Tensor gauge fields in arbitrary representations of $GL(D,\mathbf{R})$: II. Quadratic actions

Xavier BEKAERT and Nicolas BOULANGER

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

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Xavier Bekaert[a] and Nicolas Boulanger[a,b,1]

[a] Institut des Hautes Études Scientifiques
Le Bois-Marie, 35 route de Chartres, 91440 Bures-sur-Yvette (France)
[b] Université de Mons-Hainaut, Mécanique et Gravitation
6 avenue du Champ de Mars, 7000 Mons (Belgium)

Abstract

Quadratic, second-order, non-local actions for tensor gauge fields transforming in arbitrary irreducible representations of the general linear group in D-dimensional Minkowski space are explicitly written in a compact form by making use of Levi–Civita tensors. The field equations derived from these actions ensure the propagation of the correct massless physical degrees of freedom and are shown to be equivalent to non-Lagrangian local field equations proposed previously. Moreover, these actions allow a frame-like reformulation à la MacDowell–Mansouri, without any trace constraint in the tangent indices.

[1] Chargé de Recherches FNRS (Belgium); E-mail address: `nicolas.boulanger@umh.ac.be`.

Contents

1 Introduction

Combing the principle of relativity with the rules of quantum mechanics implies that linear relativistic wave equations describing the free propagation of relativistic particles in Minkowski space are in one-to-one correspondence with unitary representations of the Poincaré group. Using the method of induced representations, Wigner showed in 1939 that the unitary irreducible representations (UIRs) of the Poincaré group $ISO_0(3,1)$ are completely characterized by two real numbers : the mass-squared m^2 and the spin[2] s of the corresponding particle [1]. Physical considerations[3] further impose $m^2 \geqslant 0$ (no tachyon) and $2s \in \mathbb{N}$ (discrete spin). The *Barmann-Wigner programme* amounts to associating, with any given UIR of the Poincaré group, a manifestly covariant differential equation whose positive energy solutions transform according to the corresponding UIR. In 1948, this programme was completed in four dimensions when, for each UIR of $ISO_0(3,1)$, a relativistic wave equation was written whose positive energy solutions transform according to the corresponding UIR [2].

This programme is the first step towards the completion of the *Fierz-Pauli programme* which consists in writing a manifestly covariant quadratic action for each first-quantized elementary particle propagating in Minkowski spacetime. In four spacetime dimensions, the latter programme was initiated in 1939 [3] and completed in the seventies by Singh and Hagen for the massive case ($m^2 > 0$) [4] and by Fronsdal and Fang for the massless case ($m^2 = 0$) [5, 6]. The description of free massless (massive) gauge fields in $D = 4$ has thus been known for a long time and is tightly linked with the representation theory of $Spin(2) \cong U(1)$ (respectively $Spin(3) \cong SU(2)$). This case is very particular because all non-trivial irreducible representations (irreps) of these compact groups are exhausted by the completely symmetric tensor-spinors, pictured by a one-row Young diagram with $[s]$ columns for a spin-s particle (where $[n]$ denotes the integer part of n).

The Bargmann–Wigner programme generalizes to the Poincaré group $ISO_0(D-1,1)$. When $D > 4$, more complicated Young diagrams appear whose analysis requires appropriate mathematical tools, as introduced in [7, 8, 9, 10]. For tensorial representations, the word "spin" will denote the number of columns possessed by the corresponding Young diagram. From now on, we restrict the analysis to massless UIRs induced by representations of the "little group" $SO(D-2)$ for $D \geqslant 3$, because each massive representation in $D-1$ dimensions may actually be obtained as the first Kaluza–Klein mode in a dimensional reduction from D down to $D-1$. There is no loss of generality because the massive little group $SO(D-2)$ in $D-1$ dimension is identified with the D-dimensional massless little group. Such a Kaluza–Klein mechanism leads to a Stückelberg formulation of the massive field [11].

An analysis of the gauge structure for arbitrary mixed-symmetry tensor gauge fields ϕ_Y was undertaken in [9, 10]. The results of Dubois-Violette and Henneaux [8] for rectangle-shaped Young-diagram tensor representations were extended to arbitrary tensor representations of $GL(D,\mathbb{R})$. Guided by the duality symmetry principle, through a systematic study, in [9] we proposed a general local field equation which

[2]In the massless case, the discrete label s is more accurately called *helicity*, but we use the naming "spin" whenever the mass of the particle is positive or zero.

[3]In this paper, we will not consider infinite-dimensional representations of the little group.

applies to tensor gauge fields ϕ_Y in arbitrary irreps of $GL(D,\mathbb{R})$ and generalizes the Bargmann–Wigner equations [2] of $D=4$ to any spacetime dimension $D\geqslant 3$. The fermionic case goes along the same lines [12], for this reason, we will restrict ourselves to tensorial representations of the Poincaré group in this paper.

In a work [13] on completely symmetric higher-spin ($s>2$) tensor gauge fields ϕ_s, Francia and Sagnotti discovered that foregoing locality allows to relax the trace conditions of the Fronsdal formulation. They wrote a non-local field equation which involves the de Wit–Freedman curvature [14] and which was shown to be equivalent to Fronsdal's field equation, after gauge-fixing.[4]

The authors of [15] followed another path: For completely symmetric tensor fields ϕ_s of rank $s>0$ they constructed field equations derived from actions $S\sim\int d^Dx\;\phi_s\cdot\mathcal{G}(\phi_s)$, where the "Einstein tensor" $\mathcal{G}(\phi_s)$ is higher-derivative and divergence-free, $\partial\cdot\mathcal{G}(\phi_s)=0$. It contains $2[\frac{s+1}{2}]=s+\varepsilon(s)$ derivatives of the field (where $\varepsilon(n)$ denotes the parity of the natural number $n\in\mathbb{N}$: its values is zero if n is even, or one if n is odd).

