The map from the cyclohedron to the associahedron is left cofinal

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THE MAP FROM THE CYCLOHEDRON TO THE ASSOCIAHEDRON IS LEFT COFINAL

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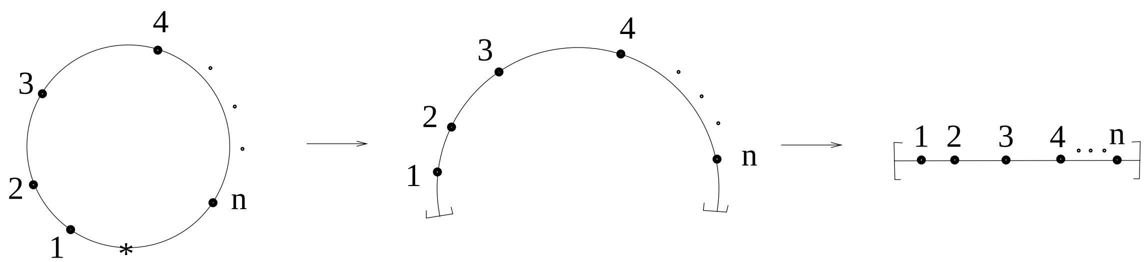
Abstract. Two natural projections from the cyclohedron to the associahedron are defined. We show that the preimages of any point via these projections might not be homeomorphic to (a cell decomposition of) a disc, but are still contractible. We briefly explain an application of this result to the study of knot spaces from the point of view of the Goodwillie-Weiss embedding calculus.

1. Introduction

The configuration space $Conf(n, [0,1])$ of n distinct points $0 < t_1 < t_2 < \ldots < t_n < 1$ in the interior of the segment $[0,1]$ is clearly homeomorphic to the configuration space $Conf_*(n, S^1)$ of $n+1$ distinct points on the circle $S^1 \simeq [0,1]/0{\sim}1$ one of which is the fixed point $* = 0{\sim}1$:

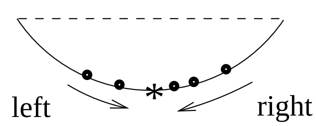


The Fulton-MacPherson compactification [5] of $Conf(n, [0,1])$ is the n-dimensional associahedron $Assoc_n$, also called the Stasheff polytope. The Fulton-MacPherson compactification of $Conf_*(n, S^1)$ is the n-dimensional cyclohedron $Cycl_n$, also called the Bott-Taubes polytope [1].

The homeomorphism $Conf_*(n, S^1) \to Conf(n, [0,1])$ induces a natural projection of compactifications

$$\pi_n \colon Cycl_n \to Assoc_n. \tag{1.1}$$

The compactification $Cycl_n$ contains more information than $Assoc_n$ since one can compare how fast configuration points approach $* = 0{\sim}1$ from the left and from the right:



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Informally speaking, the projection (1.1) forgets this information.

The poset of faces of $Assoc_n$ is the poset of planar trees (see Section 2.1) which we denote by $\Psi([n+1])$.[1] We denote the poset of faces of $Cycl_n$ by $\Phi(\underline{n})$[2] and we encode them in a way that is more geometric then the one in [10]. Namely, we associate to elements of $\Phi(\underline{n})$ certain planar trees that we call *fans* (see Section 2.2).

To any face of $Cycl_n$ one can assign a face of $Assoc_n$ which is its image via π_n. This correspondence defines a functor $\Pi_n\colon \Phi(\underline{n}) \to \Psi([n+1])$. This functor is the main object of our study.

Passing to a map of realizations $|\Phi(\underline{n})| \to |\Psi([n+1])|$ gives the projection $\bar{\pi}_n\colon Cycl_n \to Assoc_n$. However, the projections π_n and $\bar{\pi}_n$ are different because $\bar{\pi}_n$ is not a homeomorphism of interiors.

We prove the following:

Theorem 1. *The preimage of any point of $|\Psi([n+1])| = Assoc_n$ under $\bar{\pi}_n$ is contractible.*

An immediate corollary is

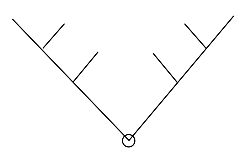
Theorem 2. *Functor $\Pi_n\colon \Phi(\underline{n}) \to \Psi([n+1])$ is left cofinal.*

Geometrically, Theorem 2 says that the preimage of any face of $Assoc_n$ under $\bar{\pi}_n$ is contractible. This result was announced in [9]. Some applications of Theorem 2 for the study of knot spaces are discussed in Section 4.

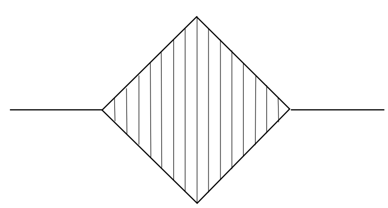
We also consider the initial projection (1.1) and describe the geometry of preimages under π_n (see Section 5). In particular we prove:

Theorem 3. *The preimage of any point of $Assoc_n$ under π_n is contractible.*

Theorems 1 and 3 are not surprising, but what is interesting is that the preimages might not be homeomorphic to a disc. For example, for the vertex of $Assoc_4$ encoded by the binary tree



(this vertex is the limit of $(t_1, t_2, t_3, t_4) = (\varepsilon^2, \varepsilon, 1-\varepsilon, 1-\varepsilon^2)$ when $\epsilon \to +0$), the preimage under both π_n and $\bar{\pi}_n$ is a square (2-disc) with two segments attached[3]:



[1] Here $[n+1]$ stands for the set $\{0, 1, \ldots, n+1\}$ of leaves.

