Matrix Representation of Renormalization in Perturbative Quantum Field Theory

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Abstract

We formulate the Hopf algebraic approach of Connes and Kreimer to renormalization in perturbative quantum field theory using triangular matrix representation. We give a Rota–Baxter anti-homomorphism from general regularized functionals on the Feynman graph Hopf algebra to triangular matrices with entries in a Rota–Baxter algebra. For characters mapping to the group of unipotent triangular matrices we derive the algebraic Birkhoff decomposition for matrices using Spitzer's identity. This simple matrix factorization is applied to characterize and calculate perturbative renormalization.

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Contents

1 Introduction

Most, if not all, of the interesting and relevant 4-dimensional quantum field theories suffer from ultraviolet divergencies and need to be renormalized [15]. The basic idea of the theory of perturbative renormalization in quantum field theory goes back to Kramers [12], and was successfully applied for the first time in a 1947 paper by Bethe [6], dealing with a concrete problem in perturbative quantum electrodynamics (QED). Five decades later, Dirk Kreimer [35] uncovered a Hopf algebra structure underlying the intricate combinatorial-algebraic structure of renormalization in general perturbative quantum field theory (QFT), hereby providing a sound and useful mathematical foundation for this most important achievement of theoretical physics. Later, Kreimer [36, 37, 38, 39] and collaborators [5, 7, 8, 9, 10, 11, 22], especially Connes and Kreimer [16, 17, 18, 19] further developed the Hopf–algebraic approach, connecting it to non-commutative geometry.

In their approach, one particle irreducible (1PI) Feynman graphs, as the building blocks of perturbative QFT and renormalization, are organized into a connected, graded, commutative, non-cocommutative Hopf $\mathbb{C}$-algebra $\mathcal{H}_{\mathcal{F}}$. The restricted dual of this Hopf algebra of Feynman graphs, denoted by $\mathcal{H}^*_{\mathcal{F}}$, contains the group $G := char(\mathcal{H}_{\mathcal{F}}, \mathbb{C})$ of characters, that is, algebra homomorphisms from $\mathcal{H}_{\mathcal{F}}$ to the underlying base field $\mathbb{C}$. Feynman rules naturally provide a special class of such characters. The group G is generated by the Lie algebra $g := \partial char(\mathcal{H}_{\mathcal{F}}, \mathbb{C})$, formed by derivations, or so-called infinitesimal characters. We refer the reader to [26, 29, 37, 40, 49] for more details.

Dealing with the ultraviolet divergencies demands a regularization plus a renormalization scheme. As a main example for the former serves dimensional regularization, where we replace the above base field $\mathbb{C}$ by Laurent series $A = \mathbb{C}[\varepsilon^{-1}, \varepsilon]]$. They form equipped with the ordinary multiplication a commutative Rota–Baxter algebra, and we consider $G_A := char(\mathcal{H}_{\mathcal{F}}, A)$, respectively $g_A := \partial char(\mathcal{H}_{\mathcal{F}}, A)$. The Hopf algebra of Feynman graphs and, via the famous Milnor–Moore Theorem [29, 43], its equivalent Lie algebra and Lie group of characters, enabled Connes–Kreimer to capture the process of renormalization in terms of a so-called algebraic Birkhoff decomposition of regularized Feynman rules characters, giving rise to a link to the Riemann–Hilbert problem [17, 18]. See also [20].

In [24, 25] the intimate link between the notion of Rota–Baxter algebras and the work of Connes–Kreimer was explored. It was shown that a non-commutative generalization of a classical result with origin in fluctuation theory of probability, known as Spitzer's identity for Rota–Baxter algebras [3, 47], lies at the heart of the Connes–Kreimer decomposition theorem for regularized Hopf algebra characters. This provides a natural way to derive Bogoliubov's recursions for the counter term and renormalized Feynman rules. This approach emphasizes the Lie algebra structure on Feynman graphs and the corresponding Lie group of characters. The former is closely related to the more general insertion and elimination Lie algebra of Feynman graphs, which was studied in detail first by Connes–Kreimer in [19], and recently in the context of matrices by Mencattini and Kreimer [41, 42].

The Hopf algebras of Feynman graphs found in renormalization theory fall into the class of so-called combinatorial Hopf algebras. In this paper, we obtain a matrix representation of the full space $\mathrm{Hom}(\mathcal{H}_{\mathcal{F}}, A)$ of regularized functionals, thus in particular of G_A and g_A, directly from the coproduct of the Hopf algebras $\mathcal{H}_{\mathcal{F}}$. In fact, the matrix representation applies to a large class of Hopf algebras $\mathcal{H}$ and gives an injective anti-homomorphism of Rota–Baxter algebras from $\mathrm{Hom}(\mathcal{H}, A)$ to the infinite size upper triangular matrices $\mathcal{M}^u_\infty(A)$. Therefore, in the particular case of QFT renormalization, i.e., when $\mathcal{H} = \mathcal{H}_{\mathcal{F}}$, the whole process of perturbative renormalization of regularized Feynman characters, presented in the context of Rota–Baxter algebras [24, 25], is translated to a parallel, but more transparent process involving matrices. The upper (or lower) triangular matrices with entries from the target space of such Hopf algebra functionals form a complete filtered non-commutative associative Rota–Baxter algebra, giving rise to Spitzer's identity and a matrix factorization, from which we easily read off the Birkhoff decomposition of Connes–Kreimer.

Our work was partly motivated by Berg and Cartier's [4], which used the pre-Lie insertion product on Feynman graphs to investigate the renormalization Hopf algebra of Connes and Kreimer in terms of a lower triangular matrix representation of the Lie group G_A and its generating Lie algebra g_A. We will compare with their work in subsection 3.6.

The following is a summary of the paper. In the next section we review the notion of complete filtered Rota–Baxter algebras and Spitzer's identity. Theorem 5 is a generalization of Spitzer's classical result to associative non-commutative Rota–Baxter algebras, giving rise to a factorization theorem with recursively defined solutions. As an interesting and useful application of this theorem we formulate explicitly the factorization of unipotent triangular matrices with entries in a commutative Rota–Baxter algebra, and provide detailed calculations of small size matrices. Section 3 deals with the application of the matrix Rota–Baxter algebras to the Hopf–algebraic approach to the process of renormalization in perturbative QFT. Subsection 3.2 is the heart to this paper. We define a representation of the dual space of the renormalization Hopf algebra of Connes–Kreimer in terms of upper triangular matrices, using the coproduct structure map. This gives rise to a (Lie) group of upper triangular matrices with unit diagonal, representing the group of regularized Hopf algebra characters. The associated generating Lie algebra of infinitesimal characters, or derivations, is represented by nilpotent upper triangular matrices. We include several explicit examples, calculating the renormalization of amplitudes of Feynman diagrams up to three loops in dimensionally regularized four dimensional φ^4-theory as a quantum field theory toy model. Further applications and calculations of our results to renormalization in perturbative QFT are detailed in the companion article with J. M. Gracia-Bondía and J. C. Várilly [23]. We finish with a comment on the work of Berg and Cartier giving matrix Lie algebra representation using the pre-Lie algebra structure on Feynman graphs.

2 Complete Rota–Baxter algebras of triangular matrices

In this paper $\mathbb{K}$ denotes a field of characteristic zero with unit denoted by $1_{\mathbb{K}}$ and often by 1. All algebras are assumed to be unital associative $\mathbb{K}$-algebras, if not stated otherwise, with unit identified with $1_{\mathbb{K}}$.

2.1 Rota–Baxter algebras

We recall basic concepts and properties of Rota–Baxter algebras. The reader may consult the following literature [2, 3, 30, 31, 44, 45, 46] for more information.

By a **Rota–Baxter algebra** in this paper, we always mean a Rota–Baxter $\mathbb{K}$-algebra of weight one, that is, a $\mathbb{K}$-algebra A with a **Rota–Baxter map** $R : A \to A$, fulfilling the relation

$$R(x)R(y) + R(xy) = R\big(R(x)y + xR(y)\big),\ \forall x, y \in A. \tag{1}$$

So we also denote a Rota–Baxter algebra by a pair (A, R). For later reference we mention the example of the well-known dimensional regularization scheme where $A = \mathbb{C}[\varepsilon^{-1}, \varepsilon]]$, the field of Laurent series. One easily shows that A is a commutative Rota–Baxter algebra, with the Rota–Baxter map $R := R_{ms}$ being the pole part projection, known under the name of minimal subtraction (MS) scheme

$$R_{ms}\left(\sum_{k=-N}^{\infty} a_k\varepsilon^k\right) := \sum_{k=-N}^{-1} a_k\varepsilon^k.$$

For a Rota–Baxter map R on A, the map $\tilde{R} := \mathrm{id}_A - R$ is also a Rota–Baxter map. Further we have the mixed relation

$$R(x)\tilde{R}(y) = R\big(x\tilde{R}(y)\big) + \tilde{R}\big(R(x)y\big),\ \ \forall x, y \in A. \tag{2}$$

The images of both Rota–Baxter maps R and $\tilde{R}$ form non-unital subalgebras of A. As a trivial observation we state the fact that every algebra A is a Rota–Baxter algebra with Rota–Baxter pair id_A and $\widetilde{\mathrm{id}_A} = 0$. A homomorphism $f : (A, R) \to (A', R')$ between two Rota–Baxter algebras is a ring homomorphism such that $f \circ R = R' \circ f$.

A **Rota–Baxter ideal** of a Rota–Baxter algebra (A, R) is an ideal I of A such that $R(I) \subseteq I$. Let (A, R) be an associative Rota–Baxter algebra. The Rota–Baxter relation extends to the Lie algebra $\mathcal{L}_A$ with commutator $[x, y] := xy - yx$, $\forall x, y \in A$. In other words,

$$[R(x), R(y)] + R\big([x, y]\big) = R\big([R(x), y] + [x, R(y)]\big), \forall\ x, y\ \in \mathcal{L}_A,$$

making $(\mathcal{L}_A, R)$ into a **Rota–Baxter Lie algebra**.

Given a Rota–Baxter algebra (A, R), we define on the $\mathbb{K}$-vector space underlying A the following so-called **double Rota–Baxter product**

$$a *_R b := R(a)b + aR(b) - ab,\ \forall\, a, b\, \in A. \tag{3}$$

Then the vector space A equipped with the product $*_R$ and operator R is again a Rota–Baxter algebra (of weight one), denoted by A_R and called the **double Rota–Baxter algebra of** A. Further, The Rota–Baxter map R becomes an algebra homomorphism from A_R to A:

$$R(a *_R b) = R(a)R(b), \quad \forall a, b \in A. \tag{4}$$

For the Rota–Baxter map $\tilde{R}$ we find $\tilde{R}(a *_R b) = -\tilde{R}(a)\tilde{R}(b)$.

2.2 Complete Rota–Baxter algebras and Spitzer's identity

A **complete filtered Rota–Baxter algebra** [25] is defined to be a Rota–Baxter algebra $(\mathcal{A}, R)$ with a complete decreasing filtration of Rota–Baxter ideals $\{\mathcal{A}_n\}_{n\geq 0}$. So we have

$$\mathcal{A}_{n+1} \subset \mathcal{A}_n, \ \mathcal{A}_m\mathcal{A}_n \subseteq \mathcal{A}_{m+n}, \ R(\mathcal{A}_n) \subset \mathcal{A}_n, \ \varprojlim \mathcal{A}/\mathcal{A}_n \cong \mathcal{A}.$$

The last equation is equivalent to saying that $\mathcal{A}$ is complete with respect to the topology on $\mathcal{A}$ defined by the ideals $\mathcal{A}_n$.

Examples 1. Let (A, R) be a Rota–Baxter algebra with Rota–Baxter map R. We have the following complete Rota–Baxter algebras.

1. The power series algebra $\mathcal{A} := A[[x]]$ where the filtration is given by the degree in x and the Rota–Baxter operator $\mathcal{R} : \mathcal{A} \to \mathcal{A}$ acts on a power series via R through the coefficients, $\mathcal{R}\left(\sum_{n\geq 0} a_n x^n\right) := \sum_{n\geq 0} R(a_n)x^n$;
2. Let $\mathcal{H}_\mathcal{F}$ be the connected graded Hopf algebra of Feynman graphs. The regularized functionals $\mathrm{Hom}(\mathcal{H}_\mathcal{F}, A)$ where the filtration is given by the grading of $\mathcal{H}_\mathcal{F}$ and the Rota–Baxter operator acts on a linear map $f : \mathcal{H}_\mathcal{F} \to A$ by acting on the target space image of f. See Theorem 16;
3. The upper triangular matrices with entries in A where the filtration is given by the number of zero subdiagonals of the matrices and the Rota–Baxter operator acts on a matrix entry by entry. See § 2.3.

By the completeness of $(\mathcal{A}, R)$, the functions

$$\exp : \mathcal{A}_1 \to 1 + \mathcal{A}_1, \ \exp(a) := \sum_{n=0}^{\infty} \frac{a^n}{n!},$$

$$\log : 1 + \mathcal{A}_1 \to \mathcal{A}_1, \ \log(1 + a) := -\sum_{n=1}^{\infty} \frac{(-a)^n}{n}$$

are well-defined and are inverse of each other.

The algebraic formulation [3, 13, 33, 45] of the **classical Spitzer identity** [47] was first given for a commutative Rota–Baxter algebra A. In terms of the complete filtered commutative Rota–Baxter algebra $\mathcal{A} = A[[x]]$ in the first example above, it takes the form

$$\exp\big(-\mathcal{R}(\log(1+ax))\big) \;=\; \sum_{n=0}^{\infty}(-1)^n(Ra)^{[n]}x^n,\ \forall\, ax \in \mathcal{A}_1. \tag{5}$$

Here we inductively define

$$(Ra)^{[n+1]} := R\big((Ra)^{[n]}\, a\big)$$

with the convention that $(Ra)^{[0]} = 1$. We note that the element $b := \sum_{n=0}^{\infty}(-1)^n(Ra)^{[n]}x^n \in \mathcal{A}$ uniquely solves the recursive equation $b = 1 - \mathcal{R}(bax)$.

If $(\mathcal{A}, R)$ is a complete filtered commutative Rota–Baxter algebra of weight zero, i.e., R fulfills

$$R(a)R(b) = R\big(R(a)b + aR(b)\big),\ \ \forall a, b \in \mathcal{A}, \tag{6}$$

then Spitzer's identity (5) reduces to the identity

$$\exp\big(-R(a)x\big) = \sum_{n=0}^{\infty}(-1)^n(Ra)^{[n]}x^n,\ \forall\, a \in \mathcal{A},$$

which is well-known in the context of linear differential equations, where R is the Riemann integral. It follows naturally from (6) since $R(a)^n = n!(Ra)^{[n]}$ for $a \in A$.

In the following theorem, we generalize the Spitzer identity to non-commutative complete filtered Rota–Baxter algebras. The essential difference with the commutative case is the map $\chi : \mathcal{A}_1 \to \mathcal{A}_1$ appearing inside the exponential. This map was introduced in [24] and is defined recursively by

$$\chi(u) := u - BCH\big(R(\chi(u)), \tilde{R}\,(\chi(u))\big),\ \forall u \in \mathcal{A}_1 \tag{7}$$

using the **Baker–Campbell–Hausdorff** (BCH) formula

$$\exp(x)\exp(y) = \exp\big(x + y + BCH(x, y)\big)$$

which is a power series in $x, y \in \mathcal{A}_1$ of degree 2. One finds a simpler recursion for χ, using the factorization property implied by the χ map on $\mathcal{A}$.

Lemma 2. *[24] Let $\mathcal{A}$ be a complete filtered $\mathbb{K}$-algebra. $K : \mathcal{A} \to \mathcal{A}$ is a linear map. The map χ in (7) solves the following recursion*

$$\chi(u) := u + BCH\big(-K(\chi(u)), u\big),\ \ u \in \mathcal{A}_1. \tag{8}$$

Proof. In general for any $u \in \mathcal{A}$ we can write $u = K(u) + (\mathrm{id}_{\mathcal{A}} - K)(u)$ using linearity of K. The map χ then implies for $u \in \mathcal{A}_1$ that $\exp(u) = \exp\big(K(\chi(u))\big)\exp\big(\tilde{K}(\chi(u))\big)$. Further,

$$\exp\big(\tilde{K}(\chi(u))\big) \;=\; \exp\big(-K(\chi(u))\big)\exp(u)$$

$$= \exp\big(-K(\chi(u)) + u + BCH(-K(\chi(u)), u)\big).$$

Bijectivity of exp map then implies, that

$$\chi(u) - K(\chi(u)) \;=\; -K(\chi(u)) + u + BCH\big(-K(\chi(u)), u\big).$$

From which Equation (8) follows. □

More details on this recursion can also be found in [25, 40]. The factorization in the complete filtered algebra $\mathcal{A}$ follows from the map χ. Replacing the linear map K by a Rota–Baxter operator R, we arrive at **Spitzer's identity for non-commutative Rota–Baxter algebra**.