Subsequently, in [16] we proved that, restricted to completely symmetric tensor gauge fields ϕ_s, the field equation proposed in [9] was equivalent to Fronsdal's field equation and we further conjectured the validity of the same field equation in the arbitrary mixed-symmetry tensor gauge field ϕ_Y case. This conjecture was verified explicitly on a simple mixed-symmetry higher-spin tensor gauge field example.[5] In the same work [16], we then showed that both works [13] and [15] were actually equivalent, provided one multiplied the higher-derivative Einstein-like tensor $\mathcal{G}(\phi_s)$ of [15] by an appropriate power of the non-local inverse d'Alembertian operator $\Box^{-1}$, thereby recovering the non-local action of [13]. At the light of this observation, the authors of [15] reconsidered their previous work in [19] and inserted the fermionic case along the lines of [13]. They also conjectured a schematic form of the Einstein-like tensor $\mathcal{G}(\phi_Y)$ where ϕ_Y transforms in an arbitrary irrep. of $GL(D,\mathbb{R})$.

In the present work we pursue this investigation and provide the explicit expression for the higher-derivative Einstein-like tensor $\mathcal{G}(\phi_Y)$ corresponding to a field transforming in an arbitrary irrep. of $GL(D,\mathbb{R})$. The field equation derived from the action ($s>0$)

$$S[\phi_Y]=\int d^Dx\;\phi_Y\cdot\frac{1}{\Box^{[\frac{s-1}{2}]}}\,\mathcal{G}(\phi_Y)\tag{1}$$

is then shown to be equivalent to the field equation of [9, 16, 17] which propagates the correct massless physical degrees of freedom. The quadratic Lagrangian is always of second order but non-local for fields of higher-spin $s>2$. The corresponding field equation sets to zero all traces of the generalized curvature tensor $\mathcal{K}_{\overline{Y}}$ introduced in [9]: Tr $\mathcal{K}_{\overline{Y}}\approx 0$, where the weak equality $X\approx 0$ means "X is equal to zero on the surface of the field equations" (or, "on-shell").

As a preliminary result of the present work, the non-local quadratic action [13] of Francia and Sagnotti is rewritten in a compact and suggestive form by using Levi-Civita tensors. Moreover, we express these

[4] Actually, Fronsdal's action $S_4[\phi_s]=\int d^4x\,\mathcal{L}^F(\phi_s)$ trivially extends to D dimensions [14]: $S_D[\phi_s]=\int d^Dx\,\mathcal{L}^F(\phi_s)$. The Lagrangian $\mathcal{L}^F(\phi_s)$ is independent of the dimension D.

[5] That the aforementioned field equation is correct for an arbitrary mixed-symmetry tensor gauge field ϕ_Y was finally proved in [17], thereby generalizing Bargmann–Wigner's programme to arbitrary dimension $D\geqslant 3$. Actually, the latter programme had previously been completed in [18] with different equations.

actions in a frame-like fashion thereby providing a bridge between the local constrained approach of Vasiliev [20] and the non-local unconstrained approach. Indeed, we show that the latter action may be obtained as a flat spacetime limit of a MacDowell–Mansouri-like action in constant-curvature background, where the gauge fields and parameters are unconstrained, in contrast with Vasiliev's formalism.

The plan of the paper is as follows. In Section 2, we first review the various approaches to higher-spin symmetric tensor gauge fields in flat spacetime. The subsection 2.2.3 proposes an extension of the non-local action for the unconstrained frame-like approach to constant-curvature spacetimes. Mixed-symmetry tensor gauge fields ϕ_Y are studied in Section 3 where we recall our results (Theorem 1) on the completion of the Bargmann–Wigner programme, writing in details most of the intermediate steps in the proof.[6] Our main result (Theorem 2) is presented in the subsection 3.2.2 where a non-local second-order covariant quadratic action is given for each inequivalent UIR of the Poincaré group, thereby completing the Fierz–Pauli programme in arbitrary dimension $D \geqslant 3$.

Three appendices follow. In the appendix A, we systematically introduce our notation by reviewing all the mathematical machinery on irreps necessary for our purpose. We also summarize some former results on the gauge structure of mixed-symmetry tensor fields. The proofs of some technical lemmas are relegated to Appendix B while the appendix C contains the proof of Theorem 1 which states that the Bargmann–Wigner equations presented in [9, 16, 17] restrict the physical components of a tensor gauge field ϕ_Y to an UIR of the little group $O(D-2)$.

2 Completely symmetric tensor gauge fields

Completely symmetric tensors $\phi_{\mu_1\dots\mu_s} = \phi_{(\mu_1\dots\mu_s)}$ of rank s correspond to a Young tableau[7] made of one row with s cells. This is the simplest case of irreducible tensors under $GL(D,\mathbb{R})$ associated with a Young diagram made of s columns, thus we fix the main ideas on this specific example since it already exhibits the prominent properties of the general case.

Einstein's gravity theory is a non-Abelian massless spin-2 field theory, the two main formulations of which are the "metric" and the "frame" approaches. In a very close analogy, there exist two main approaches to higher-spin (*i.e.* spin $s > 2$) field theories that are by-now referred to as "metric-like" [5, 14] and "frame-like" [21, 20]. In the former approach, the components of the massless field ϕ_s transform in the irreducible representation of the general linear group which is labeled by a Young diagram Y made of s columns. Both metric-like and frame-like approaches may be divided into two subclasses called the "constrained" and "unconstrained" approaches according to whether trace constraints are imposed or not on the gauge fields and parameters.

[6]Because these lemmas and other intermediate results were either spread in the literature or not yet published in full details.

[7]The reader unfamiliar with Young tableaux may read the brief introduction to the tensorial irreps of $GL(D,\mathbb{R})$ in Subsection A.1.1.

2.1 Bargmann–Wigner programme

Not all covariant wave equations that would describe proper physical degrees of freedom are Euler-Lagrange equations for some Lagrangian. Therefore, we prefer to separate the discussion of the linear field equations from the discussion on quadratic Lagrangians for symmetric tensor gauge fields.