[2] Here $\underline{n}$ stands for the set $\{1, 2, \ldots, n\}$.

[3] This preimage is the realization of the poset $X_{2,2}$; see Lemma 3.7 and Proposition 5.1.

Markl has shown in [10] that the polytope $Cycl_n$ can be obtained from $Assoc_n$ by a sequence of natural truncations by hyperplanes. Relating Theorems 1 and 3 to Markl's construction seems to be an interesting and a nontrivial problem.

2. Categories of faces

2.1. **Category of trees.** In this section we define a category $\Psi([n])$ of trees which keeps track of the faces of $Assoc_{n-1}$.

Definition 2.1. A Ψ*-tree* is an isotopy class of rooted trees embedded in the upper half-plane with the root of valence ≥ 2 at the origin. The valence of any internal vertex (i.e. a vertex that is not a leaf) except the root is at least 3.

We orient each edge of a Ψ-tree from the vertex closer to the root to that which is farther from the root. Each vertex (except the root) has exactly one incoming edge and a linearly ordered (clockwise) set of outgoing edges. The root has only outgoing edges which are linearly ordered (clockwise).

The set of leaves has a natural (clockwise) linear order. More precisely, let v_1 and v_2 be two leaves. Consider two pathes – one from the root to v_1, and another from the root to v_2. Suppose e_1 and e_2 are the first edges that are different in these paths. These edges are outgoing from some vertex and we say $v_1 < v_2$ if and only if $e_1 < e_2$. In particular we can thus speak of the minimal and maximal leaf.

Definition 2.2. A *left-most (resp. right-most) node* of a Ψ-tree is any vertex lying on the path from the root to the minimal (resp. maximal) leaf. (Neither the root, nor the extremal leaves are considered to be left-most or right-most.)

Definition 2.3. Define $\Psi([n])$ as the category whose objects are Ψ-trees with $n+1$ leaves labelled by the ordered set $[n] = \{0, 1, 2, ..., n\}$. There is a (unique) morphism in $\Psi([n])$ from T to T' if T' is obtained from T by a contraction along some set of non-leaf edges.

We will think of $\Psi([n])$ as a poset by saying $T \geq T'$ in the above situation.

Remark 1. The realization of the category $\Psi([n])$ is homeomorphic to the barycentric subdivision of the $(n-1)$-dimensional Stasheff associahedron $Assoc_{n-1}$.

Categories $\Psi([1])$, $\Psi([2])$, $\Psi([3])$ are pictured on the right side of Figure 2. The root is designated by a little circle.

2.2. **Category of fans.** In this subsection we define a category $\Phi(\underline{n})$ of fans which keeps track of the faces of $Cycl_n$.

Definition 2.4. A *fan* (or a Φ*-tree*) is an isotopy class of planar rooted trees with one marked vertex, which can be a leaf but not the root, called the *bead*. The root is of valence 1 and all the internal vertices except the bead are of valence ≥ 3.

Note that the bead can have any valence ≥ 1. A leaf of a fan is thus any vertex of valence 1. For example, the root of a fan is always a leaf. The bead might be a leaf since it might be of valence 1. The set of leaves not coinciding with the root has a natural linear order in the same way the set of leaves of a Ψ-tree does.

Definition 2.5. (i) A vertex of a fan is said to be *internal* if it is not a leaf or if it is a leaf but it is also a bead.

(ii) An edge of a fan is said to be *internal* if it joins two internal vertices.

Definition 2.6. Define $\Phi(\underline{n})$ to be the category whose objects are fans with n leaves, labeled by the ordered set $\underline{n} = \{1, 2, ..., n\}$, which are neither the root nor the bead. There is a (unique) morphism in $\Phi(\underline{n})$ from $\widehat{T}$ to $\widehat{T}'$ if $\widehat{T}'$ is obtained from $\widehat{T}$ by a contraction along some set of internal edges. A contraction of a connected set of edges produces a bead if and only if this set of edges contained the bead.

We will think of $\Phi(\underline{n})$ as a poset by saying $\widehat{T} \geq \widehat{T}'$ in the above situation.

Remark 2. The realization of the category $\Phi(\underline{n})$ is homeomorphic to the barycentric subdivision of the n-dimensional cyclohedron $Cycl_n$. This polytope was introduced by R. Bott and C. Taubes in [1].

Categories $\Phi(\underline{0})$, $\Phi(\underline{1})$, $\Phi(\underline{2})$ are pictured on the left-hand side of Figure 2. The root is designated by a black point; the bead is designated by a little circle.

2.3. **Functor Π_n.** We now define a functor $\Pi_n \colon \Phi(\underline{n}) \to \Psi([n+1])$ between the categories of fans and trees. For simplicity of notation we will sometimes omit the index n and write Π.

Definition 2.7. (i) The (only) path from the bead to the root will be called the *trunk* of the fan.

(ii) Let v be a vertex along the trunk of a fan. Let e_1 (resp. e_2) be the edge of the trunk which is adjacent to v and whose other vertex is closer to the root (resp. bead). The edges adjacent to v and lying between e_1 and e_2 (resp. e_2 and e_1) with respect to the natural clockwise order, will be called *left-going branches* (resp. *right-going branches*).

Now let $\widehat{T} \in \Phi(\underline{n})$ be a fan and cut $\mathbb{R}^2$ along the path which is the union of the trunk of $\widehat{T}$ and the ray emanating downward from the root to infinity (so this ray does not cross $\widehat{T}$). The space obtained from $\mathbb{R}^2$ by this surgery is homeomorphic to the upper half-plane. After this operation, the fan $\widehat{T}$ becomes a Ψ-tree T with $n+2$ leaves and with a root which is the former bead of $\widehat{T}$ (see Figure 1).

