Constructing conformal field theory models

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Abstract

Defining a mathematically consistent (non-free) quantum field theory in four space-time dimensions is a continuing challenge facing mathematical physicists for nearly eighty years. It justifies attempts at constructing not entirely realistic conformal invariant models. We review two complementary projects of this type based on two inequivalent notions, infinitesimal and global, of conformal symmetry. The first, based on joined work with Gerhard Mack of the 1970's, culminated in rearranging the perturbation series in terms of divergence free skeleton graphs with dressed conformal invariant vertex functions and propagators [49], and in solving the Schwinger-Dyson equations by a conformal partial wave expansion [22]. The second started (in a joined work with N. Nikolov [57]) by introducing the notion of *global conformal invariance* (GCI) and proving rationality of GCI Wightman functions. Recent and ongoing work with B. Bakalov, N. Nikolov, K.-H. Rehren and Ya. Stanev on the local field positive energy representations of the $sp(\infty, \mathbb{R})$ Lie algebra is (p)reviewed and discussed.

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Contents

1 Introduction

1.1 Local causal automorphisms versus global conformal transformations

Conformal invariance can be viewed as a local version of dilation symmetry[1]: a *rational* (real) coordinate *transformation* $g : x \to y(x)$ (with singularities) of Minkowski space $M = \mathbb{R}^{3,1}$ is said to be *conformal* if it multiplies the Lorentzian interval by a positive factor (which may depend on x):

$$dy^2 = \frac{dx^2}{\omega^2(x,g)}\,, \quad dx^2 = d\boldsymbol{x}^2 - (dx^0)^2\,, \quad d\boldsymbol{x}^2 = \sum_{i=1}^{3}(dx^i)^2\,. \tag{1.1}$$

The general *conformal factor* $\omega(x,g)$ satisfying the *cocycle condition* $\omega(x, g_1\, g_2) = \omega(g_2\, x, g_1)\, \omega(x, g_2)$ is found to be a quadratic function of x. In fact, according to (an extension to four dimensions of) *Liouville theorem* the conformal group of M is compounded by Poincaré transformations, uniform dilations ($y = \rho x$) for which ω is constant ($\omega = \rho^{-1}$) and *special conformal transformations* $x \to y(x,c)$ which can be defined as translations sandwitched between two conformal inversions ($x \to x/x^2$):

$$y\,(= y(x,c)) = \left(\frac{x}{x^2} + c\right)\left[\left(\frac{x}{x^2} + c\right)^2\right]^{-1} = \frac{x + c\,x^2}{\omega(x,c)}\,,\ \omega(x,c) = 1 + 2\,c\,x + c^2\,x^2 \tag{1.2}$$

($c\,x = \boldsymbol{c}\,\boldsymbol{x} - c^0\,x^0$). Clearly, for any $x \neq 0$, $\omega(x,c)$ changes sign with varying $c \in M$ – and vanishes on a 3-cone (which degenerates into a hyperplane for $x^2 = 0$). It follows that, unlike the infinitesimal interval (1.1), the square of a finite interval may change sign under special conformal transformation:

$$y_{12}^2 := (y_1 - y_2)^2 = \frac{x_{12}^2}{\omega(x_1,c)\,\omega(x_2,c)} \text{ for } x_{12} = x_1 - x_2\,,\ y_i = y(x_i,c)\,,\ i = 1,2\,. \tag{1.3}$$

The 15-parameter group of rational coordinate transformations thus described is isomorphic to the factor-group $\mathrm{SO}_0(4,2)/\mathbb{Z}_2$ of the connected group of pseudo-rotation $\mathrm{SO}_0(4,2)$ with respect to its centre $\mathbb{Z}_2$ (involving the reflection of all six axes). We shall use instead the spinorial *conformal group*

$$\mathcal{C} = \mathrm{SU}(2,2) \tag{1.4}$$

that is a double cover of $\mathrm{SO}_0(4,2)$ (and thus a four-fold cover of $\mathrm{SO}_0(4,2)/\mathbb{Z}_2$). $\mathcal{C}$ is not simply connected and has an infinite sheeted universal covering group $\tilde{\mathcal{C}}$ that is not a matrix group.

[1]This point of view was expounded by Hermann Weyl (1885-1955) in his 1918 paper *Gravitation und Elektrizität* which first introduced the notion of gauge symmetry – in the wrong place. The attempt to unify on this basis gravity and electromagnetism was rightly dismissed by physicists starting with Einstein. In the words of Michael Atiyah [4] the precious "infant" (gauge theory) "was nearly thrown out with the bath water".

We thus have two inequivalent notions of conformal symmetry: (1) *local causal automorphisms* of M [29] which preserve the causal order of events (and in particular, the sign of the square interval) and are compounded by Poincaré transformations, dilations and infinitesimal special conformal transformations; (2) *finite (global) conformal transformations* which only act without singularities on the conformal compactification $\bar{M}$ of Minkowski space. It was Dirac[2] [20] who realized $\bar{M}$ as a projective quadric in six dimensions

$$\bar{M} \simeq Q/\mathbb{R}^* \simeq \mathbb{S}^3 \times \mathbb{S}^1/\mathbb{Z}^2 \,, \; Q = \left\{ \vec{\xi} \in \mathbb{R}^{4,2} \,;\; \vec{\xi}^{\,2} := \sum_{\alpha=1}^{4} \xi_\alpha^2 - \xi_0^2 - \xi_{-1}^2 = 0 \right\} , \tag{1.5}$$

providing a manifestly conformal invariant description of massless wave equations. In fact, the conformal invariance of the classical Maxwell equations was recognized earlier by Cunningham and Bateman[3] [17] [9] soon after the notion of a symmetry group entered physics with the work of Poincaré and Einstein on relativity.

1.2 Conformal invariance and critical behaviour in QFT

Discrete masses of atoms and elementary particles violate "the great principle of similitude"[4] (*i.e.* scale and, *a fortiori*, conformal invariance). The situation in *quantum field theory* (QFT) is still more involved – and more interesting: dimensional parameters arise in the process of renormalization even if they are absent in the classical theory. Dilation and conformal invariance can only be preserved for a renormalization group fixed point – *i.e.*, for a critical theory, the QFT counterpart of a point of phase transition. Modern mathematical study of Feynman graphs (and renormalization) often neglect masses as an inessential complication (see *e.g.* [11]). May be the study of an idealized critical theory with no dimensional parameter will prove to be an essential step in understanding QFT – just as Galilei's law of inertia, that neglects friction, has been crucial in formulating and understanding classical mechanics?

What does matter is the fact that conformal invariance opens new avenues for attacking the central theoretical problem: constructing a consistent 4-dimensional QFT model. First, it allows to compute (dressed) propagators and vertex functions [63] [68] and to write down a skeleton graph expansion without divergences [49] which obeys a conformally invariant (bootstrap) form of the Schwinger-Dyson equations [50] [60] – see Sect. 2. Secondly, soon after

[2] P.A.M. Dirac (1902-1984) was fascinated by projective geometry ever since his student years in Bristol. He was probably the only one who praised the dismissed Weyl theory 55 years later, [21], saying that it "remains as the outstanding one, unrivalled by its simplicity and beauty".

[3] Ebenezer Cunningham (1881-1977) outlived his friend Harry Bateman (1882-1946) by over 30 years. A keenly religious man and a music lover, he did not return to major research projects after World War I.

[4] See Lord Rayleigh, The principle of similitude, *Nature* **95**:2368 (March 1915) 66-68 and 644. John William Strutt – Lord Rayleigh (1842-1919) was awarded the 1904 Nobel Prize for his discovery of the inert gas argon.

Wilson [86] introduced the notion of (small distance) *operator product expansion* (OPE) two Italian groups, [12] and [28], proposed a way to write exact (global) conformal invariant OPE. This was further developed leading to the notion of *conformal partial wave expansion* and applied to the solution of the dynamical (Schwinger-Dyson) equations, [45] [22] [23] as well as (later) to constructing conformal composite operators in $(\varphi^3)_6$ and in QCD, [16]. (The work of the 1970's is reviewed in [83] and in [30], for (alternative) subsequent developments – see [31].)

All the above developments use local causal automorphisms (see Sect. 1.1) (what is called weak conformal invariance in [38]); it implies global Euclidean conformal invariance (and allows to construct a conformal QFT on the universal cover $\tilde{M} = \mathbb{S}^3 \times \mathbb{R}$, of compactified Minkowski space, [44]). The notion of *global conformal invariance* (GCI) in Minkowski space [57], on the other hand, is much stronger. It yields a higher dimensional counterpart of a chiral *vertex algebra* (which became, in the 1-dimensional context, a rich mathematical theory starting with [13] and the subject of textbooks and monographs – see *e.g.* [40] [32] as well as the recent expository paper [18]). The work of the last five years, which explores GCI [57] [55] [56] [58] [52] [59] [53], and the possibility to construct a conformal QFT model on this basis, is reviewed in Sect. 3.

Let us stress from the outset that even the weaker assumption of local conformal invariance excludes the postulate of asymptotic completeness (except for the uninteresting case of free massless fields). Thus the usual criterion for a non-free model, the existence of a non-trivial scattering matrix, is not applicable to a conformal QFT. It seems likely that a GCI model will also share the local property of normal products of free fields discussed, say, in [14] and [15].

In order to make this survey relatively self-contained we collect some background material on the conformal Lie algebra and on the all important complex variable realization of compactified Minkowski space ([82] [59] [58]) in Appendices A and B, respectively.

To summarize: the paper reviews old and new efforts to construct non-trivial conformal QFT models. The organization of material can be read off the table of content.

2 Skeleton graph expansion for a critical Yukawa theory

2.1 Anomalous dimensions. Conformal 2- and 3-point functions. Positivity constraint

Feynman integrals produce (poly)logarithms, an observation that led to the popular catchphrase "no logs – no physics". Conformal QFT models induce us rethink this "perturbative wisdom". Indeed the leading logs in a 2-point function (or in the "wave function renormalization") sum up to a power precisely when the Callan-Symanzik β-function [79] vanishes (*i.e.* when the renormalized charge $\bar{g}(g,\mu)$ does not change under variation of the mass scale μ). This was discovered in the context of the 2D Thirring model [86] [43] and led to the introduction of anomalous dimensions[5] in QFT. It signals a change of representation, compared to the classical (or free quantum) field of canonical dimension.

Conformal invariance allows to approach the construction of a critical theory without appealing to standard perturbative expansions. The knowledge of the field dimensions allows to construct conformally invariant propagators and vertex functions ([63], [68]). These are particularly simple in a scalar field theory. The time-ordered propagator of a (hermitean) scalar conformal field ϕ of dimension d is given (for an appropriately chosen normalization) by

$$\langle 0 \mid T\phi(x_1)\,\phi(x_2) \mid 0\rangle = \frac{\Gamma(d)}{(4\pi)^2}\left(\frac{x_{12}^2}{4}+i0\right)^{-d} = -i\int \frac{\Gamma(2-d)}{(p^2-i0)^{2-d}}\, e^{ipx_{12}}\,\frac{d^4p}{(2\pi)^4}\,,$$

$$x_{12}=x_1-x_2\,,\quad x^2=\boldsymbol{x}^2-(x^0)^2\quad (\boldsymbol{x}^2=x_1^2+x_2^2+x_3^2)\,. \tag{2.1}$$

The general conformal 3-point Green's function of a triple of scalar fields $\phi_i(x)$ of dimensions d_i, $i=1,2,3$ is written as a product of three "infraparticle" propagators,

$$\langle 0 \mid T\phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3) \mid 0\rangle = C_{123}\,(x_{12}^2+i0)^{-\delta_3}\,(x_{13}^2+i0)^{-\delta_2}\,(x_{23}^2+i0)^{-\delta_1}\,, \tag{2.2}$$

where the exponents δ_i obey the conservation of dimension law

$$\delta_1+\delta_2=d_3\,,\quad \delta_1+\delta_3=d_2\,,\quad \delta_2+\delta_3=d_1\quad (\delta_3=\tfrac{1}{2}(d_1+d_2-d_3)\text{ etc.})\,. \tag{2.3}$$

The conformal invariant 2-point function of arbitrary irreducible spin-tensor fields is uniquely determined (and explicitely constructed – see [46] [82]). For reducible representations the uniqueness can be restored by demanding invariance under some discrete transformations. Thus the propagator of a conformal Dirac field ψ of dimension d' is determined (up to normalization) if

[5] According to W. Nahm [51], "In 1964 Wilson conjectured that perturbations just introduce some logarithmic corrections" to the canonical scaling, but then K. "Johnson familiarized him with the Thirring model" which led to [86]. The notion of anomalous dimension (if not the term) was encountered earlier in the critical Ising model, solved in 1944 by L. Onsager (1903-1976). It was given a field theoretic interpretation by L. Kadanoff [41].

we also demand invariance under space reflection: $\psi(x^0,\boldsymbol{x}) \to \gamma^0\psi(x^0,-\boldsymbol{x})$, $\tilde\psi(x^0,\boldsymbol{x}) \to \tilde\psi(x^0,-\boldsymbol{x})\,\gamma^{0*}$, $\tilde\psi$ standing for the Dirac conjugate field:

$$\langle 0 \mid T\psi(x_1)\,\tilde\psi(x_2) \mid 0\rangle = i\ \partial\!\!\!/_1 \ \frac{\Gamma\left(d'-\frac{1}{2}\right)}{(4\pi)^2\left(\frac{x_{12}^2}{4}+i0\right)^{d'-\frac{1}{2}}} \qquad \left(\partial\!\!\!/_1 = \gamma^\mu\,\frac{\partial}{\partial\,x_1^\mu}\right) . \quad (2.4)$$

For a pseudoscalar field ϕ the Yukawa type 3-point function in a space-reflection invariant conformal theory (for $\phi(x^0,\boldsymbol{x})$ going to $-\phi(x^0,-\boldsymbol{x})$ under space reflections) is given by (see Appendix B to [49])

$$\begin{aligned} & \langle 0 \mid T(\phi(x_1)\,\psi(x_2)\,\tilde\psi(x_3)) \mid 0\rangle \\ = \ & g\ x\!\!\!/_{12}\,\gamma_5\ x\!\!\!/_{13}\,(x_{23}^2+i0)^{\frac{d}{2}-d'}[(x_{12}^2+i0)(x_{13}^2+i0)]^{-\frac{d+1}{2}} . \end{aligned} \quad (2.5)$$

Remark 2.1. In general, the 3-point function of a tensor field is not unique (even after imposing natural discrete symmetries). For instance, the 3-point function of the stress energy tensor involves three independent structures corresponding to three free models: the massless scalar field, the massless Dirac (or the 2-component Weyl) field, the free Maxwell field – see [73].