2.1.1 Local, constrained approach of Fronsdal

The local spin-s field equation of [5, 14] states that the Fronsdal tensor $\mathcal{F}$ vanishes on-shell

$$\mathcal{F}_{\mu_1\ldots\mu_s} \equiv \Box\phi_{\mu_1\ldots\mu_s} - s\,\partial^\alpha\partial_{(\mu_1}\phi_{\mu_2\ldots\mu_s)\alpha} + \frac{s(s-1)}{2}\,\partial_{(\mu_1}\partial_{\mu_2}\mathrm{Tr}\,\phi_{\mu_3\ldots\mu_s)} \approx 0\,, \tag{2}$$

where Tr stands for the trace operator and curved (respectively square) brackets denote complete symmetrization (antisymmetrization) with strength one. The gauge transformations are

$$\delta\phi_{\mu_1\ldots\mu_s} = s\,\partial_{(\mu_1}\epsilon_{\mu_2\ldots\mu_s)}\,. \tag{3}$$

Since (3) transforms $\mathcal{F}$ as

$$\delta\mathcal{F}_{\mu_1\ldots\mu_s} = \frac{s(s-1)(s-2)}{2}\,\partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3}\,\mathrm{Tr}\,\epsilon_{\mu_4\ldots\mu_s)}\,, \tag{4}$$

the gauge parameter $\epsilon_{\mu_2\ldots\mu_s}$ is constrained to be *traceless*, $\mathrm{Tr}\,\epsilon = 0$, in order to leave the field equation (2) invariant. Eventually, the standard de Donder gauge-fixing condition

$$D_{\mu_2\ldots\mu_s} \equiv \partial^\alpha\phi_{\alpha\mu_2\ldots\mu_s} - \frac{(s-1)}{2}\,\partial_{(\mu_2}\mathrm{Tr}\,\phi_{\mu_3\ldots\mu_s)} = 0 \tag{5}$$

is used to reduce the Fronsdal equation (2) to its canonical form $\Box\phi_{\mu_1\ldots\mu_s} \approx 0$. In order that $D_{\mu_2\ldots\mu_s} = 0$ contains as many conditions as the number of independent components of the gauge parameter ϵ, the gauge potential ϕ must be *double-traceless*, $\mathrm{Tr}^2\,\phi = 0$. As shown in [14], this gauge theory leads to the correct number of physical degrees of freedom, that is, the dimension of the irrep. of the little group $O(D-2)$ corresponding to the one-row Young diagram of length s.

The main advantage of the Fronsdal approach to free massless fields is that it respects the following two requirements of orthodox quantum field theory :

(i) Locality,

(ii) Second-order field equations (for bosonic fields).

Theories for which the second requirement is violated, *i.e.* the field equations contain the nth derivatives of the bosonic field with $n > 2$, are called "higher-derivative". Roughly speaking, non-local theories are a particular case of higher-derivative theories where the order in the derivatives is infinite, $n = \infty$. Both requirements (i) and (ii) are related to the no-go theorem of Pais and Uhlenbeck on free quantum field theories with higher-derivative kinetic operator for the propagating degrees of freedom [22]. They proved that for such a kinetic operator, the quantum field theory cannot be simultaneously stable (bounded energy spectrum), unitary and causal. In modern language, one would say that the field theory contains "ghosts".

Notice that the Pais–Uhlenbeck no-go theorem does *not* imply that all higher-derivative theories are physically sick. For instance, at least three harmless violations of the requirements (i) or (ii) have been suggested in the physics literature:

- **(a)** "Gauge artifact" : The ghosts associated with the higher-derivatives correspond to spurious "gauge" degrees of freedom. More precisely, in a proper gauge, the physical degrees of freedom propagate according to local second-order field equations. For instance, the worldsheet non-local action of the non-critical bosonic string is obtained from the Polyakov action by integrating out the the massless scalar fields describing the coordinates of the string in the target space [23]. In the conformal gauge, it reduces to the local Liouville action for the scalar field associated with the conformal factor.

- **(b)** "Perturbative cure" : The theory admits a perturbative expansion with an orthodox free limit. One can prove that, if the higher-derivatives are present in the perturbative interaction terms only, then they may be replaced with lower-derivative terms order by order [24]. This perturbative cure is perfectly justified when the higher-derivative theory is the effective field theory of a more fundamental orthodox theory, the higher-derivative terms corresponding to perturbative corrections. A good example of perturbatively non-local effective field theory is Wheeler–Feynman's electrodynamics in which the degrees of freedom of the electromagnetic field are frozen out. Another one is the α'-expansion in string theory.

- **(c)** "Non-perturbative miracle" : The possibility remains that the higher-derivative quantum field theory is consistent in the non-perturbative regime but does not admit a reasonable free limit. Such a possibility has been raised for conformal gravity [25] which is of fourth order, but it has never been proved that such a scenario indeed works.

2.1.2 Curvature tensors of de Wit, Freedman and Weinberg

The main drawback of Fronsdal's approach is the presence of algebraic constraints on the fields. They introduce several technical complications and are somewhat unnatural. To get rid of these trace constraints, it is necessary to relax one of the two requirements (i) or (ii) of orthodox quantum field theory in one of the harmless ways explained in the previous subsection. This is the path followed by higher-spin gauge fields in order to circumvent the conclusions of the Pais–Uhlenbeck no-go theorem. Indeed, all known formulations of free massless higher spin fields exhibit new features with respect to lower-spin ($s \leqslant 2$) fields (e.g. trace conditions, non-locality or higher-derivative kinetic operators, auxiliary fields, etc). These unavoidable novelties of higher spins are deeply rooted in the fact that the curvature tensor, that is presumably the central object in higher-spin theory, contains s derivatives. A major progress of the recent approaches to higher-spin fields was to produce "geometric" field equations, i.e. equations written explicitly in terms of the curvature.