Note that a node of $\widehat{T}$ along the trunk can produce either one or two vertices in T. Such a node produces a left-most (resp. right-most) vertex in T if and only if it has left-going (right-going) branches. The root always produces two leaves, the minimal and the maximal one.

The following is immediate from the definition.

Lemma 2.8. *The above correspondence defines a functor* $\Pi_n \colon \Phi(\underline{n}) \to \Psi([n+1])$.

Remark 3. Passing to realizations, Π_n defines a projection $\bar{\pi}_n \colon |\Phi(\underline{n})| \to |\Psi([n+1])|$. Notice however that $\bar{\pi}_n$ is different from the projection $\pi_n \colon Cycl_n \to Assoc_n$ mentioned in Introduction. $\bar{\pi}_n$ is not a homeomorphism of interiors of $Cycl_n$ and $Assoc_n$ starting from $n \geq 2$.

Theorem 1. *The preimage of any point of* $|\Psi([n+1])| = Assoc_n$ *under* $\bar{\pi}_n$ *is contractible.*

We prove this result in the next section.

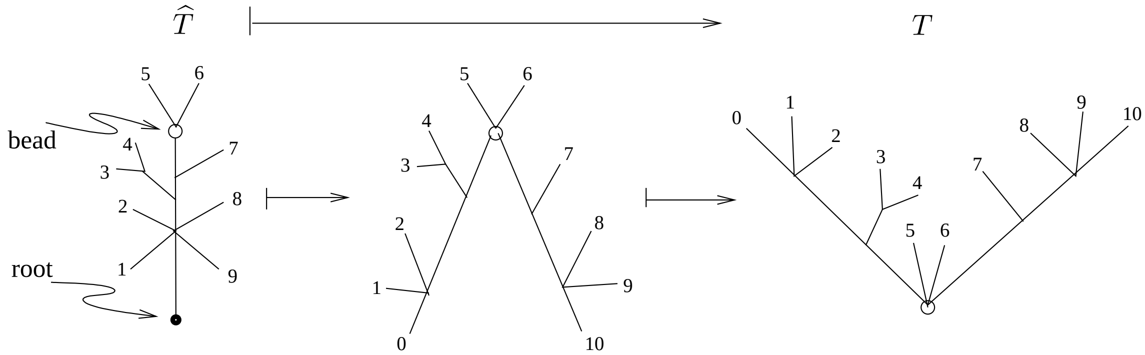


FIGURE 1. A Ψ-tree obtained from a fan

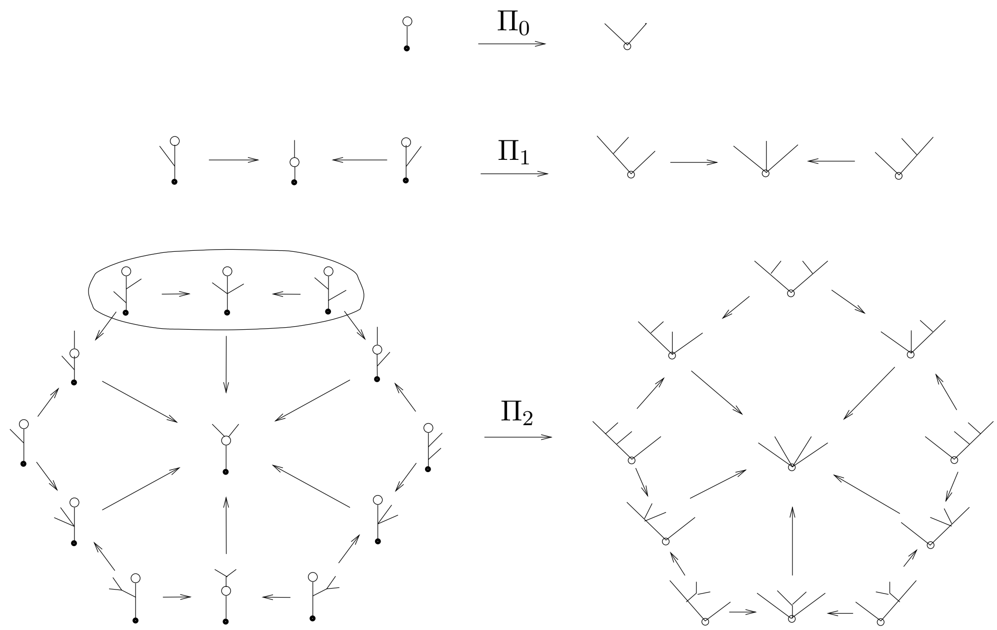


FIGURE 2. Π_0, Π_1, Π_2. The circled edge of the hexagon gets mapped to the top vertex of the pentagon.

3. Proof of Theorem 1

To prove Theorem 1, we need to define certain posets which are essential for understanding the geometry of the functor $\Pi_n \colon \Phi(\underline{n}) \to \Psi([n+1])$ and then prove that they are contractible. Theorem 1 will in the end follow from Proposition 3.10.

Definition 3.1. Define $X_{\ell,r}$, $l, r \geq 0$, to be the poset whose elements are words in a, b, and (ab) which contain exactly ℓ letters a and r letters b. Letter (ab) contributes one a and one b. We say $X < Y$ if X is obtained from Y by adding parentheses or by replacing some number of (ba)'s by (ab)'s.

