \Pi \circ \gamma^*, \quad \omega^\gamma := (\gamma^{-1})^* \circ \omega, \quad D^\gamma := (\gamma^{-1})^* \circ D \circ \gamma^*. \quad (3.2.15)$$

This action commutes with our identifications of connections, connection forms and covariant differentiations. These identifications respect also the affine structure. More precisely, putting together the results of this section, one can verify the following:

THEOREM 3.11. *Let $\mathcal{G}(X)$ be the group of gauge transformations acting on the spaces $\mathcal{P}(X)$, $\mathcal{F}(X)$ and $\mathcal{D}(X)$ of connections, connection forms and exterior covariant differentiations according to (3.2.15). The following formulas define $\mathcal{G}(X)$ -equivariant and affine isomorphisms:*

$$j_{dp} : \mathcal{D}(X) \ni D \longmapsto \Pi^D \in \mathcal{P}(X), \quad \Pi^D(df) := (d - D)f; \quad (3.2.16)$$

$$j_{pf} : \mathcal{P}(X) \ni \Pi \longmapsto \omega^\Pi \in \mathcal{F}(X), \quad (\omega^\Pi(h))(Y_x) := (\text{ver}^\Pi Y_x)(R_x^* h); \quad (3.2.17)$$

$$j_{fd} : \mathcal{F}(X) \ni \omega \longmapsto D^\omega \in \mathcal{D}(X), \quad (D^\omega(f))(Y_x) := (df - \omega(R_x^* f))(Y_x). \quad (3.2.18)$$

Here $f \in C^\infty(X)$, $h \in C^\infty(G)$, $Y_x \in T_x X$, and $\text{ver}^\Pi Y_x$ is defined by $(\text{ver}^\Pi Y_x)(f) := (\Pi(df))(Y_x)$.

3.3 Covariant derivatives and the Chern character

Let us recall that the correspondence between equivariant functions on a principal bundle and sections of an associated vector bundle extends to a correspondence between horizontal equivariant forms on the principal bundle and form-valued sections of the associated vector bundle. Since covariant differentiations on the principal bundle preserve equivariant forms, by this correspondence, they give rise to covariant derivatives (connections) on the associated vector bundle. Recall also that a connection on a vector bundle E over the base M (cf. Section 2.3) can be defined as a linear map

$$\nabla : \Gamma^\infty(E) \rightarrow \Lambda^1(M) \otimes_{C^\infty(M)} \Gamma^\infty(E) \quad (3.3.19)$$

satisfying the Leibniz rule

$$\nabla(fs) = f\nabla(s) + df \otimes_{C^\infty(M)} s, \quad f \in C^\infty(M), \quad s \in \Gamma^\infty(E). \quad (3.3.20)$$

In the first part of this section, we will analyse covariant derivatives from the point of view adopted in Section 3.1, i.e., identifying the module of smooth sections of an associated bundle with a suitable cotensor product.

To identify $\Lambda_{\text{hor}}^1(X)$ with $\Lambda^1(M) \otimes_{C^\infty(M)} C^\infty(X)$, note first that any connection, by providing us with a horizontal lift, defines a push-down:

$$\pi_* : \Lambda_{\text{hor}}^1(X) \longrightarrow \text{Hom}_{C^\infty(M)}(\Gamma^\infty(TM), C^\infty(X)), \quad (\pi_*(\alpha))(Y) := \alpha(\widehat{Y}), \quad Y \in \Gamma^\infty(TM). \quad (3.3.21)$$

It follows from Lemma 3.9 that its inverse is given by the pull-back π^* . Next, since $\Gamma^\infty(TM)$ is finitely generated projective, we have the canonical isomorphism

$$\text{Hom}_{C^\infty(M)}(\Gamma^\infty(TM), C^\infty(X)) \longrightarrow \Lambda^1(M) \otimes_{C^\infty(M)} C^\infty(X). \quad (3.3.22)$$

With the help of a finite dual bases $\{f_\mu, f^\nu\}_{\mu, \nu}$ of $\Gamma^\infty(TM)$ and $\Lambda^1(M)$, the composition of these two isomorphisms can be explicitly written as

$$\Phi : \Lambda_{\text{hor}}^1(X) \longrightarrow \Lambda^1(M) \otimes_{C^\infty(M)} C^\infty(X), \quad \Phi(\alpha) = \sum_{\nu} f^\nu \otimes_{C^\infty(M)} \alpha(\widehat{f}_\nu). \quad (3.3.23)$$

The inverse of this isomorphism is simply the multiplication. If $\alpha = f\beta$ for $f \in C^\infty(X)$ and $\beta \in \pi^*(\Lambda^1(M))$, then $\Phi(f\beta) = \beta \otimes_{C^\infty(M)} f$.

Furthermore, observe that the right action of G on X yields a right coaction Δ_R of $C^\infty(G)$ on $C^\infty(X)$, and $\Delta_R^1 := \text{id} \otimes_{C^\infty(M)} \Delta_R$ defines a right coaction on $\Lambda^1(M) \otimes_{C^\infty(M)} C^\infty(X)$. As in (3.1.10), let $\mathcal{C}^\infty(X) := \{p \in C^\infty(X) \mid \Delta_R(p) \in C^\infty(X) \otimes \mathcal{O}(G)\}$. At this point, we need the following technical lemma:

LEMMA 3.12. *If $D : C^\infty(X) \rightarrow \Lambda_{\text{hor}}^1(X)$ is a covariant differentiation, then*

$$(\Phi \circ D)(\mathcal{C}^\infty(X)) \subseteq \Lambda^1(M) \otimes_{C^\infty(M)} \mathcal{C}^\infty(X). \quad (3.3.24)$$

PROOF. For brevity, we put $\widetilde{D} := \Phi \circ D$. First we need to prove

$$(\Delta_R^1 \circ \widetilde{D})(f) \in \Lambda^1(M) \otimes_{C^\infty(M)} C^\infty(X) \otimes \mathcal{O}(G), \quad \forall f \in C^\infty(X). \quad (3.3.25)$$

Writing $\Delta_R(f)$ as a finite sum $\sum_k f_k \otimes h^k \in C^\infty(X) \otimes \mathcal{O}(G)$, for a fixed $g \in G$, we can compute:

$$\begin{aligned}
\left(\Delta_R^1(\tilde{D}(f))\right)(g) &= \sum_\nu f^\nu \otimes_{C^\infty(M)} \left(\Delta_R\left((Df)(\hat{f}_\nu)\right)\right)(g) \\
&= \sum_\nu f^\nu \otimes_{C^\infty(M)} R_g^* \left((Df)(\hat{f}_\nu)\right) \\
&= R_g^*(\tilde{D}(f)) \\
&= \tilde{D}(R_g^*f) \\
&= \tilde{D}((\Delta_R f)(g)) \\
&= \left(\sum_k \tilde{D}(f_k) \otimes h^k\right)(g). \tag{3.3.26}
\end{aligned}$$

Here we used the equivariance of D and the fact that Φ commutes with R_g^* . Since this computation works for any $g \in G$, the inclusion (3.3.25) is proven.