The curvature tensor $\mathcal{R}_{\mu_1\dots\mu_s\,;\,\nu_1\dots\nu_s}$ of de Wit and Freedman [14] and the curvature tensor $\mathcal{K}_{\mu_1\nu_1\,|\dots|\,\mu_s\nu_s}$ of Weinberg [26] are essentially the projection of $\partial_{\mu_1}\dots\partial_{\mu_s}\phi_{\nu_1\dots\nu_s}$, the sth derivatives of the gauge field, on the tensor field irreducible under $GL(D,\mathbb{R})$ with symmetries labeled by the Young tableau

$$\begin{array}{|c|c|c|c|}\hline \mu_1 & \mu_2 & \quad\cdots\quad & \mu_s \\ \hline \nu_1 & \nu_2 & \quad\cdots\quad & \nu_s \\ \hline \end{array}\ . \tag{6}$$

The Weinberg and de Wit–Freedman tensors are simply related by a choice of symmetry convention. In the case $s=2$, the de Wit–Freedman curvature tensor precisely is the Jacobi tensor while the Weinberg tensor coincides with the Riemann tensor. In the case $s=3$, they are related by

$$\mathcal{R}^{\mu_1\nu_1\rho_1}{}_{;\,\mu_2\nu_2\rho_2} = \mathcal{K}^{\mu_1}{}_{(\mu_2\,|}{}^{\nu_1}{}_{\nu_2\,|}{}^{\rho_1}{}_{\rho_2)}\,, \tag{7}$$

and

$$\mathcal{K}_{\mu_1\nu_1\,|\,\mu_2\nu_2\,|\,\mu_3\nu_3} = 2\,\mathcal{R}_{[\mu_1[\mu_2[\mu_3\,;\,\nu_1]\nu_2]\nu_3]}\,, \tag{8}$$

where the three antisymmetrizations are taken over every pair of indices (μ_i,ν_i). (We refer to Appendix A.1.1 for the notations.) The Weinberg tensor is in the antisymmetric convention for which the projection is more easy to perform because, since $\partial_{\mu_1}\ldots\partial_{\mu_s}\phi_{\nu_1\ldots\nu_2}$ is already symmetric in all indices of the two rows of the Young tableau (6), it only remains to antisymmetrize over all pairs (μ_i,ν_i). This corresponds to taking s curls of the symmetric tensor field ϕ_s. On the one hand, the Weinberg tensor is, by construction, antisymmetric in each of the s sets of two indices

$$\mathcal{K}_{[\mu_1\nu_1]\,|\ldots|\,\mu_s\nu_s} = \ldots = \mathcal{K}_{\mu_1\nu_1\,|\ldots|\,[\mu_s\nu_s]} = \mathcal{K}_{\mu_1\nu_1\,|\ldots|\,\mu_s\nu_s}\,. \tag{9}$$

Moreover, the complete antisymmetrization over any set of three indices gives zero, so that the Weinberg tensor indeed belongs to the space irreducible under $GL(D,\mathbb{R})$ characterized by a two-row rectangular Young diagram of length s. On the other hand, the de Wit–Freedman tensor is, by definition, symmetric in each of the two sets of s indices

$$\mathcal{R}_{(\mu_1\ldots\mu_s)\,;\,\nu_1\ldots\nu_s} = \mathcal{R}_{\mu_1\ldots\mu_s\,;\,(\nu_1\ldots\nu_s)} = \mathcal{R}_{\mu_1\ldots\mu_s\,;\,\nu_1\ldots\nu_s}\,. \tag{10}$$

Moreover, it obeys the algebraic identity

$$\mathcal{R}_{(\mu_1\ldots\mu_s\,;\,\nu_1)\nu_2\ldots\nu_s} = 0\,, \tag{11}$$

so that it also belongs to the space irreducible under $GL(D,\mathbb{R})$ characterized by a two-row rectangular Young diagram of length s. Both definitions of the curvature tensor are equivalent, in the sense that they define the same tensor space invariant under the action of $GL(D,\mathbb{R})$.

Due to these symmetries, the curvature tensors are strictly invariant under gauge transformations (3) with unconstrained gauge parameter $\epsilon_{\mu_1\ldots\mu_{s-1}}$. Indeed, if the indices of two partial derivatives appear in the same column, the corresponding irreducible tensor vanishes. For the same reason, the irreducible components of the partial derivative of the de Wit–Freedman tensor $\partial_\rho\mathcal{R}_{\mu_1\ldots\mu_s\,;\,\nu_1\ldots\nu_s}$ which are labeled by the Young tableau

$$\begin{array}{|c|c|c|c|}\hline \mu_1 & \mu_2 & \quad\cdots\quad & \mu_s \\ \hline \nu_1 & \nu_2 & \quad\cdots\quad & \nu_s \\ \hline \rho \\ \cline{1-1} \end{array}\ ,$$

identically vanish. In terms of the Weinberg tensor, this translates into the "Bianchi identity"

$$\partial_{[\rho}\mathcal{K}_{\mu_1\nu_1]\,|\ldots|\,\mu_s\nu_s} = 0\,. \tag{12}$$

A generalization of the Poincaré lemma states that the differential Bianchi-like identity (12) together with the previous algebraic irreducibility conditions on $\mathcal{K}$ imply that the Weinberg tensor is the sth derivative of a symmetric tensor field of rank s [7, 8]. The same theorem states that the most general pure-gauge tensor field for which the curvature vanishes identically is a symmetrized derivative of a symmetric tensor field of rank $s-1$. The gauge structure of symmetric tensor gauge fields was elegantly summarized by Dubois-Violette and Henneaux in terms of generalized cohomologies [8] (see Section A.2 for a brief review of these concepts).