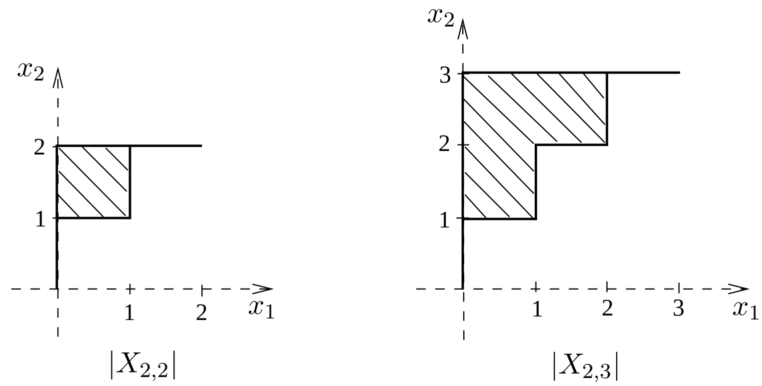


FIGURE 3. Examples of $|X_{\ell,r}|$.

$$aab \to a(ab) \leftarrow aba \to (ab)a \leftarrow baa$$

Poset $X_{2,1}$

Proposition 3.2. *For any $\ell, r \geq 0$, poset $X_{\ell,r}$ is contractible.*

In order to prove this, we will embed the realization of $X_{\ell,r}$ in $\mathbb{R}^\ell$.

Definition 3.3. An *integer cube* of $\mathbb{R}^\ell$ is a cube of any dimension s, $0 \leq s \leq \ell$, whose vertices all integers and each of its edges is of length one and is parallel to one of the axes.

The proof of the following is immediate.

Lemma 3.4. *(i) An integer cube in $\mathbb{R}^\ell$ is determined by its center. The dimension of an integer cube is the number of non-integer coordinates of the center. The set (of centers) of integer cubes is the set $\frac{1}{2} \cdot \mathbb{Z}^\ell$ of points with half-integer coordinates.*

(ii) $\mathbb{R}^\ell$ is a disjoint union of the interiors of integer cubes. Point $(x_1, \ldots, x_\ell)$ belongs to the interior of the integer cube whose center has coordinates $(\frac{\lceil x_1 \rceil + \lfloor x_1 \rfloor}{2}, \ldots, \frac{\lceil x_\ell \rceil + \lfloor x_\ell \rfloor}{2})$.

Lemma 3.5. *The realization $|X_{\ell,r}|$ of $X_{\ell,r}$ is homeomorphic to the union of integer cubes in $\mathbb{R}^\ell = \{ (x_1, \ldots, x_\ell) \}$ which are contained in the domain*

$$0 \leq x_1 \leq x_2 \leq \ldots \leq x_\ell \leq r. \tag{3.1}$$

Examples of $|X_{\ell,r}|$ are given in Figure 3.

Remark 4. It follows from Lemma 3.4 (ii), that the subspace of $\mathbb{R}^\ell$ described above is defined by the inequalities $\lceil x_{i-1} \rceil \leq \lfloor x_i \rfloor$, $i = 1 \ldots \ell + 1$, where $x_0 = 0$ and $x_{\ell+1} = r$.

Proof. We define an embedding $f \colon |X_{\ell,r}| \hookrightarrow \mathbb{R}^\ell$ on the elements of $X_{\ell,r}$. A simplex $X_0 < X_1 < \ldots < X_k$, $X_i \in X_{\ell,r}$ will then be mapped to the convex hull of $f(X_0), f(X_1), \ldots, f(X_k)$.

Let $X \in X_{\ell,r}$ be a word. We define $f(X) = (f_1(X), \ldots, f_\ell(X))$ as follows. The coordinate $f_i(X)$ is set to be the number of elements b before the ith a in X, but if this a is parenthesized with a b, then $\frac{1}{2}$ is added. For example,

$$f(\,ab(ab)bbab\,) = (0, \frac{3}{2}, 4).$$

The number of non-integer coordinates is exactly the number of letters (ab) in X.

Note that the words in $X_{\ell,r}$ without parentheses are in one-to-one correspondence (via f) with the integer points of the domain (3.1). Similarly a half-integer point $N = (\frac{n_1}{2}, \frac{n_2}{2}, \dots, \frac{n_\ell}{2})$ is in the image of f if and only if the integer cube whose center is N is contained in the domain (3.1). Consider the full subcategory $X_{\ell,r} \downarrow X$ of elements greater than or equal to some $X \in X_{\ell,r}$. The realization $|X_{\ell,r} \downarrow X|$ is homeomorphic to the barycentric subdivision of a cube (whose dimension is the number of letters (ab) in X). Space $|X_{\ell,r} \downarrow X|$ is mapped by f to the integer cube with the center $f(X)$. Thus f is an embedding, and the image is exactly the space described in the lemma. □

Proof of Proposition 3.2. We induct over ℓ. Poset $X_{0,r}$ is contractible since it is a point. Consider the description of $|X_{\ell,r}|$ given in Remark 4. The projection of $|X_{\ell,r}|$ to the first $\ell - 1$ coordinates of $\mathbb{R}^\ell$ gives $|X_{\ell-1,r}|$. The preimage of any point $(x_1, \dots, x_{\ell-1}) \in f(|X_{\ell-1,r}|)$ is the segment $x_\ell \in [\lceil x_{\ell-1} \rceil, r]$ (in the degenerate case $\lceil x_{\ell-1} \rceil = r$, this segment is a point). So $f(|X_{\ell,r}|)$ can be retracted to $f(|X_{\ell,r}|) \cap \{x_\ell = r\} \simeq |X_{\ell-1,r}|$, which is contractible by the inductive hypothesis. □

Definition 3.6. Let $Y \in \Psi([n+1])$ and define $\Pi^{-1}(Y)$ to be the full subcategory of $\Phi(\underline{n})$ with elements $\widehat{Y}$ satisfying $\Pi(\widehat{Y}) = Y$.