Next, consider the following identities:

$$\begin{aligned}
\Lambda^1(M) \otimes_{C^\infty(M)} C^\infty(X) &\ni \sum_\nu f^\nu \otimes_{C^\infty(M)} F_\nu = \sum_{\mu,\nu} f^\mu f^\nu(f_\mu) \otimes_{C^\infty(M)} F_\nu = \sum_\mu f^\mu \otimes_{C^\infty(M)} \sum_\nu f^\nu(f_\mu) F_\nu, \tag{3.3.27} \\
\Delta_R(\sum_\nu f^\nu(f_\mu) F_\nu) &= \sum_\nu f^\nu(f_\mu) \Delta_R(F_\nu) = \left((m \otimes \text{id}) \circ (f_\mu \otimes_{C^\infty(M)} \text{id} \otimes \text{id}) \circ (\text{id} \otimes_{C^\infty(M)} \Delta_R)\right)(\sum_\nu f^\nu \otimes_{C^\infty(M)} F_\nu),
\end{aligned}$$

where m is the multiplication map. Now it follows that

$$\begin{aligned}
\Delta_R^1\left(\sum_\nu f^\nu \otimes_{C^\infty(M)} F_\nu\right) &\in \Lambda^1(M) \otimes_{C^\infty(M)} C^\infty(X) \otimes \mathcal{O}(G) \\
&\Downarrow \\
\sum_\nu f^\nu(f_\mu) F_\nu &\in C^\infty(X), \quad \forall \mu. \tag{3.3.28}
\end{aligned}$$

Combining this implication with (3.3.25), we obtain the assertion of this lemma. \square

We are now ready to associate a covariant derivative to a covariant differentiation. Restricting \tilde{D} to $C^\infty(X)$, we obtain a linear map

$$\nabla := (\tilde{D}|_{C^\infty(X)} \otimes \text{id}) : C^\infty(X) \otimes_{\mathcal{O}(G)} V \longrightarrow \Lambda^1(M) \otimes_{C^\infty(M)} C^\infty(X) \otimes_{\mathcal{O}(G)} V. \tag{3.3.29}$$

It is straightforward to check that the thus defined ∇ satisfies the Leibniz rule.

Chern classes and the Chern character are fundamental topological invariants of vector bundles. Our next aim is to recall some of these constructions. Let (E, π, M) be a complex vector bundle. Then, for each $i \geq 0$, the i -th Chern class $c_i(E) \in H_{dR}^{2i}(M)$ can be defined by a certain set of axioms ([30, pp.305–312], [25, Chapter 17]). It is crucial for computations that these classes can be realized in terms of the curvatures of connections. This is the celebrated Chern-Weil construction. For a treatment in an algebraic setting we refer to [33, Section 8.1], for the classical setting to [18] and [34].

Many of the considerations below follow verbatim their counterparts in the universal differential algebra setting of Section 2.3. The graded Leibniz rule allows us to extend a covariant derivative ∇ to a linear map

$$\nabla : \Lambda^*(M) \otimes_{C^\infty(M)} \Gamma^\infty(E) \longrightarrow \Lambda^{*+1}(M) \otimes_{C^\infty(M)} \Gamma^\infty(E), \quad \nabla^1(\alpha \otimes \xi) := d\alpha \otimes \xi + (-1)^{\text{deg}\alpha} \alpha \wedge \nabla(\xi). \tag{3.3.30}$$

The composition $\nabla^2 := \nabla \circ \nabla : \Gamma^\infty(E) \rightarrow \Lambda^2(M) \otimes_{C^\infty(M)} \Gamma^\infty(E)$ is called the *curvature* of a connection ∇ . One easily checks that ∇^2 is an endomorphism of $\Lambda(M) \otimes_{C^\infty(M)} \Gamma^\infty(E)$ over the differential graded algebra $\Lambda(M)$ of de Rham differential forms. In particular, the curvature can be considered as a 2-form on M with values in the endomorphisms of $\Gamma^\infty(E)$.

Since $\Gamma^\infty(E)$ is a finitely generated projective $C^\infty(M)$ -module, we can choose finite dual bases $\{e_k, e^l\}_{k,l}$. The Graßmann connection corresponding to the idempotent $e := (e_l^k) := (e^k(e_l))$ is given by

$$\nabla^e(e_k) := \sum_j de_k^j \otimes_{C^\infty(M)} e_j. \quad (3.3.31)$$

For a general $\xi = \sum_k \xi^k e_k \in \Gamma^\infty(E)$, it follows from the Leibniz rule that $\nabla^e(\xi) = \sum_{j,k} d\xi^k e_k^j \otimes_{C^\infty(M)} e_j$. This is sometimes written in the matrix notation as $\nabla^e(\xi) = (d\xi)e$. One finds that the corresponding curvature is given by

$$(\nabla^e)^2 \left(\sum_k \xi^k e_k \right) = - \sum_{j,k,l,m} \xi^k e_k^j de_j^l de_l^m \otimes_{C^\infty(M)} e_m. \quad (3.3.32)$$

In the matrix notation one would write it $(\nabla^e)^2(\xi) = -\xi e de de$. Note that the minus sign arises from the convention to consider $\Gamma^\infty(E)$ as a left $C^\infty(M)$ -module.

Now let us view the curvature of a connection ∇ as an element of $\Lambda^2(M) \otimes_{C^\infty(M)} \text{End}_{C^\infty(M)}(\Gamma^\infty(E))$. Since the trace of an endomorphism of a finitely generated projective module over a commutative ring is well defined and linear over this ring, we can form

$$ch(\nabla^2) = \text{Tr}(e^{i\nabla^2/2\pi}) = k + ch_1(\nabla^2) + \dots + ch_k(\nabla^2) + \dots, \quad (3.3.33)$$

$$ch_k(\nabla^2) = (i/2\pi)^k \text{Tr}(\nabla^{2k})/k!. \quad (3.3.34)$$

One can show that every $ch_k(\nabla^2)$ is a closed form and that its de Rham cohomology class does not depend on the choice of a connection ∇ . The corresponding classes are denoted by $ch_k(E)$ and lie in $H_{dR}^{2k}(M)$. Their sum is the Chern character $ch(E)$. The class $ch_1(E)$ is also known as the first Chern class and is usually denoted by $c_1(E)$.

The Chern character has the following properties for two complex vector bundles E_1, E_2 :

$$ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2), \quad (3.3.35)$$

$$ch(E_1 \otimes E_2) = ch(E_1)ch(E_2). \quad (3.3.36)$$

(In the second formula the multiplication on the right is the cup product in cohomology.) For the first Chern class, one has

$$c_1(\nabla) = -\frac{1}{2\pi i} \text{Tr} \nabla^2. \quad (3.3.37)$$

If L_1, \dots, L_n are complex line bundles over a compact space, then (see [26, V.3.10])

$$c_1(L_1 \otimes \dots \otimes L_n) = \sum_{k=1}^n c_1(L_k). \quad (3.3.38)$$

For the Chern character of a line bundle L , we have $ch(L) = \exp(c_1(L))$ (see [26, V.3.23]). If the bundle lives over a 2-manifold, all higher than linear terms in $ch(L)$ vanish, so that in this case $ch(L) = 1 + c_1(L)$.

