Bubbling Geometries for Half BPS Wilson lines

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Abstract

We consider the supergravity backgrounds that correspond to supersymmetric Wilson line operators in the context of AdS/CFT correspondence. We study the gravitino and dilatino conditions of the IIB supergravity under the appropriate ansatz, and obtain some necessary conditions for a supergravity background that preserves the same symmetry as the supersymmetric Wilson lines. The supergravity solutions are characterized by continuous version of maya diagrams. This diagram is related to the eigen value distribution of the Gaussian matrix model. We also consider the similar background of 11 dimensional supergravity.

1 Introduction

In the gauge theory, the most important class of non-local operators are the Wilson loop operators. Especially, in $\mathcal{N} = 4$ super Yang-Mills theory, the Wilson loops with the scalar term are the most fundamental ones. The form of this class of operators are

$$W_R(C) = \operatorname*{Tr}_R \left[P \exp \left(i \oint_C ds (A_\mu \dot{x}^\mu + \varphi_4 |\dot{x}|) \right) \right], \tag{1.1}$$

where $\dot{x}^\mu$ denotes $\partial x^\mu / \partial s$, and φ_4 is one of the six real scalar field in the $\mathcal{N} = 4$ super Yang-Mills theory. The label R is the representation of the gauge group, in which the trace is taken[1].

In the context of AdS/CFT correspondence, the Wilson loops are described by the fundamental strings in AdS spacetime[1, 2]. The vacuum expectation value of the Wilson line has been calculated as the on-shell action of the fundamental string. In [3], the circular Wilson loop has been calculated as a on-shell action of D3-brane. This calculation by D3-brane includes some higher genus corrections.

If the gauge group is $SU(N)$, the representation label R in (1.1) is expressed by Young diagrams. In the probe picture—fundamental strings or D3-branes—, we can only see the special kind of representations. How can we see the Wilson loops with arbitrary Young diagrams in the AdS side?

Our approach in this paper to this problem is the similar one to the work of Lin, Lunin, Maldacena[4]. They have constructed supergravity solutions corresponds to half BPS local operators. In their solution, one can see the phase space of free fermions in the harmonic potential. This picture is consistent with the dynamics of eigenvalues of Gaussian matrix quantum mechanics proposed by [5, 6, 7].

In this paper, we study the supergravity solutions corresponding to the half BPS straight Wilson lines. We consider the supersymmetry conditions under the appropriate ansatz, and derive some necessary conditions.

Let us summarize the result of this paper here. The supergravity solution is characterized by one dimensional black and white pattern like figure 1. For example, $AdS_5 \times S^5$,

Figure 1: The black and white pattern that appears on the boundary of the base two dimensional space. At a black point, S^2 shrinks, and at the white point S^4 shrinks.

[1]We denote Tr_R a trace in the representation R, while tr a trace in the fundamental representation.

Figure 2: The pattern that characterizes the $AdS_5 \times S^5$ solution. The black segment expresses S^5.

is characterized by the pattern of figure 2. If this kind of pattern is discretized in some mechanism, we get a "maya diagram" and it actually is a equivalent label as a Young diagram.

On the other hand, it is conjectured that the expectation value of a circular Wilson loop is calculated by the Gaussian matrix model [8, 9]. *We claim that the black and white pattern in the geometry corresponds to the eigen value distribution of the matrix model.* The black part is where the eigen values exist.

We checked that the length of the black segment in the $AdS_5 \times S^5$ is quantized like $\sqrt{N}$, which is just the same as the length of the cut of Gaussian matrix model. As a father check, we compare $AdS_2 \times S^2$ D3-brane probe and the eigen value distribution of the saddle point of the integral $\left\langle \frac{1}{N} \operatorname{tr}[e^{-kM}] \right\rangle_{\text{mm}}$ in the matrix model. It is shown that they are completely consistent. This identification leads to "D-brane exclusion principle"; Two same $AdS_2 \times S^2$ D3-branes cannot exist at the same position at the same time.

We also investigate a similar problem in M-theory. There are half BPS surface operators in the 6 dimensional (2,0) SCFT. Also in the 3 dimensional $\mathcal{N} = 8$ SCFT there are half BPS surface operators. We derive some necessary conditions for the supergravity solutions that correspond to these surface operators.

The construction of this paper is as follows. In section 2, we study the supersymmetry conditions in IIB supergravity under the ansatz, and derive some necessary conditions. In section 3, we see the black and white pattern on the boundary of the two dimensional base space. We compare it to the eigen value distribution of Gaussian matrix model. In section 4 we investigate the similar problem in M-theory. Section 5 is devoted to conclusions and discussions.

2 1/2 BPS Wilson lines and Bubbling Geometry

In this section, we investigate the IIB geometry that corresponds to half BPS straight Wilson lines. First we consider the symmetry preserved by the Wilson lines and make the ansatz. Next, we study the supersymmetry conditions and derive the necessary conditions.

2.1 Symmetry and Ansatz for Wilson lines

In this subsection, we consider the symmetry preserved by Wilson line operators and make the ansatz respecting that symmetry. We are considering the straight Wilson line which extends to time direction. It is equivalent to introduce a test particle with infinite mass, sitting on the origin of the space. For this kind of Wilson line, (1.1) can be written as

$$W_R = \operatorname*{Tr}_R \left[P \exp\left(i \int dx^0 (A_0 + \varphi_4) \right) \right]. \tag{2.1}$$

Let us see the bosonic symmetry which leave this operator invariant.

First, the Wilson line looks like a particle sit on the origin of the space. Therefore, it preserves the rotation around the origin. This group is SO(3).

Next, this Wilson line is invariant under the time translation. This Wilson line is also scale invariant, that means, it preserves the dilatation symmetry as well as the special conformal symmetry of the time direction. These three generator of the transformations make the algebra SL(2,R).

Finally, the Wilson line operator (2.1) preserves SO(5) part of the R-symmetry, because this operator includes only φ_4 but does not include other five scalar fields $\varphi_5, \ldots, \varphi_9$.

In summary, the total bosonic symmetry is SL(2,R)×SO(3)×SO(5). The spacetime part of this symmetry SL(2,R)×SO(3) can be seen more clearly by Weyl transformation[10]. The four dimensional Minkowski space can be transformed by Weyl transformation to $AdS_2 \times S^2$. The isometry of $AdS_2 \times S^2$ is SL(2,R)×SO(3).

Let us turn to making the ansatz that respect the symmetry SL(2,R)×SO(3)×SO(5). First, the metric can be written in the form

$$ds^2 = e^{2A} d\check{\Omega}_2^2 + ds_2^2 + e^{2B} d\hat{\Omega}_2^2 + e^{2C} d\Omega_4^2, \tag{2.2}$$

where $d\check{\Omega}_2^2, d\hat{\Omega}_2^2, d\Omega_4^2$ are the metric of unit AdS_2, S^2, S^4 respectively. ds_2^2 in (2.2) is a general two dimensional metric to be determined. Let us call this two dimensional space expressed by ds_2^2 "base space." A, B, and C in (2.2) are functions on the base space.

Next, let us consider the ansatz for the form fields. We first write here some notations. Let (θ^0, θ^1) be the vielbein of unit AdS_2, (θ^4, θ^5) be the vielbein of unit S^2, and $(\theta^6, \ldots, \theta^9)$ be the vielbein of unit S^4. We also denote the vielbein of the metric (2.2) as E^M, $M = 0, 1, \ldots, 9$. This means $E^0 = e^A\theta^0$, $E^1 = e^A\theta^1$, and $E^4 = e^B\theta^4$, $E^5 = e^B\theta^5$, and $E^6 = e^C\theta^6, \ldots, E^9 = e^C\theta^9$.