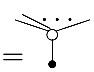
Lemma 3.7. *Let $Y \in \Psi([n+1])$ have ℓ left-most nodes and r right-most nodes (see Definition 2.2). Then $\Pi^{-1}(Y)$ is isomorphic to $X_{\ell,r}$.*

Proof. Suppose $\widehat{Y} \in \Pi^{-1}(Y)$. We assign to $\widehat{Y}$ a word in letters a, b, (ab) as follows: If we travel along the trunk (see Definition 2.7) from the bead to the root and meet a node that has only left-going branches, we write a. If we meet a node that has only right-going branches, we write b. If this node has both left-going and right-going branches, we write (ab). Proceeding like this, we get a word in $X_{\ell,r}$. (For example, the fan from Figure 1 gives $ba(ab)$.) It is easy to see that such words are in one-to-one correspondence with the elements of $\Pi^{-1}(Y)$. □

The following is a consequence of Proposition 3.2 and Lemma 3.7.

Corollary 3.8. $\Pi^{-1}(Y)$ *is contractible.*

Definition 3.9. Suppose given $Y \in \Psi([n+1])$ and $\widehat{T} \in \Phi(\underline{n})$ such that $Y \geq \Pi(\widehat{T})$. Define $\Pi^{-1}(Y| \geq \widehat{T})$ as the full subcategory of $\Phi(\underline{n})$ whose elements $\widehat{Y}$ satisfy $\Pi(\widehat{Y}) = Y$ and $\widehat{Y} \geq \widehat{T}$.

Remark 5. $\Pi^{-1}(Y) = \Pi^{-1}(Y| \geq *)$, where $* =$  is the terminal (minimal) element of $\Phi(\underline{n})$.

Proposition 3.10. *Poset $\Pi^{-1}(Y| \geq \widehat{T})$ is contractible for any $Y \in \Psi([n+1])$ and $\widehat{T} \in \Phi(\underline{n})$ satisfying $Y \geq \Pi(\widehat{T})$.*

Proof. We will prove that $\Pi^{-1}(Y| \geq \widehat{T})$ is isomorphic to $\prod_{i=0}^{s} X_{\ell_i, r_i}$ for some $s \geq 0$, $\ell_i, r_i \geq 0$, $i = 0 \dots s$. The result will follow from Proposition 3.2.

Label the vertices of the trunk of $\widehat{T}$ by $0, 1, 2, \dots, s$ (from the bead to the root), with 0 corresponding to the bead and s to the last node before the root (this vertex is the only one joined to the root by an edge). Define ℓ_0 (resp. r_0) as the number of left-most (resp. right-most) vertices of Y that are contracted to the root in $\Pi(\widehat{T})$. Analogously define ℓ_i, r_i, $i = 1, ..., s$ as follows: Denote by I_i, $i = 1, ..., s$, the ith vertex on the trunk (which is neither the bead nor the trunk). If I_i does not have any left-going (resp. right-going) branches, then set $\ell_i = 0$ (resp.

$r_i = 0$). Otherwise I_i defines a left-most (resp. right-most) node L_i (resp. R_i) of $\Pi(\widehat{T})$. Then ℓ_i (resp. r_i) is the number of left-most (resp. right-most) vertices of Y contracted to L_i (resp. R_i).

Poset $\Pi^{-1}(Y| \geq \widehat{T})$ is a subposet of $\Pi^{-1}(Y) \cong X_{\ell,r}$, where $\ell = \sum_{i=0}^{s} \ell_i$, $r = \sum_{i=0}^{s} r_i$. This subposet consists of those words in $X_{\ell,r}$ which can be broken up into $s+1$ words from X_{ℓ_0,r_0}, X_{ℓ_1,r_1},..., X_{ℓ_s,r_s}. But such a subposet is clearly the product $\prod_{i=0}^{s} X_{\ell_i,r_i}$. □

Proof of Theorem 1. Let $z = z_k$ be a point in the open simplex $\Delta^k \subset |\Psi([n+1])| = Assoc_n$ defined by a sequence

$$Y_0 < Y_1 < \ldots < Y_k, \tag{3.2}$$

where $Y_i \in \Psi([n+1])$. So z_k is of the form

$$z_k = \sum_{i=0}^{k} t_i Y_i; \quad \sum_{i=0}^{k} t_i = 1; \quad t_i > 0,\, i = 0 \ldots k.$$

We will prove that $\bar{\pi}^{-1}(z_k)$ is contractible by induction on k.

If $k = 0$, then $\bar{\pi}^{-1}(z_0)$ is the realization of $\Pi^{-1}(Y_0)$, which is contractible by Corollary 3.8.