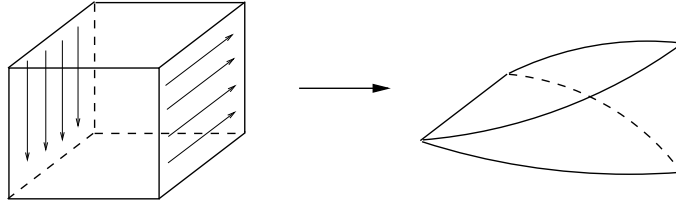
3.4 Hopf fibration and Dirac monopole

We begin this section with the description of a general topological construction of a principal bundle starting from a compact group G . When specialising to $G = U(1)$, we shall recover the Hopf fibration $S^3 \rightarrow S^2$.

Let $I = [0, 1]$ be the closed unit interval and let G be a compact group. On the space $I \times G \times G$, consider the equivalence relation where R given by

$$\begin{aligned} (0, g, h) \sim (0, g', h') &\iff h = h' \\ (t, g, h) \sim (t', g', h') &\iff t = t', g = g', h = h', t \neq 0, 1, \\ (1, g, h) \sim (1, g', h') &\iff g = g'. \end{aligned} \quad (3.4.39)$$

The quotient space $X := (I \times G \times G)/R$ is like a partial suspension of $G \times G$, with the first copy of G collapsed at the left end, and the second copy collapsed at the right end. The space X is compact Hausdorff.



The diagonal action of G on $I \times G \times G$ factorizes to the quotient, so that the formula $([(t, g, h)], k) \mapsto [(t, gk, hk)]$ makes X a right G -space. It is immediate that this action is principal. The quotient X/G is homeomorphic to the suspension $SG := (I \times G)/(\{0\} \times G \cup \{1\} \times G)$ of G . (Note that this notion of suspension differs from the one used in K -theory.) Indeed, one can verify that the map $X/G \rightarrow SG$ given by

$$[[[t, g, h]]] \mapsto \begin{cases} x_0 := [(0, k)] & \text{for } t = 0, \quad k \in G \\ [(t, gh^{-1})] & \text{for } t \neq 0, 1 \\ x_1 := [(1, k)] & \text{for } t = 1, \quad k \in G \end{cases} \quad (3.4.40)$$

is a homeomorphism. The composition of the canonical quotient map with this homeomorphism defines a principal G -bundle $\pi : X \rightarrow SG$. Explicitly, π is given by

$$\pi([[(t, g, h)]]) = \begin{cases} x_0 & \text{for } t = 0 \\ [(t, gh^{-1})] & \text{for } t \neq 0, 1 \\ x_1 & \text{for } t = 1 \end{cases} \quad (3.4.41)$$

PROPOSITION 3.13. *The principal bundle (X, π, SG, G) is trivial if and only if G is contractible.*

PROOF. Suppose that the bundle is trivial. Then there exists a continuous global section $\sigma : SG \rightarrow X$. This section is necessarily of the form:

$$\sigma(x_0) = [(0, g_0, h)], \quad (3.4.42)$$

$$\sigma([(t, g)]) = [(t, s(t, g), g^{-1}s(t, g))] \quad \text{for } t \neq 0, 1, \quad (3.4.43)$$

$$\sigma(x_1) = [(1, h, g_1)]. \quad (3.4.44)$$

Here $s :]0, 1[\times G \rightarrow G$ is a continuous map. The continuity of σ implies that $\lim_{[(t, g)] \rightarrow x_0} s(t, g) = g_0$ and $\lim_{[(t, g)] \rightarrow x_1} g^{-1}s(t, g) = g_1$. On the other hand, consider the following map:

$$H(t, g) := \begin{cases} g_0 g_1^{-1} & \text{for } t = 0 \\ s(t, g) g_1^{-1} & \text{for } t \in]0, 1[\\ g & \text{for } t = 1 \end{cases} \quad (3.4.45)$$

It is evidently continuous at any point in $]0, 1[\times G$. The continuity at any point $(0, k)$ follows from the first of the above limits. To conclude the continuity at any point $(1, k)$, note that H is continuous if and only if H' defined by $H'(t, g) := g^{-1}H(t, g)$ is continuous, and use the second limit. Hence H is a homotopy between id_G and the constant map $g \mapsto g_0g_1^{-1}$. The implication in the other direction is clear. \square

We now proceed to exhibit the local triviality of our bundle. Define $U_0 := \{[(t, g)] \in SG \mid t \in [0, 1[\}$ and $U_1 := \{[(t, g)] \in SG \mid t \in]0, 1]\}$. They form an open cover admitting local sections $\sigma_0 : U_0 \rightarrow X$ and $\sigma_1 : U_1 \rightarrow X$ are defined by

$$\sigma_0(x_0) = [(0, e, h)], \quad \sigma_0([(t, g)]) = [(t, e, g^{-1})], \quad (3.4.46)$$

$$\sigma_1(x_1) = [(1, h, e)], \quad \sigma_1([(t, g)]) = [(t, g, e)]. \quad (3.4.47)$$

These two local sections determine a transition function $\theta : U_0 \cap U_1 \cong]0, 1[\times G \rightarrow G$ via (see the end of Section 1)

$$\theta([(t, g)]) = \check{\tau}(\sigma_0([(t, g)]), \sigma_1([(t, g)])) = g. \quad (3.4.48)$$

Restricting to the cones $C_0 := \{[(t, g)] \in SG \mid t \in [0, \frac{1}{2}]\}$ and $C_1 := \{[(t, g)] \in SG \mid t \in [\frac{1}{2}, 1]\}$, we get piecewise trivialisations. The restriction of the transition function (3.4.48) to $C_0 \cap C_1 \cong G$ can be considered as the identity map of G . Thus we arrive at the following geometrical picture: The base space SG is glued from the two cones C_0 and C_1 , and the total space X is glued from the two trivial pieces $C_0 \times G$ and $C_1 \times G$ via the above transition function.

Let us now specialise to the case $G = U(1)$, and see that we obtain the Hopf fibration. To this end, first recall the usual description of this bundle (see [35]). Consider the 3-sphere S^3 as $\{(a, c) \in \mathbb{C}^2 \mid |a|^2 + |c|^2 = 1\}$. The Hopf fibration is obtained by the right action $((a, c), e^{i\varphi}) \mapsto (ae^{i\varphi}, ce^{i\varphi})$ of $U(1)$ on S^3 . Identifying S^3 with $SU(2)$ by identifying (a, c) with the matrix

$$\begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix}, \quad (3.4.49)$$

we obtain the right action of $U(1)$ as the right multiplication with $\text{diag}(e^{i\varphi}, e^{-i\varphi})$. The quotient $S^3/U(1) = SU(2)/U(1)$ is diffeomorphic to the two-sphere S^2 . The bundle projection $\pi : S^3 \rightarrow S^2$ is explicitly given by

$$\pi(a, c) = (2a\bar{c}, |a|^2 - |c|^2). \quad (3.4.50)$$

By Theorem 1.21, this bundle is a locally trivial principal bundle.