The most general ansatz for the form fields that respect the SL(2,R)×SO(3)×SO(5) symmetry is as follows.

$$H_3 = FE^0E^1 + \widehat{F}E^4E^5, \tag{2.3}$$

$$G_3 = \check{K}E^0E^1 + \widehat{K}E^4E^5, \tag{2.4}$$

$$G_5 = JE^0E^1E^4E^5 + \widetilde{J}E^6E^7E^8E^9, \tag{2.5}$$

where $F, \widehat{F}, \check{K}, \widehat{K}, J, \widetilde{J}$ are 1-forms on the base 2 dimensional space. The RR 1-form G_1 should also be a 1-form on the base space. The dilaton should be a function on the base space.

If all of these fields are excited, the background may include not only Wilson lines but also 'tHooft lines. In this paper, we concentrate on the Wilson lines. This means that the test particle has only electric charges but does not have magnetic charges. Hence we have to consider how to truncate the fields of supergravity. First of all, since the original background $AdS_5 \times S^5$ contains J and $\tilde{J}$ excitation, these two fields cannot be set to be 0. Next, in the probe picture, the Wilson line is a fundamental string or a D3-brane with electric flux. In both cases, the brane has the charges for the NSNS 3-form field strength along the AdS_2. This means F should not be 0. $\widetilde{J}$ and F excitation may become a source for the field $\widehat{K}$. Actually, the equation of motion for $\widehat{K}$ is written as

$$e^{-2B-4C}d(e^{2B+4C} *_2 \widehat{K}) = -\widetilde{J}F. \tag{2.6}$$

Here $*_2$ means the Hodge dual in the base 2-dimensional space.

As a result, we can put $0 = G_1 = \widehat{F} = \check{K} = 0$ consistently. For the convenience, let us rescale the fields $\widehat{K}, J, \widetilde{J}$ as

$$L = \frac{e^\phi}{4}J, \qquad \widetilde{L} = \frac{e^\phi}{4}\widetilde{J}, \qquad K = \frac{e^\phi}{4}\widehat{K}. \tag{2.7}$$

If we take the self-duality of the 5-form into account, L and $\widetilde{L}$ are not independent. They are Hodge dual to each other, that is $L = *_2\widetilde{L}$, $\widetilde{L} = - *_2 L$. The unknown fields on the 2-dimensional base space are the following fields.

- Metric ds_2^2.
- Scalar fields A, B, C, ϕ.
- 1-forms F, K, L.

These fields should be determined by the supersymmetry conditions and equations of motion.

2.2 Analysis of Supersymmetry conditions

In this subsection, we consider the supersymmetry conditions under the ansatz that we put in the previous subsection. These conditions lead to the necessary conditions for the backgrounds.

Let us first prepare the things needed to write down the supersymmetry conditions in terms of base two dimensional space. We use the set of the 10 dimensional gamma matrices.

$$\begin{aligned} &\Gamma^0 = \check{\sigma}^0 \otimes \sigma_C \otimes 1 \otimes 1, && \Gamma^1 = \check{\sigma}^1 \otimes \sigma_C \otimes 1 \otimes 1, \\ &\Gamma^2 = 1 \otimes \sigma_1 \otimes 1 \otimes 1, && \Gamma^3 = 1 \otimes \sigma_2 \otimes 1 \otimes 1, \\ &\Gamma^4 = \check{\sigma}^3 \otimes \sigma_C \otimes \hat{\sigma}_4 \otimes 1, && \Gamma^5 = \check{\sigma}^3 \otimes \sigma_C \otimes \hat{\sigma}_5 \otimes 1, \\ &\Gamma^a = \check{\sigma}^3 \otimes \sigma_C \otimes \hat{\sigma}_6 \otimes \gamma^a, && (a = 6, 7, 8, 9), \end{aligned} \tag{2.8}$$

where $(\sigma_1, \sigma_2, \sigma_C)$ and $(\hat{\sigma}_4, \hat{\sigma}_5, \hat{\sigma}_6)$ are sets of Pauli matrices. $(\check{\sigma}_1, \check{\sigma}_2, \check{\sigma}_3)$ is another set of Pauli matrices, and we defined $\check{\sigma}^0$ as $\check{\sigma}^0 := i\check{\sigma}_2$. γ^a, $(a = 6, 7, 8, 9)$ are gamma matrices of Euclidean 4 dimensions.

We also use some typical spinors in AdS spacetimes and spheres, which is called "Killing spinors." For example in AdS_2, we have spinors satisfying the relation

$$\overset{\circ}{\nabla}_p \check{\chi}^I_a = \frac{i}{2} a \check{\sigma}_p \check{\chi}^I_{-a}, \qquad \check{\sigma}_3 \check{\chi}^I_a = a \check{\chi}^I_a, \qquad (p = 0, 1, \qquad a = \pm 1, \qquad I = 1, 2), \tag{2.9}$$

where $\overset{\circ}{\nabla}$ is the covariant derivative of Levi-Civitá connection of unit AdS_2. As the same way, there are Killing spinors in S^2 and S^4.

$$\overset{\circ}{\nabla}_p \hat{\chi}^J_b = \frac{1}{2} b \hat{\sigma}_p \hat{\chi}^J_{-b}, \qquad \hat{\sigma}_6 \hat{\chi}^J_b = b \hat{\chi}^J_b, \qquad (p = 4, 5, \qquad b = \pm 1, \qquad J = 1, 2), \tag{2.10}$$

$$\overset{\circ}{\nabla}_p \chi^K_c = \frac{1}{2} c \gamma_p \chi^K_{-c}, \qquad \gamma_{6789} \chi^K_c = c \chi^K_c, \qquad (p = 6, \ldots, 9, \qquad b = \pm 1, \qquad K = 1, 2, 3, 4). \tag{2.11}$$

One can reduce the problem to 2-dimensions by expanding the 10-dimensional spinor pair ξ by above Killing spinors.

$$\xi = \sum_{a,b,c} \check{\chi}^I_a \otimes \epsilon_{abcIJK} \otimes \hat{\chi}^J_b \otimes \chi^K_c. \tag{2.12}$$

ϵ_{abcIJK} is a pair of 2-dimensional spinor. $\Gamma_2, \Gamma_3, \sigma_C, \tau_1, \tau_2, \tau_3$ acts on ϵ_{abcIJK}.

By using these materials, we can reduce the problem to the base two dimensional space. The supersymmetry conditions(A.5),(A.6) can be written in terms of two dimensional language as

$$e^{-A}\epsilon_{(-a)bc} = \left[-ibc\partial\!\!\!/ A - \frac{i}{2}abc F\!\!\!/\tau_3 + iac L\!\!\!/\tau_2 + c K\!\!\!/\tau_1\right]\epsilon_{abc},$$

$$e^{-B}\epsilon_{a(-b)c} = \left[c\partial\!\!\!/ B - abc L\!\!\!/\tau_2 - ibc K\!\!\!/\tau_1\right]\epsilon_{abc},$$

$$e^{-C}\epsilon_{ab(-c)} = \left[\partial\!\!\!/ C + ab L\!\!\!/\tau_2 + ib K\!\!\!/\tau_1\right]\epsilon_{abc},$$

$$\nabla_m\epsilon_{abc} = \left[-\frac{1}{4}aF_m\tau_3 + \frac{1}{2}(ab L\!\!\!/\tau_2 + ib K\!\!\!/\tau_1)\Gamma_m\right]\epsilon_{abc},$$

$$0 = \left[\partial\!\!\!/\phi + \frac{1}{2}a F\!\!\!/\tau_3 + 2ib K\!\!\!/\tau_1\right]\epsilon_{abc}. \tag{2.13}$$

In these equations, we neglect the I, J, K indices.