Hilbert space positivity constraints the Wightman 2-point functions and hence the dimensions of all fields in the theory. We shall illustrate what is going on on the example of a scalar field and will then formulate the general result.

The x-space Wightman function, $w(x_{12})$, is a boundary value of an analytic function $w(x+iy)$ holomorphic in the backward *tube* $\mathcal{T}_-$ where

$$\mathcal{T}_\pm = \left\{ x+iy \in \mathbb{C}^4\,;\ \pm y^0 > |\boldsymbol{y}| = \sqrt{y_1^2+y_2^2+y_3^2} \right\} . \quad (2.6)$$

In particular, $w(x_{12}) = \langle 0 \mid \phi(x_1)\,\phi(x_2) \mid 0\rangle$ differs from (2.1) just by the "$i\varepsilon$-prescription" (its Fourier transform having support in the future cone):

$$w(x) = \frac{\Gamma(d)}{(4\pi)^2}\left(\frac{x^2}{4}+i0x^0\right)^{-d} = \frac{2\pi}{\Gamma(d-1)}\int \theta(p^0)(-p^2)_+^{d-2}\,e^{ipx}\,\frac{d^4p}{(2\pi)^4} \quad (2.7)$$

where

$$(a)_+ = \begin{cases} a & \text{for } a>0 \\ 0 & \text{for } a<0 \end{cases} , \quad \lim_{d\to 1}\frac{(-p^2)_+^{d-2}}{\Gamma(d-1)} = \delta(p^2)\,. \quad (2.8)$$

Wightman positivity means that the Fourier transform $\tilde w(p)$ of w should be a positive measure. This is satisfied if the integral (2.7) converges at the lower limit (for $p_\mu \to 0$), that is, for d not smaller than its canonical value, $d \geq 1$.

In general, for (irreducible) conformal fields, 2-point positivity amounts to unitarity of the corresponding representations of the universal cover $\tilde C$ of the conformal group. The positive energy irreducible representations of $\tilde C$ have beeen classified by Mack [46]. They are induced by irreducible finite-dimensional representations of the (covering of the) maximal compact subgroup $S(U(2)\times U(2))$ of $G = \mathrm{SU}(2,2)$, labeled by triples $(d;j_1,j_2)$ where $d \geq 0$ stands for the minimal eigenvalue of the conformal Hamiltonian H (A.9) (see Appendix A) $j_{1,2}$ ($\in \frac{1}{2}\mathbb{N}$) are the SU(2) spins, the dimension of the inducing representation being $(2j_1+1)(2j_2+1)$. For Minkowski space fields d coincides with the scale dimension, while (j_1, j_2) label the corresponding finite-dimensional representation of $SL(2,\mathbb{C})$ (the quantum mechanical Lorentz group). For a non-trivial representation ($(d; j_1, j_2) \neq (0; 0, 0)$) the unitarity (Wightman positivity) condition reads

$$d \geq 1 + j_1 + j_2 + \theta(j_1\, j_2)\,,\ \theta(a) = 1 \text{ for } a > 0\,,\ \theta(0) = 0\,. \tag{2.9}$$

The resulting representations are single-valued on $\mathcal{C}$ for integer *twist* $d - j_1 - j_2$. The fields corresponding to the boundary of the positivity domain (where the equality sign in (2.9) takes place) satisfy free field equations for $j_1\, j_2 = 0$ or conservation laws – for $j_1\, j_2 > 0$.

2.2 Vertex functions, skeleton graphs, bootstrap equations

In order to write down a *skeleton graph* expansion – with no self-energy and vertex function corrections – we need the "amputated" vertex function, obtained from the 3-point Green's function ((2.3) or (2.5)) by taking the convolution with the inverse external lines' propagators. The result can be reproduced (up to a proportionality constant) by substituting the conformal dimensions d_i by the corresponding "shadow[6] dimensions" $4 - d_i$, or equivalently, replacing the exponents δ_i in (2.3) by $2 - \delta_i$ (see Sect. IIB and Appendix B of [49]).

The inequalities (2.9) for the conformal dimensions of interacting fields suggest that the resulting dressed propagators have worse ultraviolet behaviour than the free ones. It is all the more remarkable that the contribution corresponding to an arbitrary skeleton graph in the above Yukawa model of meson-nucleon interaction is given by a convergent integral for non-exceptional momenta provided

$$1 < d < 3 \ (d \neq 2) \quad \frac{3}{2} < d' < \frac{5}{2}\,. \tag{2.10}$$

This is the central result (proven in Sect. III) of [49]. It shows, in particular, that the skeleton graphs with four pseudoscalar external meson lines are convergent, in spite of the fact that the corresponding canonical Feynman graphs are logarithmically divergent (the dressed conformal vertex functions providing effectively an ultraviolet cutoff, which compensates for the slow decrease of propagators).

The Schwinger-Dyson equations are implemented by Migdal's bootstrap equation [50] for the vertex function and by the generalized unitarity relation of Parisi and Peliti [64] for the propagator. In our derivation ([49] Sect. IV) we start with Symanzik's[7] form [77] of the integro-differential equations of the

[6] A terminology, introduced in [28] in the context of conformal OPE.

[7] Kurt Symanzik (1923-1983) has contributed more to this field (as well as to many others) than usually recognized: see (besides the above lectures) [78] [79] (where his β-function is introduced), [80], [48] (where generalized unitarity and Ward identities involving the stress energy tensor are considered).

theory. After differentiation with respect to an external momentum (which annihilates the constant $g\gamma_5$-term) the resulting renormalized Schwinger-Dyson equation for the vertex function is the same for all values of coupling constant (and masses) and is therefore valid in the renormalization group fixed point (also called Gell-Man-Low limit theory). In the presence of anomalous dimensions the requirement of dilation invariance fixes the integration constant to zero and one recovers Migdal bootstrap equation.

We thus end up with three equations – one for the vertex function and one for each of the two (meson and nucleon) propagators – for the three dimensionless constants, g, d and d' in the theory. One can, therefore, hope that the resulting conformal theory will involve no free parameters (or that there will be a discrete set of solutions) – a hope that cannot be verified as long as we are unable to solve the system of (highly non-linear) integral equations.

2.3 Conformal partial wave expansion. Discussion

Ordinary partial wave expansion is nothing but the tensor product expansion of two irreducible (Wigner) particle representations of the Poincaré group. *Conformal partial wave expansion* is, by definition, the tensor product expansion of two irreducible positive energy representations of $\tilde{\mathcal{C}}$. For the Wightman function of a pair of conjugate scalar fields ϕ and ϕ^* of dimension d and $n-2$ arbitrary local fields ϕ_i this expansion assumes the form

$$\begin{aligned} & \langle 0 \mid \phi^*(x_1)\,\phi(x_2)\,\phi_3(x_3)\ldots\phi_n(x_n) \mid 0\rangle \\ = \; & \langle 0 \mid \phi^*(x_1)\,\phi(x_2) \mid 0\rangle\langle 0 \mid \phi_3(x_3)\ldots\phi_n(x_n) \mid 0\rangle \\ + \; & \sum_{\kappa\geq 1}\sum_{\ell=0}^{\infty}\langle 0 \mid \phi^*(x_1)\,\phi(x_2)\prod_{\kappa\ell}\phi(x_3)\ldots\phi(x_n) \mid 0\rangle \end{aligned} \qquad (2.11)$$

where $\prod_{\kappa\ell}$ is the orthogonal projection operator on the subspace of symmetric traceless tensors of rank ℓ and twist 2κ (*i.e.* of dimension $2\kappa+\ell$). For a free massless field φ the twists are even integers so that κ takes positive integer values. We assume that the "compact Hamiltonian" H (A.9) in a conformal QFT has a discrete spectrum so that (2.11) involves no integral in κ. (This is derived as a Corollary in [47] of an appropriate assumption about the existence of OPE.)

The applicability of the expansion (2.11) is enhanced by the fact that one can write rather explicitly the projected 2-particle state:

$$\langle 0 \mid \phi^*(x_1)\,\phi(x_2)\prod_{\kappa\ell} = (x_{12}^2)^{\kappa-d}\,K_{\kappa\ell}(x_{12}\,\partial, x_{12}^2\,\Box)\,O_{2\kappa,\ell}(x;x_{12})\,|_{x=x_2} \qquad (2.12)$$

where $O_{2\kappa,\ell}(x;\zeta)$ is a rank ℓ symmetric traceless tensor field contracted with ζ:

$$O_{2\kappa,\ell}(x;\zeta) = O_{2\kappa}^{\mu_1\ldots\mu_\ell}(x)\,\zeta_{\mu_1}\ldots\zeta_{\mu_\ell}\,,\; \Box_\zeta\,O_{2\kappa,\ell}(x;\zeta) = 0\,, \qquad (2.13)$$

and

$$K_{\kappa\ell}(t_1,t_2) = \sum_{n=0}^{\infty}\int_0^1 \frac{[\alpha(1-\alpha)]^{\ell+\kappa+n-1}\,e^{\alpha t_1}\,(-t_2)^n\,d\alpha}{4^n\,B(\ell+\kappa,\ell+\kappa)\,n!(2\ell+2\kappa-1)_n} \qquad (2.14)$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}\,, \quad B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\,,$$

the normalization being chosen in such a way that $K_{\kappa\ell}(0,0) = 1$.

It was Gerhard Mack [45] who realized that the infinite set of dynamical integro-differential equations for the conformal Euclidean Green's functions (incorporating generalized-off-shell-unitarity) can be solved by the Euclidean (Spin (5,1)-) partial wave expansion (*cf.* also [64]). For a scalar QFT model this program is implemented in [22] [23]. The Yukawa model is dealt with (at least formally) in [30].

The above discussion does not apply directly to gauge theories. One cannot expect correlation functions of gauge dependent fields to be conformal invariant. For instance, the general conformal invariant 2-point function of a gauge potential (a vector field A_μ of dimension 1) is purely longitudinal, proportional to $\frac{\partial}{\partial x_1^\mu} \frac{x_{12\nu}}{x_{12}^2 + i0\, x_{12}^0}$, thus giving rise to a zero field strength. Furthermore, the general conformal and space-reflection invariant 2-point function of the Maxwell field,

$$\begin{aligned} & \langle 0 \mid F_{\lambda,\mu_1}(x_1)\, F_{\lambda_2\mu_2}(x_2) \mid 0\rangle \\ = \; & \{\partial_{\lambda_1}(\partial_{\lambda_2}\,\eta_{\mu_1\mu_2} - \partial_{\mu_2}\,\eta_{\mu_1\lambda_2}) \\ - \; & \partial_{\mu_1}(\partial_{\lambda_2}\,\eta_{\lambda_1\mu_2} - \partial_{\mu_2}\,\eta_{\lambda_1\lambda_2})\}\, \frac{1}{4\pi^2(x_{12}^2 + i0\, x_{12}^0)} \end{aligned} \tag{2.15}$$

satisfies the free field equations implying that the electromagnetic current vanishes (*cf.* [1] [7]; for – so far inconclusive – attempts to use indecomposable representations of $\mathcal{C}$ in conformal quantum electrodynamics – see [62] [74]). One can hope, however, to find conformal correlation functions of gauge invariant composite operators, a hope that appears to be fulfilled in the $N = 4$ super Yang-Mills theory (see, *e.g.* [19] for a review, and [3] [34] [37] [66] for a sample of more recent publications).

Composite conserved currents and the stress-energy tensor appear in the OPE of mutually conjugate basic fields (like in (2.12) for $\kappa = 1$). Their properties are studied in [48] [73] [61] [2] among others. Such (observable) integer dimension fields are not appropriate substitutes for basic fields appearing in a skeleton expansion since the convergence result (cited in Sect. 2.2) does not hold for integer d. By contrast, we shall argue in the next section that the GCI postulate applies precisely to such fields and thus provides an alternative tool for model building.

3 Four-dimensional conformal field theories with an infinite number of conserved tensor fields

3.1 From GCI in Minkowski space to higher dimensional vertex algebras

The notion of global conformal invariance (GCI) in Minkowski space (not to be confused with the weaker property of GCI on the infinite sheeted cover $\tilde{M}$ of $\bar{M}$ [44]) has been around for many years (and even appeared sporadically in the literature [35] [38]) but was not fully explored until recently. Realizing that a special conformal transformation (1.2) of a pair of points (x_1, x_2) with non-lightlike x_{12} may change the sign of the interval x_{12}^2 (according to (1.3)) makes one skeptical on the applicability of the stronger notion of GCI to a causal field theory. The story of our first paper [57] on the subject has been a story of how my young collaborator Nikolay Nikolov, who took GCI seriously, was overcoming my skepticism.

The first important consequence of GCI exploits precisely the counterintuitive fact that spacelike and timelike pairs of points can be exchanged by special conformal transformations.