Let $k \geq 1$. Then $\bar{\pi}^{-1}(z_k)$ has a natural prismatic decomposition. Indeed, any $\widehat{z}_k \in \bar{\pi}^{-1}(z_k)$ must be in the interior of some simplex

$$(\widehat{Y}_0^0 < \widehat{Y}_0^1 < \ldots < \widehat{Y}_0^{m_1}) < (\widehat{Y}_1^0 < \ldots < \widehat{Y}_1^{m_2}) < \ldots < (\widehat{Y}_k^0 < \ldots < \widehat{Y}_k^{m_k}), \tag{3.3}$$

$$\widehat{z}_k = \sum_{\substack{i=0\ldots k\\ j=0\ldots m_i}} t_i^j \widehat{Y}_i^j,$$

where $m_i \geq 0$, $i = 0, ..., k$, $\Pi(\widehat{Y}_i^j) = Y_i$, and $t_i^j > 0$ satisfy the condition $\sum_{j=0}^{m_i} t_i^j = t_i$. (Parentheses are meant to simplify the reading of the expression.) The sequence (3.3) defines an open prism $\Delta^{m_1} \times \ldots \times \Delta^{m_k}$ in $\bar{\pi}^{-1}(z_k)$. A prism is in the boundary of another if the sequence which defines the first is a subsequence of the sequence which defines the second (however not any subsequence of (3.3) defines a prism in $\bar{\pi}^{-1}(z_k)$; a subsequence that does has to contain at least one element $\widehat{Y}_i^j$ for each $i = 0 \ldots k$).

The preimages $\bar{\pi}^{-1}(z_k)$ are naturally homeomorphic for all $z_k \in \Delta^k \subset |\Psi([n+1])|$ (recall that Δ^k is defined by the sequence (3.2)). So $\bar{\pi}^{-1}(\Delta^k) = \Delta^k \times \bar{\pi}^{-1}(z_k)$.

Now let z_{k-1} be any point in the open simplex $\Delta^{k-1} \subset |\Psi([n+1])|$ defined by the sequence

$$Y_0 < Y_1 < \ldots < Y_{k-1}$$

and consider $\bar{\pi}^{-1}(z_{k-1})$. By induction hypothesis $\bar{\pi}^{-1}(z_{k-1})$ is contractible. There is a natural map

$$p\colon \bar{\pi}^{-1}(z_k) \longrightarrow \bar{\pi}^{-1}(z_{k-1})$$

which is geometrically a boundary limit map. This is well-defined since $\bar{\pi}^{-1}(z_k)$ is always the same space as $z_k \in \Delta^k$ tends to $z_{k-1} \in \Delta^{k-1} \subset \partial\Delta^k$. We will show that p^{-1} of any point in $\bar{\pi}^{-1}(z_{k-1})$ is contractible.

In terms of prismatic decomposition, p forgets the last factor in $\Delta^{m_1} \times \ldots \times \Delta^{m_{k-1}} \times \Delta^{m_k}$ by mapping it to $\Delta^{m_1} \times \ldots \times \Delta^{m_{k-1}}$ corresponding to the sequence

$$(\widehat{Y}_0^0 < \widehat{Y}_0^1 < \ldots < \widehat{Y}_0^{m_1}) < (\widehat{Y}_1^0 < \ldots < \widehat{Y}_1^{m_2}) < \ldots < (\widehat{Y}_{k-1}^0 < \ldots < \widehat{Y}_{k-1}^{m_{k-1}}). \tag{3.4}$$

Let $\widehat{z}_{k-1}$ be a point of the open prism $\Delta^{m_1} \times \ldots \times \Delta^{m_{k-1}} \subset \bar{\pi}^{-1}(z_{k-1})$. It is easy to see that $p^{-1}(\widehat{z}_{k-1})$ is exactly the realization of the poset $\Pi^{-1}(Y_k| \geq \widehat{Y}_{k-1}^{m_k-1})$. (Informally, we need to consider all the "prolongations" of the sequence (3.4) to the sequence (3.3).) But the last poset is contractible by Proposition 3.10. Thus $\bar{\pi}^{-1}(z_k)$ is surjectively mapped by p to $\bar{\pi}^{-1}(z_{k-1})$. Space $\bar{\pi}^{-1}(z_{k-1})$ is contractible, and the preimage of p is contractible for any point of $\bar{\pi}^{-1}(z_{k-1})$. So $\bar{\pi}^{-1}(z_k)$ is contractible as well. □

4. Applications

4.1. **Cofinality.** Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor between two small categories. Recall that, for any object $d \in Ob(\mathcal{D})$, $\mathcal{F} \downarrow d$ is defined as a category whose objects are pairs (c, f), where $c \in \mathcal{C}$ and $f \in Mor_{\mathcal{D}}(\mathcal{F}(c), d)$. Morphisms are defined as

$$Mor_{\mathcal{F}\downarrow\mathcal{D}}(\,(c_1, f_1);\,(c_2, f_2)\,) = \{\, f \in Mor_{\mathcal{C}}(c_1, c_2) \,|\, f_2 \circ \mathcal{F}(f) = f_1 \,\}.$$

Functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is said to be *left cofinal* if, for any object $d \in Ob(\mathcal{D})$, the realization of $\mathcal{F} \downarrow d$ is contractible [2, Ch. XI, page 316]. This notion is important in homotopy theory since left cofinal functors preserve homotopy limits. More precisely, for $X\colon \mathcal{D} \to \mathrm{Top}$ a functor from $\mathcal{D}$ to the category of topological spaces and $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ left cofinal, we have

$$\operatorname*{holim}_{\mathcal{C}} X \circ \mathcal{F} \simeq \operatorname*{holim}_{\mathcal{D}} X.$$

The following theorem was the main motivation for this paper. We discuss its applications in Section 4.3.