It is convenient to introduce some other sets of Pauli matrices to express (2.13) in the simple way. Let $\mu_j, \nu_j, \lambda_j,\ (j = 1, 2, 3)$ are sets of Pauli matrices acts on indices a, b, c respectively. This means, for example,

$$(\mu_j\epsilon)_{abc} = (\mu_j)_{aa'}\epsilon_{a'bc}. \tag{2.14}$$

We also define a set of matrices $\rho_j,\ (j = 1, 2, 3)$ as

$$\rho_1 = \mu_3\tau_3, \qquad \rho_2 = \nu_3\tau_1, \qquad \rho_3 = \mu_3\nu_3\tau_2. \tag{2.15}$$

These three matrices ρ_j satisfy the algebra of the Pauli matrices.

Using these notations, we can express (2.13) in more convenient way.

$$-ie^{-A}\mu_1\nu_3\lambda_3\epsilon = \left[-\partial\!\!\!/ A - \frac{1}{2}\rho_1 F\!\!\!/ + \rho_3 L\!\!\!/ - i\rho_2 K\!\!\!/\right]\epsilon, \tag{2.16}$$

$$e^{-B}\nu_1\lambda_3\epsilon = \left[\partial\!\!\!/ B - \rho_3 L\!\!\!/ + i\rho_2 K\!\!\!/\right]\epsilon, \tag{2.17}$$

$$e^{-C}\lambda_1\epsilon = \left[\partial\!\!\!/ C + \rho_3 L\!\!\!/ + i\rho_2 K\!\!\!/\right]\epsilon, \tag{2.18}$$

$$\nabla_m\epsilon = \left[-\frac{1}{4}\rho_1 F_m + \frac{1}{2}\rho_3 L\!\!\!/\Gamma_m + \frac{i}{2}\rho_2 K\!\!\!/\Gamma_m\right]\epsilon, \tag{2.19}$$

$$0 = \left[\partial\!\!\!/\phi + \frac{1}{2}\rho_1 F\!\!\!/ + 2i\rho_2 K\!\!\!/\right]\epsilon. \tag{2.20}$$

The Weyl condition $\Gamma_{10}\xi = \xi$ can be written in terms of ϵ as

$$\Gamma_{23}\epsilon = \eta\epsilon, \qquad \eta := -i\mu_3\nu_3\lambda_3 \tag{2.21}$$

The standard method to solve the supersymmetry conditions is making the spinor bilinears and considering the differential equation for those bilinears. Here are part of the relevant bilinears.

$$f_0 = \epsilon^\dagger\epsilon, \qquad f_j = \epsilon^\dagger\rho_j\epsilon,$$

$$g_0 = i\epsilon^\dagger\eta\epsilon, \qquad g_j = i\epsilon^\dagger\eta\rho_j\epsilon. \tag{2.22}$$

f_j and g_j are real functions on the base 2-dimensional space. $f_0 \geq 0$ and at least for a set of (a, b, c), f_0 does not vanish because we have $\epsilon \neq 0$. We will consider in such set (a, b, c) below.

The derivative of these bilinears can be calculated by using (2.19) as

$$df_0 = -\frac{1}{2}f_1 F + f_3 L + g_2\widetilde{K}, \tag{2.23}$$

$$df_1 = -\frac{1}{2}f_0 F - g_2\widetilde{L} - f_3 K, \tag{2.24}$$

$$dg_2 = f_1\widetilde{L} + f_0\widetilde{K}, \tag{2.25}$$

$$df_3 = f_0 L + f_1 K. \tag{2.26}$$

Here, we use tilde notation as the two dimensional Hodge dual. For example,

$$\widetilde{K}_m = -\varepsilon_{mn}K_n, \qquad \varepsilon_{23} = +1, \qquad m, n = 2, 3. \tag{2.27}$$

One can also derive some relations among the bilinears from (2.16)-(2.18). For example, (2.16) multiplied by $\epsilon^\dagger\Gamma_m$ reads

$$\epsilon^\dagger\Gamma_m(-i)e^{-A}\mu_1\nu_3\lambda_3\epsilon = \epsilon^\dagger\Gamma_m\left[-\partial\!\!\!/ A - \frac{1}{2}\rho_1 F\!\!\!/ + \rho_3 L\!\!\!/ - i\rho_2 K\!\!\!/\right]\epsilon, \tag{2.28}$$

and the hermitian conjugation of this equation reads

$$\epsilon^\dagger(+i)e^{-A}\mu_1\nu_3\lambda_3\Gamma_m\epsilon = \epsilon^\dagger\left[-\partial\!\!\!/ A - \frac{1}{2}\rho_1 F\!\!\!/ + \rho_3 L\!\!\!/ + i\rho_2 K\!\!\!/\right]\Gamma_m\epsilon, \tag{2.29}$$

If we add eq.(2.28) and eq.(2.29), we obtain the relation

$$f_0 dA = -\frac{1}{2}f_1 F + f_3 L + g_2\widetilde{K} \tag{2.30}$$

Comparing this equation and eq.(2.23), we can conclude that e^A is proportional to f_0. We have the freedom of the normalization of ϵ, so we will fix this normalization by $f_0 = e^A$.

As the same way, one can derive, from (2.17) and (2.18), the relations

$$f_3 dB = f_0 L + f_1 K, \qquad g_2 dC = f_1 \widetilde{L} + f_0 \widetilde{K}, \tag{2.31}$$

and comparing these equations to (2.26) and (2.25), we can conclude that e^B is proportional to f_3 and e^C is proportional to g_2.

Can we determine the coefficient of the proportions? Actually, in the $AdS_5 \times S^5$ solution, $e^B = f_3$ and $e^C = g_2$ are satisfied. The solution we want is asymptotically $AdS_5 \times S^5$. So $e^B = f_3$ and $e^C = g_2$ should be satisfied in our solution.

To proceed the analysis, let us consider the spinor bilinear S_m defined as

$$S_m := \epsilon^\dagger \Gamma_m (-i\rho_2) \nu_1 \lambda_3 \epsilon = \epsilon^\dagger \Gamma_m \nu_2 \lambda_3 \tau_1 \epsilon. \tag{2.32}$$

The derivative of S_m can be calculated by using (2.19).

$$\nabla_n S_m = \epsilon^\dagger \nu_2 \lambda_3 \tau_1 \left[\frac{1}{2} \Gamma_m \not{L} \Gamma_n + \frac{i}{2} \rho_2 \Gamma_m \not{K} \Gamma_n \right] \epsilon + (m \leftrightarrow n). \tag{2.33}$$

Especially, $\nabla_n S_m$ is symmetric under the exchange of m and n. That means, 1-form $S := S_m E^m$ is closed.

On the other hand, S_m can be related to other bilinears. Multiply $\epsilon^\dagger \Gamma_m (-i\rho_2)$ to (2.17), you obtain the relation

$$e^{-B} S_m = -g_2 \tilde{\partial}_m B + f_1 L_m + f_0 K_m. \tag{2.34}$$

We can simplify the above relation by Hodge dual of (2.25) and $g_2 = e^C$ as

$$S = -\tilde{d} e^{B+C}. \tag{2.35}$$

As we see before, S is a closed 1-form. So we can conclude that $d\tilde{d} e^{B+C} = 0$. In other words, e^{B+C} is a harmonic function on the base two dimensional space.

Now, we can take the coordinates of the base 2-dimensional space. One coordinate is $y = e^{B+C}$. As we have shown, this y is harmonic. Thus we can define the other coordinate x orthogonal to y, that means $dx := \tilde{d} y$.