Theorem 3. [25] *Let $(\mathcal{A}, R, \mathcal{A}_n)$ be a complete filtered Rota–Baxter algebra of weight one. Let $a \in \mathcal{A}_1$ and recall that $\tilde{R} := \mathrm{id}_{\mathcal{A}} - R$.*

1. *The equation $b = 1 - R(ba)$ has a unique solution $b = \exp\big(-R(\chi(\log(1+a)))\big)$.*
2. *The equation $b' = 1 - \tilde{R}(ab')$ has a unique solution $b' = \exp\big(-\tilde{R}(\chi(\log(1+a)))\big)$.*

When $(\mathcal{A}, R)$ is commutative, the map χ reduces to the identity map, giving back Spitzer's classical identity (5). In general we have **Atkinson's theorem** [2], giving a decomposition.

Theorem 4. [2, 25] *Let $(\mathcal{A}, R)$ be a complete filtered Rota–Baxter algebra. For solutions b and b' in items* (1) *and* (2) *of Theorem 3, we have*

$$b(1+a)b' = 1, \text{ that is, } (1+a) = b^{-1}b'^{-1}. \tag{9}$$

If R is idempotent: $R^2 = R$, then this is the unique decomposition of $1+a$ into a product of an element in $1 + R(\mathcal{A}_1)$ with an element in $1 + \tilde{R}(\mathcal{A}_1)$.

Recall that $\tilde{R} := \mathrm{id}_{\mathcal{A}} - R$ is a Rota–Baxter operator if and only if R is. Further $\tilde{\tilde{R}} := \mathrm{id}_{\mathcal{A}} - \tilde{R} = R$. Thus by exchanging $\tilde{R}$ and R in the definition (7) of χ and in Theorem 3, we have the following variation which will be useful later.

Theorem 5. *Let $(\mathcal{A}, R, \mathcal{A}_n)$ be a complete filtered Rota–Baxter algebra of weight one. Define $\bar{\chi} : \mathcal{A}_1 \to \mathcal{A}_1$ by the recursion*

$$\bar{\chi}(u) := u - BCH\big(\tilde{R}(\bar{\chi}(u)), R(\bar{\chi}(u))\big), \ \forall u \in \mathcal{A}_1. \tag{10}$$

Let $a \in \mathcal{A}_1$.

1. *The equation $\bar{b} = 1 - R(a\bar{b})$ has a unique solution $\bar{b} = \exp\big(-R(\bar{\chi}(\log(1+a)))\big)$.*
2. *The equation $\bar{b}' = 1 - \tilde{R}(\bar{b}'a)$ has a unique solution $\bar{b}' = \exp\big(-\tilde{R}(\bar{\chi}(\log(1+a)))\big)$.*

3. *For solutions $\bar{b}$ and $\bar{b}'$ in item* (1) *and* (2), *we have*

$$\bar{b}'(1+a)\bar{b} = 1, \text{ that is, } (1+a) = \bar{b}'^{-1}\bar{b}^{-1}. \tag{11}$$

When R is idempotent, this gives the unique decomposition of $1+a$ into a product of an element in $1+\tilde{R}(\mathcal{A}_1)$ with an element in $1+R(\mathcal{A}_1)$.

The two decompositions in Theorem 4 and Theorem 5 are simply related as follows.

Proposition 6. *Let $\mathcal{A}^{\rm op}$ be the opposite algebra of $\mathcal{A}$, with product defined by $a \centerdot b := ba$. Let $O : \mathcal{A} \to \mathcal{A}^{\rm op}, O(a) = a$, be the canonical antihomomorphism of Rota–Baxter algebras. For $a \in \mathcal{A}_1$, let $1+a = b^{-1}b'^{-1}$ be the decomposition in Theorem 4, with b (resp. b') being solution of item (1) (resp. (2)) of Theorem 3. Then in*

$$1+a = O(1+a) = O\big(b'^{-1}\big) \centerdot O\big(b^{-1}\big) = O(b')^{-1} \centerdot O(b)^{-1},$$

the factor $O(b)$ (resp. $O(b')$) is the solution, in the opposite Rota–Baxter algebra $(\mathcal{A}^{\rm op}, R)$, from item (2) (resp. item (1)) of Theorem 5.

Proof. We just need to note that, under the anti-isomorphism $O : \mathcal{A} \to \mathcal{A}^{\rm op}$, the defining equations of χ, b and b' in Theorem 3, with multiplication in $\mathcal{A}$, are sent to the defining equations of $\bar{\chi}$, $\bar{b}$ and $\bar{b}'$ in Theorem 5, with multiplication in $\mathcal{A}^{\rm op}$. □

We give another variation of Theorem 3.

Proposition 7. *Let b and b' be solutions of the equations in item* (1) *respectively* (2) *of Theorem 3. Let $\breve{a} = (1+a)^{-1} - 1$. Then*

1. *$b^{-1} = \exp\big(R(\chi(\log(1+a)))\big)$ is the unique solution of the equation $c = 1 - R\big(\breve{a}c\big)$.*

2. *$b'^{-1} = \exp\big(\tilde{R}(\chi(\log(1+a)))\big)$ is the unique solution of the equation $c' = 1 - \tilde{R}\big(c'\breve{a}\big)$.*

3. *Further*

$$b^{-1} = 1 + R\big(a\ b'\big), \qquad b'^{-1} = 1 + \tilde{R}\big(b\ a\big).$$

Proof. (1) Since $\breve{a}$ is in $\mathcal{A}_1$, the equation $c = 1 - R(\breve{a}c)$ has a unique solution. We just need to check that the solution c is the inverse of b. Since b satisfies $b = 1 - R(ba)$, we have

$$\begin{aligned}
bc &= (1 - R(ba))(1 - R(\breve{a}c)) \\
&= 1 - R(ba) - R(\breve{a}c) + R(ba)R(\breve{a}c) \\
&= 1 - R(ba) - R(\breve{a}c) + R(baR(\breve{a}c)) + R(R(ba)\breve{a}c) - R(ba\breve{a}c) \\
&= 1 - R\big(ba(1 - R(\breve{a}c)\big) - R\big((1 - R(ba))\breve{a}c\big) - R(ba\breve{a}c) \\
&= 1 - R(bac) - R(b\breve{a}c) - R(ba\breve{a}c)
\end{aligned}$$

$$= \quad 1 - R\big(b(a + \check{a} + a\check{a})c\big),$$

since $a + \check{a} + a\check{a} = a + (1+a)\check{a} = a - a = 0$, we get $bc = 1$, as needed.

(2) The prove of the second statement is the same.

(3) Note that

$$\check{a}b^{-1} = (b'b - 1)b^{-1} = b' - b^{-1} = b' - (1+a)b' = -ab'.$$

So by item (1), we have

$$b^{-1} = 1 - R(\check{a}b^{-1}) = 1 + R(ab').$$

□

By Theorem 5.(1), the equation $c = 1 - R(\check{a}c)$ in Proposition 7 has a unique solution

$$c = \exp\big(-R(\bar{\chi}(\log(1+\check{a})))\big) = \exp\big(-R(\bar{\chi}(\log(1+a)^{-1}))\big) = \exp\big(-R(\bar{\chi}(-\log(1+a)))\big).$$

By the bijectivity of log and exp, we have

$$R\big(\chi(u)\big) = -R\big(\bar{\chi}(-u)\big), \ \forall\, u \in \mathcal{A}_1.$$

Thus we obtain

Corollary 8.

$$R\big(\chi(u) + \bar{\chi}(-u)\big) = 0, \ \forall\, u \in \mathcal{A}_1.$$

2.3 Decomposition of triangular matrices

Let A be a commutative $\mathbb{K}$-algebra. In the following one might replace "upper" by "lower" without restriction. The algebra of $n \times n$ matrices with entries in A is denoted by $\mathcal{M}_n(A)$. We have the subalgebras $\mathcal{M}_n^u(A) \subset \mathcal{M}_n(A)$, $1 \leq n < \infty$, of upper triangular matrices. We also let $\mathcal{M}_\infty^u(A)$ denote the algebra of $\infty \times \infty$ upper triangular matrices.

The subset $\mathfrak{M}_n(A)$, $1 \leq n \leq \infty$, of upper triangular matrices with unit diagonals, i.e. those $\alpha \in \mathcal{M}_n^u(A)$ such that $\alpha_{ii} = 1$, $i = 1, \ldots, n$, form a group under matrix multiplication. The inverse of $\alpha = (\alpha_{ij}) \in \mathfrak{M}_n(A)$ is given by the well-known, recursively defined inversion formula for upper triangular $n \times n$ matrices

$$(\alpha^{-1})_{ij} = -\alpha_{ij} - \sum_{k=i+1}^{j-1} (\alpha^{-1})_{ik}\alpha_{kj}. \tag{12}$$

Here commutativity of the algebra A is needed.

For each $n \leq \infty$, the algebra $\mathcal{M}_n^u(A)$ carries a natural decreasing filtration in terms of the number of zero upper subdiagonals. We denote by $\mathcal{M}_n^u(A)_1$ the upper triangular matrices with the main diagonal being zero, that is, the strict upper triangular matrices. Let $\mathcal{M}_n^u(A)_k$, $k > 1$, denote the ideal of strictly upper triangular matrices with zero on the main diagonal and on the first $k-1$ subdiagonals. We then have the decreasing filtration

$$\mathcal{M}_n^u(A) \supset \mathcal{M}_n^u(A)_1 \supset \cdots \supset \mathcal{M}_n^u(A)_{k-1} \supset \mathcal{M}_n^u(A)_k \supset \cdots, k < n,$$

with

$$\mathcal{M}_n^u(A)_k \, \mathcal{M}_n^u(A)_m \subset \mathcal{M}_n^u(A)_{k+m}.$$

We also have $\mathfrak{M}_n(A) = \mathbf{1} + \mathcal{M}_n^u(A)_1$, here $\mathbf{1}$ denotes the $n \times n$ unit matrix. It is easy to see that for any $n \leq \infty$, the filtration is complete, that is, $\mathcal{M}_n^u(A)$ is complete with respect to the topology defined by the ideals $\mathcal{M}_n^u(A)_k$, $k \geq 0$.

Thus the maps

$$\exp : \mathcal{M}_n^u(A)_1 \to \mathfrak{M}_n(A), \ \exp(Z) = \sum_{k=0}^{\infty} \frac{Z^n}{n!}, \tag{13}$$

$$\log : \mathfrak{M}_n(A) \to \mathcal{M}_n^u(A)_1, \ \log(\alpha) = -\sum_{k=1}^{\infty} \frac{(1-\alpha)^n}{n}. \tag{14}$$

are well-defined and are the inverse of each other. We denote $Z_\alpha = \log(\alpha)$ for $\alpha \in \mathfrak{M}_n(A)$.

Now let A be a Rota–Baxter algebra with Rota–Baxter operator R. We define a Rota–Baxter map $\mathcal{R}$ on $\mathcal{M}_n^u(A)$ by extending the Rota–Baxter map R entrywise, i.e. for the matrix $\alpha = (\alpha_{ij}) \in \mathcal{M}_n^u(A)$, define

$$\mathcal{R}(\alpha) = \big(R(\alpha_{ij})\big). \tag{15}$$

Theorem 9. *The triple $(\mathcal{M}_n^u(A), \mathcal{R}, \{\mathcal{M}_n^u(A)_k\}_{k\geq 1})$ forms a complete filtered Rota–Baxter algebra.*

Proof. We only need to show the Rota–Baxter relation for $\mathcal{R}$. For distinction, we use brackets [and] for matrix delimiters. Let $\alpha = [\alpha_{ij}]$ and $\beta = [\beta_{ij}]$ be in $\mathcal{M}_n^u(A)$. By the entry-wise definition of $\mathcal{R}$ in (15) and the Rota–Baxter relation, we have

$$\begin{aligned}
\mathcal{R}(\alpha)\,\mathcal{R}(\beta) &= \big[R(\alpha_{ij})\big]\big[R(\beta_{ij})\big] \\
&= \Big[\sum_k R(\alpha_{ik})R(\beta_{kj})\Big] \\
&= \Big[\sum_k \big(R(R(\alpha_{ik})\beta_{kj}) + R(\alpha_{ik}R(\beta_{kj})) - R(\alpha_{ik}\beta_{kj})\big)\Big] \\
&= \Big[\sum_k R(R(\alpha_{ik})\beta_{kj})\Big] + \Big[\sum_k R(\alpha_{ik}R(\beta_{kj}))\Big] - \Big[\sum_k R(\alpha_{ik}\beta_{kj})\Big] \\
&= \Big[R\big(\sum_k R(\alpha_{ik})\beta_{kj}\big)\Big] + \Big[R\big(\sum_k \alpha_{ik}R(\beta_{kj})\big)\Big] - \Big[R\big(\sum_k \alpha_{ik}\beta_{kj}\big)\Big] \\
&= \mathcal{R}\Big[\sum_k R(\alpha_{ik})\beta_{kj}\Big] + \mathcal{R}\Big[\sum_k \alpha_{ik}R(\beta_{kj})\Big] - \mathcal{R}\Big[\sum_k \alpha_{ik}\beta_{kj}\Big] \\
&= \mathcal{R}\big(\mathcal{R}([\alpha_{ij}])[\beta_{ij}]\big) + \mathcal{R}\big([\alpha_{ij}]\mathcal{R}([\beta_{ij}])\big) - \mathcal{R}\big([\alpha_{ij}]\,[\beta_{ij}]\big) \\
&= \mathcal{R}(\mathcal{R}(\alpha)\beta) + \mathcal{R}(\alpha\mathcal{R}(\beta)) - \mathcal{R}(\alpha\,\beta)
\end{aligned}$$

as needed. $\square$

Then we can apply Theorems 3, 4 and 5 to obtain decompositions of upper triangular matrices. For later applications to matrix Birkhoff decomposition in renormalization, we will stress the variation in Theorem 5.

Corollary 10. *Let α be in $\mathfrak{M}_n(A)$.*

1. *There is a factorization*
$$\alpha = \bar{\alpha}_+ \bar{\alpha}_-^{-1}, \tag{16}$$
of α into a product of an element in $\mathbf{1} + \tilde{\mathcal{R}}\big(\mathcal{M}_n^u(A)_1\big)$ and an element in $\mathbf{1} + \mathcal{R}\big(\mathcal{M}_n^u(A)_1\big)$ which is unique if $R^2 = R$.

2. *The factors $\bar{\alpha}_+$ and $\bar{\alpha}_-$ have the explicit expression*
$$\bar{\alpha}_+ = \exp\Big(\tilde{\mathcal{R}}\big(\bar{\chi}(Z_\alpha)\big)\Big), \quad \bar{\alpha}_- = \exp\Big(-\mathcal{R}\big(\bar{\chi}(Z_\alpha)\big)\Big). \tag{17}$$
Here $Z_\alpha = \log(\alpha)$ and $\bar{\chi}$ is defined in Equation (10) in analogy to χ.

3. *Further, $\bar{\alpha}_+^{-1}$ is the unique solution to the equation*
$$\bar{\beta}' = \mathbf{1} - \tilde{\mathcal{R}}(\bar{\beta}'(\alpha - \mathbf{1})) \tag{18}$$
and $\bar{\alpha}_-$ is the unique solution to the equation
$$\bar{\beta} = \mathbf{1} - \mathcal{R}((\alpha - \mathbf{1})\bar{\beta}). \tag{19}$$

Equations (18) and (19) are similar to the well-known recursions in renormalization theory where they are called Bogoliubov recursions [15, 16]. We will explore their connection in the next section through a matrix representation of regularized Feynman characters in renormalization in perturbative QFT.

Corollary 10 suggests that $\bar{\alpha}_-$ and $\bar{\alpha}_+$ in (16) can be calculated either by their exponential formulae (17) or directly from their recursive equations (18) and (19). We will first describe the recursive method to find the factor matrices $\bar{\alpha}_-$ and $\bar{\alpha}_+$. Later in subsection 3.5.1 we will calculate the factor matrices using the BCH-recursion $\bar{\chi}$ in (10).

As an example we first consider a straightforward 2×2 factorization of the matrix $\alpha \in \mathfrak{M}_2(A)$,
$$\alpha = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & R(a) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \tilde{R}(a) \\ 0 & 1 \end{pmatrix},$$
which simply follows from $R + \tilde{R} = \mathrm{id}_A$.