2.1.3 Non-local, unconstrained approach of Francia and Sagnotti

The field equations proposed by Francia and Sagnotti [13] for unconstrained completely-symmetric tensor gauge fields are non-local, but they are invariant under gauge transformations (3) where the trace of the completely-symmetric tensor gauge parameter ϵ is not constrained to vanish. They read

$$\begin{cases} \eta^{\mu_1\mu_2}\ldots\eta^{\mu_{s-1}\mu_s}\,\Box^{-\frac{s-2}{2}}\,\mathcal{R}_{\mu_1\ldots\mu_s\,;\,\nu_1\ldots\nu_s}\approx 0 & \text{for } s \text{ even}\,,\\ \eta^{\mu_1\mu_2}\ldots\eta^{\mu_s\mu_{s+1}}\,\Box^{-\frac{s-1}{2}}\,\partial_{\mu_{s+1}}\mathcal{R}_{\mu_1\ldots\mu_s\,;\,\nu_1\ldots\nu_s}\approx 0 & \text{for } s \text{ odd}\,,\end{cases} \tag{13}$$

where $\mathcal{R}_{\mu_1\ldots\mu_s\,;\,\nu_1\ldots\nu_s}$ is the spin-s curvature tensor introduced by de Wit and Freedman. Putting it in words, the geometric equations (13) for completely symmetric tensor gauge fields ϕ_s are easily constructed: When s is *odd* one takes one divergence together with $\frac{s-1}{2}$ trace(s) of the tensor $\mathcal{R}_{\mu_1\ldots\mu_s\,;\,\nu_1\ldots\nu_s}$ and when s is *even* one just takes $s/2$ trace(s) [13]. So one constructs a gauge-invariant object with the symmetries of the field of rank s but containing $s+\varepsilon(s)$ derivatives. Consequently, the authors of [13] further multiplied by $\Box^{1-\frac{s+\varepsilon(s)}{2}}$ in order to get second-order field equations.

Via algebraic manipulations, the field equations (13) for rank-s completely symmetric tensor fields have been shown [13] to be equivalent to

$$\mathcal{F}_{\mu_1\mu_2\mu_3\mu_4\ldots\mu_s} - \partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3}\mathcal{H}_{\mu_4\ldots\mu_s)} \approx 0\,, \tag{14}$$

where the tensor $\mathcal{H}_{\mu_1\ldots\mu_{s-3}}$ is a non-local function of the field $\phi_{\mu_1\ldots\mu_s}$ and its derivatives, whose gauge transformation is proportional to the trace of the gauge parameter. The gauge-fixing condition $\mathcal{H}_{\mu_1\ldots\mu_{s-3}}=0$ leads to the Fronsdal equation (2). Therefore, this geometric formulation of higher-spin gauge fields falls into the class (a) of harmless non-locality. Basically, the main additional subtlety arising for spin $s\geqslant 4$ is that the usual de Donder condition is reachable with a traceless gauge parameter if and only if the double trace of the field vanishes. Therefore, in the Fronsdal approach the field is constrained to have vanishing double trace (which is consistent with the invariance of the double trace of the field under gauge transformations with traceless parameter). As pointed out in [27], more work is therefore required in order to obtain the double-trace condition for spin $s\geqslant 4$ in the unconstrained approach. A solution is to take a modified *identically traceless* de Donder gauge [27]. After this further gauge-fixing, the field equation implies the vanishing of the double trace of the field, thereby recovering the usual de Donder condition.

Heuristically, one can also argue that the non-local field equations (13) are equivalent to local ones (2) by going in a traceless-transverse gauge (*i.e.* $\text{Tr}\,\phi = 0$ and $\partial \cdot \phi = 0$), because both equations reduce to the Klein–Gordon equation $\Box\phi \approx 0$ since the powers of the d'Alembertian cancel in the non-local approach. Of course, rigorously speaking, we should prove that this rule applies for the formal object $\Box^{-1}$. We take this opportunity to briefly discuss the meaning given to the inverse d'Alembertian in the non-local unconstrained approach, and in which sense local higher-derivative field equations may be equivalent to non-local second-order field equations. Regarding $\Box^{-1}$, we note that an obvious way of defining a pseudodifferential operator (such as $1/\Box$) is through its Fourier transform, because the latter simply is a non-polynomial function of the momentum (such as $-1/p^2$), a much less frightening object. The second comment is that any linear application A on a vector space V is invertible on the quotient $V/\text{Ker}A \cong \text{Im}A$ (More concretely, let $w = Av$ be in $\text{Im}A$, then one may write $v = A^{-1}w + u$ with $u \in \text{Ker}A$). The third comment is that the representatives in the quotient $\text{Ker}\Box^n/\text{Ker}\Box$ for $n > 1$ are usually called "runaway solutions" because they are unbounded at infinity. These solutions are the classical counterparts of the ghosts in the quantum theory, so one rejects them on physical ground. In mathematical terms, one requires the solutions to be in an appropriate functional class such that $\text{Ker}\Box^n = \text{Ker}\Box$ (for all $n > 1$). In this restricted sense, the non-local equations (13) and the following higher-derivative equations

$$\begin{cases} \eta^{\mu_1\mu_2}\dots\eta^{\mu_{s-1}\mu_s}\,\mathcal{R}_{\mu_1\dots\mu_s\,;\,\nu_1\dots\nu_s} \approx 0 & \text{for } s \text{ even}\,, \\ \eta^{\mu_1\mu_2}\dots\eta^{\mu_s\mu_{s+1}}\,\partial_{\mu_{s+1}}\mathcal{R}_{\mu_1\dots\mu_s\,;\,\nu_1\dots\nu_s} \approx 0 & \text{for } s \text{ odd}\,, \end{cases} \tag{15}$$

are thus equivalent at the level of sourceless free field equations. Nevertheless, this equivalence of the equations of motion does *not* imply the equivalence of the variational principle of course and, thus, does not contradict the Pais-Uhlenbeck no-go theorem on higher-derivative Lagrangians [22]. This being said, from now on we refer to (13) or (15) without any distinction.