It is straightforward to check that the following formulas define a $U(1)$ -equivariant homeomorphism $f : X = ([0, 1] \times U(1) \times U(1))/R \rightarrow S^3$ and its inverse:

$$f[(t, g, h)] = \left(\sqrt{t} g, \sqrt{1-t} h \right), \quad (3.4.51)$$

$$f^{-1}(a, c) = \left[\left(|a|^2, \frac{a}{|a|}, \frac{c}{|c|} \right) \right]. \quad (3.4.52)$$

The suspension of $U(1)$ is identified with S^2 via the following homeomorphism φ and its inverse:

$$\varphi[(t, g)] = \left(2g\sqrt{t-t^2}, 2t-1 \right), \quad (3.4.53)$$

$$\varphi^{-1}(z, s) = \left[\left(\frac{s+1}{2}, \frac{z}{|z|} \right) \right]. \quad (3.4.54)$$

(Note that the endpoints x_0 and x_1 of the suspension of $U(1)$ are identified with the poles $(0, -1)$ and $(0, 1)$ of S^2 , respectively.)

Next, consider $V_0 = S^2 \setminus \{(0, 1)\}$ and $V_1 = S^2 \setminus \{(0, -1)\}$. Then $\pi^{-1}(V_0) = \{(a, c) \in S^3 \mid c \neq 0\}$ and $\pi^{-1}(V_1) = \{(a, c) \in S^3 \mid a \neq 0\}$, and we get locally trivialising maps $\chi_0 : \pi^{-1}(V_0) \rightarrow V_0 \times U(1)$ and $\chi_1 : \pi^{-1}(V_1) \rightarrow V_1 \times U(1)$ by

$$\chi_0(a, c) = (\pi(a, c), \frac{c}{|c|}), \quad \chi_1(a, c) = (\pi(a, c), \frac{a}{|a|}). \quad (3.4.55)$$

The corresponding transition function $\theta_{01} : V_0 \cap V_1 \rightarrow U(1)$ is given by

$$\theta_{01}(\pi(a, c)) = \frac{a}{|a|} \frac{|c|}{c}. \quad (3.4.56)$$

It is straightforward to verify that in our concrete case $G = U(1)$ the open sets V_0 and V_1 are precisely the images under φ of the open sets U_0 and U_1 defined above, that the local trivialisations χ_0 , χ_1 correspond to the sections σ_0 , σ_1 , and that the transition functions (3.4.56) and (3.4.48) coincide up to the homeomorphism φ . Note that the restriction of θ_{01} to the equator $|a| = |c|$ is the identity function, which is in accordance with the above remark on piecewise triviality.

As is well-known, the 3-sphere can be considered as a gluing of two solid tori along their boundaries. Indeed, S^3 can be split into the union of $T_1 = \{(a, c) \in S^3 \mid |a|^2 \in [0, \frac{1}{2}]\}$ and $T_2 = \{(a, c) \in S^3 \mid |a|^2 \in [\frac{1}{2}, 1]\}$ (Heegaard splitting, see Picture 3.4.63). To prove that T_1 and T_2 are solid tori, we map S^3 onto

$$X := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_i| \leq 1, (1 - |z_1|^2)(1 - |z_2|^2) = 0\} \quad (3.4.57)$$

by a homeomorphism g given by

$$g(a, c) = \frac{\sqrt{2}(a, c)}{\sqrt{1 + ||a|^2 - |c|^2|}}. \quad (3.4.58)$$

Its inverse is defined by

$$g^{-1}(z_1, z_2) = \frac{(z_1, z_2)}{\sqrt{1 + |z_1|^2|z_2|^2}}. \quad (3.4.59)$$

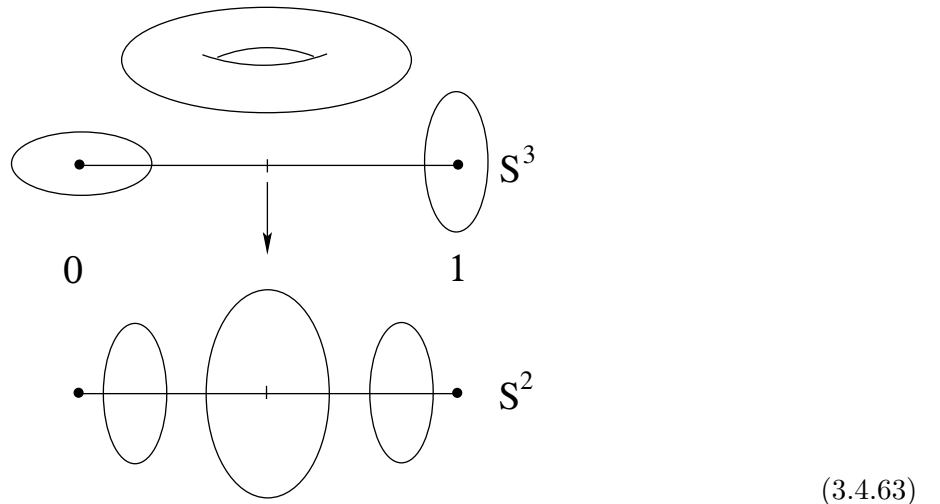
See [22, 3] for proofs and details. It is immediate to check that

$$g(T_1 \cap T_2) = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = |z_2| = 1\}, \quad (3.4.60)$$

$$g(T_1) = X_1 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = 1, |z_2| \leq 1\} = S^1 \times D^2, \quad (3.4.61)$$

$$g(T_2) = X_2 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \leq 1, |z_2| = 1\} = D^2 \times S^1. \quad (3.4.62)$$

Thus we have shown that $S^3 \cong D^2 \times S^1 \cup S^1 \times D^2$.