The case of 3×3 matrices is already more telling. For a given $\alpha = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ in $\mathfrak{M}_3(A)$, the equation that $\bar{\alpha}_- := \begin{pmatrix} 1 & \bar{a} & \bar{b} \\ 0 & 1 & \bar{c} \\ 0 & 0 & 1 \end{pmatrix}$ should fulfill is (19):

$$\begin{pmatrix} 1 & \bar{a} & \bar{b} \\ 0 & 1 & \bar{c} \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{1} - \mathcal{R}\left(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & \bar{a} & \bar{b} \\ 0 & 1 & \bar{c} \\ 0 & 0 & 1 \end{pmatrix}\right).$$

Equating the two sides entry-wise, we simply obtain,

$$\bar{a} = -R(a),\ \bar{c} = -R(c),\ \bar{b} = -R(a\bar{c} + b) = R(aR(c)) - R(b).$$

So

$$\bar{\alpha}_- = \begin{pmatrix} 1 & -R(a) & -R(b) + R(aR(c)) \\ 0 & 1 & -R(c) \\ 0 & 0 & 1 \end{pmatrix}$$

and by using Equation (12), we obtain

$$\bar{\alpha}_-^{-1} = \begin{pmatrix} 1 & R(a) & R(b) + R(aR(c)) - R(a)R(c) \\ 0 & 1 & R(c) \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & R(a) & R(b) + R(R(a)c) - R(ac) \\ 0 & 1 & R(c) \\ 0 & 0 & 1 \end{pmatrix}$$

We similarly use (18) to find

$$\bar{\alpha}_+^{-1} = \begin{pmatrix} 1 & -\tilde{R}(a) & -\tilde{R}(b) + \tilde{R}(\tilde{R}(a)c) \\ 0 & 1 & -\tilde{R}(c) \\ 0 & 0 & 1 \end{pmatrix}$$

and then use (12) to find

$$\bar{\alpha}_+ = \begin{pmatrix} 1 & \tilde{R}(a) & \tilde{R}(b) + \tilde{R}(a\tilde{R}(c)) - \tilde{R}(ac) \\ 0 & 1 & \tilde{R}(c) \\ 0 & 0 & 1 \end{pmatrix}$$

We thus obtain the unique factorization in (16):

$$\alpha = \bar{\alpha}_+\bar{\alpha}_-^{-1} = \begin{pmatrix} 1 & \tilde{R}(a) & \tilde{R}(b) + \tilde{R}(a\tilde{R}(c)) - \tilde{R}(ac) \\ 0 & 1 & \tilde{R}(c) \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & R(a) & R(b) + R(R(a)c) - R(ac) \\ 0 & 1 & R(c) \\ 0 & 0 & 1 \end{pmatrix} \tag{20}$$

To compute $\bar{\alpha}_\pm$ effectively in general, we have

Theorem 11. *Let $\alpha \in \mathfrak{M}_n(A)$.*

1. *The equation $\bar\beta = \mathbf{1} - \mathcal{R}\big((\alpha - \mathbf{1})\,\bar\beta\big)$ has a unique solution $\bar\beta = (\bar\beta_{ij})$ where*

$$\bar\beta_{ij} = -R(\alpha_{ij}) - \sum_{k=2}^{j-i} \sum_{i<l_1<l_2<\cdots<l_{k-1}<j} (-1)^{k+1} R\big(\alpha_{il_1} R(\alpha_{l_1l_2} R(\alpha_{l_2l_3} \cdots R(\alpha_{l_{k-1}j})\cdots))\big). \quad (21)$$

2. *The equation $\bar\beta' = \mathbf{1} - \tilde{\mathcal{R}}\big(\bar\beta'\,(\alpha - \mathbf{1})\big)$ has a unique solution $\bar\beta' = (\bar\beta'_{ij})$ where*

$$\bar\beta'_{ij} = -\tilde R(\alpha_{ij}) - \sum_{k=2}^{j-i} \sum_{i<l_1<l_2<\cdots<l_{k-1}<j} (-1)^{k+1} \tilde R\big(\tilde R(\cdots \tilde R(\tilde R(\alpha_{il_1})\alpha_{l_1l_2})\cdots \alpha_{l_{k-2}l_{k-1}})\alpha_{l_{k-1}j}\big). \quad (22)$$

Proof. We will prove Equation (21). The proof for the second equation is the same. Comparing two sides of $\bar\beta = \mathbf{1} - \mathcal{R}((\alpha - \mathbf{1})\bar\beta)$, we have

$$\bar\beta_{ij} = \begin{cases} 0, & i > j, \\ 1, & i = j, \\ -R\Big(\alpha_{ij} + \sum_{i<u<j} \alpha_{iu}\bar\beta_{uj}\Big), & i < j \end{cases}$$

So we just need to prove Equation (21) for $i < j$. For this we use induction on $j - i \geq 1$. There is nothing to prove when $j - i = 1$. Assuming the equation holds for $j - i \leq m$, then for $j - i = m+1$, we have

$$\begin{aligned}\bar\beta_{ij} &= -R\Big(\alpha_{ij} + \sum_{i<u<j} \alpha_{iu}\bar\beta_{uj}\Big) \\ &= -R(\alpha_{ij}) - R\bigg(\sum_{i<u<j} \alpha_{iu}\Big(-R(\alpha_{uj}) - \sum_{k=2}^{u-i} \sum_{u<l_1<l_2<\cdots<l_{k-1}<j} (-1)^{k+1} R\big(\alpha_{ul_1} R(\alpha_{l_1l_2} \cdots R(\alpha_{l_{k-1}j})\cdots)\big)\Big)\bigg).\end{aligned}$$

This is what we need. $\square$

The same formulae hold when R and $\tilde R$ are exchanged. These formulae will be useful in the next section when we consider anti-homomorphisms of Rota–Baxter algebras. Let us also remark at this stage that the simplicity of the formulae in Theorem 11 suggests a computational advantage of working with matrices over working directly with regularized Feynman characters.

3 Matrix calculus in perturbative QFT

The perturbative approach to quantum field theory provides theoretical predictions which match the experimental data with very high precision. At the heart of this scheme lies the idea to approximate physical quantities by power series expansions in terms of a supposed to be small parameter called the coupling constant, which measures the strength of interaction in the physical system under investigation. Of course, this idea raises immediately the question of convergence of such power series, but for such issues we refer the interested reader to more sophisticated presentations of perturbative QFT. The power series expansion is organized by Feynman graphs which serve to labels the terms in the power series. A Feynman graph is a collection of vertices and edges, reflecting the interaction and propagation of particles, respectively. For each term in the power series the number of loops of the corresponding Feynman graph fixes the order in the power series. The terms of the power series, called Feynman amplitudes, follow from Feynman rules which provide a way to translate a Feynman graph into a Feynman integral.

In general this perturbative ansatz is plagued with so-called ultraviolet divergencies, i.e. the Feynman integrals corresponding to Feynman graphs diverge in the limit of large momenta or equivalently small distances, and therefore seem to be useless in physics. For now we ignore the infrared problems appearing in theories with massless particles.

Renormalization theory in perturbative QFT was developed to cure these divergencies in a meaningful way. Actually, it consists of two steps, first a regularization prescription meant to control the divergences by extending the target space of the Feynman rules from the base field $\mathbb{C}$ to a particular algebra of regularized Feynman amplitudes. As a main example we mention here dimensional regularization, where this algebra is that of the Laurent series. Second, on the new target space algebra, a renormalization scheme is introduced, that isolates the problematic pieces, i.e., the divergent part. The renormalization process itself is made up in a recursive manner based on the self similar structure of Feynman graphs. Hereby we mean the fact that graphs of lower order –in terms of the number of loops– appear inside Feynman graphs of higher order.

Unfortunately, despite its impressive successes, renormalization was stigmatized, especially for its lack of a sound mathematical underpinning. One reason for this weakness might have been the fact that Feynman graphs in itself appeared to be unrelated to any mathematical structure that may possibly underlie the renormalization prescription.

This changed to a great extend with the original paper by Kreimer [35], followed by the work of Connes and Kreimer [16, 17, 18, 19]. The combinatorial-algebraic side of the process for renormalization is captured via a combinatorial Hopf algebra of Feynman graphs, essentially characterized by the coproduct map which organizes the decomposition of a Feynman graph into its subgraphs in a firm way. The analytical side of renormalization, i.e. the Feynman rules providing the Feynman integrals, is imbedded as algebra homomorphisms in the dual space of this Feynman graph Hopf algebra.

3.1 Connes–Kreimer renormalization Hopf algebra

Let us briefly summarize the main results of the Hopf algebra description of renormalization theory. For a general introduction to Hopf algebras the reader might want to consult the standard references like for instance [1, 29, 48]. Details about Hopf algebras in renormalization theory can be found in the survey articles [26, 40].

3.1.1 Feynman graphs and decorated rooted trees

Let us mention for completeness how Feynman graphs and decorated rooted trees are related. We assume that the reader is somewhat familiar with the former. The latter will be introduced in the next section. For more details on Feynman graphs, such as the notion of ultraviolet (UV) subgraphs, we refer to the standard literature such as [15]. Kreimer [37], and then Connes–Kreimer [16, 17] well-developed this link.

A Feynman graph is a collection of internal and external lines, or edges, and vertices of several types. A proper subgraph of a graph is determined by proper subsets of the set of internal edges and vertices. Of vital importance are so-called one particle irreducible (1PI) Feynman graphs, a connected graph that cannot be made disconnected by removing one of its internal edges. In general, a Feynman graph Γ beyond one-loop order, is characterized essentially by the appearance of its UV Feynman subgraphs $\gamma_i \subset \Gamma$. For instance, two proper Feynman subgraphs $\gamma_1,\ \gamma_2 \subset \Gamma$ might be nested, $\gamma_1 \subset \gamma_2 \subset \Gamma$, or disjoint, $\gamma_1 \cap \gamma_2 = \emptyset$. This hierarchy in which subgraphs appear inside another subgraph is best represented by a decorated rooted tree. The example below is borrowed from φ^4_{4dim}-theory.

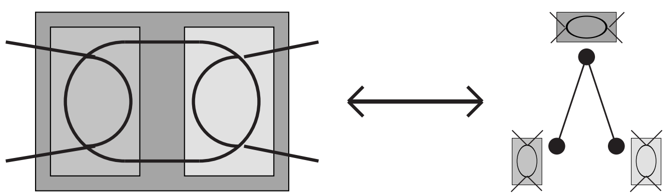
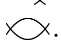


Figure 1: Example of a 3-loop graph with two nested disjoint UV subgraphs from φ^4_{4dim}-theory.

The two UV subgraphs  (little boxes inside) are disjoint, but nested inside another Feynman graph of the same type, . The rooted tree on the right represents this hierarchy. There is a third possibility, that subgraphs might be overlapping. Such Feynman graphs are represented by linear combinations of decorated rooted trees. Let us illustrate this with an example from φ^3_{6dim}-theory.

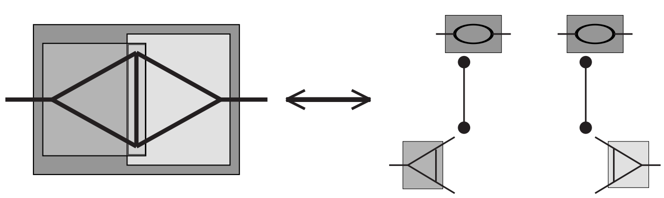
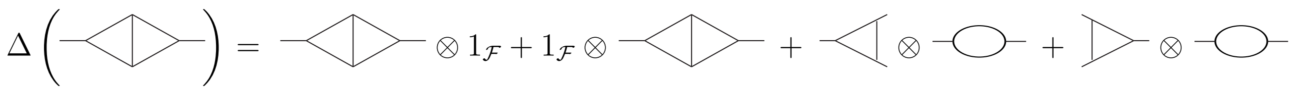
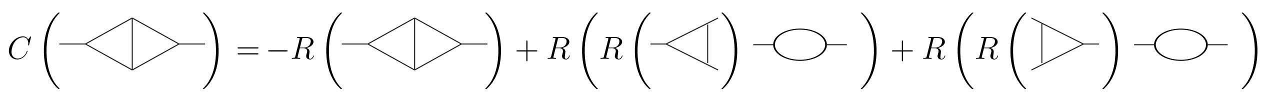


Figure 2: 2-loop example with overlapping UV subgraphs from φ^3_{6dim}-theory.

For a detailed treatment of this special and important case of overlapping structures we refer to [37]. The essential combinatorial operation on the set $\mathcal{F}$ of (equivalence classes of) Feynman graphs in the process of renormalization is a particular decomposition of such a Feynman graph into its UV subgraphs, well-known to the practitioners under the name of Bogoliubov's $\bar{\mathrm{R}}$-operation, or its solution by Zimmermann's forrest formula [50]. The concept of the combinatorial Hopf algebra of Feynman graphs, $\mathcal{H}_{\mathcal{F}}$, enters via its coproduct map, denoted $\Delta : \mathcal{H}_{\mathcal{F}} \to \mathcal{H}_{\mathcal{F}} \otimes \mathcal{H}_{\mathcal{F}}$, which organizes such a decomposition in a mathematical sound way, e.g. for the last examples from φ^3_{6dim}-theory we find



which must be compared with Bogoliubov's formula for the counter term $C(\Gamma)$ of a Feynman graph Γ. Only here we apply this notation for the counter term used in [16], and we use a symbolic graph notation. R denotes the renormalization scheme map

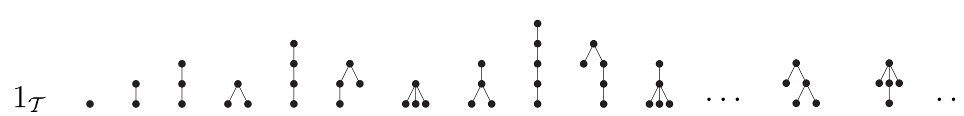


Kreimer [35] was the first to realize in renormalization this underlying Hopf algebra structure.

From a conceptual point of view, and having the non-specialist in mind, we feel that it is useful to work with decorated rooted trees. Also, this underlines the generality of our results, applicable to any theory in perturbative QFT and its renormalization by choosing the particular set of decorations dictated by the theory itself. Nevertheless, in terms of applications of Connes–Kreimer's Hopf algebra of renormalization to physics it is most naturally to formulate the Hopf algebra directly on Feynman graphs, once a theory has been specified. We included a simple calculation in section 3.4.3 using an example from φ^4_{4dim}-theory. Also, we refer the reader to the companion paper [23] with J. M. Gracia-Bondía and J. C. Várilly, which contains detailed applications of the presented results to renormalization in perturbative QFT. We should warn the reader, that in [23] a slightly different notation is used.

3.1.2 Connes–Kreimer Hopf algebra of rooted trees

A rooted tree t is made out of vertices and nonintersecting oriented edge, such that all but one vertex have exactly one incoming edge. We denote the set of vertices and edges of a rooted tree by $V(t)$, $E(t)$ respectively. The root is the only vertex with no incoming line. The empty tree is denoted by $1_{\mathcal{T}}$. Each rooted tree is a representant of an isomorphism class, and the $\mathbb{K}$-vector space freely generated by the set of all isomorphism classes will be denoted by $\mathcal{T}$.



Definition 1. The $\mathbb{K}$-algebra of non-planar rooted trees, denoted by $\mathcal{H}_{\mathcal{T}}$, is the polynomial algebra, generated by the symbols t, each representing an isomorphism class in $\mathcal{T}$. The unit is the empty

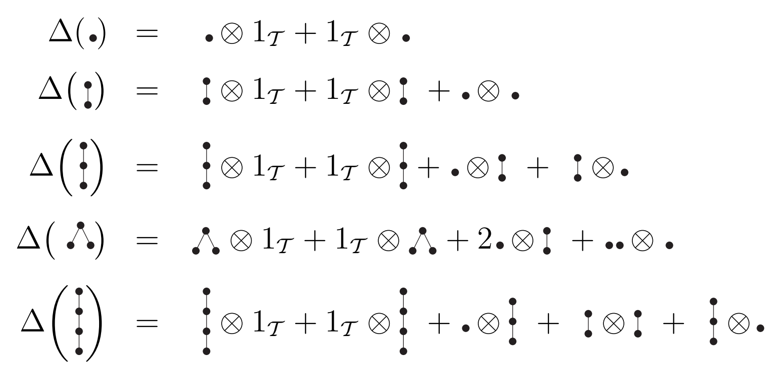
tree $1_{\mathcal{T}}$. The disjoint union of rooted trees serves as a product, denoted by juxtaposition, i.e. $m_{\mathcal{A}_{\mathcal{T}}}(t', t) =: t't$.