To determine the metric of the base 2-dimensional space, let us consider the norm of dy. If we add (2.17) and (2.18), we will obtain

$$\left[e^{-B} \nu_1 \lambda_3 + e^{-C} \lambda_1 - \not{\partial}(B + C) \right] \epsilon = 0. \tag{2.36}$$

Multiplying this equation $\left[e^{-B}\nu_1\lambda_3 + e^{-C}\lambda_1 + \partial\!\!\!/(B+C)\right]$ reads $|d(B+C)|^2 = e^{-2B}+e^{-2C}$. Consequently the norm of $dy = de^{B+C}$ can be written as $|dy|^2 = e^{2B} + e^{2C}$. Now, we can express the metric of the base 2-dimensional space in the simple form

$$ds_2^2 = \frac{1}{e^{2B}+e^{2C}}(dy^2 + dx^2). \tag{2.37}$$

From the dilatino condition (2.20), by multiplying $\epsilon^\dagger$ and $\epsilon^\dagger\rho_j$, we obtain the following set of relations

$$0 = f_0 d\phi + \frac{1}{2}f_1 F - 2g_2\widetilde{K}, \qquad 0 = f_1 d\phi + \frac{1}{2}f_0 F - 2f_3 K,$$

$$0 = g_2 d\phi + \frac{1}{2}f_3\widetilde{F} - 2f_0\widetilde{K}, \qquad 0 = f_3 d\phi - \frac{1}{2}g_2\widetilde{F} + 2f_1 K. \tag{2.38}$$

These relations lead to the algebraic relation among f_0, f_1, g_2, f_3 like

$$-f_0^2 + f_1^2 + g_2^2 + f_3^2 = 0. \tag{2.39}$$

In summary, we can express every unknown quantity in terms of A, B, C, by using (2.23)-(2.26),(2.37), and (2.38)

$$ds_2^2 = \frac{1}{e^{2B}+e^{2C}}(dy^2 + dx^2), \tag{2.40}$$

$$F = \frac{2}{f_0^2 - f_1^2}\left[f_1 df_0 - f_0 df_1 + f_3\tilde{d}g_2 - g_2\tilde{d}f_3\right], \tag{2.41}$$

$$K = \frac{1}{f_0^2 - f_1^2}\left[-f_0\tilde{d}g_2 - f_1 df_3\right], \tag{2.42}$$

$$L = \frac{1}{f_0^2 - f_1^2}\left[f_1\tilde{d}g_2 + f_0 df_3\right], \tag{2.43}$$

$$d\phi = \frac{1}{(f_0^2 - f_1^2)^2}\left[f_0^2 g_2 dg_2 + f_1^2 f_3 df_3 + f_0 f_1(f_3\tilde{d}g_2 - g_2\tilde{d}f_3)\right], \tag{2.44}$$

where

$$y = e^{B+C}, \qquad f_0 = e^A, \qquad f_3 = e^B, \qquad g_2 = e^C, \qquad f_1 = \sqrt{e^{2A} - e^{2B} - e^{2C}}. \tag{2.45}$$

3 Interpretation of the geometry and the Gaussian matrix model

We study the detail of the geometry which we obtain in the previous section. We will find the continuous version of maya diagram at the boundary of the base two dimensional space. We also compare this pattern with the eigen value distribution of the Gaussian matrix model.

3.1 The structure of the geometry

Let us see the detail of the geometry (2.40)-(2.44).

As the same way as in [4], one of the coordinate y is the multiplication of the radius of S^2 and S^4. So y is greater than 0 and the line $y = 0$ is the boundary of the base 2-dimensional space. On this boundary(let us call it x-axis), $e^B = 0$ or $e^C = 0$ should be satisfied because $y := e^B e^C = 0$. Actually at a point on the x-axis, if one of e^B and e^C vanish and the other remain finite, the geometry is smooth at the point.

If the geometry is given, we can draw a 1 dimensional black and white pattern like figure 1 as follows. Take a point on the x-axis. If at that point $e^B = 0$ and $e^C \neq 0$ are satisfied, mark that point by black. If $e^C = 0$ and $e^B \neq 0$, mark that point by white. Then one get the one dimensional black and white pattern.

For example, $AdS_5 \times S^5$ is a solution that satisfy the ansatz we are considering. The metric of the $AdS_5 \times S^5$ can be written as

$$ds^2 = e^{2A} d\check{\Omega}_2^2 + R^2(du^2 + d\theta^2) + e^{2B} d\hat{\Omega}_2^2 + e^{2C} d\Omega_4^2,$$

$$e^A = R\cosh u, \qquad e^B = R\sinh u, \qquad e^C = R\sin\theta,$$

$$u \geq 0, \qquad 0 \leq \theta \leq \pi, \qquad R = (4\pi g_s N)^{1/4}. \tag{3.1}$$

where $d\check{\Omega}_2^2, d\hat{\Omega}_2^2, d\Omega_4^2$ are the metric of unit AdS_2, S^2, S^4 respectively. In this metric, $y := e^B e^C = R^2 \sinh u \sin\theta$. x is defined by $dx = \tilde{d}y$, so x is written as $x = -R^2 \cosh u \sin\theta$ in the $AdS_5 \times S^5$ solution. On the x-axis, S^2 shrinks in the region $-R^2 \leq x \leq R^2$, and S^4 shrinks in the region $x \leq -R^2$ or $R^2 \geq x$. One can draw a diagram like figure 2 for $AdS_5 \times S^5$ geometry.

3.2 Relation to Gaussian matrix model

It is conjectured in [8, 9] that a vacuum expectation value of a circular Wilson loop are calculated by Gaussian matrix integral. For example, the following partition function corresponds to the Wilson loop of trivial representation i.e. identity operator.

$$Z = \int dM \exp\left(-\frac{1}{\hbar}\,\mathrm{tr}[M^2]\right). \tag{3.2}$$

Here M is a $N \times N$ Hermitian matrix and the measure of integral $\int dM$ is a component-wise one.

We claim that the x-axis in our geometry corresponds to the eigen value space of the matrix model. In order to see this, let us first give a short review the "steepest decent" method to evaluate the integral (3.2).

By diagonalization of the matrix M, the integral can be rewritten as a integral of the eigen values λ_i, $(i = 1, \ldots, N)$ as

$$Z = \int (\prod_{i=1}^{N} d\lambda_i) \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j)^2 \exp\left(-\frac{1}{\hbar}\sum_{i=1}^{N} \lambda_i^2\right), \tag{3.3}$$

where $\prod_{1\le i<j\le N}(\lambda_i - \lambda_j)^2$ is the Jacobian. This is the square of the Vandermonde determinant. We evaluate (3.3) by the saddle point. In that case, due to the Vandermonde determinant, there is repulsive force between the eigen values. The task to evaluate the integral is to calculate the classical distribution of the eigen values.

In order to obtain the distribution, it is convenient to consider the resolvent $\omega(z) := \mathrm{tr}\left[\frac{1}{z-M}\right] = \sum_i \frac{1}{z-\lambda_i}$. From the equation of motion, $\omega(z)$ satisfies the following differential equation

$$0 = \frac{2N}{\hbar} + \frac{2}{\hbar} z\omega(z) + \omega(z)^2 - \omega'(z). \tag{3.4}$$

In the large N limit, the last term $\omega'(z)$ is negligible. So (3.4) becomes an algebraic equation and we can solve it easily.

$$\omega(z) = -\frac{z}{\hbar} \pm \sqrt{\frac{z^2}{\hbar^2} - \frac{2N}{\hbar}}. \tag{3.5}$$

This function has a cut between $z = \pm\sqrt{2\hbar N}$. The density of the eigen values are expressed as

$$\begin{aligned} \rho(\lambda) &= \frac{1}{2\pi i}(\omega(\lambda - i\varepsilon) - \omega(\lambda + i\varepsilon)) \\ &= \begin{cases} 0, & (\lambda < -\sqrt{2\hbar N} \text{ or } \lambda > \sqrt{2\hbar N}), \\ \frac{1}{\pi}\sqrt{\frac{2N}{\hbar} - \frac{\lambda^2}{\hbar^2}}, & (-\sqrt{2\hbar N} \le \lambda \le \sqrt{2\hbar N}). \end{cases} \end{aligned} \tag{3.6}$$

We can compare the x-axis in the $AdS_5 \times S^5$ geometry and the eigen value of Gaussian matrix model. In the figure 2 , the length of the black segment is $2\sqrt{4\pi g_s N}$. In the matrix model partition function, the length of the cut, i.e. the distance between the smallest eigen value and the largest eigen value is $2\sqrt{2\hbar N}$. *These two are completely the same if we identify* $\hbar = 2\pi g_s$.