Theorem 2. *The functor* $\Pi_n\colon \Phi(\underline{n}) \to \Psi([n+1])$ *is left cofinal for any* $n \geq 0$.[4]

Proof. For any $T \in \Psi([n+1])$, the realization of the category $\Pi_n \downarrow T$ is the preimage under $\bar{\pi}_n$ of the face of $Assoc_n$ encoded by T. Any face, being a convex polytope, is contractible. Theorem 1 then finishes the proof. □

4.2. **Bimodules over an operad and associated fanic diagrams.** In the same way as categories of trees encode the structure of an operad, the categories of fans encode the structure of bimodules over an operad.

Let $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 2}$ be a non-Σ operad in a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$. Notice that we assume that $\mathcal{O}$ does not have operations of arity $n = 0, 1$ (if it does, we ignore them).

A sequence $\{\mathcal{X}(n)\}_{n\geq 0}$ of objects in $\mathcal{C}$ is a *bimodule* over $\mathcal{O}$ if it is endowed with structure composition maps

$$\begin{aligned} \circ_i\colon \mathcal{X}(n) \otimes \mathcal{O}(m) \to \mathcal{X}(n+m-1),\ i = 1 \ldots n \\ \circ^i\colon \mathcal{O}(n) \otimes \mathcal{X}(m) \to \mathcal{X}(n+m-1),\ i = 1 \ldots n \end{aligned} \tag{4.1}$$

that satisfy natural associativity conditions.

For example, if $\mathcal{X}$ is an operad endowed with a morphism from $\mathcal{O}$,

$$\mathcal{O} \to \mathcal{X},$$

then $\mathcal{X}$ is naturally a bimodule over $\mathcal{O}$.

To any such bimodule one can assign a fanic diagram, i.e. a functor

$$\widehat{\mathcal{X}}^n\colon \Phi(\underline{n}) \longrightarrow \mathcal{C}$$

[4] Note that Π_n is clearly right cofinal. It preserves homotopy colimits simply because both of the categories $\Phi(\underline{n})$ and $\Psi([n+1])$ have a terminal object, and Π_n maps the terminal object to the terminal object.

from the category $\Phi(\underline{n})$ of fans to $\mathcal{C}$ as follows.

We first define the value of the diagram $\widehat{\mathcal{X}}^n$ on objects. Let $T \in \Phi(\underline{n})$ be an n-fan. Recall that each edge of T is oriented in the unique way such that its origin is on the path joining the root to its end. Let $|v|$ be the number of edges emanating from v. Since each vertex except the root has exactly one incoming edge, we have $|v| = \text{valence}(v) - 1$.

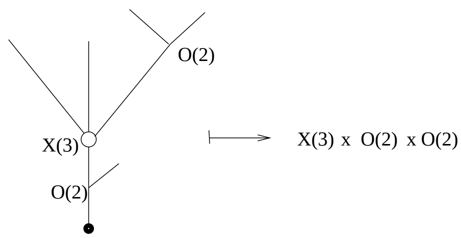
Recall from Definition 2.5 that the inner vertices of T are its non-leaves and the bead (even if the bead is a leaf). Let b denote the bead. For any inner vertex v of T set

$$\mathcal{X}(T : v) := \begin{cases} \mathcal{O}(|v|), & \text{if } v \neq b \\ \mathcal{X}(|v|), & \text{if } v = b \end{cases}$$

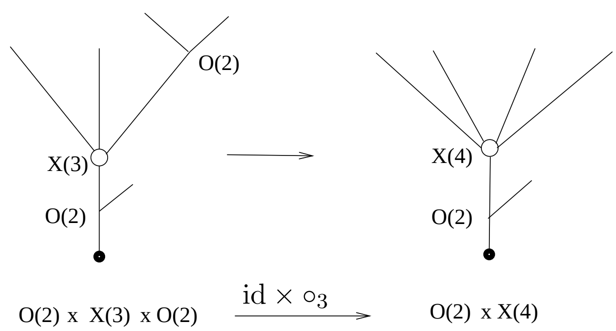
and define

$$\widehat{\mathcal{X}}^n(T) = \bigotimes_{v \text{ inner vertex of } T} \mathcal{X}(T : v).$$

For example,



To define $\widehat{\mathcal{X}}^n$ on morphisms, a contraction of an edge is sent to the corresponding composition (4.1). For example,



Example 1. The cyclohedra $\{Cycl_n\}_{n\geq 0}$ form a bimodule over the Stasheff operad $\{Assoc_{n-2}\}_{n\geq 2}$. In the corresponding fanic diagram

$$\widehat{Cycl}^n : \Phi(\underline{n}) \longrightarrow \text{Top},$$

all the maps are inclusions. This diagram describes the stratification of $Cycl_n$ by its faces.

Another example (corresponding to a morphism of operads) is given in the next section.

4.3. **Formal cyclohedral model for the Goodwillie-Weiss embedding tower for spaces of knots.** The space of long knots Emb_d, $d \geq 3$, is the space of smooth embeddings $f\colon \mathbb{R} \hookrightarrow \mathbb{R}^d$ that coincide with a fixed linear embedding $t \mapsto (t, 0, \ldots, 0)$ outside a compact subset of $\mathbb{R}$. T. Goodwillie and M. Weiss defined a tower of spaces

$$P_0 Emb_d \leftarrow P_1 Emb_d \leftarrow \ldots \leftarrow P_n Emb_d \leftarrow \ldots \tag{4.2}$$

converging to Emb_d for $d \geq 4$ [7]. Each space $P_n Emb_d$ is defined as a homotopy limit over a subcubical diagram, i.e. over the category of faces of the n-simplex.