There exists a natural grading on $\mathcal{H}_{\mathcal{T}}$ in terms of the number of vertices of a rooted tree, $\#(t) := |V(t)|$. On forests of rooted trees, we extend it to $\#(t_1 \cdots t_n) := \sum_{i=1}^{n} \#(t_i)$. So that $\mathcal{H}_{\mathcal{T}} = \mathbb{K} \oplus \bigoplus_{n>0} \mathcal{H}_{\mathcal{T}}^{(n)}$ becomes a graded commutative $\mathbb{K}$-algebra.

To define a coproduct on $\mathcal{H}_{\mathcal{T}}$, we introduce the notion of admissible cuts on a rooted tree. A cut c_t on a rooted tree $t \in \mathcal{T}$ is a subset of edges $c_t \subset E(t)$. It becomes an admissible cut if and only if along a path from the root to any of the leaves of the tree t, one encounters at most one element of c_t. By removing the subset c_t of edges, each admissible cut c_t produces a forrest of pruned trees, denoted by P_{c_t}. The remaining part, which is a single rooted tree linked to the original root, is denoted by R_{c_t}. We exclude the cases, where $c_t = \emptyset$, such that $R_{c_t} = t$, $P_{c_t} = \emptyset$ and the full cut, such that $R_{c_t} = \emptyset$, $P_{c_t} = t$. The rooted tree algebra $\mathcal{H}_{\mathcal{T}}$ is equipped with a bialgebra structure by defining the coproduct $\Delta : \mathcal{H}_{\mathcal{T}} \to \mathcal{H}_{\mathcal{T}} \otimes \mathcal{H}_{\mathcal{T}}$ in terms of all admissible cuts C_t of a rooted tree t:

$$\Delta(t) = t \otimes 1_{\mathcal{T}} + 1_{\mathcal{T}} \otimes t + \sum_{c_t \in C_t} P_{c_t} \otimes R_{c_t}. \tag{23}$$

For example,



(24)

Latter we will use the shortened notation $\Delta(t) = \sum_{(t)} t_{(1)} \otimes t_{(2)}$. For products of trees we demand the compatibility $\Delta(t_1 \ldots t_n) := \Delta(t_1) \ldots \Delta(t_n)$.

Remark 12. *It is important to notice that the right hand side of $\Delta(t) \in \mathcal{H}_{\mathcal{T}} \otimes \mathcal{H}_{\mathcal{T}}$ is linear. Therefore we can write, for the $\mathbb{K}$-vector space $\mathbb{K}\mathcal{T}$ spanned by $\mathcal{T}$,*

$$\mathbb{K}\mathcal{T} \xrightarrow{\Delta} \mathcal{H}_{\mathcal{T}} \otimes \mathbb{K}\mathcal{T} \tag{25}$$

The counit $\epsilon : \mathcal{H}_{\mathcal{T}} \to \mathbb{K}$ simply maps the empty tree $1_{\mathcal{T}}$ to $1_{\mathbb{K}} \in \mathbb{K}$ and the rest to zero

$$\epsilon(t_1 \cdots t_n) := \begin{cases} 1_{\mathbb{K}} & t_1 \cdots t_n = 1_{\mathcal{T}}, \\ 0 & t_1 \cdots t_n \neq 1_{\mathcal{T}}. \end{cases} \tag{26}$$

Theorem 13. [16, 35] *The algebra $\mathcal{H}_{\mathcal{T}}$ equipped with the above defined compatible coproduct $\Delta : \mathcal{H}_{\mathcal{T}} \to \mathcal{H}_{\mathcal{T}} \otimes \mathcal{H}_{\mathcal{T}}$ and counit $\epsilon : \mathcal{H}_{\mathcal{T}} \to \mathbb{K}$ forms a connected, graded, commutative, non-cocommutative bialgebra. In fact it is a Hopf algebra, with antipode $S : \mathcal{H}_{\mathcal{T}} \to \mathcal{H}_{\mathcal{T}}$ defined recursively by $S(1_{\mathcal{T}}) = 1_{\mathcal{T}}$, $S(t) = -t - \sum_{c_t \in C_t} S(P_{c_t}) R_{c_t}$.*

We denote the restricted dual of $\mathcal{H}_{\mathcal{T}}$ by $\mathcal{H}^*_{\mathcal{T}}$. It contains linear maps from $\mathcal{H}_{\mathcal{T}}$ into $\mathbb{K}$. Let A be a commutative $\mathbb{K}$-algebra, we also consider the linear functionals $\mathrm{Hom}(\mathcal{H}_{\mathcal{T}}, A)$. Equipped with the convolution product:

$$f \star g := m_A(f \otimes g)\Delta : \mathcal{H}_{\mathcal{T}} \xrightarrow{\Delta} \mathcal{H}_{\mathcal{T}} \otimes \mathcal{H}_{\mathcal{T}} \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m_A} A \tag{27}$$

$\mathrm{Hom}(\mathcal{H}_{\mathcal{T}}, A)$ becomes a non-commutative $\mathbb{K}$-algebra with unit given by the counit ϵ of $\mathcal{H}_{\mathcal{T}}$.

A linear map $\phi : \mathcal{H}_{\mathcal{T}} \to A$ is called a **character** if ϕ is an algebra homomorphism. We denote the set of characters by $G_A := char_A \mathcal{H}_{\mathcal{T}}$.

Proposition 14. *The set of characters G_A forms a group with respect to the convolution product (27). The inverse of $\phi \in char_A \mathcal{H}_{\mathcal{T}}$ is given by $\phi^{-1} := \phi \circ S$.*

A linear map $Z : \mathcal{H}_{\mathcal{T}} \to A$ is called a **derivation**, or **infinitesimal character** if

$$Z(t_1 t_2) = Z(t_1)\epsilon(t_2) + \epsilon(t_1) Z(t_2), \;\; t_i \in \mathcal{H}_{\mathcal{T}}, \; i = 1, 2. \tag{28}$$

We have $Z(1_{\mathcal{T}}) = 0$. The set of infinitesimal characters is denoted by $g_A := \partial char_A \mathcal{H}_{\mathcal{T}}$. For any $Z \in \partial char_A \mathcal{H}_{\mathcal{T}}$ and $t \in \mathcal{H}_{\mathcal{T}}$ of degree $\#(t) = n < \infty$, we have $Z^{\star m}(t) = 0$ for $m > n$. This implies that the exponential $\exp^{\star}(Z)(t) := \sum_{k \geq 0} \frac{Z^{\star k}}{k!}(t)$, $Z \in \partial char_A \mathcal{H}_{\mathcal{T}}$, is finite, ending at $k = \#(t)$. Given the explicit base of rooted trees generating $\mathcal{H}_{\mathcal{T}}$, the A-module of derivations $\partial char_A \mathcal{H}_{\mathcal{T}}$ is generated by the dually defined infinitesimal characters, indexed by rooted trees

$$Z_t(t') := \delta_{t,t'} 1_{\mathbb{K}}, \; t, t' \in \mathcal{T}. \tag{29}$$

Proposition 15. *The set $g_A = \partial char_A \mathcal{H}_{\mathcal{T}}$ defines a Lie algebra when equipped with the commutator:*

$$\begin{aligned} [Z_{t'}, Z_{t''}] &:= Z_{t'} \star Z_{t''} - Z_{t''} \star Z_{t'} \\ &= \sum_{t \in \mathcal{T}} \big(n(t', t''; t) - n(t'', t'; t) \big) Z_t, \end{aligned} \tag{30}$$

where the $n(t', t''; t) \in \mathbb{N}$ denote so-called section coefficients, which count the number of single admissible cuts, $|c_t| = 1$, such that $P_{c_t} = t'$ and $R_{c_t} = t''$.

Ultraviolet divergencies demand for a regularization of the theory by replacing the base field $\mathbb{K}$ in $\mathcal{H}^*_{\mathcal{T}}$ as target space, by a commutative, unital Rota–Baxter $\mathbb{K}$-algebra (A, R). As an example we mentioned earlier the commutative Rota–Baxter algebra of Laurent series $\mathcal{A} := \mathbb{C}[\varepsilon^{-1}, \varepsilon]]$.

The following theorem recalls the Birkhoff decomposition of Hopf algebra characters from the work of Connes and Kreimer in the context of complete filtered Rota–Baxter algebras.

Theorem 16. [24] *Let $\mathcal{H}$ be the connected graded Hopf algebra of rooted trees, $\mathcal{H}_{\mathcal{T}}$ or Feynman graphs, $\mathcal{H}_{\mathcal{F}}$, and let (A, R) be a commutative Rota–Baxter algebra with Rota–Baxter operator R of weight 1. Let $\mathcal{A} := Hom(\mathcal{H}, A)$. Define a decreasing filtration $\{\mathcal{A}_n\}_n$ on $\mathcal{A}$ in duality to the increasing filtration on $\mathcal{H}$ by grading. Define $\mathcal{R} : \mathcal{A} \to \mathcal{A}$ by $\mathcal{R}(f) = R \circ f$, $f \in \mathcal{A}$. For a regularized character $\phi \in G_A$, let*

$$\phi = \phi_-^{-1} \star \phi_+ \tag{31}$$

be the algebraic Birkhoff decomposition of Connes–Kreimer [17] that gives the counter term ϕ_- and renormalization ϕ_+.

1. *The triple $(\mathcal{A}, \mathcal{R}, \{\mathcal{A}_k\}_k)$ is a complete Rota–Baxter algebra under convolution (27).*

2. *The decomposition (31) is the factorization of ϕ in Theorem 4, with $\phi_- \in \epsilon + \mathcal{R}(\mathcal{A}_1)$ and $\phi_+^{-1} \in \epsilon + \tilde{\mathcal{R}}(\mathcal{A}_1)$. For idempotent Rota–Baxter map R we have uniqueness of the factorization.*

3. *Explicitly we have*

$$\phi_- = \exp^\star\big(-\mathcal{R}\big(\chi(\log^\star(\phi))\big)\big), \qquad \phi_+^{-1} = \exp^\star\big(-\tilde{\mathcal{R}}\big(\chi(\log^\star(\phi))\big)\big)$$

   where $Z_\alpha := \log^\star(\phi) \in g_A$ with $\log^\star$ and $\exp^\star$ defined by convolution product (27), and χ is the BCH-recursion (7).

4. *Further, ϕ_- and ϕ_+^{-1} solve Bogoliubov's recursions, i.e. item (1) respectively item (2) of Theorem 5 on Spitzer's identity, for $\phi - \epsilon \in \mathcal{A}_1$:*

$$\begin{aligned} \phi_- &= \epsilon - \mathcal{R}\big(\phi_- \star (\phi - \epsilon)\big) & \qquad \phi_+^{-1} &= \epsilon - \tilde{\mathcal{R}}\big((\phi - \epsilon) \star \phi_+^{-1}\big) \\ &= \epsilon + \mathcal{R}\big(\exp^{\star_\mathcal{R}}\big(-\chi(\log^\star(\phi))\big) - \epsilon\big) & &= \epsilon - \tilde{\mathcal{R}}\big(\exp^{\star_\mathcal{R}}\big(\chi(\log^\star(\phi))\big) - \epsilon\big) \end{aligned}$$

Proposition 17. [24] *Let $\phi \in G_A$ be a Feynman rules character, with $Z_\alpha := \log^\star(\phi) \in g_A$. The map*

$$b[\phi] := \exp^{\star_\mathcal{R}}\big(-\chi(Z_\phi)\big) - \epsilon \tag{32}$$

defined in terms of the exponential with respect to the double Rota–Baxter product represents Bogoliubov's preparation map, also known under name $\bar{R}$-operation.

Here $\exp^{\star_\mathcal{R}}$ denotes the exponential map with respect to the Rota–Baxter double product (3) defined in the Rota–Baxter algebra $(\mathcal{A}, \mathcal{R}, \{\mathcal{A}_k\}_k)$.

3.2 Matrix representation of linear functionals

Recall that a Hopf algebra $\mathcal{H}$ is called filtered if there are $\mathbb{K}$-subvector spaces $\mathcal{H}^{(n)}$, $n \geq 0$ of $\mathcal{H}$ such that

1. $\mathcal{H}^{(n)} \subseteq \mathcal{H}^{(n+1)}$;

2. $\bigcup_{n\geq 0} \mathcal{H}^{(n)} = \mathcal{H}$;

3. $\mathcal{H}^{(p)}\mathcal{H}^{(q)} \subseteq \mathcal{H}^{(p+q)}$;

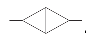
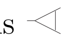
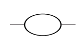
4. $\Delta(\mathcal{H}^{(n)}) \subseteq \bigoplus_{p+q=n} \mathcal{H}^{(p)} \otimes \mathcal{H}^{(q)}$.

$\mathcal{H}$ is called connected, if in addition $\mathcal{H}^{(0)} = \mathbb{K}$. Then for any $x \in \mathcal{H}^{(n)}$, we have

$$\tilde{\Delta}(x) := \Delta(x) - x \otimes 1 - 1 \otimes x \in \oplus_{p+q=n,p>0,q>0}\mathcal{H}^{(p)} \otimes \mathcal{H}^{(q)} \tag{33}$$

Definition 2. A subset X of $\mathcal{H}$ is called a **(left) subcoset** if X is $\mathbb{K}$-linearly independent and if $\mathbb{K}\,X$ is a left subcomodule of $\mathcal{H}$. A subcoset X is called **filtration ordered** if X is given an order that is compatible with the order from the filtration of $\mathcal{H}$. In other words, if $i \leq j$ and x_i is in $\mathcal{H}_n$, then x_j is in $\mathcal{H}_n$. A subcoset X is called a **1-subcoset** if $1_{\mathcal{H}}$ is in X.

It is clear that X is a left subcoset of $\mathcal{H}$ means that the coproduct Δ on $\mathcal{H}$ restricts to $\Delta_X : X \to \mathcal{H} \otimes X$. In other words, for any $x \in X$, with Sweedler's notation $\Delta_X(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, we have $x_{(2)} \in \mathbb{K}\,X$.

This definition reflects the combinatorial character of certain filtered Hopf algebras. Objects, mostly of graphical type, with specific substructures are disentangled. For an example see the coproduct example of the Feynman graph, , in subsection 3.1.1. The two subgraphs  and  are replaced by a vertex,  , with the cograph  as a result.

Our matrix representation applies to any connected filtered Hopf algebra $\mathcal{H}$ with a comodule. It naturally generalizes the classical construction of modules from comodules [40, 48].

Examples 18. Examples of such Hopf algebras with a left 1-subcoset X include

1. the Hopf algebra $\mathcal{H}_{\mathcal{T}}$ of rooted trees with $X = \mathcal{T}$ being the set of rooted trees. See Remark 12;

2. the Hopf (sub)algebra of ladder trees $\mathcal{H}_{\ell d} \subset \mathcal{H}_{\mathcal{T}}$ with $X = \ell d \subset \mathcal{T}$ being the set of ladder trees, i.e. trees whose vertices (except the root vertex) have only one incoming and at most one outgoing edge;

3. the Hopf algebra $\mathcal{H}_{\mathcal{F}}$ of Feynman graphs with $X = \mathcal{F}$ being the set of one particle irreducible Feynman graphs. This follows from a remark similar to Remark 12;

4. one of the above Hopf algebras with X being the first n ($n \geq 1$) elements in the subset there, with an ordering compatible with the filtration of the Hopf algebra.

3.2.1 The representation anti-homomorphism map

Let A be a commutative $\mathbb{K}$-algebra. Denote $A\,X = A \otimes \mathbb{K}X$ which is a free A-module with basis X. For a Hopf algebra $\mathcal{H}$ with a left 1-subcoset X, we will define a linear map $\Psi_{X,A}$: $\mathrm{Hom}(\mathcal{H}, A) \to \mathrm{End}(AX)$, eventually giving rise to a natural representation of $\mathrm{Hom}(\mathcal{H}, A)$ in terms of upper triangular matrices with entries in A. In the following the subscript X will be suppressed when possible.

Definition 3. Let $\mathcal{H}$ be a Hopf algebra with a filtration ordered left 1-subcoset X. A is a commutative $\mathbb{K}$-algebra. The linear map $\Psi_{A,X}$: $\mathrm{Hom}(\mathcal{H}, A) \to \mathrm{End}(A\,X)$ is defined by taking the composition

$$\Psi_{A,X}[f] : A \otimes \mathbb{K}X \xrightarrow{\mathrm{id}_A \otimes \Delta} A \otimes \mathcal{H} \otimes \mathbb{K}X \xrightarrow{\mathrm{id}_A \otimes f \otimes \mathrm{id}_{\mathbb{K}X}} A \otimes A \otimes \mathbb{K}X \xrightarrow{m_A \otimes \mathrm{id}_{\mathbb{K}X}} A \otimes \mathbb{K}X \tag{34}$$

for $f \in \mathrm{Hom}(\mathcal{H}, A)$.