Next, let us turn to another check. We will consider the operator $\frac{1}{N}\mathrm{tr}[U^k]$, $U := P\exp(i\int dx^0(A_0 + \varphi_4))$, for the positive integer k. According to [3], the operator $\frac{1}{N}\mathrm{tr}[U^k]$ corresponds to a $AdS_2 \times S^2$ D3-brane with k unit of electric flux. This $AdS_2 \times S^2$ D3-brane is an analogue of giant graviton[11, 12, 13]. In our base 2 dimensional space, this D3-brane sit at a point on the x-axis. The position of this $AdS_2 \times S^2$ D3-brane is

$$x = \sqrt{4\pi g_s N + \frac{k^2(2\pi g_s)^2}{4}}, \tag{3.7}$$

Figure 3: Figure (A) expresses the x-axis of the $AdS_5 \times S^5$. The $AdS_2 \times S^2$ D3-brane looks like a point on the x-axis. It sits on the point $x = \sqrt{4\pi g_s N + \frac{k^2(2\pi g_s)^2}{4}}$. On the other hand, figure (B) represents the eigen value distribution of the matrix model. The black bar is the $\lambda_2, \ldots, \lambda_N$. Only λ_1 separates from the other eigen values. Its position is $\lambda_1 = \sqrt{2\hbar N + \frac{k^2\hbar^2}{4}}$.

as shown in figure 3 (A).

On the other hand in the matrix model side, the vacuum expectation value of $\frac{1}{N}\,\mathrm{tr}[U^k]$ can be calculated by the matrix model like

$$\left\langle \frac{1}{N}\,\mathrm{tr}[U^k] \right\rangle = \frac{1}{Z}\int dM \frac{1}{N}\,\mathrm{tr}[e^{-kM}]\exp\left(-\frac{1}{\hbar}\,\mathrm{tr}[M^2]\right). \tag{3.8}$$

Diagonalizing the matrix M leads to

$$\begin{aligned} &\frac{1}{Z}\int (\prod_{i=1}^{N} d\lambda_i) \prod_{1\le i<j\le N} (\lambda_i - \lambda_j)^2 \sum_{i=1}^{N} \frac{1}{N} e^{-k\lambda_i} \exp\left(-\frac{1}{\hbar}\sum_{i=1}^{N}\lambda_i^2\right) \\ &= \frac{1}{Z}\int (\prod_{i=1}^{N} d\lambda_i) \prod_{1\le i<j\le N} (\lambda_i - \lambda_j)^2 e^{k\lambda_1} \exp\left(-\frac{1}{\hbar}\sum_{i=1}^{N}\lambda_i^2\right), \end{aligned} \tag{3.9}$$

Here the equations of motion for $\lambda_2, \ldots, \lambda_N$ are the same as before. Since N is large now, we use the solution for $\lambda_2, \ldots, \lambda_N$ neglecting λ_1 and take it as a background for λ_1. In this case, the equation of motion for λ_1 becomes

$$0 = \frac{2}{\hbar}\lambda_1 - 2\sum_{i=2}^{N} \frac{1}{\lambda_1 - \lambda_i} - k. \tag{3.10}$$

The second term of the right-hand side of the above equation becomes $\omega(\lambda_1)$. We replace this term with the expression of 3.5. Then, we can solve this equation

$$\lambda_1 = \sqrt{2\hbar N + \frac{k^2\hbar^2}{4}}, \tag{3.11}$$

as shown in the right-hand side of figure 3. *The position of particle in* (3.7) *and* (3.11) *completely match.*

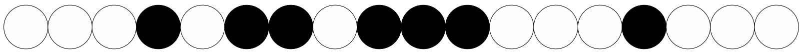

Figure 4: An example of maya diagram.

3.3 The Maya diagram and the Young diagram

In this subsection, we propose a rule of correspondence between the pattern of x-axis and the Young diagram: label of the Wilson line[2].

First, let us explain how to discretize the pattern of x-axis. Since the black segment expresses a S^5, we can replace a black segment with a sequence of black dots of the same number as the 5-form flux through the S^5. As the same way we will replace white segment with white dots of the same number as the 3-form flux through the S^3. The semi-infinite white lines should be replaced by infinite number of white dots. Then we will obtain a kind of maya diagram.

This is not the ordinary maya diagram. In ordinary maya diagram, every dots are black after a certain position in the left, while every dots are white after a certain position in the right. This kind of maya diagram correspond to a general Young diagram. On the other hand, in our maya diagram here, every dots are white after a certain position in the both direction. This kind of maya diagram is associated to a Young diagram with rows less than N: the number of black dots.

We associate the maya diagram to Young diagram as the following way. First, as the same way as the ordinary maya diagram, replace white dot with "go right" and black dot with "go up", and draw a path as figure 5 (A). Next, cut the upper area of the path by the line like figure 5 (B). Then we obtain the Young diagram with rows less then N.

If we take the matrix model eigen value picture into account, we find that this Young diagram does not correspond to the label of "representation basis" of the Wilson line operators. Actually this Young diagram corresponds to the label of the following "monomial basis" of the Wilson line operators.

In order to explain this basis, let us define the $N \times N$ unitary matrix $U := P\exp(i\int dx^0(A_0+\varphi_4))$, whose eigen values are $u_1,\ldots,u_N$. It is convenient to express a Young diagram with rows less then N by $(\mu_1,\ldots,\mu_N)$, where μ_j is the number of boxes in the j-th row. The monomial basis for the Young diagram μ is $m_\mu = u_1^{\mu_1}u_2^{\mu_2}\ldots u_N^{\mu_N} + (\mathrm{sym})$, where (sym) means the terms for symmetrization. On the other hand, the representation basis is expressed by the Schur polynomial $\mathrm{Tr}_R[U] = S_R(u_1,\ldots,u_N)$ for a Young diagram R. Of

[2]Kazuo Hosomichi kindly explained to me about the basic idea of this subsection. I appreciate it very much.

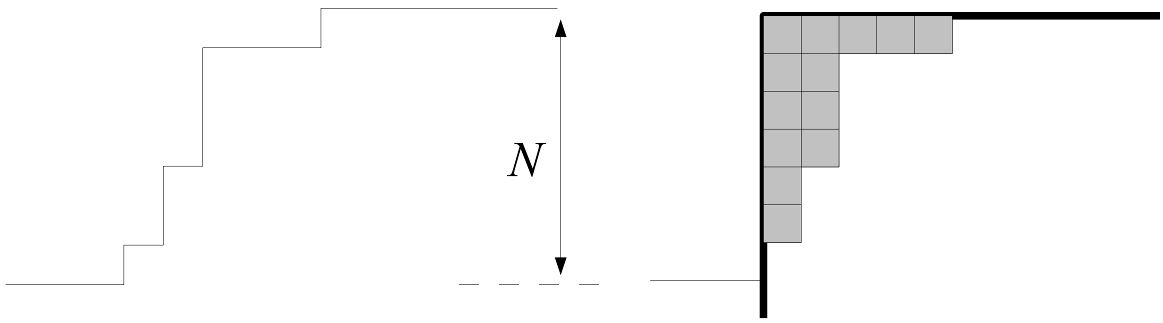

Figure 5: Correspondence between maya diagrams and Young diagrams. Here we show the Young diagram which corresponds to the example of figure 4. The figure of (A) is the line made by replacing a white dot with horizontal segment and a black dot with vertical segment of unit length. The "height" of this figure is equal to the number of black dots N. The figure (B) shows how to make the Young diagram from the figure (A).

course we can express one basis in terms of some linear combination of the other.