It is convenient to rewrite the Francia–Sagnotti equations (15) in terms of the Weinberg tensor in order to generalize them to mixed-symmetry tensor gauge fields more easily:

$$\begin{cases} \eta^{(\nu_1\nu_2}\dots\eta^{\nu_{s-1}\nu_s)}\,\mathcal{K}_{\mu_1\nu_1\,|\dots|\,\mu_s\nu_s} \approx 0 & \text{for } s \text{ even}\,, \\ \eta^{(\nu_1\nu_2}\dots\eta^{\nu_s\nu_{s+1})}\,\partial_{\nu_{s+1}}\mathcal{K}_{\mu_1\nu_1\,|\dots|\,\mu_s\nu_s} \approx 0 & \text{for } s \text{ odd}\,, \end{cases} \tag{16}$$

where the symmetrization over all indices ν of the Minkowski metrics is important in order to have the proper symmetries on the free indices μ_i, $1 \leqslant i \leqslant s$.

2.1.4 Higher-derivative, unconstrained approach

The compensator field equation for symmetric tensor fields [13, 27] (generalized later to completely symmetric tensor-spinor fields [28])

$$\mathcal{F}_{\mu_1\mu_2\mu_3\mu_4\dots\mu_s} - \frac{s(s-1)(s-2)}{2}\,\partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3}\alpha_{\mu_4\dots\mu_s)} \approx 0 \tag{17}$$

is the same as (14) except that the symmetric tensor $\alpha_{\mu_1\ldots\mu_{s-3}}$ of rank $s-3$ is an independent field, called "compensator". It is a pure-gauge field whose gauge transformation

$$\delta\alpha_{\mu_1\ldots\mu_{s-3}} = (\mathrm{Tr}\,\epsilon)_{\mu_1\ldots\mu_{s-3}} \tag{18}$$

precisely cancels the contribution (4) coming from the Fronsdal tensor so that (17) is invariant under gauge transformations with unconstrained gauge parameter. The compensator field may be gauged away by using the freedom (18), which gives the Fronsdal equation (2). Fixing $\alpha = 0$ is called the "Fronsdal gauge", where the constraint $\mathrm{Tr}\,\epsilon = 0$ is imposed on the gauge parameter. Again, in order to recover the double trace constraint $\mathrm{Tr}^2\phi = 0$ on the gauge field more work is necessary [28].

The "Ricci curvature tensor" $(\mathrm{Tr}\,\mathcal{R})_{\mu_1\ldots\mu_s\,;\,\nu_1\ldots\nu_{s-2}}$ is the trace of the de Wit–Freedman tensor. Its symmetries are encoded in the Young tableau

$$\begin{array}{|c|c|c|c|c|c|}\hline \mu_1 & \mu_2 & \quad\cdots\quad & \mu_{s-2} & \mu_{s-1} & \mu_s \\ \hline \nu_1 & \nu_2 & \quad\cdots\quad & \nu_{s-2} & \multicolumn{2}{c}{} \\ \cline{1-4} \end{array} \quad . \tag{19}$$

The Damour–Deser identity [29] schematically written $\mathrm{Tr}\mathcal{K} = d^{s-2}\mathcal{F}$ relates the Ricci-like tensor $\mathrm{Tr}\mathcal{R}$ to the $(s-2)$th curl of the Fronsdal tensor $\mathcal{F}$. These curls are obtained by projecting the $(s-2)$th partial derivative $\partial_{\nu_1}\ldots\partial_{\nu_{s-2}}\mathcal{F}_{\mu_1\ldots\mu_s}$ of the Fronsdal tensor on the irreducible component labeled by (19) *via* the antisymmetrization over the pairs (μ_i, ν_i) for $1 \leqslant i \leqslant s-2$. Consequently, the compensator equation (17) implies the higher-derivative "Ricci-flat" equation

$$(\mathrm{Tr}\mathcal{R})_{\mu_1\ldots\mu_s\,;\,\nu_1\ldots\nu_{s-2}} \approx 0 \quad \Longleftrightarrow \quad (\mathrm{Tr}\mathcal{K})_{\mu_1\nu_1\,|\ldots\ldots|\,\mu_{s-2}\nu_{s-2}\,|\,\mu_{s-1}\,|\,\mu_s} \approx 0 \quad . \tag{20}$$

Conversely, the equation (20) and the Damour–Deser identity imply that the $s-2$th curl of the Fronsdal tensor $\mathcal{F}$ vanishes on-shell. As was explained in [16], the generalized Poincaré lemma of [7, 8] shows[8] the equivalence of this "closure" condition $d^{s-2}\mathcal{F} \approx 0$ of the Fronsdal tensor to its "exactness" expressed by the compensator equation (17). In other words, the field equations (17) and (20) are strictly equivalent in a flat spacetime with trivial topology. Notice that both of them are higher-derivative when $s > 2$, the compensator equation being of third order and the Ricci-flat-like equation being of sth order.

Furthermore, the Ricci-flat-like equation (20) is equivalent to a set of first-order field equations. In $D = 4$, they correspond to the Bargmann–Wigner equations [2], originally expressed in terms of two-component tensor-spinors in the representation $(s, 0) \oplus (0, s)$ of $SL(2, \mathbb{C})$. They were generalized to $D > 4$ in [9, 16] for arbitrary tensorial UIRs of the Poincaré group, and in [12] for spinoral UIRs. The main idea is to start with a tensor field that is (on-shell) irreducible under the Lorentz group $O(D-1, 1)$ with symmetries labeled by the Young tableau depicted by (6). The antisymmetric convention proves to be more convenient so one considers a (on-shell) traceless tensor field whose components $\mathcal{K}_{\mu_1\nu_1\,|\ldots|\,\mu_s\nu_s}$ obey the $GL(D, \mathbb{R})$ irreducibility conditions explained in Subsection 2.1.2. One then requires that it also obeys the Bianchi-like identity (12),

[8] We insist on the fact that it was not necessary to make use of the de Wit–Freedman connections to derive this result since the Poincaré lemma allows a direct jump from the Ricci-flat-like equation to the compensator equation.

which is equivalent to the fact that the tensor $\mathcal{K}$ is the Weinberg curvature of a completely-symmetric tensor gauge field ϕ_s of rank s. The on-shell tracelessness $\mathrm{Tr}\,\mathcal{K}\approx 0$ of the irreducible tensor is therefore equivalent to the Ricci-flat-like equation (20) if the tensor $\mathcal{K}$ obeys the differential Bianchi identity (12). Finally, one can also show that the (on-shell) $O(D-1,1)$-irreducibility conditions combined with the differential Bianchi identity imply that the tensor field is divergenceless on-shell $\partial\cdot\mathcal{K}\approx 0$.