In particular, for $x \in \mathbb{K}X$ with $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, we have

$$\Psi_A[f](x) = \sum_{(x)} f(x_{(1)}) x_{(2)}. \tag{35}$$

This justifies the short-hand notation $f \star \mathrm{id}_{\mathbb{K}X}$ for the composition (34) defining $\Psi_A[f]$.

As an example, consider the Hopf algebra $\mathcal{H}_{\mathcal{T}}$ of rooted trees with $X = \mathcal{T}$. Using the notation in (23), we have, for $f \in \mathrm{Hom}(\mathcal{H}_{\mathcal{T}}, A)$ and $t \in \mathcal{T}$,

$$\Psi_A[f](t) = f(1_{\mathcal{T}})t + f(t)1_{\mathcal{T}} + \sum_{c_t \in C_t} f(P_{c_t}) R_{c_t} \tag{36}$$

which is in $A\,\mathcal{T}$.

3.2.2 Coproduct matrix for filtered Hopf algebras

With the natural basis X of $A\,X$, the map Ψ_A gives a matrix representation of $\mathrm{Hom}(\mathcal{H}, A)$. Explicitly, fix a linear order $\{x_i\}_{i\geq 1}$ of the left 1-subcoset X that is compatible with the filtration of the Hopf algebra $\mathcal{H}$. Then the coproduct Δ of $\mathcal{H}$ restricted to X writes

$$\Delta(x_j) = \sum_{i=1}^{\infty} X_{ij} \otimes x_i \tag{37}$$

for uniquely determined X_{ij} in $\mathcal{H}$. Note the order of i and j. For all $i < j$, $\#(X_{ij}) < \#(x_j)$. This leads to the

Definition 4. Let $\mathcal{H}$ be a filtered Hopf algebra. Fix a linear order $\{x_i\}_{i\geq 1}$ of the filtration ordered left 1-subcoset X. We define the $|X| \times |X|$ matrix

$$M_{\mathcal{H}} := M_{\mathcal{H},X} := (X_{ij})$$

in $\mathcal{M}_{|X|}(\mathcal{H})$, called the **coproduct matrix of** $\mathcal{H}$ (with respect to X).

Then $M_{\mathcal{H}}$ is upper triangular since $\mathcal{H}$ is filtered. For a different choice of basis, we get conjugate matrices. Under a fixed ordering $(x_1, \cdots, x_n, \cdots)$ of X, we obtain an isomorphism

$$A\,X \to A^{|X|}$$

sending x_n to the unit column vector $|x_n\rangle$ with 1 in the n-th entry and zero elsewhere. Here we use the familiar bra-ket notation to denote abstract vectors as column vectors. We likewise obtain the isomorphism

$$\mathrm{End}(AX) \to \mathcal{M}^u_{|X|}(A)$$

sending $\Psi_A[f]$ to the upper triangular matrix

$$\widehat{f} = (f_{ij}) := \big(f(X_{ij})\big). \tag{38}$$

We further have an isomorphism

$$\mathrm{Hom}(A\,X, A) \to A^{|X|}$$

sending the dual basis x_n^*, defined by $x_n^*(x_m) = \delta_{n,m}$, to the row vector $\langle x_n|$ with 1 in the n-th entry and zero elsewhere. The image of $f \in \mathrm{Hom}(A\,X, A)$ under this isomorphism is denoted by $\langle f|$. So we have

$$\langle f| = (f(x_1), \cdots, f(x_n), \cdots).$$

We will often use these isomorphisms as identifications when there is no danger of confusion. In particular, $\widehat{f}$ is often identified with $\Psi[f]$.

As an illustrating example we consider again the Hopf algebra $\mathcal{H}_{\mathcal{T}}$ to rooted trees with a truncated X being

$$\mathcal{T}_{(6)} := \Big\{ e_1 := 1_{\mathcal{T}},\ e_2 := \bullet,\ e_3 := \ ,\ e_4 := \ ,\ e_5 := \ ,\ e_6 := \ \Big\} \tag{39}$$

with the displayed linear order. Then we have the corresponding column vectors

$$\Big\{ |1_{\mathcal{T}}\rangle,\ |\bullet\rangle,\ |\ \rangle,\ |\ \rangle,\ |\ \rangle,\ |\ \rangle \Big\} \tag{40}$$

In particular,

$$|1_{\mathcal{T}}\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\langle 1_{\mathcal{T}}| = (1, 0, \cdots, 0).$$

From the coproducts in (24), we obtain the truncated unipotent coproduct matrix from Definition 4

$$M_{\mathcal{H}} = \begin{pmatrix} 1_{\mathcal{T}} & e_2 & e_3 & e_4 & e_5 & e_6 \\ 0 & 1_{\mathcal{T}} & e_2 & e_3 & e_2e_2 & e_4 \\ 0 & 0 & 1_{\mathcal{T}} & e_2 & 2e_2 & e_3 \\ 0 & 0 & 0 & 1_{\mathcal{T}} & 0 & e_2 \\ 0 & 0 & 0 & 0 & 1_{\mathcal{T}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_{\mathcal{T}} \end{pmatrix} \tag{41}$$

For an $f \in \mathrm{Hom}(\mathcal{H}_{\mathcal{T}}, A)$ we then obtain its representation $\widehat{f}_{(6)}$ in terms of an upper triangular matrix by applying f to $M_{\mathcal{H}}$ entry by entry. In each column we just see the image of the left hand side of $\Delta(e_i)$ under f.

For an infinitesimal character $Z \in g$ we find particularly simple nilpotent matrices

$$\Psi[Z](t) = Z(t)1_{\mathcal{T}} + \sum_{\substack{c_t \in C_t \\ |c_t|=1}} Z(t')t''. \tag{42}$$

Here, due to Leibniz' relation (28) for such infinitesimal characters

$$Z(t't'') = \epsilon(t')Z(t'') + Z(t')\epsilon(t'')$$

we only need to consider single admissible cuts, $|c_t| = 1$. The sum on the right hand side of (42) therefore goes over all decompositions of the tree t, resulting from the removal of exactly one edge. The tree t' denotes the pruned subtree of tree t, and t'' the denotes the tree, which is still connected to the root of t. For the generators Z_t we find

$$\Psi[Z_{t_1}](t_2) = \delta_{t_1,t_2}1_{\mathcal{T}} + \sum_{\substack{c_{t_2} \in C_{t_2} \\ |c_{t_2}|=1}} \delta_{t_1,t_2'}t_2'' = \delta_{t_1,t_2}1_{\mathcal{T}} + \sum_t n(t_1,t;t_2)t.$$

As an example, let us calculate

$$\begin{aligned} \Psi[Z_\bullet]\Big(\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \end{smallmatrix}\Big) &= 2Z_\bullet(\bullet)\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} = 2\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}, \qquad \Psi[Z_\bullet](e_4) = Z_\bullet(\bullet)\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} = \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}, \qquad \Psi[Z_\bullet](e_6) = Z_\bullet(\bullet)\begin{smallmatrix}\bullet\\|\\\bullet\\|\\\bullet\end{smallmatrix} = \begin{smallmatrix}\bullet\\|\\\bullet\\|\\\bullet\end{smallmatrix} \\ \Psi\Big[Z_{\begin{smallmatrix}\bullet\\|\\\bullet\\|\\\bullet\end{smallmatrix}}\Big](e_6) &= Z_{\begin{smallmatrix}\bullet\\|\\\bullet\\|\\\bullet\end{smallmatrix}}(e_4)\bullet = \bullet, \qquad \Psi\Big[Z_{\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}}\Big](e_4) = Z_{\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}}\Big(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\Big)\bullet = \bullet, \qquad \Psi\Big[Z_{\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}}\Big](e_6) = Z_{\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}}\Big(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\Big)\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} = \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}. \end{aligned}$$

For the generators Z_t of the Lie algebra of derivations we find in this particular example the following matrices.

$$\widehat{Z}_{\begin{smallmatrix}\bullet\\|\\\bullet\\|\\\bullet\end{smallmatrix}} = \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \quad \widehat{Z}_{\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \end{smallmatrix}} = \begin{pmatrix} 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \quad \widehat{Z}_{\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}} = \begin{pmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \tag{43}$$

$$\widehat{Z}_{\substack{\bullet\\[-2pt] \mid \\[-2pt] \bullet}} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \widehat{Z}_{\bullet} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{44}$$

For a character $\phi \in \mathrm{Hom}(\mathcal{H}_\mathcal{T}, A)$, i.e. an algebra homomorphism from $\mathcal{H}_\mathcal{T}$ to A, $\phi(1_\mathcal{T}) = 1_A = 1_\mathbb{K}$, we find the upper triangular matrix with unit diagonal $\widehat{\phi} \in \mathfrak{M}_6(A) \subset \mathcal{M}^u_6(A)$

$$\widehat{\phi} = \begin{pmatrix} 1_\mathbb{K} & \phi(e_2) & \phi(e_3) & \phi(e_4) & \phi(e_5) & \phi(e_6) \\ 0 & 1_\mathbb{K} & \phi(e_2) & \phi(e_3) & \phi(e_2)^2 & \phi(e_4) \\ 0 & 0 & 1_\mathbb{K} & \phi(e_2) & 2\phi(e_2) & \phi(e_3) \\ 0 & 0 & 0 & 1_\mathbb{K} & 0 & \phi(e_2) \\ 0 & 0 & 0 & 0 & 1_\mathbb{K} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_\mathbb{K} \end{pmatrix} \tag{45}$$

The counit, which is a character by definition (26), is represented by the unit matrix $\widehat{\epsilon} = \mathbf{1}$. The structure of the representation matrices shows that the row vector

$$\langle 1_\mathcal{T} |\, \widehat{\phi} = (1, 0, 0, 0, 0, 0)\, \widehat{\phi}$$

is

$$\langle \phi | := (1_\mathbb{K}, \phi(e_2), \cdots, \phi(e_6)).$$

Returning to the general case, where $\mathcal{H}$ contains a (filtration ordered left) 1-subcoset X, and (A, R) is a Rota–Baxter algebra, we have the following key property of $\Psi_{A,X}$ that establishes the connection between the Rota–Baxter algebras $\mathrm{Hom}(\mathcal{H}, A)$ and $\mathcal{M}^u_n(A)$. Remember that we suppressed the subindex X if there is no danger of confusion.

Theorem 19. *Let $\mathcal{H}$ be a connected filtered Hopf algebra with a filtration ordered (left) 1-subcoset $X \subset \mathcal{H}$. Let A be a Rota–Baxter algebra (of weight 1).*

1. *The linear map*
$$\Psi_A := \Psi_{A,X} : \mathrm{Hom}(\mathcal{H}, A) \to \mathcal{M}^u_{|X|}(A)$$
is an anti-homomorphism of Rota–Baxter algebras that is continuous with respect to the topologies defined by the filtrations on the filtered Rota–Baxter algebras. More precisely, for any $m \geq 1$, there is $N \geq 1$, such that for all $k \geq N$ and $f \in \mathrm{Hom}(\mathcal{H}, A)$ with $f(\mathcal{H}^k) = 0$, we have $\Psi_A[f] \in \mathcal{M}^u_{|X|}(A)_m$.

2. *The first row of $\widehat{f} := \Psi_{A,X}[f] \in \mathcal{M}^u_{|X|}(A)$ is $\langle f | := \langle 1_X |\, \widehat{f} = (f(x))_{x \in X}$.*

3. *If X is a generating set of the algebra $\mathcal{H}$, then the map $\Psi_{A,X}$ restricts to an injective map from the multiplicative group $G_A = \mathrm{char}_A\mathcal{H}$ of algebra homomorphism $\mathcal{H} \to A$ to* $\mathrm{End}(A\,X)$.

Proof. (1) We denote the composition in $\mathrm{End}(AX)$ by concatenation. Let $f, g \in \mathrm{Hom}(\mathcal{H}, A)$, and $x \in X$. We will use Sweedler's notation of $\Delta(x) = x_{(1)} \otimes x_{(2)}$ and the short hand notation $\Psi_A[f] = f \star \mathrm{id}_{AX}$ in (35). By the definition of Ψ_A, the coassociativity of the coproduct Δ and the definition of the convolution product $\star$, we have

$$
\begin{aligned}
\Psi_A[f]\Psi_A[g](x) &= (f \star \mathrm{id}_{AX})\big((g \star \mathrm{id}_{AX})(x)\big) \\
&= (f \star \mathrm{id}_{AX}(g(x_{(1)})x_{(2)}) \\
&= g(x_{(1)})\big(f(x_{(2)(1)})x_{(2)(2)}\big) \\
&= \big(g(x_{(1)(1)})f(x_{(1)(2)})\big)x_{(2)} \\
&= (g \star f)(x_{(1)})\, x_{(2)} \\
&= \big((g \star f) \star \mathrm{id}_{AX}\big)(x) \\
&= \Psi_A[g \star f](x)
\end{aligned}
\tag{46}
$$

This proves the anti-homomorphism property.

Further, $\mathcal{R}$ is the Rota–Baxter map on $\mathrm{Hom}(\mathcal{H}, A)$, and we let $\widehat{\mathcal{R}}$ denote the Rota–Baxter operator on $\mathcal{M}^u_{|X|}(A)$ by acting entry-wise. Let $x \in X$ with $\Delta(x) = x_{(1)} \otimes x_{(2)}$. We have, for $f \in \mathrm{Hom}(\mathcal{H}, A)$,

$$
\begin{aligned}
\Psi_{A,X}[\mathcal{R}(f)](x) &= \mathcal{R}(f)(x_{(1)})x_{(2)} \\
&= R\big(f(x_{(1)})\big)x_{(2)} \\
&= \widehat{\mathcal{R}}(f(x_{(1)})x_{(2)}) \\
&= \widehat{\mathcal{R}}\big(\Psi_{A,X}[f]\big)(x).
\end{aligned}
$$

This proves that $\Psi_{A,X}$ is compatible with the Rota–Baxter operators.

The continuity is verified using the fact that the linear order on X is compatible with the filtration of $\mathcal{H}$.

(2) Since X is assumed to contain $1 = 1_X$ and any linear order of X is assumed to be compatible with the filtration of $\mathcal{H}$, we have

$$X = (x_1 := 1, x_2, \cdots)$$

for any choice of such linear ordering. So by Equation (33), we have

$$\Delta(x_j) = x_j \otimes 1 + \sum_{i \geq 2} X_{ij} \otimes x_i.$$

Therefore the first row vector of the coproduct matrix $M_{\mathcal{H}}$ is $(x_1, x_2, \cdots)$. So the first row of $\widehat{f}$ is $(f(x_1), f(x_2), \cdots)$.

(3) Suppose $\Psi_A[f]=0$. Then $\widehat{f}=0$. By item (2), $f(x)=0$ for all $x\in X$. Since $f:\mathcal{H}\to A$ is an algebra homomorphism and X is an algebra generating set of $\mathcal{H}$, we have $f=0$. This proves the injectivity. □

3.3 Renormalization of Feynman rules matrices

We now consider the Hopf algebra $\mathcal{H}_{\mathcal{F}}$ of Feynman graphs. Let $\mathcal{F}$ be the set of (equivalence classes of) Feynman graphs with a fixed linear ordering $\Gamma_1:=1,\Gamma_2,\cdots$ compatible with the filtration $\mathcal{H}_{\mathcal{F}}^{(n)}$ of $\mathcal{H}_{\mathcal{F}}$. Then with X being either $\mathcal{F}$ or the subset $\mathcal{F}_n$ of the first n-elements in $\mathcal{F}$, the results of last subsection apply.

With the representation of a regularized Feynman character $\phi:\mathcal{H}_{\mathcal{F}}\to A$ by an upper triangular matrix obtained in Theorem 19, we can apply the complete filtered Rota–Baxter algebra structure on the matrices to decompose the representation $\widehat{\phi}$, via Corollary 10, into the inverse of the counter term matrix $\widehat{\phi}_-$ and the renormalized matrix $\widehat{\phi}_+$, giving rise to an analog of Connes–Kreimer's algebraic Birkhoff decomposition for regularized Feynman characters. Then by Theorem 19.3, these later matrices recover the counter term and renormalization of the regularized Feynman character ϕ.

In light of the anti-homomorphism Ψ_A in Theorem 19, we expect that Ψ_A exchanges the order of the Birkhoff decomposition of ϕ in (31). The precise relation between these two decompositions is provided by the following theorem by applying Theorem 19 and Proposition 6. We will use $\mathcal{R}$ to denote both, the Rota–Baxter map of $\mathrm{Hom}(\mathcal{H}_{\mathcal{F}},A)$ and $\mathcal{M}_n^u(A)$, since from the context it will be clear which one is used.