All the above can be written in a dual algebraic way, using the coordinate functions α and γ , which assign to an $SU(2)$ -matrix its components a and c . Let us recall that the right action of $U(1)$ on $SU(2)$ gives rise to a right coaction $\Delta_R : C(SU(2)) \rightarrow C(SU(2)) \otimes C(U(1))$, and that $C(S^2)$ can be identified with the subalgebra of invariants, $C(S^2) = \{f \in C(SU(2)) \mid \Delta_R(f) = f \otimes 1\}$. Note also that the $*$ -subalgebras $\mathcal{O}(SU(2)) \subseteq C(SU(2))$ (respectively $\mathcal{O}(U(1)) \subseteq C(U(1))$) can be identified with the free $*$ -algebras generated by α, γ modulo commutativity and the relation $\alpha^* \alpha + \gamma^* \gamma = 1$ (respectively by one element u modulo commutativity and the relation $u^* u = 1$). The comultiplication of $C(SU(2))$ is given on the generators by

$$\Delta(\alpha) = \alpha \otimes \alpha - \gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma, \quad (3.4.64)$$

and the antipode and the counit by

$$S(\alpha) = \alpha^*, \quad S(\gamma) = -\gamma, \quad \varepsilon(\alpha) = 1, \quad \varepsilon(\gamma) = 0. \quad (3.4.65)$$

The right coaction on the generators is

$$\Delta_R(\alpha) = \alpha \otimes u, \quad \Delta_R(\gamma) = \gamma \otimes u. \quad (3.4.66)$$

Note that here all the maps (also the antipode) are $*$ -homomorphisms, so that it suffices to define them on generators. Using the Poincaré-Birkhoff-Witt basis $\alpha^k \gamma^{*l} \gamma^m, \gamma^{*p} \gamma^r \alpha^{*s}, k, l, m, p, r, s \in \mathbb{N}, k > 0$, of $\mathcal{O}(SU(2))$, one can easily derive that the subalgebra $\mathcal{O}(S^2) \subseteq C(S^2)$ of invariant polynomials is spanned by the elements $\alpha^k \gamma^{*l} \gamma^m$ with $k + m - l = 0$ and the elements $\gamma^{*p} \gamma^r \alpha^{*s}$ with $r - p - s = 0$.

Our goal is now to describe vector bundles associated to the $U(1)$ -Hopf principal bundle, and to compute Chern numbers for these bundles. The irreducible representations (all one-dimensional) of $U(1)$ are parametrized by \mathbb{Z} and given by $e^{i\varphi} \mapsto e^{i\mu\varphi}, \mu \in \mathbb{Z}$. For any of these representations we have a line bundle L_μ associated to the Hopf fibration. Since the integer μ functions as the winding number of a representation defining L_μ , we call it the *winding number* of the bundle L_μ . Let P_μ denote the space of continuous sections of L_μ , i.e., $P_\mu := \Gamma(L_\mu)$. Then we have the following isomorphisms, coming from Lemma 1.16 and Theorem 3.3:

$$P_\mu \cong \text{Hom}_{U(1)}(SU(2), \mathbb{C}_\mu) \cong \mathcal{C}(SU(2)) \square_{\mathcal{O}(U(1))} {}^\mu \mathbb{C}. \quad (3.4.67)$$

Here \mathbb{C}_μ is the symbol for the space \mathbb{C} viewed as a right $U(1)$ -space with action $(g, z) \mapsto g^{-\mu} z$, and ${}^\mu \mathbb{C}$ signifies the space \mathbb{C} as a left $\mathcal{O}(U(1))$ -comodule with left coaction ${}_{\mathbb{C}} \Delta(1) = u^{-\mu} \otimes 1$. By the first isomorphism, P_μ is identified with the $C(S^2)$ -submodule $\{f \in C(SU(2)) \mid f(xg) = g^{-\mu} f(x)\}$ of $C(SU(2))$, and by the second one with the $C(S^2)$ -submodule $\{f \in \mathcal{C}(SU(2)) \mid \Delta_R(f) = f \otimes u^{-\mu}\}$ of $\mathcal{C}(SU(2)) = \bigoplus_{\mu \in \mathbb{Z}} P_\mu$ (algebraic direct sum). The last equality is immediate from the fact that the elements $u^\mu, \mu \in \mathbb{Z}$, form a linear basis of $\mathcal{O}(U(1))$.

Our next aim is to obtain explicit idempotents determining the line bundles. We will follow the lines of [21], where this was done for the quantum group $SU_q(2)$ (cf. [32]). With the help of the Poincaré-Birkhoff-Witt basis $\alpha^k \gamma^{*l} \gamma^m, \gamma^{*p} \gamma^r \alpha^{*s}, k, l, m, p, r, s \in \mathbb{N}, k > 0$, of $\mathcal{O}(SU(2))$, one can show that $f \in \mathcal{O}(SU(2)) \cap P_\mu$ is an element of $\sum_{k=0}^{-\mu} \mathcal{O}(S^2) \alpha^{-\mu-k} \gamma^k$ for $\mu \leq 0$, and of $\sum_{k=0}^{\mu} \mathcal{O}(S^2) \gamma^{*k} \alpha^{*\mu-k}$ for $\mu \geq 0$. This suggests that we should have

$$P_\mu = \begin{cases} \sum_{k=0}^{-\mu} C(S^2) \alpha^{-\mu-k} \gamma^k & \text{for } \mu \leq 0 \\ \sum_{k=0}^{\mu} C(S^2) \gamma^{*k} \alpha^{*\mu-k} & \text{for } \mu \geq 0. \end{cases} \quad (3.4.68)$$

It is obvious that the right hand side of (3.4.68) is contained in P_μ . In order to prove the reverse inclusion, let $f \in P_\mu$, i.e., $\Delta_R(f) = f \otimes u^{-\mu}$. Let us consider the case $\mu \leq 0$. Then the functions

$f_k := f\alpha^{*-\mu-k}\gamma^{*k}$, $k = 0, \dots, -\mu$, are in $C(S^2)$. On the other hand,

$$\sum_{k=0}^{-\mu} f_k \binom{-\mu}{k} \alpha^{-\mu-k} \gamma^k = f \sum_{k=0}^{-\mu} \alpha^{*-\mu-k} \gamma^{*k} \binom{-\mu}{k} \alpha^{-\mu-k} \gamma^k = f, \quad (3.4.69)$$

because the last sum coincides with $(m \circ (S \otimes \text{id}) \circ \Delta)(\alpha^{-\mu}) = \varepsilon(\alpha^{-\mu}) = 1$. This proves the first equation (3.4.68). The second equation is derived similarly.

The foregoing argument gives us a key for finding idempotents. Indeed (again for the case $\mu \leq 0$), the two column vectors $\tilde{v}_\mu, \tilde{w}_\mu$ defined by

$$\tilde{v}_\mu^\top = \left(\alpha^{*-\mu}, \dots, \binom{-\mu}{k} \alpha^{*- \mu - k} \gamma^{*k}, \dots, \gamma^{*-\mu} \right), \quad (3.4.70)$$

$$\tilde{w}_\mu^\top = \left(\alpha^{-\mu}, \dots, \alpha^{-\mu-k} \gamma^k, \dots, \gamma^{-\mu} \right), \quad (3.4.71)$$

satisfy $\tilde{v}_\mu^\top \tilde{w}_\mu = 1$. Therefore, $(\tilde{w}_\mu \tilde{v}_\mu^\top)^2 = \tilde{w}_\mu \tilde{v}_\mu^\top \tilde{w}_\mu \tilde{v}_\mu^\top = \tilde{w}_\mu \tilde{v}_\mu^\top$. For the case $\mu \geq 0$ we similarly use

$$\tilde{v}_\mu^\top = \left(\alpha^\mu, \dots, \binom{\mu}{k} \alpha^{\mu-k} \gamma^k, \dots, \gamma^\mu \right), \quad (3.4.72)$$

$$\tilde{w}_\mu^\top = \left(\alpha^{*\mu}, \dots, \alpha^{*\mu-k} \gamma^{*k}, \dots, \gamma^{*\mu} \right). \quad (3.4.73)$$

Let us put $\tilde{E}_\mu = \tilde{w}_\mu \tilde{v}_\mu^\top$. The entries of this $(|\mu| + 1) \times (|\mu| + 1)$ -matrix are elements of $C(S^2)$.