In summary, the equations

$$\left\{\begin{array}{c}\partial_{[\rho}\mathcal{K}_{\mu_1\nu_1]\,|\,\mu_2\nu_2\,|\ldots|\,\mu_s\nu_s}=0\\ \partial^{\rho}\mathcal{K}_{\rho\nu_1\,|\,\mu_2\nu_2\,|\ldots|\,\mu_s\nu_s}\approx 0\end{array}\right.\,, \tag{21}$$

imposed on a tensor field $\mathcal{K}$ taking values in an irreducible representation of the group $O(D-1,1)$, are equivalent to the Ricci-flat-like equations (20) and thereby to all other field equations of symmetric tensor gauge fields alike.

2.2 Fierz–Pauli programme

Fronsdal was able to write down a local second-order action, quadratic in the double-traceless gauge field ϕ and invariant under the gauge transformations (3) with traceless parameter ϵ [5]. Moreover, Curtright pointed out that these requirements fix the Lagrangian uniquely, up to an overall factor [30]. The Euler-Lagrange equation derived from Fronsdal's action is equivalent to (2).

Notice that by introducing a pure gauge field (sometimes refered to as "compensator"), it is possible to write a local (but higher-derivative) action for spin-3 [13, 27] that is invariant under unconstrained gauge transformations. Very recently, this action was generalized to the completely symmetric spin-s case by further adding an auxiliary field associated with the double trace of the gauge field [31]. Retrospectively, the reference [32] may be interpreted as an older "non-minimal" version of it, as explained in more details in [28] (see also [33] for the fermionic counterpart of [32]).

2.2.1 Non-local actions of Francia and Sagnotti

In this subsection, we introduce a compact expression for the "Einstein tensors" of [13] by using Levi-Civita "epsilon" tensors. In this way, it is much simpler to write the Einstein-like tensor, and the Noether (sometimes referred to as "Bianchi") identity is automatically satisfied without explicitly introducing the trace expansion as in [13].

Since the Levi-Civita tensors are involved it is natural to use the antisymmetric convention for Young tableaux, so the starting point are the Francia–Sagnotti equations (16) in terms of the Weinberg tensor $\mathcal{K}$. It turns out to be convenient to introduce the symmetric tensor $\eta_{\mu_1\ldots\mu_{2n}}$ of rank $2n$ defined by

$$\eta_{\mu_1\,\mu_2\,\mu_3\,\mu_4\,\ldots\,\mu_{2n-1}\mu_{2n}}:=\eta_{(\mu_1\mu_2}\eta_{\mu_3\mu_4}\ldots\eta_{\mu_{2n-1}\mu_{2n})}\,, \tag{22}$$

for all integers $n\in\mathbb{N}$, corresponding to the product of n metrics with all indices symmetrized. The Einstein-like tensor

$$\mathcal{G}^{\mu_1\mu_2\dots\mu_{s-1}\mu_s} := \begin{cases} \varepsilon^{\mu_1\nu_1\dots\rho_1\sigma_1\tau_1}\dots\varepsilon^{\mu_s\nu_s\dots\rho_s\sigma_s\tau_s}\,\eta_{\nu_1\dots\nu_s}\dots\eta_{\rho_1\dots\rho_s}\,\mathcal{K}_{\sigma_1\tau_1\,|\dots|\,\sigma_s\tau_s} & s \text{ even}\,, \\ \varepsilon^{\mu_1\nu_1\dots\rho_1\sigma_1\tau_1}\dots\varepsilon^{\mu_{s+1}\nu_{s+1}\dots\rho_{s+1}\sigma_{s+1}\tau_{s+1}}\,\eta_{\nu_1\dots\nu_{s+1}}\dots\eta_{\rho_1\dots\rho_{s+1}}\,\eta_{\mu_{s+1}\tau_1}\,\partial_{\sigma_1}\mathcal{K}_{\sigma_2\tau_2\,|\dots|\,\sigma_{s+1}\tau_{s+1}} & s \text{ odd}\,, \end{cases} \tag{23}$$

is defined *via* traces of the Hodge dual on every set of antisymmetric indices of the Weinberg tensor. In the even spin case, the symmetry under the exchange of two μ_i indices is a consequence of the symmetry properties of the curvature tensor $\mathcal{K}$ under the exchange of pairs (σ_i, τ_i) of antisymmetric indices together with the symmetry properties of the tensor η defined in (22). In the odd spin case, the symmetry is not automatic and, actually, one must understand that there is an implicit symmetrization over the μ indices in the second line of (23). By taking traces, *etc*, one may show that the Einstein-like equations $\mathcal{G}^{\mu_1\mu_2\dots\mu_s} \approx 0$ are algebraically equivalent to the equations (16) of Francia and Sagnotti [13]. The Einstein-like tensor (23) is automatically gauge invariant under (3) because it is a linear combination of the curvature tensor. The Noether identity corresponding to the gauge transformations (3) with unconstrained parameters is the divergencelessness of the Einstein-like tensor, $\partial_{\mu_1}\mathcal{G}^{\mu_1\mu_2\dots\mu_s} = 0$, which follows from the Bianchi-like identity (12) obeyed by the Weinberg tensor. The Einstein-like tensor contains a product of $D-3$ symmetric tensors $\eta_{\nu_1\dots\nu_{s+\varepsilon(s)}}$. One may rewrite the traces over the Levi-Civita tensors as products of Kronecker symbols

$$\delta^{\nu_1\dots\nu_p}_{\mu_1\dots\mu_p} \equiv \delta^{[\nu_1}_{\mu_1}\dots\delta^{\nu_p]}_{\mu_p} = \delta^{\nu_1}_{[\mu_1}\dots\delta^{\nu_p}_{\mu_p]}$$

via the identity

$$\varepsilon_{\mu_1\dots\mu_p\,\rho_1\dots\rho_{D-p}}\,\varepsilon^{\nu_1\dots\nu_p\,\rho_1\dots\rho_{D-p}} = -\,p\,!\,(D-p)!\,\delta^{\nu_1\dots\nu_p}_{\mu_1\dots\mu_p}\,. \tag{24}$$