Theorem 20. *Let ϕ be a regularized character from $\mathcal{H}_{\mathcal{F}}$ to (A,R) and $Z_\phi=\log^\star(\phi)\in g_A$. For a fixed $1\le n\le\infty$, let $X=\mathcal{F}_n$ be the first n Feynman graphs and let $\widehat{\phi}\in\mathfrak{M}_n(A)\subset\mathcal{M}_n^u(A)$ be the upper triangular matrix representation of ϕ given by (38). Let*

$$\phi=\phi_-^{-1}\star\phi_+$$

be the Birkhoff decomposition of $\phi\in G_A$ in (31). Let $\widehat{(\phi_-)}$ and $\widehat{(\phi_+)}$ be the matrix representations of $\phi_-\in G_A$ respectively $\phi_+\in G_A$.

1. *Then $\widehat{(\phi_-)}$ is the unique solution of the equation*

$$\beta=\mathbf{1}-\mathcal{R}\big((\widehat{\phi}-\mathbf{1})\,\beta\big) \tag{47}$$

explicitly given by

$$\widehat{(\phi_-)}=\exp\Big(-\mathcal{R}(\bar{\chi}(\widehat{Z}_\phi)\Big),$$

and $\widehat{(\phi_+)}^{-1}$ is the unique solution of the equation

$$\beta'=\mathbf{1}-\tilde{\mathcal{R}}(\beta'(\widehat{\phi}-\mathbf{1})) \tag{48}$$

explicitly given by

$$\widehat{(\phi_+)}^{-1} = \exp\left(-\tilde{\mathcal{R}}(\bar{\chi}(\widehat{Z}_\phi)\right).$$

Here $\bar{\chi}$ is the BCH-recursion defined in equation (10).

2. *$\widehat{\phi}$ factorizes as*

$$\widehat{\phi} = \widehat{(\phi_+)}\widehat{(\phi_-)}^{-1}. \tag{49}$$

3. *The first row vector of $\widehat{(\phi_\pm)}$ is $\left(\phi_\pm(\Gamma)\right)_{\Gamma\in\mathcal{F}_n}$. This can be summarized by the linear renormalization matrix-vector equation for $\langle\phi_+| := \langle 1_\mathcal{F}|\,\widehat{(\phi_+)}$ and $\langle\phi| := \langle 1_\mathcal{F}|\,\widehat{\phi}$:*

$$\langle\phi_+| = \langle\phi|\ \widehat{(\phi_-)}, \tag{50}$$

following from the matrix Birkhoff decomposition in (49).

Remark 21. The renormalized matrix $\widehat{(\phi_+)}$ uniquely solves the equation

$$c = \mathbf{1} - \tilde{\mathcal{R}}\big((\widehat{\phi}^{\,-1} - \mathbf{1})\ c\big), \tag{51}$$

where we used Propositions 6 and 7. Instead of inverting $\widehat{(\phi_+)}^{-1} \longrightarrow \widehat{(\phi_+)}$ at the end of calculation, here we first invert the Feynman rules character matrix $\widehat{\phi}$, which we can do simply and efficiently using the inverse coproduct matrix, see below.

3.4 Summary/Algorithm and examples

We now discuss applications of our matrix approach to renormalization in Theorem 20. One advantage of the matrix approach lies in the efficiency to calculate the matrices $\widehat{(\phi_-)}$ and $\widehat{(\phi_+)}$ of the matrix Birkhoff factorization directly in the matrix algebra, see Corollary 10, using the recursions of Theorem 11.

We outline an algorithm which is followed by examples.

3.4.1 Calculation of the coproduct matrix and its inverse

Let $X := \mathcal{T}$ be the set of rooted trees (or Feynman graphs) and fix an ordering of $\mathcal{T}$ that is compatible with the grading of $\mathcal{T}$, $t_1 := 1_\mathcal{T}, t_2, \cdots, t_n, \cdots$. Then the coproduct Δ on $\mathcal{H}_\mathcal{T}$ is given by

$$\Delta(t) = \sum_{(t)} t' \otimes t'',$$

or in our ordering,

$$\Delta(t_j) = \sum_{i=1}^{j} T_{ij} \otimes t_i \tag{52}$$

where T_{ij} is in $\mathcal{H}_\mathcal{T}$, t_i is in $\mathcal{T}$, see Remark 12, and $\#(T_{ij}) \leq \#(t_j)$. Define

$$M_\mathcal{H} = (T_{ij})$$

to be the coproduct matrix of $\mathcal{H}_\mathcal{T}$. $M_\mathcal{H}$ is an $\infty \times \infty$ upper triangular matrix with entries in $\mathcal{H}_\mathcal{T}$ and unit on the diagonal. The entries T_{ij} can be obtained by the B^+ operator of Connes and Kreimer providing a recursive way to calculate the coproduct. One can also consider the truncated coproduct matrix by restriction to the subspace of the first n graphs $t_1, \cdots, t_n$ and obtain a finite upper triangular matrix in $\mathcal{M}_n^u(\mathcal{H})$.

The direct calculation of the renormalization matrix $\widehat{\phi_+}$ in Theorem 20 respectively Remark 21 demands an inversion of character matrices. In Proposition 14 we stated the fact that the inverse of a Hopf algebra character $\phi \in G$ is given by composition with the antipode anti-homomorphism, $\phi^{-1} = \phi \circ S$. The representation matrix of ϕ loosely speaking follows from (38), $\widehat{\phi} := \phi \circ M_\mathcal{H}$. As we want to calculate the inverse coproduct matrix, $M_\mathcal{H}^{-1} = (T_{ij}^{-1})$, such that $\widehat{\phi}^{-1} := \phi \circ M_\mathcal{H}^{-1}$, we may take another look at the coproduct matrix $M_\mathcal{H}$.

For this purpose we go back to Definition 3, where we define the commutative $\mathbb{K}$-algebra $A := \mathcal{H}$. For a filtered Hopf algebra $\mathcal{H}$ with a filtration ordered (left) 1-subcoset X this provides us with an upper triangular matrix representation of $\mathrm{Hom}(\mathcal{H}, \mathcal{H})$. The coproduct matrix thereby represents the identity homomorphism $\mathrm{id}_\mathcal{H} : \mathcal{H} \to \mathcal{H}$

$$\Psi_\mathcal{H}[\mathrm{id}_\mathcal{H}] = \widehat{\mathrm{id}_\mathcal{H}} = M_\mathcal{H}. \tag{53}$$

We suppressed the 1-subcoset $X \subset \mathcal{H}$ in the notation.

As an upper triangular matrix, the inverse of $M_\mathcal{H}$ follows immediately in a recursive manner from Equation (12). But we want to make a little detour, using the aforementioned representation point of view in (53). This will provide us with a non-recursive simple formula for calculating $M_\mathcal{H}^{-1}$.

The antipode S in a Hopf algebra $\mathcal{H}$ is an anti-homomorphism, defined as the solution of the equation $S \star \mathrm{id}_\mathcal{H} = \epsilon = \mathrm{id}_\mathcal{H} \star S$. From a convolution product point of view the antipode is the inverse of the identity map, $S = \mathrm{id}_\mathcal{H}^{\star -1}$. For connected filtered Hopf algebras the identity can simply be written as $\mathrm{id}_\mathcal{H} = \exp^\star(\log^\star(\mathrm{id}_\mathcal{H}))$, using the bijectivity of $\exp^\star$ and $\log^\star$, and therefore we have $S = \exp^\star(-\log^\star(\mathrm{id}_\mathcal{H}))$. We already mentioned the trivial fact that every algebra is a Rota–Baxter algebra, with Rota–Baxter operator pair id and $\tilde{\mathrm{id}} = 0$. The space $\mathrm{End}(\mathcal{H})$ is equipped with two products, composition and convolution, both forming an associative algebra. Spitzer's identity for the non-commutative algebra $(\mathrm{End}(\mathcal{H}), \star)$ with Rota–Baxter map $\mathrm{id}_\mathcal{H} : \mathrm{End}(\mathcal{H}) \to \mathrm{End}(\mathcal{H})$ then implies for the antipode

$$S = \exp^\star\big(-\mathrm{id}_\mathcal{H}\big(\log^\star(\epsilon + (\mathrm{id}_\mathcal{H} - \epsilon))\big)\big)$$

to be a solution of the equation $b = \epsilon - \mathrm{id}_\mathcal{H}\big(b \star (\mathrm{id}_\mathcal{H} - \epsilon)\big)$. Explicitly

$$S = \epsilon - (\mathrm{id}_\mathcal{H} - \epsilon) + (\mathrm{id}_\mathcal{H} - \epsilon) \star (\mathrm{id}_\mathcal{H} - \epsilon) - (\mathrm{id}_\mathcal{H} - \epsilon) \star (\mathrm{id}_\mathcal{H} - \epsilon) \star (\mathrm{id}_\mathcal{H} - \epsilon) + \cdots \tag{54}$$

The BCH-recursion χ (7) obviously does not enter in this particular case as $\widetilde{\mathrm{id}_{\mathcal{H}}} = 0$. See [27] for the geometric series ansatz. Applying the matrix representation anti-homomorphism $\Psi_{\mathcal{H}}$, we find the **inverse coproduct matrix**

$$\begin{aligned}\Psi_{\mathcal{H}}[S] &= \widehat{\mathrm{id}_{\mathcal{H}}^{\star-1}} = M_{\mathcal{H}}^{-1} &(55)\\ &= \mathbf{1} - (M_{\mathcal{H}} - \mathbf{1}) + (M_{\mathcal{H}} - \mathbf{1})(M_{\mathcal{H}} - \mathbf{1}) - (M_{\mathcal{H}} - \mathbf{1})(M_{\mathcal{H}} - \mathbf{1})(M_{\mathcal{H}} - \mathbf{1}) + \cdots &(56)\\ &= \mathbf{1} - \sum_{n>0} (M_{\mathcal{H}} - \mathbf{1})^n. &(57)\end{aligned}$$

The inverse character matrix of $\phi \in G$ therefore is given by

$$\widehat{\phi}^{-1} = \phi \circ M_{\mathcal{H}}^{-1} = \mathbf{1} - \sum_{m>0} (\widehat{\phi} - \mathbf{1})^m. \tag{58}$$

For a truncated coproduct matrix $M_{\mathcal{H}} \in \mathcal{M}_n^u(\mathcal{H})$, $n < \infty$, the series on the right breaks up at order n, $(\widehat{\phi} - \mathbf{1})^n = 0$, as $\widehat{\phi} - \mathbf{1} \in \mathcal{M}_n^u(A)_1$ is nilpotent. In components, the formula for $\widehat{\phi}^{-1}$ is

$$(\widehat{\phi}^{-1})_{ij} = -\widehat{\phi}_{ij} + \sum_{k=1}^{j-i-1} \sum_{i<l_1<l_2<\cdots<l_k<j} (-1)^{k+1} \widehat{\phi}_{il_1} \widehat{\phi}_{l_1 l_2} \ldots \widehat{\phi}_{l_k l_j}. \tag{59}$$

3.4.2 Matrix renormalization by factorization

Now let $\phi : \mathcal{H}_{\mathcal{T}} \to A$ denote a regularized Feynman rules character with image in a Rota–Baxter algebra (A, R). Applying ϕ to $M_{\mathcal{H}}$ gives the **Feynman rules matrix**

$$\widehat{\phi} := \phi(M_{\mathcal{H}}) = \big(\phi(T_{ij})\big).$$

Let β be the unique solution of the recursion

$$\beta = \mathbf{1} - \mathcal{R}\big((\widehat{\phi} - \mathbf{1})\beta\big),$$

as in Theorem 20. The matrix β can be effectively computed by Theorem 11.1. The first row vector of β is the counter term vector $(\phi_-(t_1), \cdots, \phi_-(t_n), \cdots)$. Then the first row vector of the matrix product $\widehat{\phi}\,\beta$ is the renormalization vector, i.e., we have the linear renormalization matrix-vector equation (50) of item (3) in Theorem (20)

$$\langle 1_{\mathcal{T}} |\, \widehat{\phi}\, \beta = \langle \phi |\ \beta = (\phi_+(t_1), \cdots, \phi_+(t_n), \cdots).$$

Alternatively, let β' be the unique solution of the recursion

$$\beta' = \mathbf{1} - \tilde{\mathcal{R}}(\beta'(\widehat{\phi} - \mathbf{1})),$$

as in Theorem 20, again effectively computable by Theorem 11.2. Then find β'^{-1} which can be computed by Equation (58) (or recursively (12)). Then the first row of β'^{-1} is the renormalization vector

$$\langle 1_{\mathcal{T}} |\, \beta'^{-1} = \langle \phi_+ | = (\phi_+(t_1), \cdots, \phi_+(t_n), \cdots).$$

Equivalently, using the inverse coproduct matrix to calculate the inverse Feynman rules character

$$\widehat{\phi}^{-1} := \phi(M_{\mathcal{H}}^{-1}) = \big(\phi(T_{ij}^{-1})\big),$$

we find directly the renormalized character matrix as solution of the equation $c = \mathbf{1} - \tilde{\mathcal{R}}\big((\widehat{\phi}^{-1} - \mathbf{1})\ c\big)$ of Remark 21.

3.4.3 Examples in φ^4_{4dim}-theory

This subsection serves to show how the above Hopf algebra consideration nicely applies to standard Feynman graph calculations of perturbative renormalization. In [17] Connes and Kreimer showed in full generality that the set of Feynman graphs $\mathcal{F}$ for any perturbatively treated QFT can be made into a combinatorial Hopf algebra $\mathcal{H}_{\mathcal{F}}$ of the above type.

We will use a simplified version of φ^4-theory in four dimensions as our Feynman graph toy model physics theory. A more detailed and refined treatment can be found in the companion paper [23].

The Feynman graph Hopf algebra is denoted by $\mathcal{H}_{\mathcal{F}}$. As regularization scheme we choose dimensional regularization. So that the space of linear functionals $\mathrm{Hom}(\mathcal{H}_{\mathcal{F}}, A)$ contains maps into the commutative Rota–Baxter algebra $A := \mathbb{C}[\varepsilon^{-1}, \varepsilon]]$ with Rota–Baxter map R_{ms}.

We work up to 3-loop order, by taking X to be the set of graphs

$$\mathcal{F}_{(4)} := \Big\{ e_1 := |1_{\mathcal{F}}\rangle\ ,\ e_2 := |\ \rangle\ ,\ e_3 := |\ \rangle,\ e_4 := \Big|\ \Big\rangle \Big\} \tag{60}$$

identified with the corresponding column vectors. The graph is obtained by substituting a wine-cup into the divergent 1-loop graph .

The Feynman graphs we consider here are of purely combinatorial type in the sense that the external legs are not decorated by external structure, such as external momenta, spin indices, etc. This frees us from symmetry consideration, which otherwise would demand a bigger representation space $\mathcal{F}$.

The Feynman diagram has a primitive divergence and would correspond to the one vertex tree $\bullet$ decorated by this graph. the wine-cup diagram contains exactly one nested subdivergence of type and corresponds to the ladder graph of length 2, , where the root and leaf both are decorated by the graph . The coproducts of these two graphs therefore are given in analogous forms to the first two expressions in (24). The graph contains three nested subdivergences and correspondence to the ladder graph of length 3, with each vertex decorated by , its coproduct is given by

$$\Delta\Big(\ \Big) = \ \otimes 1_{\mathcal{F}} + 1_{\mathcal{F}} \otimes \ + \ \otimes \ + \ \otimes \ .$$

Let $\phi \in G_A$ denote the Feynman rules for four dimensional φ^4-theory in dimensional regularization, together with minimal subtraction scheme, i.e. Rota–Baxter map $R := R_{ms}$. The corresponding coproduct matrix $M_{\mathcal{F}_{(4)}}$, respectively character matrix $\Psi_A[\phi] = \widehat{\phi}$ are given by

$$\widehat{\phi} := \phi \circ M_{\mathcal{F}_{(4)}} = \phi \circ \begin{pmatrix} 1 & \; & \; & \; \\ 0 & 1 & \; & \; \\ 0 & 0 & 1 & \; \\ 0 & 0 & 0 & 1 \end{pmatrix} := \begin{pmatrix} 1 & \phi(\;) & \phi(\;) & \phi(\;) \\ 0 & 1 & \phi(\;) & \phi(\;) \\ 0 & 0 & 1 & \phi(\;) \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This matrix has the special property that along any subdiagonal we find the same entry. This situation always appears if (and only if) we deal with strictly nested diagrams only. In the picture of decorated rooted trees this corresponds to the Hopf subalgebra of ladder trees.