Proposition 3.1. (a) *The conformal group acts transitively on compactified Minkowski space* $\bar{M}$*. The stability subgroup of a point is conjugate to the Poincaré subgroup* $\mathcal{P}$ *with dilations, which can be identified with the* 11*-parameter automorphism group* $\mathrm{Aut}\,\mathcal{P}$ *of* $\mathcal{P}$*, so that* $\bar{M}$ *can be viewed as the coset space*

$$\bar{M} = \mathcal{C}/\mathrm{Aut}\,\mathcal{P}\,. \tag{3.1}$$

(b) *Any pair* $(p_0 = \{\lambda\,\vec{\xi}_0\},\ p_1 = \{\lambda\,\vec{\xi}_1\})$ *of non-isotropic points of* $\bar{M}$ $(\vec{\xi}_0^{\,2} = 0 = \vec{\xi}_1^{\,2}$ *– see* (1.5); $\vec{\xi}_0 \cdot \vec{\xi}_1 \neq 0)$ *can be mapped onto any other such pair by a conformal transformation.*

A complete *proof* of this Proposition is given in the PhD thesis of Nikolov. For a proof of (b) assuming (a) – see [57] Proposition 1.1.

Remark 3.1. The tip $p_\infty = \{\lambda\,\vec{\xi}_\infty\}$ of the cone at infinity ($\vec{\xi}_\infty = (-1, 0, 0, 0, 0, 1)$ for the embedding (B.9) of M in $\bar{M}$ – see Appendix B) is left invariant precisely by the Poincaré transformations and dilations of $\mathrm{Aut}\,\mathcal{P}$. The stability subgroup of the origin $x = 0$ in Minkowski space M (corresponding to $p_0 = \lambda(1, 0, 0, 0, 0, 1)$ in $\bar{M}$) is the 11-parameter subgroup spanned by Lorentz transformations, dilations and the special conformal transformations (1.2) which are conjugate to translations by the Weyl inversion w of Appendix A.

Local commutativity combined with the fact that timelike and spacelike pairs of points can be transformed into each other (Proposition 3.1(b)) implies the *Huygens principle:* local GCI fields commute for non-light-like separations. For a field ψ of conformal weight $(d; j_1, j_2)$ (see Sect. 2.1) the Huygens principle within the Wightman framework [75] assumes the form of a *strong locality* *condition:*

$$(z_{12}^2)^N \, \{\psi(z_1)\, \psi^*(z_2) - (-1)^{2j_1+2j_2}\, \psi^*(z_2)\, \psi(z_1)\} = 0\,, \text{ for } N \geq d + j_1 + j_2\,. \quad (3.2)$$

Here $z_{1,2}$ refer to the global complex variable parametrization of compactified Minkowski space $\bar{M}$ reviewed in Appendix B. In fact, as the z's are related to the real Minkowski space coordinates x by the complex conformal transformation, (B.1), so that

$$z_{12}^2 := \sum_{\alpha=1}^{4} (z_1^\alpha - z_2^\alpha)^2 = \frac{x_{12}^2}{\omega(x_1)\,\omega(x_2)}\,,\ x_{12}^2 = (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 - (x_1^0 - x_2^0)^2 \quad (3.3)$$

where the conformal factor $\omega(x) = \frac{1}{2}\,(x^2 + 1) - i x^0$ does not vanish for real x, an equation identical to (3.2) would be also valid for the fields $\psi(x)$ in M. What does make the z-picture special is the fact that each field can be represented by a formal power series of the type

$$\psi(z) = \sum_{n \in \mathbb{Z}} \sum_{m \geq 0} (z^2)^n \, \psi_{\{n,m\}}(z)\,, \quad (3.4)$$

where $\psi_{\{n,m\}}(z)$ are (in general, multicomponent) operator valued homogeneous (of degree m) harmonic polynomials. (The uniqueness of such a decomposition and its relation to the representation theory of the rotation group are displayed in [8].) Strong locality for such series can be interpreted as an algebraic relation.

The expansion (3.4) diagonalizes the conformal Hamiltonian H (A.9):

$$[H, \psi(z)] = \left(z \cdot \frac{\partial}{\partial z} + d \right) \psi(z) \Rightarrow [H, \psi_{\{n,m\}}(z)] = (2n + m + d)\, \psi_{\{n,m\}}(z)\,. \quad (3.5)$$

The relativistic spectral conditions (including energy positivity), which ensure stability of the vacuum, imply the analyticity of the vector valued function $\psi(z) \mid 0\rangle$ in the compact picture *tube domain* T_+ (B.5). It follows that

$$\psi_{\{n,m\}}(z) \mid 0\rangle = 0 \text{ for } n < 0\,. \quad (3.6)$$

The invariance of the vacuum vector under (complex) z-translations T_α,

$$[T_\alpha, \psi(z)] = \frac{\partial}{\partial z^\alpha}\, \psi(z)\,,\ T_\alpha \mid 0\rangle = 0\,, \quad (3.7)$$

together with (3.6) allows to establish the *state field correspondence:* there is a one-to-one correspondence between finite energy states v and translation covariant strongly local fields ("vertex operators") $Y(v, z)$ such that

$$Y(v, z) \mid 0\rangle = e^{T \cdot z}\, v \quad (v = Y(v, 0) \mid 0\rangle)\,. \quad (3.8)$$

The properties (3.2) (3.4) (3.6) (3.7) allow a straightforward derivation of the central result of [57] (Theorem 3.1).

Proposition 3.2. *For any set $\phi_1, \dots, \phi_n$ of strongly local GCI fields the formal power series*

$$F_{1\dots n}(z_1, \dots, z_n) = \prod_{1 \le i < j \le n} (z_{ij}^2)^{n_{ij}} \langle 0 \mid \phi_1(z_1) \dots \phi_n(z_n) \mid 0 \rangle \,, \ n_{ij} \ge \Delta_{ij} \quad (3.9)$$

is a (translation invariant, homogeneous) polynomial in $z_1, \dots, z_n$, Δ_{ij} being natural numbers only depending on the conformal weights of the fields $\phi_i(z)$ and $\phi_j(z)$.

The fact that the Huygens principle together with the Wightman axioms yields rationality of correlation functions has been noted earlier by Baumann [10]. A more precise statement and a proof of Proposition 3.2 in the vertex algebra formalism is given in Theorem 3.3 of [59] (see also [52]).

The mathematical properties of the above defined higher dimensional vertex algebras are investigated in [52] and [5]. Thermal elliptic correlation functions of GCI fields and associated modular forms are introduced and studied in [59].

3.2 Model building: correlation functions and the twist-two bilocal field

As reviewed in Sect. 2.1, 2- and 3-point functions are essentially uniquely determined by conformal invariance. By contrast, the 4-point function may depend, in general, arbitrarily on two conformally invariant cross-ratios

$$s = \frac{z_{12}^2 \, z_{34}^2}{z_{13}^2 \, z_{24}^2} \,, \quad t = \frac{z_{14}^2 \, z_{23}^2}{z_{13}^2 \, z_{24}^2} \,. \quad (3.10)$$

(In a D-dimensional space-time there are $\frac{n(n-3)}{2}$ independent n-point cross-ratios for $4 \le n \le D+2$; for all $n \ge \mathcal{D}+1$ the number of the algebraically independent ones is $n\,D - \binom{D+2}{2}$.) Due to the rationality of correlation functions (Proposition 3.2), however, including the restrictions on the degree of singularity coming from Wightman positivity (the precise values of Δ_{ij} – see Proposition 4.3 of [57]) all correlation functions are determined within a finite number of free parameters.

We shall illustrate this fact by writing down the 4-point functions of a scalar GCI field of integral dimension d, which involves $\left[\frac{d^2}{3}\right]$ ($= 2d-3$ for $d = 2, 3, 4$) independent parameters. In proving this (for $D = 4$ – see [55]) one uses the fact that, as a consequence of Wightman positivity, the truncated n-point functions ($n \ge 3$) of a scalar field have strictly weaker singularities than its 2-point function. One finds, in particular, that the truncated 4-point function of a $d = 1$ scalar field vanishes identically (without having to use the free field equation and the Reeh-Schlieder theorem [75]).

We are looking for an *irreducible model* of a scalar field, *i.e.* a model whose field algebra cannot be represented as the tensor product of two (commuting) local algebras. In particular, we assume that there is a unique stress-energy tensor – *i.e.*, a unique twist-two, rank-two (necessarily conserved) symmetric traceless tensor $T(z;v) = T_{\alpha\beta}(z)\, v^\alpha\, v^\beta$ (*cf.* (2.13)) that generates its own translations. We then define the *central charge* c by the normalization of its 2-point function ([53]; *cf.* also [2])

$$\langle 0 \mid T(z_1;v_1)\, T(z_2;v_2) \mid 0 \rangle = \frac{2\,c}{3(z_{12}^2)^4}\, \left[4\,(v_1\, r(z_{12})\, v_2)^2 - v_1^2\, v_2^2\right] \tag{3.11}$$

where

$$v_1\, r(z)\, v_2 = v_1 \cdot v_2 - 2\, \frac{(z\cdot v_1)(z \cdot v_2)}{z^2} \qquad (r^2(z) = \mathbb{1}) \tag{3.12}$$

and the normalization is chosen in such a way that $c = 1$ for the stress tensor of a free hermitean massless scalar field φ. With c so defined we fix the normalization of $\phi(z)$ setting

$$\langle 12 \rangle \equiv \langle 0 \mid \phi(z_1)\, \phi(z_2) \mid 0 \rangle = \frac{c}{d}\, (z_{12}^2)^{-d}\,. \tag{3.13}$$

Eqs. (3.11) (3.13) and the Ward-Takahashi identities for T then imply

$$\langle 0 \mid \phi(z_1)\, \phi(z_2)\, T(z_3, v) \mid 0 \rangle = \frac{2\,c\,(z_{12}^2)^{1-d}}{3\; z_{13}^2\, z_{23}^2} \left[4\,(X_{12}^3 \cdot v)^2 - \frac{z_{12}^2}{z_{13}^2\, z_{23}^2} \right], \tag{3.14}$$

where

$$X_{12}^3 := \frac{z_{13}}{z_{13}^2} - \frac{z_{23}}{z_{23}^2} \qquad \left(\text{so that } (X_{12}^3)^2 = \frac{z_{12}^2}{z_{13}^2\, z_{23}^2}\right). \tag{3.15}$$

We shall review the study of the simplest model of a scalar field ϕ of dimension $d = 2$ ([55] [6]) and will only mention in passing the most promising model with $d = 4$. The uniqueness of T implies that the scalar field of dimension 2 necessarily appearing in the OPE of two ϕ's is again ϕ, and moreover[8]

$$\langle 123 \rangle \equiv \langle 0 \mid \phi(z_1)\, \phi(z_2)\, \phi(z_3) \mid 0 \rangle = \frac{c}{z_{12}^2\, z_{23}^2\, z_{13}^2}\,,\ \ i.e.\ c = \frac{8\,\langle 12\rangle\,\langle 13\rangle\,\langle 23\rangle}{|\langle 123 \rangle|^2}\,. \tag{3.16}$$

We shall reproduce the truncated 4-point function of a hermitean scalar field of dimension d ($w_4^t(\cdot;d) = \langle 1234\rangle - \langle 12\rangle\,\langle 34\rangle - \langle 14\rangle\,\langle 23\rangle - \langle 13\rangle\,\langle 24\rangle$) for $d = 2,3,4$. For $d = 2$, w_4^t is again expressed in terms of the central charge c (as a consequence of the argument referred to above for a unique T):

$$\begin{aligned} w_4^t(\cdot;2) &= \frac{c}{z_{12}^2\, z_{23}^2\, z_{34}^2\, z_{14}^2}\,(1+s+t) \\ &= c\left(\frac{1}{z_{12}^2\, z_{34}^2\, z_{14}^2\, z_{23}^2} + \frac{1}{z_{13}^2\, z_{24}^2\, z_{14}^2\, z_{23}^2} + \frac{1}{z_{12}^2\, z_{34}^2\, z_{13}^2\, z_{24}^2} \right). \end{aligned} \tag{3.17}$$

For $d = 3$ the (rational) 3-point function vanishes while

$$w_4^t(\cdot;3) = \left(\prod_{1\le i<j\le 4} z_{ij}^2 \right)^{-1} c\,(b_0\,\tilde{J}_0(s,t) + b_1\,\tilde{J}_1(s,t) + b_2) \tag{3.18}$$

[8] See [55], Sect. 2; a more general statement - without the uniqueness assumption - is also valid [54].

where $\tilde{J}_\lambda(s,t)$ are the crossing symmetric rational functions

$$\tilde{J}_0(s,t) = s + t + \frac{t+1}{s} + \frac{s+1}{t}\,,$$
$$\tilde{J}_1(s,t) = \frac{(1-t)(1-t^2)}{st} - t^{-1} - t - s\,(1+t^{-1}) + \frac{s^2}{t}\,. \qquad (3.19)$$

Finally, for $d=4$, we can write

$$w_4^t(\cdot;4) = \frac{c}{(z_{12}^2\, z_{23}^2\, z_{34}^2\, z_{14}^2)^2} \left\{ \sum_{\nu=0}^{2} a_\nu \; \frac{J_\nu(s,t)}{st} + bD(s,t) + b'Q(s,t) \right\} \qquad (3.20)$$

where $J_\nu(s,t)$ are polynomials in s and t of overall degree 5,

$$\begin{aligned}
\frac{1}{st}\, J_0(s,t) &= \frac{t+t^2}{s} + s(t^{-1}+t^2) + s^2(t^{-1}+t) \\
\frac{1}{st}\, J_1(s,t) &= \frac{(1-t)(1-t^2)}{s} + t^{-1} - 2 - 2\,t^2 + t^3 - s\,(t^{-1}+t^2) \\
&- \frac{s^2}{t}\,(1+t)^2 + s^3(t^{-1}+1) \\
\frac{1}{st}\, J_2(s,t) &= \frac{(1-t)(1-t^2)}{s}\,(t^{-1}-1+t) - 2(t^{-1}+t^3) + 1 + t^2 + s(t^{-1}+t^2) \\
&+ s^2(t^{-1}+1+t) - 2\,s^3(t^{-1}+1) + \frac{s^4}{t}\,; \qquad (3.21)
\end{aligned}$$

D and Q are second degree polynomials:

$$D(s,t) = (1-t)^2 - 2s(1+t) + s^2\,, \quad Q(s,t) = s + t + st\,. \qquad (3.22)$$

These expressions are characterized by three conditions: 1) they are rational GCI functions; 2) their poles in z_{ij}^2 $(i<j)$ have strictly lower degrees than the poles of the corresponding 2-point function $\langle ij \rangle$; 3) they are crossing symmetric – *i.e.*, symmetric under any permutation of $(z_1, \ldots, z_4)$.