Let us comment on the translation and the left-right flip of the maya diagram. In the geometry side, the system is symmetric under the translation and the left-right flip of the maya diagram. How can we see these symmetry in the label of Wilson line operator? First, translation means inserting or removing the determinant $\det U = u_1 \dots u_N$. In the language of the Young diagram, translation means $(\mu_1, \dots, \mu_N) \to (\mu_1 + a, \dots, \mu_N + a)$ for a integer a. Usually using this degrees of freedom we can set $\mu_N = 0$ and draw a Young diagram with rows less than N. As for the left-right flip, it corresponds to the complex conjugation. The monomial basis satisfy $m_\mu(u^*) = m_{-\mu}(u)$, since each u_j is a phase, that is to say, $u_j^* = u_j^{-1}$. Here Young diagram $(-\mu)$ means $(-\mu)_j = -\mu_{N-j}$. Also in this case, using translation symmetry, we can draw a Young diagram with rows less than N.

4 1/2 BPS Geometry in M-theory

In this section, we discuss the similar problem in M-theory. For example, in the 6-dimensional (2,0) superconformal field theory, we have a kind of surface operators. It preserves half of the supersymmetry if the shape is flat. We first consider the symmetry preserved by this kind of operators, and make the ansatz. Then, we study the supersymmetry conditions and derive the necessary conditions. We also show a few examples.

4.1 Ansatz for the surface operator in 6-dimensional SCFT and 3-dimensional SCFT

In the AdS/CFT correspondence context, 6-dimensional (2,0) SCFT corresponds to M-theory on the $AdS_7 \times S^4$ background. The surface operator corresponds to a membrane in the probe picture. In this picture, we can easily see that the bosonic symmetry which is preserved by this surface operator (or the membrane) is $SO(2,2) \times SO(4) \times SO(4)$.

We can also consider a 3-dimensional SCFT which corresponds to $AdS_4 \times S^7$. In this theory, we can also consider the wall like defect operators, which corresponds to $AdS_3 \times S^3$ M5-brane. The symmetry which is preserved by this operator is also $SO(2,2) \times SO(4) \times SO(4)$, which can be seen by the probe picture.

Let us turn to making the ansatz that respect the symmetry $SO(2,2)\times SO(4)\times SO(4)$. The metric can be written in the form

$$ds^2 = e^{2A} d\check{\Omega}_3^2 + ds_2^2 + e^{2B} d\hat{\Omega}_3^2 + e^{2C} d\Omega_3^2, \tag{4.1}$$

where $d\check{\Omega}_3^2, d\hat{\Omega}_3^2, d\Omega_3^2$ are the metric of unit AdS_3, S^3, S^3 respectively. ds_2^2 in (4.1) is general two dimensional metric to be determined. As in the IIB case, we call this two dimensional space "base space." A, B, and C in (4.1) are functions on the base space.

As for the flux, the most general ansatz which preserves the $SO(2,2)\times SO(4)\times SO(4)$ symmetry is

$$G_4 = 6FE^0E^1E^2 + 6JE^5E^6E^7 + 6KE^8E^9E^{10}, \tag{4.2}$$

where $E^a, a = 0, \ldots, 10$ are the vielbein and F, J, K are 1-forms on the base 2-dimensional space. (E^0, E^1, E^2) are the vielbein of AdS_3 part, (E^5, E^6, E^7) are the vielbein of one S^3 part, and (E^8, E^9, E^{10}) are the vielbein of the other S^3 part.

We study the supersymmetry conditions under this ansatz in the next subsection. There is some related work. In [14], they have obtained some necessary and sufficient conditions for $AdS_3 \times X$ geometry. So our problem is the special case of them. In [15], they obtained the solutions which includes $\mathbb{R}^{1,2} \times S^3 \times S^3$. Our problem is to obtain the geometry that includes AdS_3 instead of $\mathbb{R}^{1,2}$.

4.2 Gravitino condition, spinor bilinears

In this subsection, we use the convention of eleven dimensional gamma matrices of the following.

$$\Gamma^0 = \gamma_5 \otimes \check{\sigma}^0 \otimes 1 \otimes 1 \otimes \sigma_1, \qquad \Gamma^1 = \gamma_5 \otimes \check{\sigma}^1 \otimes 1 \otimes 1 \otimes \sigma_1, \qquad \Gamma^2 = \gamma_5 \otimes \check{\sigma}^2 \otimes 1 \otimes 1 \otimes \sigma_1,$$
$$\Gamma^3 = \gamma^3 \otimes 1 \otimes 1 \otimes 1 \otimes 1, \qquad \Gamma^4 = \gamma^4 \otimes 1 \otimes 1 \otimes 1 \otimes 1,$$
$$\Gamma^5 = \gamma_5 \otimes 1 \otimes \hat{\sigma}^5 \otimes 1 \otimes \sigma_2, \qquad \Gamma^6 = \gamma_5 \otimes 1 \otimes \hat{\sigma}^6 \otimes 1 \otimes \sigma_2, \qquad \Gamma^7 = \gamma_5 \otimes 1 \otimes \hat{\sigma}^7 \otimes 1 \otimes \sigma_2,$$
$$\Gamma^8 = \gamma_5 \otimes 1 \otimes 1 \otimes \tilde{\sigma}^8 \otimes \sigma_3, \qquad \Gamma^9 = \gamma_5 \otimes 1 \otimes 1 \otimes \tilde{\sigma}^9 \otimes \sigma_3, \qquad \Gamma^{10} = \gamma_5 \otimes 1 \otimes 1 \otimes \tilde{\sigma}^{10} \otimes \sigma_3. \tag{4.3}$$

Here $(\gamma^3, \gamma^4, \gamma^5)$, $(\hat{\sigma}^5, \hat{\sigma}^6, \hat{\sigma}^7)$, and $(\tilde{\sigma}^8, \tilde{\sigma}^9, \tilde{\sigma}^{10})$ are sets of Pauli matrices.

We also use Killing spinors in AdS spacetimes and spheres. For AdS_3, they can be written as

$$\overset{\circ}{\nabla}_p \check{\chi}^I_a = \frac{i}{2} a \check{\sigma}_p \check{\chi}^I_a, \qquad (p = 0, 1, 2, \qquad a = \pm 1, \qquad I = 1, 2), \tag{4.4}$$

where $\overset{\circ}{\nabla}$ is the covariant derivative of Levi-Civitá connection of unit AdS_3. As the same way, there are Killing spinors in S^3.

$$\overset{\circ}{\nabla}_p \hat{\chi}^J_b = \frac{1}{2} b \hat{\sigma}_p \hat{\chi}^J_b, \qquad (p = 5, 6, 7, \qquad b = \pm 1, \qquad J = 1, 2), \tag{4.5}$$

where $\overset{\circ}{\nabla}$ is the covariant derivative of Levi-Civitá connection of unit S^3. We also prepare another set of S^3 Killing spinor χ^K_c for the other S^3.

We can expand the 11 dimensional spinor ξ as $\xi = \sum_{abcIJK} \epsilon_{abcIJK} \otimes \check{\chi}^I_a \otimes \hat{\chi}^J_b \otimes \chi^K_c$. In this expansion, each of the coefficient ϵ_{abcIJK} is a element of (2 dim spinor)$\otimes\mathbb{C}^2$. γ_m, $(m = 3, 4, 5)$ acts on the (2 dim spinor) part, and σ_j, $(j = 1, 2, 3)$ acts on the $\mathbb{C}^2$ part.