PROPOSITION 3.14. *The left $C(S^2)$ -modules P_μ and $C(S^2)^{|\mu|+1} \tilde{E}_\mu$ are isomorphic.*

PROOF. We define a map $C(S^2)^{|\mu|+1} \tilde{E}_\mu \rightarrow P_\mu$ by $x^\top \tilde{E}_\mu \mapsto x^\top \tilde{w}_\mu$. This map is well defined if and only if $x^\top (1 - \tilde{E}_\mu) \mapsto 0$. To prove the latter, we observe that, due to $\tilde{v}_\mu^\top \tilde{w}_\mu = 1$,

$$x^\top (1 - \tilde{w}_\mu \tilde{v}_\mu^\top) \mapsto x^\top \tilde{w}_\mu - x^\top \tilde{w}_\mu \tilde{v}_\mu^\top \tilde{w}_\mu = 0. \quad (3.4.74)$$

The surjectivity of this map follows from (3.4.68). Moreover, from $x^\top \tilde{w}_\mu = 0$ we infer that $x^\top \tilde{w}_\mu \tilde{v}_\mu^\top = x^\top \tilde{E}_\mu = 0$, so that the map is injective. It is obviously a homomorphism of left $C(S^2)$ -modules. \square

REMARK 3.15. The idempotent \tilde{E}_μ is not hermitian, which is due to the fact that the binomial coefficients appear only in \tilde{v}_μ . However, hermiticity can be achieved by a similarity transformation. Let

$$A_\mu := \text{diag} \left(1, \dots, \sqrt{\binom{|\mu|}{k}}, \dots, 1 \right). \quad (3.4.75)$$

Then the vectors $w_\mu = A_\mu \tilde{w}_\mu$, $v_\mu^\top = \tilde{v}_\mu^\top A_\mu^{-1}$ give rise to $E_\mu := w_\mu v_\mu^\top = A_\mu \tilde{E}_\mu A_\mu^{-1}$. Hence, from $v_\mu^\top = w_\mu^*$, we conclude that $E_\mu^* = E_\mu$. \diamond

Next, let us denote the real coordinates in $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ by x_1, x_2, x_3 , and consider x_1, x_2, x_3 as continuous functions on $S^2 \subseteq \mathbb{C} \times \mathbb{R}$. Then the formula (3.4.50) means

$$\pi^*(1 + x_3) = 2\alpha^* \alpha, \quad \pi^*(1 - x_3) = 2\gamma^* \gamma, \quad \pi^*(x_1 + ix_2) = 2\alpha^* \gamma, \quad \pi^*(x_1 - ix_2) = 2\alpha \gamma^*. \quad (3.4.76)$$

Omitting the pull-back π^* in these formulas we can write the following explicit expressions for the projectors $E_{\pm 1}$:

$$E_{-1} = \tilde{E}_{-1} = \begin{pmatrix} \alpha^* \alpha & \alpha \gamma^* \\ \alpha^* \gamma & \gamma^* \gamma \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & 1 - x_3 \end{pmatrix} \quad (3.4.77)$$

and

$$E_1 = \tilde{E}_1 = \begin{pmatrix} \alpha^* \alpha & \alpha^* \gamma \\ \alpha \gamma^* & \gamma^* \gamma \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & 1 - x_3 \end{pmatrix}. \quad (3.4.78)$$

Let us now see which of these idempotents corresponds to the tautological line bundle. Consider S^2 as $\mathbb{C}P^1$ with homogeneous coordinates $[z_1 : z_2]$. Then the fibre of the tautological line bundle at $[z_1 : z_2]$ is the complex line $\{(\lambda z_1, \lambda z_2) \mid \lambda \in \mathbb{C}\}$, and

$$\frac{1}{|z_1|^2 + |z_2|^2} \begin{pmatrix} z_1^* \\ z_2^* \end{pmatrix} \begin{pmatrix} z_1 & z_2 \end{pmatrix} \quad (3.4.79)$$

is an idempotent defining the left module of sections of this bundle. Restricting to $S^3 \subseteq \mathbb{C}^2$, we have $(|z_1|^2 + |z_2|^2)^{-1/2} z_1 = \alpha$, $(|z_1|^2 + |z_2|^2)^{-1/2} z_2 = \gamma$, and the above idempotent coincides with E_1 . This means that the tautological line bundle coincides with the associated line bundle L_1 coming from the identity representation.

From the proof of the Serre-Swan Theorem, together with the fact that S^2 is covered by two trivialising neighbourhoods, we know that there exist idempotents in $M_2(C(S^2))$ defining the modules P_μ . (See [26, p.34–35] for a general formula for an idempotent in terms of a partition of unity and transition functions subordinate to local trivialisations, and [16, p.77] for concrete formulas.) Let us give now some heuristic arguments leading to such idempotents. From (3.4.68) we know that $\alpha^{-\mu}, \dots, \alpha^{-\mu-k} \gamma^k, \dots, \gamma^{-\mu}$ are generators of the $C(S^2)$ -module P_μ (for $\mu \leq 0$). If we want to choose only two of these as generators, we immediately see that the only possibility is to take $\alpha^{-\mu}$ and $\gamma^{-\mu}$, because any other pair has common zeros. Now, for sure we have $C(S^2)\alpha^{-\mu} + C(S^2)\gamma^{-\mu} \subseteq P_\mu$. To prove the reverse inclusion, it is sufficient to find $g_1, g_2 \in P_{-\mu}$ such that $g_1\alpha^{-\mu} + g_2\gamma^{-\mu} = 1$. Indeed, then $f \in P_\mu$ can be written as $f = f g_1 \alpha^{-\mu} + f g_2 \gamma^{-\mu}$, with $f g_1, f g_2 \in C(S^2)$. It is immediate that

$$g_1 = \frac{\alpha^{*-\mu}}{|\alpha|^{-2\mu} + |\gamma|^{-2\mu}} \quad \text{and} \quad g_2 = \frac{\gamma^{*-\mu}}{|\alpha|^{-2\mu} + |\gamma|^{-2\mu}} \quad (3.4.80)$$

satisfy these requirements. Thus we have the equality $C(S^2)\alpha^{-\mu} + C(S^2)\gamma^{-\mu} = P_\mu$.