We shall make explicit the meaning of the third property. The permutation group $\mathcal{S}_4$ has a normal subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$, generated by the double transpositions $s_{12}\, s_{34}$ and $s_{14}\, s_{23}$ $(s_{12}\, s_{34}\, s_{14}\, s_{23} = s_{13}\, s_{24})$, leaving invariant the cross-ratios s and t. The factor group $\mathcal{S}_4/\mathbb{Z}_2 \times \mathbb{Z}_2$, isomorphic to $\mathcal{S}_3$, acts faithfully on the "amplidues" $\tilde{J}_\lambda$, J_ν, D, Q:

$$s_{12}\, \tilde{J}_\lambda(s,t) := \tilde{J}_\lambda\left(\frac{s}{t}, \frac{1}{t}\right) = \tilde{J}_\lambda(s,t)\,,$$
$$s_{23}\, \tilde{J}_\lambda(s,t) := \tilde{J}_\lambda\left(\frac{1}{s}, \frac{t}{s}\right) = \tilde{J}_\lambda(s,t)\,,\ \lambda = 0,1\,; \qquad (3.23)$$

if $f(s,t)$ is any of the five functions $\frac{1}{st}\, J_\nu(s,t)$, $D(s,t)$, $Q(s,t)$ then

$$(s_{12}\, f)(s,t) := t^2\, f\left(\frac{s}{t}, \frac{1}{t}\right) = f(s,t)\,,\ (s_{23}\, f)(s,t) := s^2\, f\left(\frac{1}{s}, \frac{t}{s}\right)\,. \qquad (3.24)$$

We shall analyse our models by writing down the OPE of two ϕ's in the form

$$(z_{12}^2)^{d-1}\,[\phi(z_1)\,\phi(z_2) - \langle 12\rangle] = V(z_1,z_2) + z_{12}^2\,\tilde V(z_1,z_2) \tag{3.25}$$

where $V(z_1,z_2) = V(z_2,z_1)$ is a *bilocal conformal scalar field* of dimension $(1,1)$ whose OPE only involves twist-two conserved symmetric traceless tensor currents $T_{2\ell}(z;v)$,

$$T_{2\ell}(z;v) = T_{2\ell}(z)_{\alpha_1\ldots\alpha_{2\ell}}\, v^{\alpha_1}\ldots v^{\alpha_{2\ell}}\,, \quad \frac{\partial}{\partial v}\cdot\frac{\partial}{\partial z}\; T_{2\ell}(z;v) = 0 = \frac{\partial}{\partial v}\cdot\frac{\partial}{\partial v}\; T(z,v)\,, \tag{3.26}$$

and every such current enters the OPE of V in local fields (so that the remainder, $\tilde V(z_1,z_2)$, is expanded in twist four and higher fields only). This definition of V has two immediate consequences: the orthogonality relations

$$\langle 0 \mid V(z_1,z_2)\,\tilde V(z_3,z_4) \mid 0\rangle = 0 = \langle 0 \mid V(z_1,z_2) \mid 0\rangle \tag{3.27}$$

hold (as the vacuum expectation value of the product of two fields transforming under two inequivalent irreducible representations of $\mathcal{C}$ vanishes) and V is harmonic in each argument,

$$\Delta_{z_1}\, V(z_1,z_2) := \frac{\partial}{\partial z_1}\cdot\frac{\partial}{\partial z_1}\; V(z_1,z_2) = 0 = \Delta_{z_2}\, V(z_1,z_2) \tag{3.28}$$

(as a consequence of the conservation law for all $T_{2\ell}$ (3.26) – see Proposition 2.1 of [56]).

The bilocal field $V(z_1,z_2)$ has thus the properties of a normal product of free fields and is, in fact, a simpler object than the original field ϕ. It is therefore important that for $d=2$, V can be expressed in terms of ϕ, a result announced in [55] (Proposition 2.3); two versions of the proof were sketched one in Sect. 2 of [55] and another in Appendix A.1 of [56].

Proposition 3.3. (a) *The truncated n-point function of ϕ ($n \ge 3$) is a sum of $\frac{1}{2}\,(n-1)!$ 1-loop graph contributions obtained from $c\,[z_{12}^2\, z_{23}^2 \ldots z_{n-1n}^2\, z_{1n}^2]^{-1}$ by permutations σ of $(2,\ldots,n)$ identifying $\sigma(2)\ldots\sigma(n)$ with $\sigma(n)\ldots\sigma(2)$.*
(b) *Given* (a) *for $3 \le n \le 6$ we can define $V(z_1,z_2)$ by*

$$V(z_1,z_2) = \lim_{\substack{z_{13}^2\to 0\\ z_{23}^2\to 0}} \{z_{13}^2\, z_{23}^2\,[\phi(z_1)\,\phi(z_2)\,\phi(z_3) - \langle 123\rangle - \langle 23\rangle\,\phi(z_1) - \langle 13\rangle\,\phi(z_2)]\} \tag{3.29}$$

proving that the limit is independent of z_3.

Sketch of proof. The statement (a) can be proven by induction, using the OPE (3.25) and the harmonicity of V (3.28). We shall indicate here, on the example of the 5-point function, a streamlined way to using the latter. Expanding the 5-point function $\langle 12345\rangle$ with respect to z_{45} and keeping only the singular terms we find

$$\langle 12345\rangle = \langle 123\rangle\,\langle 45\rangle + \langle 0 \mid \phi(z_1)\,\phi(z_2)\,\phi(z_3)\; \frac{V(z_4,z_5)}{z_{45}^2} \mid 0\rangle + 0(1)\,.$$

The space of GCI 5-point functions of weight $(2,2,2;1,1)$ with the properties of

$$F(z_1,z_2,z_3;z_4,z_5) = \langle 0 \mid \phi z_1)\,\phi(z_2)\,\phi(z_3)\,V(z_4,z_5) \mid 0\rangle \tag{3.30}$$

is spanned by graded monomials of the type

$$\prod_{1\le i<j\le 5} \rho_{ij}^{\mu_{ij}}\,,\ \ \rho_{ij} = z_{ij}^2\,,\ \ \mu_{ij}(\equiv \mu_{ji}) \in \mathbb{Z}\,,\ \sum_{j=1}^{5} \mu_{ij} = -\,d_i\ \ (\mu_{ii} = 0)$$

$$i = 1,\ldots,5\,,\quad d_1 = d_2 = d_3 = 2\,,\quad d_4 = d_5 = 1\,. \tag{3.31}$$

The harmonicity of V further implies

$$\begin{aligned}
&\frac{\partial}{\partial z_k}\cdot\frac{\partial}{\partial z_k}\,F(z_1,z_2,z_3;z_4,z_5)\\
=\ &\left\{ 8\sum_{j\neq k}\frac{\partial}{\partial\rho_{jk}} + 2\sum_{i,j(\neq k)}(\rho_{ik}+\rho_{jk}-\rho_{ij})\,\frac{\partial^2}{\partial\rho_{ik}\,\partial\rho_{jk}}\right\} F\\
=\ &-2\sum_{i,j(\neq k)}\rho_{ij}\,\frac{\partial^2 F}{\partial\rho_{ik}\,\partial\rho_{jk}} = 0 \quad \text{for} \quad k = 4,5\,.
\end{aligned} \tag{3.32}$$

(In the second equation we have used the homogeneity condition (3.31).) We find as a corollary that a monomial of type (3.31) cannot appear if there is a pair (i,j), $i\neq j$ such that $\mu_{ik}<0$ and $\mu_{jk}<0$ for $k=4$ or $k=5$. Taking further the growth and the symmetry conditions into account we find the following general form of F:

$$F(z_1,\ldots,z_5) = c\left(\frac{\langle 12\rangle}{\rho_{34}\,\rho_{35}} + \frac{\langle 13\rangle}{\rho_{24}\,\rho_{25}} + \frac{\langle 23\rangle}{\rho_{14}\,\rho_{15}}\right) + \langle 123\rangle \sum_{i,j=1}^{3}\frac{\rho_{ij}}{\rho_{i4}\,\rho_{j5}}\,. \tag{3.33}$$

The statement (b) is established by a direct computation noting that to evaluate $\langle 0 \mid V(z_1,z_2)\,V(z_3,z_4) \mid 0\rangle$ from (3.29) one only needs the n-point function of ϕ for $n = 2,3,4,6$.

Note that Proposition 3.3 is not valid for 1-dimensional $z_{1,2}$.

3.3 Four-point functions and partial wave expansions of twist-two bilocal fields

In order to study systematically the properties of harmonic bilocal fields we begin with the obvious generalization of the OPE (3.25) to the case of a pair of hermitean conjugate complex scalar fields $\psi(z)$ and $\psi^*(z)$ of (integer) dimension d:

$$(z_{12}^2)^{d-1}[\psi^*(z_1)\,\psi(z_2) - \langle 12\rangle] = W(z_1,z_2) + z_{12}^2\,\tilde W(z_1,z_2) \tag{3.34}$$

where $W(z_1,z_2)$ is again harmonic (of dimension $(1,1)$) but no longer symmetric, satisfying instead

$$W^*(z_1,z_2) = W(z_2,z_1)\,. \tag{3.35}$$

The general 4-point function of W can be written in the form

$$\langle 0 \mid W(z_1, z_2)\, W(z_3, z_4) \mid 0 \rangle = \frac{1}{z_{13}^2\, z_{24}^2} \sum_{\nu=0}^{d-2} a_\nu\, f_\nu(s,t) \tag{3.36}$$

where the *amplitudes* $f_\nu(s,t)$ are solutions of the conformal Laplace equation

$$\left\{ s\, \frac{\partial^2}{\partial s^2} + t\, \frac{\partial^2}{\partial t^2} + (s+t-1)\, \frac{\partial^2}{\partial s\, \partial t} + 2 \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t} \right) \right\} f_\nu(s,t) = 0 \tag{3.37}$$

such that $f_\nu(0,t) = \frac{(1-t)^\nu}{t^{\nu+1}}$. (The upper limit in the sum (3.36) comes from the restriction on the degree of the pole in $z_{14}^2\, z_{23}^2$, allowed by Wightman positivity.) The solution of (3.37) assumes a simple form in the higher dimensional generalization [24] of the *chiral conformal variables*, $u, \bar{u}$ related to s and t by:

$$s = u\, \bar{u}\,,\ t = (1-u)(1-\bar{u})\,;\ u + \bar{u} = 1 + s - t\,,\ (u-\bar{u})^2 = D(s,t) \tag{3.38}$$

where the discriminant $D(s,t)$ is given by (3.22). Regarded as a function of u and $\bar{u}$, $f_\nu(s,t)$ is given by

$$(u-\bar{u})\, f_\nu(s,t) = \frac{u^{\nu+1}}{(1-u)^{\nu+1}} - \frac{\bar{u}^{\nu+1}}{(1-\bar{u})^{\nu+1}} \qquad \nu = 0, 1, 2, \dots\,. \tag{3.39}$$

(The variables u and $\bar{u}$ are complex conjugate to each other for real euclidean z_i (for which $s \geq 0$, $t \geq 0$, $D(s,t) \leq 0$).)

Proposition 3.4. *The amplitudes f_ν admit an expansion in twist-two conformal partial waves*

$$\beta_L(u, \bar{u}) = \frac{G_{L+1}(u) - G_{L+1}(\bar{u})}{u - \bar{u}}\,,\ G_\nu(u) = u^\nu\, F(\nu, \nu; 2\nu; u) \tag{3.40}$$

computed from the following identities for hypergeometric functions:

$$\frac{u^{\nu+1}}{(1-u)^{\nu+1}} = \sum_{n=0}^{\infty} \frac{(n+2\nu)!}{n!\, (\nu!)^2}\, \frac{G_{n+\nu+1}(u)}{\binom{2n+2\nu}{n+\nu}}\,. \tag{3.41}$$

The lowest angular momentum contributing to the expansion of f_ν is $L = \nu$.

The *proof* is given in Appendix B to [53].

Remark 3.2. The solution f_ν can be recovered if we identify $W_\nu(z_1, z_2)$ by a product of hermitean conjugate free massless fields of spin $\nu/2$; in particular,

$$W_0(z_1, z_2) =: \varphi^*(z_1)\, \varphi(z_2) :\,,\ W_1(z_1, z_2) =: \psi^*(z_1)\, \not{z}_{12}\, \psi(z_2) :\,,$$

$$W_2(z_1, z_2) =: F^*_{\dot{A}_1 \dot{B}_1}(z_1)\, \not{z}_{12}^{\dot{A}_1 A_2}\, \not{z}_{12}^{\dot{B}_1 B_2}\, F_{A_2 B_2}(z_2) :\,, \tag{3.42}$$

where $\psi(z) = \{\psi_A(z),\, A = 1,2\}$ is a 2-component Weyl spinor, $\not{z}$ is the 2×2 (quaternionic) matrix (B.14) – see Appendix B, $F_{AB}(z)$ $(= F_{BA}(z))$ is a complex spin 1 field. Bilinear combinations of higher rank free spin-tensor fields do not contain a stress-energy tensor in their OPE. Therefore, we shall not be interested in W_ν (and V_ν) for $\nu > 2$.