With these notations, we can rewrite the supersymmetry conditions as

$$0 = \left[ae^{-A}\sigma_1 - \partial\!\!\!/ A\gamma_5 + 2F\!\!\!/\sigma_1 - J\!\!\!/ i\sigma_2 - K\!\!\!/ i\sigma_3\right]\epsilon, \tag{4.6}$$

$$0 = \left[ibe^{-B}\sigma_2 - \partial\!\!\!/ B\gamma_5 - F\!\!\!/\sigma_1 + 2J\!\!\!/ i\sigma_2 - K\!\!\!/ i\sigma_3\right]\epsilon, \tag{4.7}$$

$$0 = \left[ice^{-C}\sigma_3 - \partial\!\!\!/ C\gamma_5 - F\!\!\!/\sigma_1 - J\!\!\!/ i\sigma_2 + 2K\!\!\!/ i\sigma_3\right]\epsilon, \tag{4.8}$$

$$0 = \nabla_m \epsilon - \frac{i}{2}\left[\widetilde{F}_m\sigma_1 + \widetilde{J}_m i\sigma_2 + \widetilde{K}_m i\sigma_3\right]\epsilon - \gamma_5\left[F_m\sigma_1 + J_m i\sigma_2 + K_m i\sigma_3\right]\epsilon, \tag{4.9}$$

where tilde is the 2-dimensional Hodge dual. In these equations, we omit the index of ϵ_{abcIJK}.

Let us consider the spinor bilinears like

$$f_0 := \epsilon^\dagger \epsilon, \quad f_j := \epsilon^\dagger \sigma_j \epsilon, \quad g_0 := \epsilon^\dagger \gamma_5 \epsilon, \quad g_j := \epsilon^\dagger \gamma_5 \sigma_j \epsilon. \tag{4.10}$$

The derivative of these functions can be calculated by using (4.9).

$$df_0 = 2g_1 F - f_2 \widetilde{J} - f_3 \widetilde{K}, \tag{4.11}$$

$$df_2 = f_3 \widetilde{F} - f_0 \widetilde{J} - 2g_1 K, \tag{4.12}$$

$$df_3 = -f_2 \widetilde{F} + 2g_1 J - f_0 \widetilde{K}, \tag{4.13}$$

$$dg_1 = 2f_0 F - 2f_3 J + 2f_2 K. \tag{4.14}$$

On the other hand, from eqs.(4.6)-(4.8), we can derive the relations

$$f_0 dA = 2g_1 F - f_2 \widetilde{J} - f_3 \widetilde{K}, \tag{4.15}$$

$$f_3 dB = -f_2 \widetilde{F} + 2g_1 J - f_0 \widetilde{K}, \tag{4.16}$$

$$f_2 dC = f_3 \widetilde{F} - f_0 \widetilde{J} - 2g_1 K. \tag{4.17}$$

This means e^A, e^B, e^C are proportional to f_0, f_3, f_2 respectively. By adjusting the normalization of ϵ, we can set $f_0 = e^A$. f_2 and f_3 can be written with constant coefficients p, q as

$$f_3 = pe^B, \qquad f_2 = qe^C. \tag{4.18}$$

Next, let us show $y := e^{A+B+C}$ is a harmonic function on the base 2-dimensional space. Adding eqs.(4.6)-(4.8) gives

$$0 = \left[ae^{-A}\sigma_1 + ibe^{-B}\sigma_2 + ice^{-C}\sigma_3 - \partial\!\!\!/ (A+B+C)\gamma_5 \right] \epsilon. \tag{4.19}$$

From this equation and its hermitian conjugation, we obtain

$$\tilde{dy} = -be^C P^{(2)} - ce^B P^{(3)}, \tag{4.20}$$

where $P^{(j)}$'s are spinor bilinears defined as

$$P^{(0)}_m := \epsilon^\dagger \gamma_m \epsilon, \qquad P^{(j)}_m := \epsilon^\dagger \sigma_j \gamma_m \epsilon, \tag{4.21}$$

and 1-form is defined for example $P = P_m E^m$. In order to show the right-hand side of (4.20) is a closed 1-form, let us first see $e^B P^{(3)}$ is closed. From eq.(4.7), we have the relation

$$0 = be^{-B}P^{(3)} + \frac{1}{2}dg_1 + g_1 dB + 3f_3 J. \tag{4.22}$$

This equation and eq.(4.14) read

$$0 = be^{B}P^{(3)} + d\left(\frac{1}{2}e^{2B}g_1\right) + 3pe^{3B}J. \tag{4.23}$$

The second term of the left-hand side is exact 1-form, and the last term is closed because of the Bianchi identity for the 4-form field strength of the 11 dimensional supergravity. So we can conclude that $e^B P^{(3)}$ is a closed 1-form. As the same way, we can show that $e^C P^{(2)}$ is closed, so $\tilde{d}y$ is closed because of eq.(4.20). As a result, we can define the other coordinate x by $dx = \tilde{d}y$.

We can express the norm $|dy|^2$ by A, B, C. Multiply $\big[ae^{-A}\sigma_1 + ibe^{-B}\sigma_2 + ice^{-C}\sigma_3 + \not{\partial}(A+B+C)\gamma_5$ to (4.19), and we obtain

$$0 = e^{-2A} - e^{-2B} - e^{-2C} - |dy|^2/y^2. \tag{4.24}$$

This equation implies $|dy|^2 = -e^{2B+2C} + e^{2A+2B} + e^{2A+2C}$. Hence the metric of two dimensional space can be written as

$$ds_2^2 = \frac{1}{-e^{2B+2C} + e^{2A+2B} + e^{2A+2C}}(dy^2 + dx^2). \tag{4.25}$$

From eq.(4.19), we can show that $P^{(1)} = 0$. This fact and the Fierz identity read

$$0 = -f_0^2 + f_2^2 + f_3^2 + g_1^2. \tag{4.26}$$

Eq.(4.19) also lead to the relation between constants,

$$0 = a - bp + cq. \tag{4.27}$$

In summary, we obtain the necessary conditions of supersymmetry.

$$ds_2^2 = \frac{1}{-e^{2B+2C} + e^{2A+2B} + e^{2A+2C}}(dy^2 + dx^2), \tag{4.28}$$

$$6F = 4\frac{df_0}{g_1} - \frac{f_0 dg_1}{g_1^2} + \frac{2}{g_1^2}(f_2\tilde{d}f_3 - f_3\tilde{d}f_2), \tag{4.29}$$

$$6J = 4\frac{df_3}{g_1} - \frac{f_3 dg_1}{g_1^2} + \frac{2}{g_1^2}(-f_0\tilde{d}f_2 + f_2\tilde{d}f_0), \tag{4.30}$$

$$6K = -4\frac{df_2}{g_1} + \frac{f_2 dg_1}{g_1^2} + \frac{2}{g_1^2}(-f_0\tilde{d}f_3 + f_3\tilde{d}f_0), \tag{4.31}$$

Figure 6: x-axis of $AdS_4 \times S^7$. It is painted in black $x < 0$ and in white $x > 0$.

where

$$f_0 = e^A, \qquad f_3 = pe^B, \qquad f_2 = qe^C, \qquad (p, q : \text{constants}), \qquad g_1 = \sqrt{f_0^2 - f_2^2 - f_3^2}. \tag{4.32}$$

4.3 Continuous maya diagram and examples

As in the IIB case, the base two dimensional space has a boundary expressed by $y = 0$. On this boundary $y = e^{A+B+C} = 0$ is satisfied. In order to make the geometry regular, one of e^B and e^C vanishes at the a point on the boundary but the other does not vanish at that point. Let us make a one dimensional pattern as the same way as the IIB case. Paint the point $e^B = 0$ in black, and paint the point $e^C = 0$ in white.