We can also prove an analogue of Proposition 3.14 for 2×2 -idempotents coming from the above considerations. Furthermore, we can argue as in Remark 3.15 to arrive at a hermitian 2×2 -idempotent. It can be constructed as follows. Define column vectors r_μ by

$$r_\mu^* = \frac{1}{\sqrt{|\alpha|^{2|\mu|} + |\gamma|^{2|\mu|}}} \begin{cases} (\alpha^{*-\mu}, \gamma^{*-\mu}) & \text{for } \mu < 0 \\ (\alpha^\mu, \gamma^\mu) & \text{for } \mu > 0 \end{cases}. \quad (3.4.81)$$

They satisfy $r_\mu^* r_\mu = 1$ and give a hermitian idempotent $p_\mu := r_\mu r_\mu^*$ such that $P_\mu \cong C(S^2)^2 p_\mu$. We have the following explicit formulas for the projectors p_μ (the pull-back π^* omitted):

$$p_\mu = \frac{1}{(\alpha^* \alpha)^{|\mu|} + (\gamma^* \gamma)^{|\mu|}} \begin{pmatrix} (\alpha^* \alpha)^{|\mu|} & (\alpha \gamma^*)^{|\mu|} \\ (\alpha^* \gamma)^{|\mu|} & (\gamma^* \gamma)^{|\mu|} \end{pmatrix} \quad (3.4.82)$$

$$= \frac{1}{(1+x_3)^{|\mu|} + (1-x_3)^{|\mu|}} \begin{pmatrix} (1+x_3)^{|\mu|} & (x_1 - ix_2)^{|\mu|} \\ (x_1 + ix_2)^{|\mu|} & (1-x_3)^{|\mu|} \end{pmatrix}, \mu < 0, \quad (3.4.83)$$

$$p_\mu = \frac{1}{(\alpha^* \alpha)^\mu + (\gamma^* \gamma)^\mu} \begin{pmatrix} (\alpha^* \alpha)^\mu & (\alpha^* \gamma)^\mu \\ (\alpha \gamma^*)^\mu & (\gamma^* \gamma)^\mu \end{pmatrix} \quad (3.4.84)$$

$$= \frac{1}{(1+x_3)^\mu + (1-x_3)^\mu} \begin{pmatrix} (1+x_3)^\mu & (x_1 + ix_2)^\mu \\ (x_1 - ix_2)^\mu & (1-x_3)^\mu \end{pmatrix}, \mu > 0. \quad (3.4.85)$$

In particular we obtain $p_{\pm 1} = E_{\pm 1}$. Finally, one can see directly that the projectors E_μ and p_μ are Murray-von Neumann equivalent for any μ . Indeed, the $(|\mu| + 1) \times 2$ -matrix $F_\mu := r_\mu w_\mu^*$ has entries from $C(S^2)$ and satisfies $F_\mu F_\mu^* = p_\mu$ and $F_\mu^* F_\mu = E_\mu$.

Our next and main goal is to compute the Chern class $c_1(L_\mu)$ for any integer μ . First, let us make some general remarks about computations with 2×2 -idempotents. Any 2×2 -matrix e with entries from an algebra A can be written uniquely as $e = \sum_{k=0}^3 s_k \sigma_k$, where σ_k are the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (3.4.86)$$

and $s_k \in A$. The matrices σ_k , $k = 1, 2, 3$, are traceless and satisfy $\sigma_k^2 = I$ and $\sigma_j \sigma_k = i \varepsilon_{jkl} \sigma_l$. Here (j, k, l) is any cyclic permutation of $(1, 2, 3)$, and ε_{jkl} is the completely antisymmetric tensor. Assume now that $A := C(M)$ is the algebra of continuous functions on a compact Hausdorff space M . It is immediate from these formulas and the linear independence of the σ_k 's that the condition $e^2 = e$ is equivalent to the equations

$$s_0 = s_0^2 + s_1^2 + s_2^2 + s_3^2, \quad s_1 = 2s_0 s_1, \quad s_2 = 2s_0 s_2, \quad s_3 = 2s_0 s_3. \quad (3.4.87)$$

Moreover, it follows that $s_0 = \frac{1}{2}$ at every point of M where at least one of the s_k , $k = 1, 2, 3$, does not vanish. (We disregard the trivial case $e = 0$.) Also, it follows that $s_0 = 1$ at every point where all the other s_k vanish. Thus, if M is connected, either $s_0 = 1$ and $s_k = 0$, $k = 1, 2, 3$, everywhere (i.e., $e = I$), or $s_0 = \frac{1}{2}$ and $s_1^2 + s_2^2 + s_3^2 = \frac{1}{4}$. If M is a manifold and the entries of e are differentiable, then

$$de = ds_1 \sigma_1 + ds_2 \sigma_2 + ds_3 \sigma_3. \quad (3.4.88)$$

An easy calculation making use of the above properties of the Pauli matrices (in particular regarding the trace), leads to

$$\text{Tr}(e de de) = 4i (s_1 ds_2 \wedge ds_3 + s_2 ds_3 \wedge ds_1 + s_3 ds_1 \wedge ds_2). \quad (3.4.89)$$

On the other hand, for the line bundles over the 2-sphere, let us notice that L_μ is the tensor product of $|\mu|$ copies of L_1 (for $\mu > 0$) or L_{-1} (for $\mu < 0$). Also, $L_\mu \otimes L_{-\mu} \cong L_0$ for any $\mu \in \mathbb{Z}$ because the tensor product of a representation and its conjugate is the trivial representation. Hence, it follows from (3.3.38) that $c_1(L_\mu) = \mu c_1(L_1)$. In view of (3.3.37) and (3.3.32), the right hand side is determined by $c_1(L_1) = \frac{1}{2\pi i} \text{Tr}(E_1 dE_1 dE_1)$. Furthermore, from (3.4.78) we have

$$E_1 = \frac{1}{2}(I + x_3 \sigma_1 + x_1 \sigma_2 - x_2 \sigma_3), \quad (3.4.90)$$

so that $s_1 = \frac{1}{2}x_3$, $s_2 = \frac{1}{2}x_1$, $s_3 = -\frac{1}{2}x_2$. Putting this into (3.4.89), we immediately obtain

$$c_1(L_1) = -\frac{1}{4\pi}(x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2). \quad (3.4.91)$$

Integrating $c_1(L_1)$ in the spherical coordinates (for the orientation with outward-pointing normal vector) gives -1 . Consequently, $\int_{S^2} c_1(L_\mu) = -\mu$.