Using the counter term recursion (47) of Theorem 20 for $\widehat{\phi}$,

$$\beta = \mathbf{1} - \mathcal{R}\left((\widehat{\phi} - \mathbf{1})\,\beta\right)$$

and applying formula (21) for its solution, we find

$$\beta := \begin{pmatrix} 1 & -R(\phi(\;)) & \beta_{13} & \beta_{14} \\ 0 & 1 & -R(\phi(\;)) & \beta_{24} \\ 0 & 0 & 1 & -R(\phi(\;)) \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\beta_{13} = -R\Big(\phi(\;)\Big) + R\Big(\phi(\;)R\Big(\phi(\;)\Big)\Big),$$

$$\beta_{24} = \beta_{13},$$

and

$$\beta_{14} = -R\Big(\phi(\;)\Big) + R\Big(\phi(\;)R\Big(\phi(\;)\Big)\Big) + R\Big(\phi(\;)R\Big(\phi(\;)\Big)\Big) \tag{61}$$

$$-R\Big(\phi(\;)R\Big(\phi(\;)R\Big(\phi(\;)\Big)\Big)\Big) \tag{62}$$

$$= \phi_-\Big(\;\Big). \tag{63}$$

So in the end we have the following counter term matrix

$$\widehat{\phi}_- = \begin{pmatrix} 1 & -R\Big(\phi(\;)\Big) & -R\Big(\phi(\;) - \phi(\;)R\Big(\phi(\;)\Big)\Big) & \phi_-\Big(\;\Big) \\ 0 & 1 & -R\Big(\phi(\;)\Big) & -R\Big(\phi(\;) - \phi(\;)R\Big(\phi(\;)\Big)\Big) \\ 0 & 0 & 1 & -R\Big(\phi(\;)\Big) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

giving the counter terms for the graphs , and in its first row. The renormalized character matrix $\widehat{\phi_+}$ follows from the matrix Birkhoff factorization (49) in Theorem 20, $\widehat{\phi}_+ = \widehat{\phi}\,\widehat{\phi}_-$. According to the matrix-vector equation (50) of Theorem 20 we find

$$\langle \phi_+ | = \langle 1_{\mathcal{F}} |\, \widehat{\phi}_+ = \langle \phi |\, \widehat{\phi}_-,$$

yielding the renormalized amplitudes for the graphs , , and in its first row, which we write in transposed form

$$\langle \phi_+ |^{\top} = \begin{pmatrix} 1 \\ \phi() - R\Big(\phi()\Big) \\ \phi() - \phi() R\Big(\phi()\Big) - R\Big(\phi()\Big) + R\Big(\phi() R\Big(\phi()\Big)\Big) \\ \begin{Bmatrix} \phi\Big(\Big) - \phi() R\Big(\phi()\Big) - \phi() R\Big(\phi()\Big) + \phi() R\Big(\phi() R\Big(\phi()\Big)\Big) \\ -R\Big(\phi\Big(\Big) - \phi() R\Big(\phi()\Big) - \phi() R\Big(\phi()\Big) + \phi() R\Big(\phi() R\Big(\phi()\Big)\Big)\Big) \end{Bmatrix} \end{pmatrix}$$

Another example up to 3-loop order, including a graph with two disjoint 1-loop subdivergences is provided by taking X to be the set of graphs

$$\mathcal{F}'_{(4)} := \Big\{ e_1 := |1_{\mathcal{F}'}\rangle\,,\ e_2 := |\ \rangle\,,\ e_3 := \Big|\ \Big\rangle,\ e_4 := \Big|\ \Big\rangle \Big\} \tag{64}$$

The new graph is made of the two disjoint fish graphs as subdivergences sitting inside of such a graph. Remember that our graphs carry no external structure. The coproduct of this graph is given by

$$\Delta\Big(\Big) = \otimes 1_{\mathcal{F}'} + 1_{\mathcal{F}'} \otimes + 2 \otimes + \otimes .$$

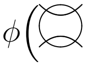
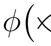
Let $\phi \in G_A$ be the Feynman rules character. The coproduct matrix, respectively the character matrix are given by

$$\widehat{\phi} := \phi \circ M_{\mathcal{F}'_{(4)}} = \phi \circ \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} := \begin{pmatrix} 1 & \phi() & \phi() & \phi\Big(\Big) \\ 0 & 1 & \phi() & \phi()^2 \\ 0 & 0 & 1 & 2\phi() \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

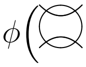
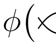
In this example the only new counter term matrix entry we need to calculate is position $(1,4)$ in $\widehat{\phi}_-$. Applying formula (21) for its solution, we find

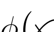
$$\begin{aligned} \beta_{14} \ &= \ -R\Big(\phi\Big(\Big)\Big) + R\Big(\phi() R\Big(\phi()^2\Big)\Big) + R\Big(\phi() R\Big(2\phi()\Big)\Big) \\ &\qquad -R\Big(\phi()\ 2R\Big(\phi() R\Big(\phi()\Big)\Big)\Big). \end{aligned} \tag{65}$$

The identity $R(a)^2 = 2R(aR(a)) - R(a^2)$ following from (1), and which is true only for commutative Rota–Baxter algebras, immediately implies

$\beta_{14} = -R\Big(\phi\big($  $\big)\Big) - R\Big(\phi($  $)R\big(\phi($  $)\big)^2\Big) + 2R\Big(\phi($  $)R\big(\phi($  $)\big)\Big).$ (66)

Likewise for the renormalized expression we find in entry $(1,4)$ of the renormalized matrix $\widehat{\phi}_+$, respectively the 4th component of the vector $\langle\phi_+|^\top$

$(\langle\phi_+|^\top)_4 = \phi\big($  $\big) + \phi($  $)R\big(\phi($  $)\big)^2 - 2\phi($  $)R\big(\phi($  $)\big)$ (67)

$-R\Big(\phi\big($  $\big) + \phi($  $)R\big(\phi($  $)\big)^2 - 2\phi($  $)R\big(\phi($  $)\big)\Big).$

3.5 More examples and comments on matrix factorization

As another illustration, let us consider the case of the truncated space $\mathcal{T}_{(6)}$ in (39) of undecorated rooted trees. For a given regularized character $\phi : \mathcal{H}_\mathcal{T} \to A$, we have the corresponding matrix $\widehat{\phi}$ in Equation (45) which we record below for easy reference.

$$\begin{aligned} \widehat{\phi} &= \phi \circ M_\mathcal{H} \qquad (68)\\ &= \begin{pmatrix} 1 & \phi(e_2) & \phi(e_3) & \phi(e_4) & \phi(e_5) & \phi(e_6) \\ 0 & 1 & \phi(e_2) & \phi(e_3) & \phi(e_2)^2 & \phi(e_4) \\ 0 & 0 & 1 & \phi(e_2) & 2\phi(e_2) & \phi(e_3) \\ 0 & 0 & 0 & 1 & 0 & \phi(e_2) \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

where $M_\mathcal{H}$ is the coproduct matrix in (41). Recall that the unit diagonal upper triangular matrix

$$\beta := \widehat{\phi_-} = \exp\Big(-\mathcal{R}(\bar\chi(\widehat{Z}_\phi))\Big) = \mathbf{1} + \mathcal{R}\left(\exp^{*_\mathcal{R}}\Big(-\bar\chi(\widehat{Z}_\phi)\Big) - \mathbf{1}\right) \qquad (69)$$

is solution to the equation (47)

$$\beta = \mathbf{1} - \mathcal{R}\left((\widehat{\phi} - \mathbf{1})\,\beta\right).$$

Remember that the second equality in (69) follows from Proposition 32 respectively its matrix representation, with $\bar\chi$ in place of χ, see Theorem 20. .

Equation (47) is solved by the formula (21), and we find

$$\beta := \begin{pmatrix} 1 & -R(\phi(e_2)) & \beta_{13} & \beta_{14} & \beta_{15} & \beta_{16} \\ 0 & 1 & -R(\phi(e_2)) & \beta_{24} & \beta_{25} & \beta_{26} \\ 0 & 0 & 1 & -R(\phi(e_2)) & \beta_{35} & \beta_{36} \\ 0 & 0 & 0 & 1 & 0 & \beta_{46} \\ 0 & 0 & 0 & 0 & 1 & \beta_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where, in abbreviating $\phi(e_i)$ by e_i, $1 \leq i \leq 6$,

$$\begin{aligned}
\beta_{13} &= -R(e_3) + R\big(e_2R(e_2)\big),\\
\beta_{14} &= -R(e_4) + R\big(e_2R(e_3)\big) + R\big(e_3R(e_2)\big) - R\big(e_2R(e_2R(e_2))\big),\\
\beta_{15} &= -R(e_5) + R\big(e_2R(e_2e_2)\big) + R\big(e_3R(2e_2)\big) - R\big(e_2R(e_2R(2e_2))\big),\\
\beta_{16} &= -R(e_6) + R\big(e_2R(e_4)\big) + R\big(e_3R(e_3)\big) + R\big(e_4R(e_2)\big)\\
&\quad -R\big(e_2R(e_2R(e_3))\big) - R\big(e_2R(e_3R(e_2))\big) - R\big(e_3R(e_2R(e_2))\big) + R\big(e_2R(e_2R(e_2R(e_2)))\big),\\
\beta_{24} &= -R(e_3) + R\big(e_2R(e_2)\big),\\
\beta_{25} &= -R(e_2^2) + R\big(e_2R(2e_2)\big) = R(e_2)R(e_2),\\
\beta_{26} &= -R(e_4) + R\big(e_2R(e_3)\big) + R\big(e_3R(e_2)\big) - R\big(e_2R(e_2R(e_2))\big),\\
\beta_{35} &= -2R(e_2),\\
\beta_{36} &= -R(e_3) + R\big(e_2R(e_2)\big),\\
\beta_{46} &= -R(e_2), \text{ and } \beta_{56} = 0.
\end{aligned}$$

The counter term expressions for the graphs e_i, $i = 2, \dots, 6$ are given in the first row of β,

$$\langle \phi_- | = \langle 1_{\mathcal{T}} | \, \beta = (1, -R(e_2), -R(e_3) + R(e_2R(e_2)), \beta_{14}, \beta_{15}, \beta_{16}).$$

These calculations for the β_{ij} match the results of Bogoliubov's counter term recursion applied to the coproduct matrix. For the renormalized matrix character $\widehat{\phi}\widehat{\phi}_- = \widehat{\phi}_+$ we find

$$\widehat{\phi}_+ \;=\; \exp\Big(\tilde{\mathcal{R}}(\bar{\chi}(\widehat{Z}_\phi))\Big) = \mathbf{1} - \tilde{\mathcal{R}}\left(\exp^{*_{\mathcal{R}}}\Big(-\bar{\chi}(\widehat{Z}_\phi)\Big) - \mathbf{1}\right)$$

Then the renormalized expressions $\phi_+(e_i)$ for $i = 2, \dots, 6$ are obtained as components of the vector

$$\langle \phi_+ | = \langle 1_{\mathcal{T}} | \, \widehat{\phi}_+ = \langle 1_{\mathcal{T}} | \, \widehat{\phi}\,\widehat{\phi}_- = \langle \phi | \, \widehat{\phi}_-.$$

As a remark for the practitioner we mentioned that from Proposition 7 we derive the more familiar equation for the renormalized character

$$\widehat{\phi}_+ \;=\; \mathbf{1} + \tilde{\mathcal{R}}\big((\widehat{\phi} - \mathbf{1})\,\widehat{\phi}_-\big), \tag{70}$$

which is just Bogoliubov's classical $\bar{\mathrm{R}}$-operation giving the renormalized Feynman rules (matrix).

3.5.1 Exponential approach of matrix factorizations

In this subsection we leave the realm of combinatorial Hopf algebras of renormalization and come back to the results of Section 2.3, where we discussed the decomposition of upper triangular matrices with entries in a commutative Rota–Baxter algebra.

The following dwells on the exponential approach to the calculation of the factor matrices $\bar{\alpha}_\pm$ in the factorization of α in Corollary 10. Other than its theoretical significance, it also relates to the exponential approach of the Birkhoff decomposition of Connes and Kreimer.

The reader should remember Theorem 9 asserting that with (A,R) being a commutative Rota–Baxter algebra, the triple $(\mathcal{M}_n^u(A), \mathcal{R}, \{\mathcal{M}_n^u(A)_k\}_{k\geq 1})$ forms a complete filtered Rota–Baxter algebra.

Let us start with some properties of the exponential and logarithm functions for complete filtered algebras of upper triangular matrices defined in (13) respectively (14). For any $1 \leq n \leq \infty$, a natural basis of $\mathcal{M}_n^u(A)$ is given by the matrices $E_{ij} \in \mathcal{M}_n^u(A)$, $0 < i \leq j \leq n$, where the entry at position (i,j) is 1, the rest zero. These matrices multiply according to $E_{ij}E_{kl} = \delta_{jk}E_{il}$. We can express the logarithm and exponential in terms of this basis $\{E_{ij}\}$. For $\alpha \in \mathfrak{M}_n(A)$, we have $\log(\alpha) \in \mathcal{M}_n^u(A)_1$. So we have

$$Z_\alpha := \log(\alpha) = (\tilde{\alpha}_{ij}) = \sum_{0<i<j\leq n} \tilde{\alpha}_{ij}E_{ij}.$$

These $\tilde{\alpha}_{ij} \in A$ are called *matrix normal coordinates* (of the second kind). The concept of normal coordinates in the context of Connes–Kreimer renormalization theory appeared in [21].

For example, let the 3×3 matrix α in $\mathfrak{M}_3(A)$ be

$$\alpha = \sum_{i=1}^{3} E_{ii} + aE_{12} + bE_{13} + cE_{23}$$

with $a,b,c \in A$. Note that $\alpha - \mathbf{1}$ is strictly upper triangular and so $(\alpha - \mathbf{1})^k = 0$ for $k \geq 3$. Therefore, we have

$$Z_\alpha \;:=\; \log(\alpha) = \alpha - \mathbf{1} - \frac{1}{2}(\alpha - \mathbf{1})^2 = aE_{12} + \left(b - \frac{1}{2}ac\right)E_{13} + cE_{23} \in \mathcal{M}_3^u(A)_1, \qquad (71)$$

giving normal coordinates

$$\tilde{\alpha}_{12} = a,\ \tilde{\alpha}_{13} = b - \frac{1}{2}ac,\ \tilde{\alpha}_{23} = c.$$

Thus

$$\alpha = \exp\left(aE_{12} + \left(b - \frac{1}{2}ac\right)E_{13} + cE_{23}\right). \qquad (72)$$

In general, for given $\alpha = (\alpha_{ij}) \in \mathfrak{M}_n(A)$ these matrix normal coordinates can be calculated by the formula

$$\tilde{\alpha}_{ij} = \alpha_{ij} + \sum_{k=1}^{j-i} \sum_{i<l_1<l_2<\cdots<l_k<j} \frac{(-1)^k}{k+1}\alpha_{il_1}\alpha_{l_1l_2}\ldots\alpha_{l_kj},\quad 0<i<j\leq n. \qquad (73)$$

These new coordinates allow us to write any $n \times n$-matrix $\alpha \in \mathfrak{M}_n(A)$ as

$$\alpha = \exp\big(Z_\alpha\big) = \exp\left(\sum_{0<i<j\leq n} \tilde{\alpha}_{ij}E_{ij}\right).$$

In order to obtain the factorization $\alpha = \bar{\alpha}_+\bar{\alpha}_-^{-1}$, we need the BCH-recursion (10)

$$\bar{\chi}(Z_\alpha) := Z_\alpha - BCH\left(\tilde{\mathcal{R}}(\bar{\chi}(Z_\alpha)), \mathcal{R}(\bar{\chi}(Z_\alpha))\right),$$

which allows us to calculate $\mathcal{R}(\bar{\chi}(Z_\alpha))$ and $\tilde{\mathcal{R}}(\bar{\chi}(Z_\alpha))$ in $\mathcal{M}_n^u(A)_1$. This is valid for any $n \leq \infty$.