The sums $V_\nu(z_1, z_2) = W_\nu(z_1, z_2) + W_\nu(z_2, z_1)$, $\nu = 0, 1, \ldots$, represent special cases of symmetric bilocal fields with 4-point functions obtained from (3.36) (3.39) by symmetrization in z_1, z_2:

$$\langle 0 \mid V_\nu(z_1, z_2)\, V_\nu(z_3, z_4) \mid 0 \rangle = \frac{j_\nu(s,t)}{z_{13}^2\, z_{24}^2}\,,\ j_\nu(s,t) = f_\nu(s,t) + \frac{1}{2}\, f_\nu\left(\frac{s}{t}, \frac{1}{t}\right), \tag{3.43}$$

which gives

$$(u - \bar{u})\, j_\nu(s,t) = \left(\frac{u}{1-u}\right)^{\nu+1} - (-u)^{\nu+1} - \{u \to \bar{u}\}\,. \tag{3.44}$$

We have, in particular,

$$\begin{aligned} & j_0(s,t) = 1 + t^{-1}\,,\ j_1(s,t) = (1-t)(t^{-2} - 1) - s(t^{-2} + 1)\,, \\ & j_2(s,t) = (1 + t^{-3})\,[(1-t)^2 + s^2] - \frac{s}{t^3}\,(2 - t - t^3 + 2\,t^4)\,. \end{aligned} \tag{3.45}$$

The three amplitudes (3.45) appear in the general 4-point function of a hermitean scalar field $\mathcal{L}(z)$ of dimension $d = 4 (= D)$ which has the properties of a (gauge invariant) Lagrangian density and provides the most attractive possibility for a physically interesting model [56]. The functions $\frac{1}{st}\, J_\nu(s,t)$ (3.21) appear as symmetrizations of $\frac{t^2}{s}\, j_\nu(s,t)$:

$$\frac{1}{st}\, J_\nu(s,t) = \lambda_\nu (1 + s_{23} + s_{13}) \left(\frac{t^2}{s}\, j_\nu(s,t)\right) \quad \lambda_0 = \lambda_1 = 1\,,\ \lambda_2 = \frac{1}{2} \tag{3.46}$$

where s_{23} is defined in (3.24) and s_{13} $(= s_{12}\, s_{23}\, s_{12})$ exchanges s and t; the values of λ_ν in (3.46) are chosen in such a way that $\frac{t^2}{s}\, j_\nu$ provides the leading term in the small s expansion of $\frac{1}{st}\, J_\nu$.

Remark 3.3. In general, the contribution of all tensor fields of a given twist to the OPE of two scalar fields is *not* a (strongly) bilocal field: its correlation functions are not rational (they involve hypergeometric functions of type (3.40) which can be expressed in terms of logs - see [56]). It is all the more remarkable that the twist two contribution $W(z_1, z_2)$ or $V(z_1, z_2)$ is a bilocal field - as a consequence of the wave equation (3.28) - with rational correlation functions as displayed above. We shall use this fact in our analysis of the $d = 2$ case below.

3.4 The $d = 2$ case; the $\mathrm{sp}(\infty, \mathbb{R})$-algebra and its positive energy representations

The theory of a (unique) hermitean scalar field ϕ of dimension two (normalized to satisfy (3.16) (3.17)) is reduced, in view of Proposition 3.3, to the study of a biharmonic bilocal field $V(z_1, z_2)$ such that

$$\phi(z_1)\,\phi(z_2) - \frac{c}{2(z_{12}^2)^2} = \frac{1}{z_{12}^2}\,V(z_1,z_2) + :\phi(z_1)\,\phi(z_2):\,, \quad V(z,z) = 2\,\phi(z)\,, \tag{3.47}$$

and whose $2n$-point function can be read off the sum of 1-loop graphs giving the truncated Wightman functions of ϕ (Proposition 3.3 (a)).

We shall see (following [55]) that V generates the infinite dimensional Lie algebra $\mathrm{sp}(\infty,\mathbb{R})$ and that its general vacuum representation corresponds to positive integral central charge $c = N$ for which V has the properties of a sum of normal products of N free massless scalar fields $\varphi_i(z)$:

$$V(z_1,z_2) = \sum_{i=1}^{N} :\varphi_i(z_1)\,\varphi_i(z_2):\,, \quad [\varphi_i(z_1), \varphi_j(z_2)] = 0 \text{ for } i \neq j\,. \tag{3.48}$$

We shall also preview the results of [6] concerning general positive energy representations of the vacuum "algebra of observables" [36] generated by local fields.

As the commutation relations of V are representation independent (except for the value of the central charge) they can be deduced, in view of (3.48), from those of the free massless field φ. The power series expansion (3.4) for φ coincides with its decomposition into irreducible representations of type $(n; j, j)$, $2\,j + 1 = |n|$, of the maximal compact subgroup $S(U(2) \times U(2))$ of $\mathcal{C}$:

$$\varphi(z) = \sum_{n\in\mathbb{Z}^\times} \varphi_n(z)\,, \quad \varphi_n(\lambda z) = \lambda^{-n-1}\varphi_n(z) \tag{3.49}$$

where φ_n are homogeneous harmonic functions of z:

$$\varphi_{-n}(z) = \sum_{\sigma=1}^{n^2} \varphi_{-n\sigma}\, h_{n\sigma}(z)\,, n \in \mathbb{Z}^\times\,, h_{-n,\sigma}(z) = \frac{h_{n\sigma(z)}}{(z^2)^n} \;\; for \;\; n > 0; \tag{3.50}$$

$\{h_{n\sigma}(z), \sigma = 1, \ldots, n^2\}$ is an orthonormal basis of homogeneous of degree $n-1$ harmonic polynomials in z_α, (with real coefficients) and $\varphi_{\pm n\sigma}$ can be expressed in terms of the *residue functional* of [5] and are hermitean conjugate to each other:

$$\varphi_{n\sigma} = Res_z(\frac{\varphi(z)}{z^2} h_{n\sigma}(z)),\;\; \varphi_{-n\sigma} = \varphi_{n\sigma}^*,\;\; n \in \mathbb{Z}^\times \tag{3.51}$$

($\mathbb{Z}^\times$ is the set of non-zero integers). Here $Res_z f(z)$ is the unique translation invariant functional normalized by $Res_z(z^2)^{-2} = 1$ so that

$$2\pi^3 i Res_z f(z) = \int_{\bar{M}} f(z) d^4 z \tag{3.52}$$

Let $u_a = (u_{a\alpha})$ $a = 1, 2$ be real euclidean unit 4-vectors:

$$u_a \in \mathbb{S}^3\,, \quad i.e.\; u_a^2 = \boldsymbol{u}_a^2 + u_{a4}^2 = 1\,,\; a = 1,2\,,\; u_1 \cdot u_2 = \cos\alpha\,. \tag{3.53}$$

Then the canonical (Heisenberg) commutation relations for $\varphi_n(u)$ and $\varphi_{n\sigma}$ assume the form:

$$[\varphi_n(u_1), \varphi_{-m}(u_2)] = \delta_{nm} \frac{\sin n\alpha}{\sin \alpha}\,, \;\; n[\varphi_{n\sigma}, \varphi^*_{m\tau}] = \delta_{nm}\, \delta_{\sigma\tau}\,. \tag{3.54}$$

The sum in σ of $h_{n\sigma}(z_1)h_{n\sigma}(z_2)$ is independent of the choice of basis:

$$\sum_{\sigma=1}^{n^2} h_{n\sigma}(z_1)h_{n\sigma}(z_2) = nH_n(z_1, z_2) \tag{3.55}$$

where H_n is the unique $O(4)$-invariant biharmonic bihomgeneopus polynomial of degree $(n-1, n-1)$ normalized by $H_n(u,u) = n$ for $u \in \mathbb{S}^3$. According to (3.54) its restriction to the product of unit 3-spheres is expressed in terms of the hyperspherical (Gegenbauer) polynomial:

$$H_n(u_1, u_2) = C^1_{n-1}(cos\alpha) = \frac{\sin n\alpha}{\sin \alpha}\; (u_1.u_2 = cos\alpha) \tag{3.56}$$

($H_1 = 1$, $H_2 = 2z_1.z_2$, $H_3 = 4(z_1.z_2)^2 - z_1^2 z_2^2$, ...). The free 2-point function of φ provides the generating function for H_n:

$$(z^2)^{-1}\langle 0 \mid \varphi(\frac{z}{z^2})\varphi(w) \mid 0\rangle = (1 - 2z.w + z^2w^2)^{-1} = \sum_{n=1}^{\infty} H_n(z,w) \tag{3.57}$$

Proceeding to the bilocal field V, we shall write its mode expansion in the form

$$\begin{aligned} V(z_1, z_2) = \sum_{n_1,n_2>0} \{ & (z_1^2)^{-n_1}(z_2^2)^{-n_2} X_{n_1\sigma_1,n_2\sigma_2} + X^*_{n_1\sigma_1,n_2\sigma_2} + \\ & (z_2^2)^{-n_2} N_{n_1\sigma_1,n_2\sigma_2} + (z_1^2)^{-n_1} N_{n_2\sigma_2,n_1\sigma_1}\} \,\frac{h_{n_1\sigma_1}(z_1)\, h_{n_2\sigma_2}(z_2)}{\sqrt{n_1 n_2}}. \end{aligned} \tag{3.58}$$

The operator valued coefficients $X^{(*)}_{\chi_1\chi_2}$ and $N_{\chi_1\chi_2}$ where χ_i stands for the pair (n_i, σ_i) span the Lie algebra $\mathrm{sp}(\infty, \mathbb{R})$ characterized by the commutation relations

$$[X_{\chi_1\chi_2}, X_{\chi_3\chi_4}] = 0 = [X^*_{\chi_1\chi_2}, X^*_{\chi_3\chi_4}]\,, \tag{3.59}$$

$$\begin{aligned} [X_{\chi_1\chi_2}, X^*_{\chi_3\chi_4}] \;&=\; \delta_{\chi_2\chi_3}\, E_{\chi_4\chi_1} + \delta_{\chi_2\chi_4}\, E_{\chi_3\chi_1} + \delta_{\chi_1\chi_3}\, E_{\chi_4\chi_2} \\ &+\; \delta_{\chi_1\chi_4}\, E_{\chi_3\chi_2}\,,\; E_{\chi_1\chi_2} = N_{\chi_1\chi_2} + \frac{c}{2}\,\delta_{\chi_1\chi_2}\,, \end{aligned} \tag{3.60}$$

where $\delta_{\chi_1\chi_2} \equiv \delta_{n_1n_2}\,\delta_{\sigma_1\sigma_2}$ (for $\chi_i = (n_i, \sigma_i)$);

$$[E_{\chi_1\chi_2}, E_{\chi_3\chi_4}] = \delta_{\chi_2\chi_3}\, E_{\chi_1\chi_4} - \delta_{\chi_1\chi_4}\, E_{\chi_3\chi_2} = [N_{\chi_1\chi_2}, N_{\chi_3\chi_4}]\,; \tag{3.61}$$

$$\begin{aligned} [E_{\chi_1\chi_2}, X^*_{\chi_3\chi_4}] =\;& \delta_{\chi_2\chi_3}\, X^*_{\chi_1\chi_4} + \delta_{\chi_2\chi_4}\, X^*_{\chi_1\chi_3}\,, \\ [E_{\chi_1\chi_2}, X_{\chi_3\chi_4}] =\;& \delta_{\chi_2\chi_3}\, X_{\chi_1\chi_4} + \delta_{\chi_1\chi_3}\, X_{\chi_2\chi_4}\,. \end{aligned} \tag{3.62}$$

Note that the central extension of $\mathrm{sp}(\infty,\mathbb{R})$ (hidden in the $\frac{c}{2}$ term in (3.60)) is trivial and is incorporated in the redefinition $N_{\chi_1\chi_2} \to E_{\chi_1\chi_2}$. The operators $X_{\chi_1\chi_2}$ and $X^*_{\chi_1\chi_2}$ are conjugate to each other and symmetric ($X_{\chi_1\chi_2} = X_{\chi_2\chi_1}$), while the E's satisfy

$$(E_{\chi_1\chi_2})^* = E_{\chi_2\chi_1} \tag{3.63}$$

and span an $u(\infty)$ subalgebra. In view of (3.61) and (3.62) the *Cartan elements*

$$h_\chi = N_{\chi\chi} = E_{\chi\chi} - \frac{c}{2}\,,\ \chi = (\ell,\sigma)\,,\ h_\ell = \sum_{\sigma=1}^{\ell^2} h_{\ell\sigma} \tag{3.64}$$

satisfy

$$[h_\chi, E_{\chi_1\chi_2}] = (\delta_{\chi\chi_1} - \delta_{\chi\chi_2})\, E_{\chi_1\chi_2}\,,\ [h_\chi, X^*_{\chi_1\chi_2}] = (\delta_{\chi\chi_1} + \delta_{\chi\chi_2})\, X^*_{\chi_1\chi_2}\,. \tag{3.65}$$

Their sum, $\sum\limits_{\ell=1}^{\infty} h_\ell$ generates the centre of the $u(\infty)$ subalgebra while the conformal Hamiltonian H is given by

$$H = \sum_{\ell=1}^{\infty} \ell\, h_\ell = \sum_{\chi=(n,\sigma)} n(E_{\chi\chi} - \frac{c}{2})\,. \tag{3.66}$$

We are interested in unitary irreducible $\mathrm{sp}(\infty,\mathbb{R})$ modules $\mathcal{H}_{c\underline{m}}$, $\underline{m} = (m_{11}, m_{21}, \ldots, m_{24}, \ldots, m_{\ell 1}, \ldots, m_{\ell\ell^2}, \ldots)$ that possess a minimal energy normalized ground state $|c;\underline{m}\rangle$ defined as follows. We fix an order in the set of pairs χ setting

$$\chi_1 = (n_1,\sigma_1) < \chi_2 = (n_2,\sigma_2) \text{ if either } n_1 < n_2 \text{ or } n_1 = n_2,\ \sigma_1 < \sigma_2\,. \tag{3.67}$$

We then require

$$X_{\chi_1\chi_2} \mid c;\underline{m}\rangle = 0 \text{ for all } \chi_1,\chi_2\,;\ E_{\chi_1\chi_2} \mid c;\underline{m}\rangle = 0 \text{ for } \chi_1 < \chi_2\,;$$
$$(h_{\ell\sigma} - m_{\ell\sigma}) \mid c;\underline{m}\rangle = 0\,, \tag{3.68}$$

and convergence of the series $E_{\underline{m}}$ which gives the energy of $|c;\underline{m}\rangle$,

$$E_{\underline{m}} = \sum_{\ell=1}^{\infty} \ell \sum_{\sigma=1}^{\ell^2} m_{\ell\sigma}\,. \tag{3.69}$$

The following result of [6] is a straightforward consequence of the definition.