D. Sinha gave several different models for $P_n Emb_d$ [12, 13]. In one of these models, the homotopy limit is taken over the category $\Psi([n+1])$ of faces of $Assoc_n$. We will call this model *associahedral*. On the other hand, the authors defined a *cyclohedral* model for the tower (4.2) in [9] using the construction of a fanic diagram assigned to a morphism of operads. More precisely, let $FM_d = \{FM_d(n)\}_{n \geq 0}$ denote the Fulton-MacPherson operad introduced in [6, 8]. Each space in this operad is the Fulton-MacPherson compactification of the configuration space $F(n, \mathbb{R}^d)$ of n distinct points in $\mathbb{R}^d$ quotiented out by translations and scalings. Let $FM_1^0(n)$ be the main connected component of $FM_1(n)$. The non-Σ operad $FM_1^0 = \{FM_1^0(n)\}_{n \geq 2}$ is in fact the Stasheff operad ($FM_1^0(n) = Assoc_{n-2}$).

Theorem ([9])**.** *For any $n \geq 2$ and $d \geq 3$, there is a weak homotopy equivalence*

$$\operatorname*{holim}_{\Phi(\underline{n})} \widehat{FM}_d^{\,n} \simeq P_n Emb_d \times \Omega^2 S^{d-1},$$

where $\widehat{FM}_d^{\,n}\colon \Phi(\underline{n}) \to \mathrm{Top}$ is the fanic diagram associated to the morphism of operads $FM_1^0 \to FM_d$.

The advantage of this model is that the diagram $\widehat{FM}_d^{\,n}$ is $\mathbb{Q}$-formal, which allows one to determine the rational homotopy type of $P_n Emb_d$ and Emb_d for $d \geq 4$ [9].

The proof of the cited theorem does not directly use the Theorem 2. In that proof, it was enough that the composite map from $\Phi(\underline{n})$ to the poset of faces of the n-simplex is left cofinal. However, D. Sinha and the second author are planning to use Theorem 2 to give a more direct geometric relation between the cyclohedral model and the associahedral model from [12] in [14]. This is a part of the program aimed at relating Bott-Taubes integrals [1] and their generalizations [3, 4] to the Goodwillie-Weiss calculus of knots. This program was partially realized by the third author in [15] on the level of finite-type invariants of knots in $\mathbb{R}^3$.

5. Projection π_n.

In this section we briefly study the projection $\pi_n\colon Cycl_n \to Assoc_n$ mentioned in the Introduction.

Theorem 3. *The preimage of any point of $Assoc_n$ under π_n is contractible.*

To prove this, we will need the following proposition. For any $T \in \Psi([n+1])$ let $Assoc_n(T)$ denote the face of $Assoc_n$ encoded by T. Similarly let $Cycl_n(\widehat{T})$ stand for the face of $Cycl_n$ encoded by $\widehat{T} \in \Phi(\underline{n})$.

Proposition 5.1. *Let t be any point in the interior of the face $Assoc_n(T)$. Then $\pi_n^{-1}(t)$ is homemorphic to the realization of $\Pi_n^{-1}(T)$.*

Recall from Lemma 3.7 that $\Pi_n^{-1}(T)$ is isomorphic to $X_{\ell,r}$ for some ℓ and r. Thus Corollary 3.8 and Proposition 5.1 imply Theorem 3. The proof of Proposition 5.1, on the other hand, will follow from the two lemmas below.

Denote by $p_{n,m}\colon Assoc_{n+m+1} \to Assoc_n \times Assoc_m$ the projection which sends any point $0 < t_1 < \ldots < t_{n+m+1} < 1$ from the interior of $Assoc_{n+m+1}$ to the pair

$$\left(0 < \frac{t_1}{t_{i+1}} < \frac{t_2}{t_{i+1}} < \ldots < \frac{t_i}{t_{i+1}} < 1\,;\, 0 < \frac{t_{i+2}-t_{i+1}}{1-t_{i+1}} < \frac{t_{i+3}-t_{i+1}}{1-t_{i+1}} < \ldots < \frac{t_{n+m+1}-t_{i+1}}{1-t_{i+1}} < 1\right).$$

The following is immediate:

Lemma 5.2. *The preimage of any point in the interior of $Assoc_n \times Assoc_m$ under $p_{n,m}$ is a segment.*

To understand the geometry of $\pi_n^{-1}(t)$ one first has to describe the preimage of t inside any face of $Cycl_n$ which is mapped onto $Assoc_n(T)$.

Lemma 5.3. *The intersection of $\pi_n^{-1}(t)$ (point t is in the interior of $Assoc_n(T)$) with $Cycl_n(\widehat{T})$, $\Pi_n(\widehat{T}) = T$, is a k-dimensional cube, where k is the number of vertices of the trunk of $\widehat{T}$ having both left and right outgoing edges.*

Proof. This lemma follows from Lemma 5.2. Any face of an associahedron is a product of associahedra. Any face of a cyclohedron is a product of a cyclohedron and a number of associahedra. The projection π_n restricted on $Cycl_n(\widehat{T})$ sends this face onto $Assoc_n(T)$. This map is a product of identity maps of associahedra, projections $p_{i,j}$, and a projection $\pi_{|b|}$ (which is a homeomorphism on the interior), where $|b|$ is the number of emanating edges from the bead b of the fan $\widehat{T}$. □

Proof of Proposition 5.1. To finish the proof of Proposition 5.1, notice that the cubes from Lemma 5.3 glue together exactly in the way the integer cubes do in the image of $f\colon |X_{\ell,r}| \hookrightarrow \mathbb{R}^\ell$ (see the proof of Lemma 3.5).

□

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