In the framework of noncommutative geometry, the integration of $c_1(L_\mu)$ can be understood as a pairing between the cyclic 2-cocycle given by $\varphi_{S^2}(f_0, f_1, f_2) := \frac{1}{2\pi i} \int_{S^2} f_0 df_1 \wedge df_2$ and the K_0 -class of the projector E_μ . Therefore, we can summarise our computation as:

THEOREM 3.16. *The pairing of the K_0 -class of the projector E_μ and the cyclic cohomology class of the cyclic 2-cocycle φ_{S^2} (fundamental class) is equal to minus the winding number of the line bundle L_μ :*

$$\langle [E_\mu], [\varphi_{S^2}] \rangle = \frac{1}{2\pi i} \int_{S^2} \text{Tr}(E_\mu dE_\mu \wedge dE_\mu) = \int_{S^2} c_1(L_\mu) = -\mu. \quad (3.4.92)$$

We proceed now to the discussion of connections. In the general situation of a connected Lie group G with a closed subgroup H giving rise to a principal bundle $\pi : G \rightarrow G/H$ with the structure group H , it is natural to look for connections that are invariant under the left action of G on itself. Such connections are characterized by $\text{ad}(H)$ -invariant complements of the Lie algebra \mathfrak{h} of H in the Lie algebra \mathfrak{g} of G (see [29, II, Theorem 11.1]). Equivalently, a G -invariant connection is given by a projection $\mathfrak{g} \rightarrow \mathfrak{h}$ that commutes with $\text{ad}(H)$. In the semisimple case, a complement can be defined by orthogonality with respect to the Killing metric. The corresponding connection is then called canonical.

In the dual formulation in terms of function algebras, this can be rephrased as an appropriate splitting of the pull-back of the imbedding of a closed subgroup into a compact group. To be more precise, let G be a compact group with closed subgroup H , and let $p : \mathcal{O}(G) \rightarrow \mathcal{O}(H)$ be the pull-back of the inclusion map $H \rightarrow G$. Denote by Δ^G and Δ^H the coproducts of the Hopf algebras $\mathcal{O}(G)$ and $\mathcal{O}(H)$, respectively. If $i : \mathcal{O}(H) \rightarrow \mathcal{O}(G)$ is a unital linear splitting of p , then the desired invariance properties of i can be given as follows:

$$(i \otimes \text{id}) \circ \Delta^H = (\text{id} \otimes p) \circ \Delta^G \circ i, \quad (\text{id} \otimes i) \circ \Delta^H = (p \otimes \text{id}) \circ \Delta^G \circ i. \quad (3.4.93)$$

We want to show that the connection defining the Dirac monopole can be obtained along these lines.

To this end, let us imbed $U(1)$ in $SU(2)$ as the subgroup of diagonal matrices, so that its pull-back reads $p(\alpha) = u$, $p(\gamma) = 0$. Now, observe that the unital linear map $i : \mathcal{O}(U(1)) \rightarrow \mathcal{O}(SU(2))$ defined by

$$i(u^n) = \alpha^n, \quad i(u^{*n}) = \alpha^{*n}, \quad n \in \mathbb{N}, \quad (3.4.94)$$

enjoys all the aforementioned properties (see [11, 2.58]). Starting from i , we can define $\omega : \mathcal{O}(U(1)) \rightarrow \Lambda^1(SU(2))$ by $\omega = m \circ (S \otimes \text{d}) \circ \Delta \circ i$. Here d denotes the de Rham differential on $SU(2)$ and m is the multiplication. In particular, we have

$$\omega(u) = \alpha^* \text{d}\alpha + \gamma^* \text{d}\gamma. \quad (3.4.95)$$

We want to prove now that ω is a connection form in the sense of Definition 3.6. First, ω is constructed as the canonical projection on the de Rham differential forms of a universal-calculus connection form (see [11, Example 2.14]). It follows from the general theory of such connection forms that their projections on the de Rham calculus always satisfy the Leibniz rule [11, (2.62)] that is the first defining property of a connection form. Therefore, ω is determined by its values on u . With this in mind, a verification of the remaining defining properties is straightforward.

On the other hand, it is known that the traditionally defined Dirac monopole connection form comes from the canonical invariant connection (see [29, p.110]), and is given by the formula (cf. [36, pp.38-40])

$$\tilde{\omega} = \alpha^* \text{d}\alpha + \gamma^* \text{d}\gamma. \quad (3.4.96)$$

The very form of equations (3.4.95) and (3.4.96) already strongly suggests that ω and $\tilde{\omega}$ correspond to each other in the sense of Proposition 3.7. Taking into account that both ω and $\tilde{\omega}$ satisfy the Leibniz rule, it suffices to verify the equation of Proposition 3.7 for $h = u$, which is straightforward.

Finally, let us see that the ω -defined covariant derivative ∇ on L_{-1} coincides with the Graßmann connection given by the projector p_{-1} . More precisely, on one hand side the connection form ω determines a covariant differentiation D by Proposition 3.7, and the latter induces a covariant derivative ∇ by the formula (3.3.29). On the other hand, the dual bases $e_1 := \alpha$, $e_2 := \gamma$, $e^1 := \alpha^*$, $e^2 := \gamma^*$, of P_{-1} define the idempotent p_{-1} by $e^j(e_i) = (p_{-1})_i^j$ (see (3.4.81) and (3.4.82)).

On any $f \in \mathcal{O}(SU(2))$ we have the right coaction of the Hopf algebra $\mathcal{O}(U(1))$, so that (3.2.18) can be written as $Df = \text{d}f - f_{(0)}\omega(f_{(1)})$. Hence, for the values of D on the generators, we obtain

$$D\alpha = \text{d}\alpha - \alpha\alpha^*\text{d}\alpha - \alpha\gamma^*\text{d}\gamma = \gamma^*\gamma\text{d}\alpha - \alpha\gamma^*\text{d}\gamma = \gamma\text{d}(\alpha\gamma^*) - \text{ad}(\gamma^*\gamma). \quad (3.4.97)$$

Similarly, $D\gamma = \alpha d(\alpha^* \gamma) - \gamma d(\alpha^* \alpha)$. The right hand sides of these two formulas are already written as elements of $C^\infty(S^3)\pi^*(\Lambda^1(S^2))$, so that the isomorphism Φ defined in (3.3.23) takes very simple form. Plugging it into the formula (3.3.29) yields:

$$\nabla(\alpha) = d(\alpha\alpha^*) \underset{C^\infty(S^2)}{\otimes} \alpha + d(\alpha\gamma^*) \underset{C^\infty(S^2)}{\otimes} \gamma, \quad (3.4.98)$$

$$\nabla(\gamma) = d(\gamma\alpha^*) \underset{C^\infty(S^2)}{\otimes} \alpha + d(\gamma\gamma^*) \underset{C^\infty(S^2)}{\otimes} \gamma. \quad (3.4.99)$$

This coincides with the Graßmann connection $\nabla^e(e_k) = d(p_{-1})_k^j \otimes_{C^\infty(S^2)} e_j$. An analogous reasoning can be carried out for the tautological line bundle L_1 .

The considerations of this section manifest a well-known fact that the K_0 -invariants of vector bundles associated to a principal bundle can be computed using constructions originating from the principal bundle. In the setting of noncommutative geometry, strong connections on a principal extension are used to obtain explicit idempotents representing finitely generated projective modules associated to the principal extension. Further along this line, the Chern-Galois character is used to produce a cyclic homology class out of a quantum-group representation defining the associated module [6]. Pairing this class with a cyclic cocycle replaces the integration of characteristic classes given in terms of the de Rham cohomology.

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