Proposition 3.5. *Hilbert space positivity and convergence of the series* (3.69) *imply that* $\underline{m}$ *is a finite decreasing sequence of positive integers:*

$$m_{11} \geq m_{21} \geq m_{22} \geq m_{23} \geq m_{24} \geq m_{31} \geq \cdots \geq 0\,,\ m_{\ell\sigma} = 0 \text{ for } \ell > r\,. \tag{3.70}$$

for some finite non-negative r*).*

Proof. For any $\chi = (\ell, \sigma)$ we have

$$0 \le \| X^*_{\chi\chi} \mid c; \underline{m}\rangle \|^2 \quad = \quad \langle c; \underline{m} \mid X_{\chi\chi}\, X^*_{\chi\chi} \mid c; \underline{m}\rangle$$
$$= 4 \quad \left\langle c; \underline{m} \left| h_\chi + \frac{c}{2} \right| c; \underline{m} \right\rangle = 4\, m_{\ell\sigma} + 2c\,.$$

On the other hand for $\chi_1 = (n_1, \sigma_1) < \chi_2 = (n_2, \sigma_2)$ we have

$$\| (E_{\chi_2\chi_1})^\ell \mid c; \underline{m}\rangle \|^2 = \ell! \prod_{k=0}^{\ell-1} (m_{n_1\sigma_1} - m_{n_2\sigma_2} - k)\,. \tag{3.71}$$

Unitarity implies that the difference $m_{n_1\sigma_1} - m_{n_2\sigma_2}$ should be a non-negative integer. Convergence of (3.69) says that all but a finite number of $m_{n\sigma}$ should vanish as asserted.

We see as a corollary that the minimal energy $E_{\underline{m}}$ of an unitary irreducible module is non-negative. It is only zero for the *vacuum module* for which all $m_{\ell\sigma}$ vanish. Furthermore, the vacuum spans a 1-dimensional representation of $u(\infty)$:

$$E_{\chi_1\chi_2} \mid c; \underline{0}\rangle = \frac{c}{2}\, \delta_{\chi_1\chi_2} \mid c; \underline{0}\rangle\,. \tag{3.72}$$

The following result of [55] amounts to computing the *Kac determinant* [39] for the vacuum Verma module of $\mathrm{sp}(\infty, \mathbb{R})$ and restricting as a result the central charge c to positive integer values. The null vectors for general highest weight modules of both $\mathrm{sp}(\infty, \mathbb{R})$ and $\mathrm{u}(\infty, \infty)$ are displayed in [6].

Theorem 3.6. *The vacuum irreducible representation* (IR) *(with a minimal energy state* $|c; \underline{0}\rangle$ *satisfying* (3.72)) *is unitary iff* c *is a positive integer.*

Sketch of proof. Let $\chi_1 < \chi_2 < \cdots < \chi_n$; the norm square of the vector

$$\langle c; D(\chi_1, \ldots, \chi_n)| := \langle c; \underline{0}| \begin{vmatrix} X_{\chi_1\chi_1} & \ldots & X_{\chi_1\chi_n} \\ \ldots & \ldots & \ldots \\ X_{\chi_n\chi_1} & \ldots & X_{\chi_n\chi_n} \end{vmatrix} \tag{3.73}$$

has the form

$$\langle c; D(\chi_1, \ldots, \chi_n) \mid c; D(\chi_1, \ldots, \chi_n)\rangle = B(\chi_1, \ldots, \chi_n)\, c(c-1) \ldots (c-n+1) \tag{3.74}$$

where B is a positive constant (independent of c). Indeed, the commutation relations (3.60) imply that the scalar square (3.74) is a polynomial in c of degree not exceeding n. On the other hand, the determinant in (3.73) vanishes identically for $V(z_1, z_2)$ given by (3.48) with $N < n$.

The right hand side of (3.74) is non-negative for all n iff c is a non-negative integer. In that case the "bra" (3.73) is a null vector for $n = c + 1$.

The unitarity of the resulting IR for $c \in \mathbb{N}$ follows from the expression (3.48) of V and from the unitarity of the Fock space representation of free fields.

Thus the vacuum representation of a $d = 2$ field $\phi(z)$ – and hence of a bilocal field $V(z_1, z_2)$ related to ϕ by (3.29) (3.47) – satisfies Wightman positivity for positive integer c only. For $c = N$ the QFT generated by ϕ (or V) is an $O(N)$ invariant subtheory of the N-fold tensor product of the Fock space representations of a free massless scalar field. According to a fundamental result of Doplicher and Roberts [25] one expects that the QFT representations of $\mathrm{sp}(\infty, \mathbb{R})$ for $c = N$ are in one-to-one correspondence with the representations of the *gauge group* $O(N)$. In order to establish a result of this type one needs a polylocal version of the determinant in (3.73). That is achieved [6] by considering an antisymmetrized normal product of $N + 1$ V-factors and applying the Reeh-Schlieder theorem [65] that allows to extend the null vector condition to any unitary positive energy representation generated by fields relatively local to the observables (in accord with the philosophy of Haag [36]).

Alternatively, we can recover this result by reintroducing c in the definition of the conformal Hamiltonian H by the second equation (3.66) and looking for positive energy representations of $\mathrm{sp}(\infty, \mathbb{R})$ (i.e. demanding positivity of H) and using just Lie algebraic methods [26] [6]. In the mathematical work on lowest weight representations of infinite dimensional groups on Fock spaces (see, e.g. [27] [67] and references therein) the integrality of c is derived from the assumption that the highest weight representations of $\mathrm{sp}(\infty, \mathbb{R})$ can be integrated to representations of the metaplectic group.

As a result all QFT representations of $\mathrm{sp}(\infty, \mathbb{R})$ provide local extensions of the observable algebra generated by $V(z_1, z_2)$. For instance, for $c = 2$, they are of two sorts: (1) an infinite series of charge field extensions; (2) a d=3 real("neutral") extension. The representations of the first type are generated by a pair of conjugate charged fields $\psi^{(*)}$ of charge $\pm d$ and dimension d:

$$\psi(z) = \frac{1}{d}\sqrt{\frac{2}{(d-1)!}} : \varphi^d(z) : , \;\; \varphi(z) = \frac{1}{\sqrt{2}} \left(\varphi_1(z) + i\, \varphi_2(z) \right). \tag{3.75}$$

Both ψ and ψ^* generate a unitary IR of $\mathrm{sp}(\infty, \mathbb{R})$ with lowest weight vector $|c; \underline{m}\rangle = |2; d, 0, \ldots\rangle$ $(= \psi^{(*)}(0) \mid 2; \underline{0}\rangle)$. The single nontrivial neutral extension corresponds to a lowest weight vector $|2; 1, 1, 0, ...\rangle = J_1(0)|2; \underline{0}\rangle$ where $J_\alpha(z)$ is the U(1) current

$$J_\alpha(z) = \varphi_1(z) \partial_\alpha \varphi_2(z) - \varphi_2(z) \partial_\alpha \varphi_1(z). \tag{3.76}$$

An SO(2) rotation $(R_\theta\, \varphi)_1 = \varphi_1 \cos\theta + \varphi_2 \sin\theta$, $(R_\theta\, \varphi)_2 = -\varphi_1 \sin\theta + \varphi_2 \cos\theta$ yields a $U(1)$ gauge transformation $R_\theta\, \psi = e^{-i\theta d}\, \psi$, $R_\theta\, \psi^* = e^{i\theta d}\, \psi^*$. The reflection $(\varphi_1, \varphi_2) \to (\varphi_1, -\varphi_2)$ exchanges ψ and ψ^* and changes the sign of J. Clearly, the $O(2)$ action commutes with $\mathrm{sp}(\infty, \mathbb{R})$ as it leaves $V(z_1, z_2)$ invariant. Note that the field algebra, generated by ψ and ψ^*, involves a wider extension of the charge-zero algebra, which now involves the field $W(z_1, z_2)$ defined by (3.34) and its conjugate on top of the $U(1)$ current J (3.76). The $\mathrm{sp}(\infty, \mathbb{R})$ Lie algebra is thus extended to $\mathrm{u}(\infty, \infty)$.

4 Outlook and concluding remarks

Conformal QFT is not expected to be a realistic interacting theory with a non-trivial scattering matrix. Yet, it is hoped to display some essential characteristics of such a theory, in particular the small distance behaviour, which may help resolve the omnipresent ultraviolet problem. Not least important, it appears to offer a better chance for constructing a viable non-perturbative QFT model than a direct assault, using constructive QFT methods, at a more realistic asymptotically free theory like quantum chromodynamics. This has served as a motivation for the work on both (complementary) attempts, surveyed here, to construct a conformal QFT.

The developments of the 1970's (triggered at the time by the interest in the Bjorken scaling in deep inelastic scattering[9]), briefly reviewed in Sect. 2, are in our view of more than a historical interest. If for the method of OPE and conformal partial wave expansion this is rather obvious[10], it is perhaps not superfluous to recall the role of anomalous dimensions in making sense of skeleton graph expansion and in solving the conformal (bootstrap) Schwinger-Dyson equations. We refer in this connection to D. Kreimer's suggestion (in Sect. 4.2 of [42]) to use the simplified Schwinger-Dyson equation in a critical gauge QFT, at the point where the β-function vanishes. It may be also of interest to apply these methods to the conformal super-Yang-Mills theory, regarding the Konishi field ([19]) of scale dimension in the interval between 2 and 3 as fundamental.

The major part of the paper (Sect. 3 and Appendices A and B) is devoted to a survey of recent developments in GCI QFT and to the ensuing concept of higher dimensional vertex algebras. The chief objection which sceptics voice towards this work is the suspicion that it may turn out to be just an exercise in free field theory (if one insists from the outset on Wightman positivity). As we saw in Sect. 3.4, proving that the local field extensions of the vacuum representation of $\mathrm{sp}(\infty, \mathbb{R})$ are generated by normal products of free fields in the tensor product of Fock spaces is a rather non-trivial task related, as it turns out, to a long term mathematical problem[11]. The study of thermal equilibrium states with associated elliptic correlation functions and modular invariant energy mean values [59] (a subject not touched upon in the present survey) provides another topic in which free fields display nontrivial properties that may be of interest to both physicists and mathematicians.

On the other hand, the study of local field extensions of the theory of a dimension-two scalar field solves a problem, set up in [53]: to factor out such a scalar field contribution from the most promising candidate for a nontrivial

[9] For a survey of this work and references to the original papers – see [83], as well as [49].

[10] For applications to (extended) superconformal gauge theories see, e.g. [24] as well as the review [19]; concerning their role in the study of Wightman positivity in GCI models – see [56] and [53].

[11] For the latest development on the Kashiwara-Vergne (1978) decomposition of tensor products of the Segal-Shale-Weil (1962) representations and for references to earlier work – see [27] [67].

GCI QFT – the theory of a $d = 4$ Lagrangian field ([56] [53]).

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A The conformal Lie algebra. Bases of vector fields

A natural basis of $so(4,2)$ is provided by the pseudorotations X_{ab} in $\mathbb{R}^{4,2}$ obeying the commutation relations (CR)

$$[X_{ab}, X_{cd}] = \eta_{ac}\, X_{bd} - \eta_{bc}\, X_{ad} + \eta_{bd}\, X_{ac} - \eta_{ad}\, X_{bc} \tag{A.1}$$

for $a, b, c, d = -1, 0, 1, 2, 3, 4$ and flat space metric tensor η of signature $(4, 2)$:

$$\eta_{11} = \eta_{22} = \eta_{33} = \eta_{44} = 1 = -\eta_{00} = -\eta_{-1-1}\,,\ \ \eta_{ab} = 0 \text{ for } a \neq b\,. \tag{A.2}$$

Note that the correspondence $X \to O(X)$ between the commutator action of X_{ab} on (operator, spin-tensor valued) fields $\phi(\xi)$ and the differential operators

$$O(X_{ab}) = \xi_a \frac{\partial}{\partial\, \xi^b} - \xi_b \frac{\partial}{\partial\, \xi^a} + S_{ab} \tag{A.3}$$

(where S_{ab} are constant matrices spanning a finite dimensional representation of $su(2,2)$) given by

$$[X, \phi(\xi)] = O(X)\, \phi(\xi) \tag{A.4}$$

provides a Lie algebra *antihomomorphism*

$$[O(X), O(Y)] = -\, O([X, Y]) \tag{A.5}$$

(see for details Appendix B of [58]).