We continue with our example of $\mathcal{M}_3^u(A)$. Note that $\mathcal{M}_3^u(A)_1^k = 0$ for $k \geq 3$ and that commutators of higher order in $\bar{\chi}$ are identically zero in $\mathcal{M}_3^u(A)_1$, due to the decreasing filtration. Thus by (17), we find

$$\begin{aligned}
\bar{\alpha}_- &= \exp\left(-\mathcal{R}(\bar{\chi}(Z_\alpha))\right. \\
&= \exp\left(-\mathcal{R}\Big(Z_\alpha - \frac{1}{2}[\tilde{\mathcal{R}}(Z_\alpha), \mathcal{R}(Z_\alpha)]\Big)\right) \\
&= \mathbf{1} - \mathcal{R}\Big(Z_\alpha - \frac{1}{2}[\tilde{\mathcal{R}}(Z_\alpha), \mathcal{R}(Z_\alpha)]\Big) + \frac{1}{2}\mathcal{R}(Z_\alpha)\mathcal{R}(Z_\alpha)
\end{aligned}$$

We would like to underline, that up to this point we have not used the condition that $\mathcal{R}$ is a Rota–Baxter map. Actually, for the factorization, we only needed that $\mathcal{R} + \tilde{\mathcal{R}} = \mathrm{id}_{\mathcal{M}_n^u(A)}$, which is true for any linear map, and the BCH-recursion $\bar{\chi}$ (or χ) in a suitable topology, such as a complete filtered algebra. The Rota–Baxter structure only enters in the next step, when we replace the last term $\mathcal{R}(Z_\alpha)\mathcal{R}(Z_\alpha)$ via Rota–Baxter relation. Keeping in mind that, for the Lie bracket $[-\,,-]$, we have

$$[\tilde{\mathcal{R}}(Z_\alpha), \mathcal{R}(Z_\alpha)] = [Z_\alpha - \mathcal{R}(Z_\alpha), \mathcal{R}(Z_\alpha)] = [Z_\alpha, \mathcal{R}(Z_\alpha)],$$

thus

$$\begin{aligned}
\bar{\alpha}_- &= \mathbf{1} - \mathcal{R}(Z_\alpha) + \frac{1}{2}\big(\mathcal{R}(Z_\alpha\mathcal{R}(Z_\alpha)) - \mathcal{R}(\mathcal{R}(Z_\alpha)Z_\alpha)\big) + \\
&\qquad + \frac{1}{2}\big(\mathcal{R}(\mathcal{R}(Z_\alpha)Z_\alpha) + \mathcal{R}(Z_\alpha\mathcal{R}(Z_\alpha)) - \mathcal{R}(Z_\alpha Z_\alpha)\big) \qquad (74) \\
&= \mathbf{1} - \mathcal{R}(Z_\alpha) + \mathcal{R}(Z_\alpha\mathcal{R}(Z_\alpha)) - \frac{1}{2}\mathcal{R}(Z_\alpha Z_\alpha). \qquad (75)
\end{aligned}$$

In matrix form we therefore get

$$\begin{aligned}
\bar{\alpha}_- &= \begin{pmatrix} 1 & -R(a) & \big(-R(b) + \frac{1}{2}R(ac)\big) \\ 0 & 1 & -R(c) \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & R(aR(c)) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \frac{1}{2}R(ac) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -R(a) & \big(-R(b) + R(aR(c))\big) \\ 0 & 1 & -R(c) \\ 0 & 0 & 1 \end{pmatrix} \qquad (76)
\end{aligned}$$

The inverse of $\bar{\alpha}_-$ can be calculated, using the recursive formula (12) or directly from (17)

$$\begin{aligned}
\bar{\alpha}_-^{-1} &= \exp\left(\mathcal{R}(\bar{\chi}(Z_\alpha))\right. \\
&= \mathbf{1} + \mathcal{R}\Big(Z_\alpha - \frac{1}{2}[\tilde{\mathcal{R}}(Z_\alpha), \mathcal{R}(Z_\alpha)]\Big) + \frac{1}{2}\mathcal{R}(Z_\alpha)\mathcal{R}(Z_\alpha) \\
&= \mathbf{1} + \mathcal{R}(Z_\alpha) + \mathcal{R}(\mathcal{R}(Z_\alpha)Z_\alpha) - \frac{1}{2}\mathcal{R}(Z_\alpha Z_\alpha)
\end{aligned}$$

$$=\begin{pmatrix} 1 & R(a) & R(b)-\frac{1}{2}R(ac) \\ 0 & 1 & R(c) \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & R(R(a)c) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \frac{1}{2}R(ac) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & R(a) & R(b)+R\big(R(a)c)-R(ac)\big) \\ 0 & 1 & R(c) \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & R(a) & R(b)-R\big(\tilde{R}(a)c\big) \\ 0 & 1 & R(c) \\ 0 & 0 & 1 \end{pmatrix}$$

We similarly calculate $\bar{\alpha}_+ = \exp\Big(\tilde{\mathcal{R}}\big(\bar{\chi}(Z_\alpha)\big)\Big)$ and reach the following factorization for example of 3×3 matrix $\alpha \in \mathfrak{M}^u_3(A)$.

$$\alpha = \bar{\alpha}_+\bar{\alpha}_-^{-1} = \begin{pmatrix} 1 & \tilde{R}(a) & \tilde{R}(b)-\tilde{R}(aR(c)) \\ 0 & 1 & \tilde{R}(c) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & R(a) & R(b)-R(\tilde{R}(a)c) \\ 0 & 1 & R(c) \\ 0 & 0 & 1 \end{pmatrix} \tag{77}$$

This recovers the factorization in (20).

We finally make a remark on the normal coordinates for the example in (68). For the matrix representation of the character ϕ, $\Psi_A[\phi] = \widehat{\phi} = \exp\Big(\widehat{Z}_\phi\Big)$, with $\widehat{Z}_\phi = \log(\widehat{\phi})$, the strictly upper triangular matrix $\widehat{Z}_\phi \in \mathcal{M}^u_6(A)_1$ follows by using the formula for the matrix normal coordinates (73)

$$\widehat{Z}_\phi := \sum_{0<i<j\le 6} \tilde{\phi}_{ij}E_{ij} \tag{78}$$

$$\begin{aligned} &= \tilde{\phi}_{12}E_{12}+\tilde{\phi}_{13}E_{13}+\tilde{\phi}_{14}E_{14}+\tilde{\phi}_{15}E_{15}+\tilde{\phi}_{16}E_{16}+ \\ &\quad \tilde{\phi}_{23}E_{23}+\tilde{\phi}_{24}E_{24}+\tilde{\phi}_{25}E_{25}+\tilde{\phi}_{26}E_{26}+ \\ &\quad \tilde{\phi}_{34}E_{34}+\tilde{\phi}_{35}E_{35}+\tilde{\phi}_{36}E_{36}+\tilde{\phi}_{45}E_{45}+\tilde{\phi}_{46}E_{46}+\tilde{\phi}_{56}E_{56} \end{aligned}$$

$$= \begin{pmatrix} 0 & \phi(\tilde{e}_2) & \phi(\tilde{e}_3) & \phi(\tilde{e}_4) & \phi(\tilde{e}_5) & \phi(\tilde{e}_6) \\ 0 & 0 & \phi(\tilde{e}_2) & \phi(\tilde{e}_3) & 0 & \phi(\tilde{e}_4) \\ 0 & 0 & 0 & \phi(\tilde{e}_2) & 2\phi(\tilde{e}_2) & \phi(\tilde{e}_3) \\ 0 & 0 & 0 & 0 & 0 & \phi(\tilde{e}_2) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}^u_6(A)_1. \tag{79}$$

We used the fact that $\phi \in G_A$ is a character, i.e. an algebra homomorphism. We have the following simple polynomial expressions for $\tilde{e}_i$, $i=1,\dots,6$, following from (73)

$$\tilde{e}_2 = e_2, \quad \tilde{e}_3 = e_3 - \frac{1}{2}e_2e_2 \tag{80}$$

$$\tilde{e}_4 = e_4 - e_2e_3 + \frac{1}{3}e_2e_2e_2, \quad \tilde{e}_5 = e_5 - e_2e_3 + \frac{1}{6}e_2e_2e_2 \tag{81}$$

$$\tilde{e}_6 = e_6 - e_2e_4 - \frac{1}{2}e_3e_3 + e_2e_2e_3 - \frac{1}{4}e_2e_2e_2e_2. \tag{82}$$

These are exactly the rooted tree normal coordinates as they appear in [21], and [26, 29]. They can be calculated as well using the convolution product and the logarithmic map, $\tilde{e}_i = \log^\star\big(\mathrm{id}_{\mathcal{H}_\mathcal{T}}\big)(e_i)$, $i = 1, \ldots, 6$.

We hope that these examples provided some insight into the underlying structure and calculational simplicity of the matrix factorization in the context of complete filtered Rota–Baxter algebras.

Let us briefly summarize what we have found in this section. Upper triangular $n \times n$ matrices, for any $1 \leq n \leq \infty$, with entries in a commutative Rota–Baxter algebra (A, R), $\mathcal{M}^u_n(A)$ form a complete filtered Rota–Baxter algebra $(\mathcal{M}^u_n(A), \mathcal{R}, \{\mathcal{M}^u_n(A)\}_{k\geq 1})$. The complete filtration allows us to define a Baker–Campbell–Hausdorff based recursion relation, denoted by $\bar{\chi} : \mathcal{M}^u_n(A)_1 \to \mathcal{M}^u_n(A)_1$, which in turn gives rise to a decomposition of the group $\mathfrak{M}_n(A)$ of upper triangular matrices with unit diagonal. The linear map $\mathcal{R}$ appearing in the definition of the recursion for $\bar{\chi}$ can be any linear map in $\mathrm{End}(\mathcal{M}^u_n(A))$. Choosing it to be a Rota–Baxter map gives rise to solutions of the matrix group factorization in terms of recursion equations, Theorem 20.

3.6 Berg–Cartier's ansatz using the grafting operation on rooted trees

Berg–Cartier [4] used a different but related approach to encode the derivations in $\mathcal{H}^*_\mathcal{T}$ in terms of lower triangular matrices. For this they made use of the pre-Lie insertion operation on Feynman graphs.

For the sake of convenience, let us state briefly the definition of pre-Lie algebra. Let A be a not necessarily associative $\mathbb{K}$-algebra. We denote the multiplication $m_A : A \otimes A \to A$ in A by concatenation, $m_A(a \otimes b) = a\ b$, $a, b \in A$. The associator is defined as $(-,-,-)_A : A \times A \times A \to A$,

$$(a, b, c)_A := a\ (b\ c) - (a\ b)\ c \tag{83}$$

for $a, b, c \in A$. For A being an associative $\mathbb{K}$-algebra we have $(a, b, c)_A = 0$ for all $a, b, c \in A$.

A (left) pre-Lie $\mathbb{K}$-algebra $(P, \diamond)$ is a $\mathbb{K}$-vector space P, together with a bilinear pre-Lie product $\diamond : P \times P \to P$, fulfilling the (left) pre-Lie relation

$$(a, b, c)_P = (b, a, c)_P, \quad \forall a, b, c \in P. \tag{84}$$

or explicitly

$$a \diamond (b \diamond c) - (a \diamond b) \diamond c = b \diamond (a \diamond c) - (b \diamond a) \diamond c, \quad \forall a, b, c \in P.$$

The pre-Lie property is weaker than associativity, i.e. every associative $\mathbb{K}$-algebra is evidently pre-Lie. The commutator $[a, b] := a \diamond b - b \diamond a$ for $a, b \in P$ fulfills the Jacobi identity, making the $\mathbb{K}$-vector space underlying P a Lie algebra.

In the rooted tree setting the process of insertion of Feynman graphs into other graphs becomes a grafting operation. The derivations (29) Z_t form a pre-Lie algebra $Z_{t'} \diamond Z_{t''} := \sum_{t \in \mathcal{T}} n(t', t''; t) Z_t$ [16, 17, 19], which by anti-symmetrization defines the commutator of the Lie algebra g of derivations (30). This pre-Lie composition is used to define an action of the Z_t on the vector space $\mathcal{T}$ freely spanned by the rooted trees (or Feynman graphs). The operator representing the action is denoted by $s(t) : \mathcal{T} \to \mathcal{T}$, for all $t \in \mathcal{T}$ and defined as follows

$$s_{t'} \, |t''\rangle := \sum_{t \in \mathcal{T}} n(t', t''; t) \, |t\rangle \, .$$

As an example we calculate

$s(\bullet)\,|\bullet\rangle = \left|\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}\right\rangle$, $\quad s(\bullet)\left|\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}\right\rangle = \left|\begin{smallmatrix}\bullet\\ \vert\\ \bullet\\ \vert\\ \bullet\end{smallmatrix}\right\rangle + 2\left|\begin{smallmatrix} & \bullet & \\ \diagup & & \diagdown \\ \bullet & & \bullet\end{smallmatrix}\right\rangle$, $\quad s\left(\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}\right)|\bullet\rangle = \left|\begin{smallmatrix}\bullet\\ \vert\\ \bullet\\ \vert\\ \bullet\end{smallmatrix}\right\rangle$, and by definition $s(t)\,|1_{\mathcal{T}}\rangle := |t\rangle \;\; \forall t \in \mathcal{T}$.

We used the ket-notation for the rooted tree vectors introduced in an earlier section. The rule for calculating the vector $s(t)\,|t'\rangle \in \mathcal{T}$ is to graft the tree t in all possible ways to the tree t', and to multiply each tree in the resulting linear combination by its symmetry factor. This action can be used to define a representation of the Z_t's in terms of lower triangular matrices, which are just the transposed of our upper ones (42). The difference between lower and upper triangular matrix representation reflects the fact that the former increases the degree by grafting trees, whereas the latter reduces them by "elimination" of subtrees. The matrix representation approach using the pre-Lie structure on Feynman graphs or rooted trees appears to be limited to representations of infinitesimal characters respectively characters. Relation (36) instead works for arbitrary elements in $\mathrm{Hom}(\mathcal{H}_{\mathcal{T}}, A)$.

It also appears that the ansatz chosen in [4] for the counter term matrix, denoted by $C_{1/\epsilon}$ in [4],

$$(C_{1/\epsilon})^{-1} = \exp\Big(- \sum_{t \in \mathcal{T}} C(\phi(t)) s(t) \Big), \tag{85}$$

using the operator $s(t)$ for $t \in \mathcal{T}$ is not sufficient for several reasons. First, the representation of characters $\phi \in \mathrm{Hom}(\mathcal{H}_{\mathcal{T}}, A)$ via the exponential map demands the use of normal coordinates. Second, the counter term character ϕ_-, or its matrix representation $\widehat{\phi}_-$, follows from the factorization of characters in the sense of Atkinson, see Theorem (4) respectively Spitzer's identity for non-commutative associative Rota–Baxter algebras, Theorem (3). Therefore one must include the particular properties of the Rota–Baxter relation as well as the BCH-recursion (7). This shortcoming in [4] becomes particularly evident when comparing Equations (18), (19) and the one following (19), therein (see Equation (67) above for the correct expression). Equation (18) is plagued with unwanted coefficients. The derivation of the expression after Equation (19) in [4] is problematic as it seems to assume the subtraction scheme map, denoted by C in [4], to be an idempotent algebra homomorphism. This is not true in general, e.g. in the MS scheme $C = R_{ms}$, which keeps only the pole part of Laurent series, is a projector of Rota–Baxter type. Berg and Cartier earlier in their paper (page 18 in [4]) proposed a special rule for products in the image of the map

C, e.g. $C(a)C(a)$ is supposed to read as $\frac{1}{2}C(C(a)a)$, which in general is not sufficient to resolve the aforementioned inconsistency. Instead, in a commutative Rota–Baxter algebra (A, R) we find $\frac{1}{2}R(a)^2 = R(R(a)a) - \frac{1}{2}R(a^2)$, $a \in A$.

The simple calculation of the counter term matrix (85) used in [4] would only apply to the subset of ladder trees (or Feynman graphs) denoted by $\ell d \subset \mathcal{T}$, if normal coordinates were properly used[1]. This follows from the fact that linearly ordered ladder trees form a cocommutative Hopf subalgebra $\mathcal{H}_{\ell d} \subset \mathcal{H}_{\mathcal{T}}$, or equivalently, the dually defined derivations $Z_{t_{\ell n}} =: Z_n$, indexed by the number of vertices of a ladder graph $t_{\ell n}$ form a commutative Lie subalgebra $g_{\ell d} \subset g$, on which the BCH-recursion reduces to the identity map, $\chi|_{g_{\ell d}} = \mathrm{id}$. Also, a consistent motivation for reducing the algebraic Birkhoff decomposition of the character matrix $\widehat{\phi} = \widehat{\phi}_+ \widehat{\phi}_-^{-1}$ into a linear matrix-vector equation of the form (50) was not given.

4 Conclusion

We have shown that the Connes–Kreimer Hopf algebra approach to renormalization in pQFT can be entirely represented as a simple and efficient triangular matrix calculus. Decomposing an $n \times n$ upper triangular unital matrix as described above, provides us with the counter term matrix as well as the renormalized matrix. The matrix calculus allows for an efficient calculation of counter terms, and henceforth renormalized Feynman amplitudes.

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