Here, we mention some examples. $AdS_7 \times S^4$, $AdS_4 \times S^7$ and $AdS_3 \times S^3 \times R^4 \times S^1$.

First let us see the $AdS_7 \times S^4$ solution. It can be written as

$$ds_2^2 = R^2(du^2 + d\theta^2),$$

$$e^A = R\cosh u, \qquad e^B = R\sinh u, \qquad e^C = \frac{R}{2}\sin 2\theta, \tag{4.33}$$

$$y := e^{A+B+C} = \frac{R^3}{4}\sinh 2u \sin 2\theta, \qquad x = -\frac{R^3}{4}\cosh 2u \cos 2\theta. \tag{4.34}$$

Here R is a constant. The x-axis of this solution looks just the same as figure 2.

Second let us see the $AdS_4 \times S^7$ solution

$$ds_2^2 = R^2(du^2 + d\theta^2),$$

$$e^A = \frac{R}{2}\cosh 2u, \qquad e^B = R\cos\theta, \qquad e^C = R\sin\theta, \tag{4.35}$$

$$y = \frac{R^3}{4}\cosh 2u \sin 2\theta, \qquad x = -\frac{R^3}{4}\sinh 2u \cos 2\theta. \tag{4.36}$$

The pattern of x-axis of this solution looks like figure 6.

Figure 7: The eigen value distribution with hole. The thick line represents where the eigen values distributed. The white dot represents the hole where the eigen value is not distributed. In the $AdS_5 \times S^5$ picture, the thick line is the place on the x-axis where the S^4 has finite size. So we can guess a hole corresponds to $AdS_2 \times S^4$ D5-brane. On the other hand in Yang-Mills theory, the hole corresponds to the Wilson line of anti-symmetric representation.

Finally let us see the $AdS_3 \times S^3 \times R^4 \times R^1$

$$ds_2^2 = du^2 + dv^2, \tag{4.37}$$

$$e^A = R, \qquad e^B = R, \qquad e^C = u, \tag{4.38}$$

$$y = R^2 u, \qquad x = R^2 v. \tag{4.39}$$

In this solution, the pattern of x-axis is just a white line.

5 Conclusions and discussions

In this paper, we investigate the geometry that corresponds to the half BPS straight Wilson line. In this geometry, we can see the distribution of eigen values of Gaussian matrix model, which is supposed to describe the theory of half BPS Wilson lines. We also consider similar half BPS geometry in M-theory.

In this paper, we identify an isolated eigen value in matrix model to an $AdS_2 \times S^2$ D3-brane. Then, what is the isolated "hole" in the matrix model?(see figure 7) We guess that an $AdS_2 \times S^4$ D5-brane corresponds to a hole in the eigen value distribution of the Gaussian matrix model. To investigate this probe brane and its relation to the Wilson line operator of anti-symmetric representation is a future problem.

We could not obtain so far necessary and sufficient condition of supersymmetric background satisfy the ansatz studied here. It is a important problem to obtain the necessary and sufficient condition and make nontrivial examples of supergravity solutions.

The solutions of the other kind of extended defect operators also seem interesting to study. For example, the general Wilson-'tHooft lines will corresponds to the most general ansatz that preserves SL(2,R)×SO(3) ×SO(5). In probe picture, they correspond to (p,q)-strings or $AdS_2 \times S^2$ D3-branes with electro-magnetic flux or $AdS_2 \times S^4$ (p,q) 5-brane with electric flux. Another interesting class of operators are wall-like defect operator. The probe picture is typically $AdS_4 \times S^2$ D5-brane[16, 17]. This class of operators preserve $SO(2,3) \times SO(3) \times SO(3)$ and half of the supersymmetry.

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A Convention for supergravity

A.1 Gamma matrices

We use 10 and 11 dimensional gamma matrices satisfying

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}, \qquad \eta^{MN} = \mathrm{diag}(-1, +1, \ldots, +1), \qquad M, N = 0, \ldots, 10, \tag{A.1}$$

$$\Gamma^0\Gamma^1 \ldots \Gamma^9\Gamma^{10} = +1. \tag{A.2}$$

Hodge dual for a p-form G is defined by the anti-symmetric tensor $\varepsilon^{M_0 \ldots M_n}$, $n = 9$ or $n = 10$ satisfying $\varepsilon^{01\ldots n} = +1$ as

$$(*G)_{N_1 N_2 \ldots N_{n+1-p}} := \frac{1}{p!} \varepsilon_{N_1 N_2 \ldots N_{n+1-p} M_1 M_2 \ldots M_p} G^{M_1 \ldots M_p}. \tag{A.3}$$

We also use the slash notation defined as

$$\not{G} := \frac{1}{p!} G_{M_1 \ldots M_p} \Gamma^{M_1 \ldots M_p}. \tag{A.4}$$

A.2 IIB supergravity

We use the following convention for IIB supergravity. This theory contains the metric g_{MN}, the dilaton ϕ, NSNS 3-form field strength H_3, and RR field strength G_1, G_3, G_5 as bosonic fields. This theory also contains gravitino ψ_M which is the pair of vectorial

Majorana-Weyl spinor of positive chirality $\Gamma_{10}\psi_M = +\psi_M$ as well as dilatino λ which is the pair of Majorana-Weyl spinor of negative chirality $\Gamma_{10}\lambda = -\lambda$ The parameter of the SUSY transformation is a doublet of the Majorana-Weyl spinor $\xi = (\xi_1, \xi_2)$ with positive chirality as the fermionic fields. τ_j is a set of Pauli matrices that acts as $(\tau_j\xi)_\alpha = (\tau_j)_{\alpha\beta}\xi_\beta$. The SUSY transformation for gravitino ψ_M and dilatino λ for a bosonic configuration is

$$\delta\psi_M = \nabla_M\xi + \frac{1}{8}H_{MAB}\Gamma^{AB}\tau_3\xi + \frac{e^\phi}{8}(i\not{G}_1\tau_2 - \not{G}_3\tau_1 + \frac{i}{2}\not{G}_5\tau_2)\Gamma_M\xi, \tag{A.5}$$

$$\delta\lambda = (\not{\partial}\phi)\xi + \frac{1}{2}\not{H}_3\tau_3\xi + e^\phi(-i\not{G}_1\tau_2 + \frac{1}{2}\not{G}_3\tau_1)\xi. \tag{A.6}$$

Bianchi identities for the form fields are

$$dH_3 = 0, \qquad dG_1 = 0, \qquad dG_3 = H_3 \wedge G_1, \qquad dG_5 = H_3 \wedge G_3. \tag{A.7}$$

G_5 satisfies the self-duality $G_5 = - * G_5$. The equations of motion for form fields are

$$d(e^{-2\phi} * H_3) = -G_3 \wedge G_5 + G_1 \wedge *G_3, \qquad d * G_1 = *G_3 \wedge H_3, \qquad d * G_3 = -G_5 \wedge H_3. \tag{A.8}$$

A.3 Eleven dimensional supergravity

This theory contains the metric g_{MN} and 4-form field strength G_4 as the bosonic fields. This theory also contains gravitino ψ_M which is a vectorial Majorana spinor as the fermionic fields. The parameter of SUSY transformation ξ is a Majorana spinor in 11 dimensions. The SUSY transformation of gravitino in a bosonic configuration is

$$\delta\psi_M = \nabla_M\xi + \frac{1}{12\times 4!}G_{NPQR}\Gamma_M{}^{NPQR}\xi - \frac{1}{6\times 3!}G_{MPQR}\Gamma^{PQR}\xi. \tag{A.9}$$

The Bianchi identity and the equation of motion for the 4-form are

$$dG_4 = 0, \qquad d * G_4 = G_4 \wedge G_4. \tag{A.10}$$

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