The Minkowski space infinitesimal translations $\frac{\partial}{\partial\, x^\mu}$ and special conformal transformations $x^2 \frac{\partial}{\partial\, x^\mu} - 2\, x_\mu\, x \cdot \frac{\partial}{\partial\, x}$ are images of the $so(4,2)$ generators $i\, P_\mu$, $i\, K_\mu$, respectively,

$$i\, P_\mu = -X_{-1\mu} - X_{\mu 4}\,,\ \ i\, K_\mu = -X_{-1\mu} + X_{\mu 4} \tag{A.6}$$

under this correspondence (P_μ being the *energy-momentum operator;* both P_μ and K_μ are self-adjoint for a unitary representation of $\mathcal{C}$). The CR (A.1) imply

$$[P_\mu, K_\nu] = 2\, X_{\mu\nu} - 2\, \eta_{\mu\nu}\, X_{-14}\,, \tag{A.7}$$

where X_{-14} is the *dilation generator*,

$$[X_{-14}, P_\mu] = P_\mu\,,\ \ [X_{-14}, K_\mu] = -K_\mu\,. \tag{A.8}$$

The *conformal Hamiltonian*

$$H := i\, X_{-10} = \frac{1}{2}\, (P_0 + K_0) \tag{A.9}$$

is the generator of the centre of the Lie algebra $so(2) \times so(4) \approx s(u(2) \times u(2))$ of the maximal compact subgroup $S(U(2) \times U(2))$ of $\mathcal{C}$. There is a proper conformal transformation, the *Weyl inversion* $w = e^{\pi X_{-10}}$, that is a rotation on

π in the $(-1,0)$-plane, such that $wX_{04}\,w^{-1} = -X_{04}$ and hence $w\,P_0\,w^{-1} = K_0$. It follows that the spectrum of K_0 coincides with the spectrum of P_0. This yields Segal's observation [71] that the conformal Hamiltonian (A.9) is bounded below whenever the Minkowski space energy operator P_0 is. (It can be demonstrated that the converse is also true.) On the other hand, H unlike P_0 has discrete (in fact, integer spaced) spectrum in any unitary irreducible representation of $\tilde{\mathcal{C}}$.

We also use the complex Lie algebra generators T_α and C_α, corresponding to z-translations and special conformal transformations where the z's are the complex variables in the realization of the future tube and the compactified Minkowski space described in Appendix B:

$$T_\alpha = i\,X_{0\alpha} - X_{-1\alpha}\,,\ C_\alpha = -i\,X_{0\alpha} - X_{-1\alpha}\,,\ \alpha = 1,2,3,4\,. \tag{A.10}$$

It is readily verified that T_α, C_β, $X_{\alpha\beta}$ and H satisfy the same commutation relations (A.7) (A.8) as $i\,P_\mu$, $i\,K_\nu$ (A.6) $X_{\mu\nu}$ and X_{-14}:

$$[T_\alpha, C_\beta] = 2\,(\delta_{\alpha\beta}\,H - X_{\alpha\beta})\,, \tag{A.11}$$

$$[H, T_\alpha] = T_\alpha\,,\ [H, C_\alpha] = -\,C_\alpha\,. \tag{A.12}$$

In fact, the generators T_α, C_β, H and $X_{\alpha\beta}$ span the real euclidean conformal Lie algebra spin$\,(5,1)$. They are conjugate in the complex conformal Lie algebra to iP_μ, iK_ν, X_{-14} and $X_{\mu\nu}$ by the same complex conformal transformation g_c (that corresponds to a real euclidean rotation) which defines the compactification map $x \to z$ of Appendix B.

B Global complex-variable parametrization of compactified Minkowski space

There is a remarkable complex conformal transformation [84] [82], a rotation on $\frac{\pi}{2}$ in the $(0,4)$ plane of $\mathbb{C}^6$, which plays the role of a (Cayley) compactification map for real Minkowski space:

$$g_c : x = (x^0, \boldsymbol{x}) \to z = (\boldsymbol{z}, z_4)\,,\ \boldsymbol{z} = \frac{\boldsymbol{x}}{\omega(x)}\,,\ z_4 = \frac{1 - x^2}{2\,\omega(x)}\,, \tag{B.1}$$

$$z^2 := \boldsymbol{z}^2 + z_4^2 = \frac{1 + x^2 + 2\,i\,x^0}{1 + x^2 - 2\,i\,x^0}\,,\ 2\,\omega(x) = 1 + x^2 - 2\,i\,x^0\,,\ x^2 = \boldsymbol{x}^2 - x_0^2\,. \tag{B.2}$$

The complex conjugation in complexified Minkowski space $M_{\mathbb{C}}$ (which leaves M fixed) corresponds to the involution

$$z \to z^* = \frac{\bar{z}}{\bar{z}^2} \tag{B.3}$$

which keeps $\bar{M}$ fixed. A real parametrization of $\bar{M}$ is provided by the polar decomposition of z:

$$\begin{aligned} \bar{M} &= \{z_\alpha = e^{it}\,u_\alpha\,;\ t, u_\alpha \in \mathbb{R}\,,\ \alpha = 1,2,3,4\,;\ u^2 = \boldsymbol{u}^2 + u_4^2 = 1\} \\ &\cong \mathbb{S}^3 \times \mathbb{S}^1/\mathbb{Z}_2 \end{aligned} \tag{B.4}$$

($\mathbb{Z}_2 = \mathbb{Z}/2\,\mathbb{Z}$). The properties of the transformation (B.1), viewed as a map from $M_{\mathbb{C}}$ to $E_{\mathbb{C}}$ ($= \mathbb{C}^4$ equipped with a complex euclidean metric) are summarized as follows ([58] [52]).

Proposition B.1. *The rational coordinate transformation* $g_c : M_{\mathbb{C}} \to E_{\mathbb{C}}$ (B.1) *is a complex conformal map such that* $z_{12}^2 = \frac{x_{12}^2}{\omega(x_1)\,\omega(x_2)}$, $dz^2 = \frac{dx^2}{\omega^2(x)}$ *and the image of the tube domain* $\mathcal{T}_+$ (2.6) *is the precompact symmetric space*

$$T_+ := \left\{ z \in \mathbb{C}^4\,;\ |z^2| < 1\,,\ |z|^2 = \sum_{\alpha=1}^{4} |z_\alpha|^2 < \frac{1}{2}\,(1 + |z^2|^2)\,(< 1) \right\}. \tag{B.5}$$

The image of M *in* $\bar{M}$ *under the map* (B.1) *is the open dense subset*

$$2\,\omega_z := 1 + z^2 + 2\,z_4 = 2\,e^{it}(\cos t + u_4) \neq 0\,, \tag{B.6}$$

on which x *is expressed conversely as a function of* z *(or* u *and* t*):*

$$\boldsymbol{x} = \frac{\boldsymbol{z}}{\omega_z} = \frac{\boldsymbol{u}}{\cos t + u_4}\,,\ -i\,x^0 = \frac{1 - z^2}{2\,\omega_z} = \frac{-\,i \sin t}{\cos t + u_4}$$
$$\left(x^2 = \frac{1 + z^2 - 2\,z_4}{1 + z^2 + 2\,z_4} = \frac{\cos t - u_4}{\cos t + u_4} \right). \tag{B.7}$$

Note that the complex euclidean interval $dz^2 = d\mathbf{z}^2 + dz_4^2$ has in fact Lorentzian signature when expressed in terms of real coordinates (like t and the spherical angles for $\mathbf{u} = sin\rho\,(sin\theta cos\phi,\ sin\theta sin\phi,\ cos\theta)$, $u_4 = cos\rho$):

$$\frac{dz^2}{z^2} = \frac{dx^2}{|\omega(x)|^2} = d\rho^2 + sin^2\rho\,(d\theta^2 + sin^2\theta\,d\phi^2) - dt^2. \tag{B.8}$$

Remark B.1. The tube domain T_+ (B.5) is biholomorphically equivalent to the classical Cartan domain of the fourth type - see e.g. [72] (pp. 182-192) and references therein.

Remark B.2. If we define the embedding $M \hookrightarrow \bar{M}$ in the manifestly covariant picture (1.5) of $\bar{M}$ by

$$x^\mu = \frac{\xi^\mu}{\xi^{-1} + \xi^4} = \frac{\xi^\mu}{\xi_4 - \xi_{-1}} \quad \left(x^2 = \frac{\xi^{-1} - \xi^4}{\xi^{-1} + \xi^4} = \frac{\xi_{-1} + \xi_4}{\xi_{-1} - \xi_4} \right) \tag{B.9}$$

then the complex 4-vector is expressed in terms of ξ as

$$z_\alpha = \frac{\xi_\alpha}{i\,\xi_0 - \xi_{-1}} \Longrightarrow z^2 = e^{4\pi i \zeta} = \frac{\xi_{-1} + i\,\xi_0}{\xi_{-1} - i\,\xi_0}\,. \tag{B.10}$$

Introducing the euclidean variable $x_E^0 = -i\,x^0$, $\xi_E^0 = -i\,\xi^0$ and comparing (B.9) and (B.10) we see that (B.1) indeed amounts to a real rotation in $\frac{\pi}{2}$ in the euclidean plane $(\xi_E^0, \xi_E^4 (= \xi_4))$.

It is a simple corollary of the counterpart of Wightman axioms for vertex algebras (Sect. 3.1) and of Proposition B.1 that the series

$$\psi(z) \mid 0\rangle = \sum_{n,m=0}^{\infty} (z^2)^n \, \psi_{\{n,m\}}(z) \mid 0\rangle \,, \qquad z \in T_+ \tag{B.11}$$

is norm convergent and analytic.

We shall describe the free (2-component) Weyl spinor thus providing an example of a non-scalar z-picture field. To this end we introduce a 2×2 matrix realization of the quaternion units Q_α ($\alpha = 1,2,3,4$) satisfying $Q_1\, Q_2 = Q_3 = -\, Q_2\, Q_1$,

$$Q_\alpha^+ \, Q_\beta + Q_\beta^+ \, Q_\alpha = 2\, \delta_{\alpha\beta} = Q_\alpha \, Q_\beta^+ + Q_\beta \, Q_\alpha^+ \tag{B.12}$$

where $Q_i^+ = -\, Q_i$ for $i = 1,2,3$; $Q_4^+ = Q_4$. In this notation a free Weyl field $\psi(z)$ is characterized by its 2-point function,

$$\langle 0 \mid \psi(z_1)\, \psi^*(z_2) \mid 0\rangle = Q^+ \cdot \frac{\partial}{\partial z_2} \, \frac{1}{z_{12}^2} = 2\, \frac{\not{z}_{12}^+}{(z_{12}^2)^2} \tag{B.13}$$

where $\not{z}^+$ is a short-hand for $Q^+ \cdot z$:

$$\not{z} = Q \cdot z = \begin{pmatrix} z_4 - iz_3 & -iz_1 - z_2 \\ z_2 - iz_1 & z_4 + iz_3 \end{pmatrix} \,, \; z^+ = Q^+ \cdot z = \begin{pmatrix} z_4 + iz_3 & iz_1 + z_2 \\ iz_1 - z_2 & z_4 - iz_3 \end{pmatrix} . \tag{B.14}$$

As a consequence, $\psi(z)$ satisfies the Weyl equation

$$Q \cdot \frac{\partial}{\partial z} \, \psi(z) = 0 = \frac{\partial}{\partial z_\alpha} \, \psi^*(z)\, Q_\alpha \tag{B.15}$$

which, in turn, implies the conservation of the current

$$J_\alpha(z) = i : \psi^*(z)\, Q_\alpha \, \psi(z) : \, . \tag{B.16}$$

References

[1] S. Adler, Short distance behaviour of quantum electrodynamics and an eigenvalue condition for α, *Phys. Rev.* **D5** (1972) 3021-3047; Erratum, *ibid.* **D7** (1973) 3821.

[2] D. Anselmi, Central functions and their physical implications, JHEP 9805 (1998) 005; hep-th/9702056.

[3] G. Arutyunov, S. Penati, A. Santabrogio, E. Sokatchev, Four-point correlators of BPS operators in $N = 4$ SYM at order g^4, *Nucl. Phys.* **B670** (2003) 103-147.

[4] M. Atiyah, Hermann Weyl, *Biographical Memoirs* **82** (2002) National Academy of Sciences, Washington D.C.

[5] B. Bakalov, N.M. Nikolov, Jacobi identities for vertex algebras in higher dimensions, math-ph/0601012.

[6] B. Bakalov, N.M. Nikolov, K.-H. Rehren, I. Todorov, Unitary positive energy representations of scalar bilocal quantum fields, math-ph/0604069.

[7] M. Baker, K. Johnson, Applications of conformal symmetry in quantum electrodynamics, *Physica* **96A** (1979) 120-180.

[8] V. Bargmann, I.T. Todorov, Spaces of analytic functions on a complex cone as carriers for symmetric tensor representations of SO (n), *J. Math. Phys.* **18** (1977) 1141-1148.

[9] H. Bateman, The transformation of electrodynamical equations, *Proc. London Math. Soc.* **8** (1910) 223-264.

[10] K. Baumann, All massless scalar fields with a trivial S-matrix are Wick polynomials, *Commun. Math. Phys.* **86** (1982) 247-256.

[11] S. Bloch, H. Esnault, D. Kreimer, On motives associated to graph polynomials, math.AG/0510011.

[12] L. Bonora, G. Sartori, M. Tonin, Conformal covariant operator product expansions, *Nuovo Cim.* **10A** (1972) 667-681.

[13] R.E. Borcherds, Vertex algebras, Kac-Moody algebras and the Monster, *Proc. Nat. Acad. Sci. USA* **83** (1986) 3068-3071.

[14] R. Brunetti, K. Fredenhagen, Interacting quantum fields on a curved background, *Proceedings of the XII-ICMP*, Briskane 1997, International Press, Singapore; hep-th/9709011.

[15] R. Brunetti, K. Fredenhagen, R. Verch, The generally covariant locality principle – A new paradigm for local quantum physics, *Commun. Math. Phys.* **237** (2003) 31-68.

[16] N.S. Craigie, V.K. Dobrev, I.T. Todorov, Conformal covariant composite operators in quantum chromodynamics, *Ann. Physics* **159** (1985) 411-444.

[17] E. Cunningham, The principle of relativity in electrodynamics and an extension thereof, *Proc. London Math. Soc.* **8** (1910) 77-98.

[18] A. De Sole, V.G. Kac, Finite vs affine W-algebras, math-ph/0511055.

[19] E. D'Hoker, D.Z. Freedman, Supersymmetric gauge theories and the AdS/CFT correspondence, TASI 2001 Lecture Notes, hep-th/0201253.

[20] P.A.M. Dirac, Wave equations in conformal space, *Ann. Math.* **37** (1936) 429-442.

[21] P.A.M. Dirac, Long range forces and broken symmetries, *Proc. Roy. Soc. of London* **A333** (1973) 403-418.

[22] V.K. Dobrev, G. Mack, V.B. Petkova, S.G. Petrova, I.T. Todorov, *Harmonic Analysis on the n-Dimensional Lorentz Group and Its Application to Conformal Quantum Field Theory*, Lecture Notes in Physics **63**, Springer, Berlin 1977.

[23] V.K. Dobrev, V.B. Petkova, S.G. Petrova, I.T. Todorov, Dynamical derivation of vacuum operator product expansion in Euclidean conformal field theory, *Phys. Rev.* **D13** (1976) 887-912.

[24] F.A. Dolan, H. Osborn, Conformal four point functions and operator product expansion, *Nucl. Phys.* **B599** (2001) 459-496; Conformal partial waves and operator product expansion, *Nucl. Phys* **B678** (2004) 491-507.

[25] S. Doplicher, J. Roberts, Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics, *Commun. Math. Phys.* **131** (1990) 51-107.

[26] T.J. Enright, R. Parthasarathy, A proof of a conjecture of Kashiwara and Vergne, in *Non Commutative Harmonic Analysis and Lie Groups*, Lecture Notes in Mathematics **880**, Springer 1981, pp. 74-90.

[27] T. Enright, R. Howe, N. Wallach, A classification of unitary highest weight modules, in *Representation Theory of Reductive Groups*, Progress in Mathematics **40**, Birkhäuser 1983.

[28] S. Ferrara, R. Gatto, A. Grillo, G. Parisi, The shadow operator formalism for conformal algebra vacuum operator values and operator products, *Nuovo Cim. Lett.* **4** (1972) 115-120.

[29] M. Flato, D. Sternheimer, Remarques sur les automorphismes causals de l'espace temps, *Comptes Rendus Acad. Sci., Paris* **263A** (1966) 935-936.

[30] E.S. Fradkin, M.Ya Palchik, Recent developments in conformal invariant QFT, *Phys. Reports* **44C** (1978) 249-349.

[31] E.S. Fradkin, M.Ya. Palchik, New developments in d-dimensional conformal quantum field theory, *Phys. Reports* **300** (1998) 1-112.

[32] E. Frenkel, D. Ben-Zvi, *Vertex Algebras and Algebraic Curves*, AMS, Mathematical Surveys and Monographs **88**, 2001; 2nd ed. 2004.

[33] M. Gell-Mann, F. Low, Quantum electrodynamics at small distances, *Phys. Rev.* **95** (1954) 1300-1312.

[34] L. Genovese, Ya.S. Stanev, Rationality of the anomalous dimensions in $N = 4$ SYM theory, *Nucl. Phys.* **B721** (2005) 212-228.

[35] F. Gürsey, S. Orfanidis, Conformal invariance and field theory in two dimensions, *Phys. Rev.* **D7** (1973) 2414-2437.

[36] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras,* Springer, N.Y. 1992.

[37] J. Henn, C. Jarczak, E. Sokatchev, On twist-two operators in $N = 4$ SYM, *Nucl. Phys.* **B730** (2005) 191-209.

[38] M. Hortascu, R. Seiler, B. Schroer, Conformal symmetry and reverberations, *Phys. Rev.* **D5** (1972) 2519-2534.

[39] V.G. Kac, Contravariant form for infinite dimensional Lie algebras and superalgebras, in: *Lecture Notes in Physics*, Springer, N.Y. 1979, pp. 441-445.

[40] V.G. Kac, *Vertex Algebras for Beginners*, AMS, ULS **10**, Providence, R.I., 2nd ed. 1998.

[41] L.P. Kadanoff, Operator algebra and the determination of critical indices, *Phys. Rev. Lett.* **23** (1969) 1430-1433.

[42] D. Kreimer, Anatomy of gauge theory, hep-th/0509135; *Ann. Physics* (to appear).

[43] J.H. Lowenstein, Normal products in the Thirring model, *Commun. Math. Phys.* **16** (1970) 265-289.

[44] M. Lüscher, G. Mack, Global conformal invariance in quantum field theory, *Commun. Math. Phys.* **41** (1975) 203-234.

[45] G. Mack, Group theoretical approach to conformal invariant quantum field theory, *in: Renormalization and Invariance in Quantum Field Theory*, ed. E.R. Caianiello, Plenum Press, NY 1974, pp. 123-157.

[46] G. Mack, All unitary representations of the conformal group $SU(2,2)$ with positive energy, *Commun. Math. Phys.* **55** (1977) 1-28.

[47] G. Mack, Convergence of operator product expansions on the vacuum in conformal invariant quantum field theory, *Commun. Math. Phys.* **53** (1977) 155-184.

[48] G. Mack, K. Symanzik, Currents, stress tensor and generalized unitarity in conformal invariant quantum field theory, *Commun. Math. Phys.* **27** (1972) 247-281.

[49] G. Mack, I.T. Todorov, Conformal invariant Green functions without ultraviolet divergences, *Phys. Rev.* **D8** (1973) 1764-1787.

[50] A.A. Migdal, Conformal invariance and bootstrap, *Phys. Lett.* **37B** (1971) 386-388.

[51] W. Nahm, Conformal field theory: a bridge over troubled waters, *in* : A.N. Mitra (ed.) *Quantum Field Theory – A Twentieth Century Profile*, Hindustani Book Agency and Indian Nat. Sci. Acad. 2000, pp. 571-604.

[52] N.M. Nikolov, Vertex algebras in higher dimensions and globally conformal invariant quantum field theory, *Commun. Math. Phys.* **253** (2005) 283-322.

[53] N.M. Nikolov, K.-H. Rehren, I.T. Todorov, Partial wave expansion and Wightman positivity in conformal field theory, *Nucl. Phys.* **B722** (2005) 266-296.

[54] N.M. Nikolov, K.-H. Rehren, I.T. Todorov, Harmonic bilocal fields generated by globally conformal invariant scalar fields (in preparation)

[55] N.M. Nikolov, Ya.S. Stanev, I.T. Todorov, Four dimensional CFT models with rational correlation functions, *J. Phys. A: Math Gen.* **35** (2002) 2985-3007.

[56] N.M. Nikolov, Ya.S. Stanev, I.T. Todorov, Globally conformal invariant gauge field theory with rational correlation functions, *Nucl. Phys.* **B670** **[FS]** (2003) 373-400.

[57] N.M. Nikolov, I.T. Todorov, Rationality of conformally invariant local correlation functions on compactified Minkowski space, *Commun. Math. Phys.* **218** (2001) 417-436.

[58] N.M. Nikolov, I.T. Todorov, Lectures on elliptic functions and modular forms in conformal field theory, *Proceedings of the 3rd Summer School in Modern Mathematical Physics*, Zlatibor, Serbia and Montenegro (August 20-31, 2004), Eds. B. Dragovich, Z. Rakić, B. Sazdović, Institute of Physics, Belgrade 2005, pp. 1-93; math-ph/0412039.

[59] N.M. Nikolov, I.T. Todorov, Elliptic thermal correlation functions and modular forms in a globally conformal invariant QFT, *Rev. Math. Phys.* **17** (2005) 613-667.

[60] G. Parisi, G. Peliti, Calculation of critical indices, *Nuovo Cim. Lett.* **2** (1971) 627-628.

[61] A.C. Petkou, Conserved currents, consistency relations and operator product expansions in the conformally invariant $O(N)$ vector model, *Ann. Physics* **249** (1996) 180-221.

[62] V.B. Petkova, G.M. Sotkov, I.T. Todorov, Conformal gauges and renormalized equations of motion in massless quantum electrodynamics, *Commun. Math. Phys.* **97** (1985) 227-256.

[63] A.M. Polyakov, Conformal invariance of critical fluctuations, *ZhETF Pis. Red.* **12** (1970) 538-540 (*JETP Lett.* **12** (1970) 381-383).

[64] A.M. Polyakov, Non-Hamiltonian approach in the conformal invariant quantum field theory, *Zh.Eksp. Teor. Fiz* **66** (1974) 23-42 (English transl.: *JETP* **39** (1974) 10-18).

[65] H. Reeh, S. Schlieder, Bemerkungen zur Unitäräquivalenz von Loprentz-invarianten Feldern, *Nuovo Cim.* **22** (1961) 1051-1068.

[66] G.C. Rossi, E. Sokatchev, Ya.S. Stanev, New results in the deformed $N = 4$ SYM theory, *Nucl. Phys.* **B729** (2005) 581-593; hep-th/0507113.

[67] M.U. Schmidt, Lowest weight representations of some infinite dimensional groups on Fock spaces, *Acta Appl. Math.* **18** (1990) 59-84.

[68] S. Schreier, Conformal symmetry and three-point functions, *Phys. Rev.* **D3** (1971) 980-988.

[69] B. Schroer, A necessary and sufficient condition for the softness of the trace of the energy-momentum tensor, *Nuovo. Cim. Lett.* **2** (1971) 867-872.

[70] B. Schroer, J.A. Swieca, A.H. Völkel, Global operator expansion in conformally invariant relativistic theory, *Phys. Rev.* **D11** (1975) 1509-1520.

[71] I.E. Segal, Causally oriented manifolds and groups, *Bull. Amer. Math. Soc.* **77** (1971) 958-959.

[72] A.G.Sergeev, V.S. Vladimirov, IV. Complex analysis in the future tube, in: *Several Complex Variables II. Function Theory in Classical Domains and Complex Potential Theory*, G.M. Khenkin, A.G. Vitushkin (Eds.)*Encyclopaedia of Mathematical Sciences* Vol. **8**, Springer, Berlin et al. 1994, pp.179-253.

[73] Ya.S. Stanev, Stress energy tensor and $U(1)$ current operator product expansion in conformal QFT, *Bulg. J. Phys.* **15** (1988) 93-107.

[74] Ya.S. Stanev, I.T. Todorov, Towards a conformal QED4 with a non-vanishing current 2-point function, *Int. J. Mod. Phys.* **A3** (1988) 1023-1049.

[75] R.F. Streater, A.S. Wightman, *PCT, Spin and Statistics, and All That*, W.A. Benjamin, 1964; Princeton Univ. Press, Princeton, N.J. 2000.

[76] J.A. Swieca, A.H. Völkel, Remarks on conformal invariance, *Commun. Math. Phys.* **29** (1973) 319-342.

[77] K. Symanzik, Green's functions method and renormalization of renormalizable quantum field theory, in: *Lectures in High Energy Physics*, ed. by B. Jaksič, Zagreb 1961, Gordon and Breach, N.Y. 1965.

[78] K. Symanzik, Small distance behaviour in field theory and power counting, *Commun. Math. Phys.* **18** (1970) 227-246.

[79] K. Symanzik, Small distance behaviour and Wilson expansions, *Commun. Math. Phys.* **23** (1971) 49-86.

[80] K. Symanzik, On calculations in conformal field theories, *Nuovo Cim. Lett.* **3** (1972) 734-739.

[81] I.T. Todorov, Local field representations of the conformal group and their applications, in: *Mathematics + Physics, Lectures on Recent Results*, ed. L. Streit, Vol. 1, World Scientific, Singapore 1985, pp. 195-338.

[82] I.T. Todorov, Infinite dimensional Lie algebras in conformal QFT models. In: A.O. Barut and H.-D. Doebner (eds.) *Conformal Groups and Related Symmetries. Physical Results and Mathematical Background*, *Lecture Notes in Physics* **261**, pp. 387-443, Springer, Berlin et al. 1986.

[83] I.T. Todorov, M.C. Mintchev, V.B. Petkova, *Conformal Invariance in Quantum Field Theory*, Scuola Normale Superiore, Pisa 1978.

[84] A. Uhlmann, Remarks on the future tube, *Acta Phys. Pol.* **24** (1963) 293; The closure of Minkowski space, *ibid.* 295-296.

[85] K.G. Wilson, Non-Lagrangian models of current algebra, *Phys. Rev.* **179** (1969) 1499-1512.

[86] K.G. Wilson, Operator product expansions and anomalous dimensions in the Thirring model, *Phys. Rev.* **D2** (1970) 1473-1477; Anomalous dimension and break down of scale invariance in perturbation theory, *ibid*, 1478-1493.