This leads to an expansion of the Einstein-like tensor as a sum of product of metrics times traces of the $[\frac{s}{2}]$th trace of the curvature tensor written in (16):

$$\mathcal{G}_{\mu_1\dots\mu_s} \propto \begin{cases} \eta^{\nu_1\dots\nu_s}\,\mathcal{K}_{\mu_1\nu_1\,|\dots|\,\mu_s\nu_s} + \dots & \text{for } s \text{ even}\,, \\ \eta^{\nu_1\dots\nu_{s+1}}\,\partial_{\nu_{s+1}}\mathcal{K}_{\mu_1\nu_1\,|\dots|\,\mu_s\nu_s} + \dots & \text{for } s \text{ odd}\,. \end{cases} \tag{25}$$

The coefficients in the expansion of the Einstein-like tensor were determined uniquely in [13] by imposing that the Noether identity be obeyed. Therefore, the Einstein-like tensor (23) must correspond to the one of Francia and Sagnotti, up to an overall coefficient.

The conclusion of the discussion on the negative powers of the d'Alembertian in Subsection 2.1.3 is that one *cannot* remove them in the Lagrangian of the non-local approach without introducing ghosts, but that one *can* remove them in the Euler-Lagrange equations provided that the ghosts are eliminated "by hand" by choosing an appropriate functional space of allowed solutions. The authors of [13] proposed an action of the form $\int d^Dx\,\phi\cdot\frac{1}{\Box^{[\frac{s-1}{2}]}}\,\mathcal{G}(\phi)$. In the form chosen here, this prescription leads to

$$S[\phi_s] = \int d^Dx\,\varepsilon^{\mu_1\nu_1\dots\rho_1\sigma_1\tau_1}\dots\varepsilon^{\mu_s\nu_s\dots\rho_s\sigma_s\tau_s}\eta_{\nu_1\dots\nu_s}\dots\eta_{\rho_1\dots\rho_s}\,\phi_{\mu_1\dots\mu_s}\,\frac{1}{\Box^{\frac{s}{2}-1}}\,\partial_{\sigma_1}\dots\partial_{\sigma_s}\phi_{\tau_1\dots\tau_s}\,, \tag{26}$$

for even spin s, and to

$$S[\phi_s] = \int d^Dx\, \varepsilon^{\mu_1\nu_1\ldots\rho_1\sigma_1\tau_1} \ldots \varepsilon^{\mu_{s+1}\ldots\tau_{s+1}}\, \eta_{\tau_1\mu_{s+1}}\, \eta_{\nu_1\ldots\nu_{s+1}} \ldots \eta_{\rho_1\ldots\rho_{s+1}}\, \phi_{\mu_1\ldots\mu_s}\, \frac{1}{\Box^{\frac{s-1}{2}}}\, \partial_{\sigma_1} \ldots \partial_{\sigma_{s+1}} \phi_{\tau_2\ldots\tau_{s+1}}\,, \quad (27)$$

for odd spin s.

The kinetic operator is self-adjoint, thus the Einstein-like equations $\mathcal{G}_{\mu_1\ldots\mu_s} \approx 0$ are the Euler-Langrange equations of the quadratic action, and the action is manifestly gauge invariant. The fact that these properties are manifest allows a straightforward generalization to any mixed-symmetry tensor gauge field, as we explain in Section 3.

2.2.2 Non-local actions in terms of differential forms

Introducing letters from the beginning of the Latin alphabet in order to denote tangent space indices, one may rewrite the actions (26) and (27) in a frame-like fashion. In flat spacetime of course, the distinction between tangent and curved indices is somewhat irrelevant since the background coframe reads, in components, $(e_0)^a_\mu = \delta^a_\mu$. However, making this distinction may suggest a natural generalization of the quadratic actions to curved spacetimes by using differential forms.

To start with, we write the action for a symmetric spin-s field ϕ_s featuring only "tangent" indices except for D suitably chosen "exterior form" indices:

$$\begin{aligned} S[\phi] \;&=\; \int d^Dx\, \varepsilon^{\mu\nu\ldots\rho\sigma\tau} \varepsilon^{a_1b_1\ldots c_1d_1f_1} \ldots \varepsilon^{a_{s-1}b_{s-1}\ldots c_{s-1}d_{s-1}f_{s-1}}\, \eta_{\nu b_1\ldots b_{s-1}} \ldots \eta_{\rho c_1\ldots c_{s-1}} \;\times \\ &\quad \times\, \phi_{\mu\, a_1\ldots a_{s-1}}\, \frac{1}{\Box^{\frac{s}{2}-1}} \mathcal{K}_{\sigma\tau\,|\,d_1f_1|\ldots|d_{s-1}f_{s-1}}\,, \end{aligned} \quad (28)$$

for s even, and

$$\begin{aligned} S[\phi_s] \;&=\; -\int d^Dx\, \varepsilon^{a_1b_1\ldots c_1d_1f_1}\, \varepsilon^{\mu\nu\ldots\rho\sigma\tau}\, \varepsilon^{a_2b_2\ldots c_2d_2f_2} \ldots \varepsilon^{a_sb_s\ldots c_sd_sf_s}\, \eta_{f_1a_s}\, \eta_{\nu b_1b_2\ldots b_s} \ldots \eta_{\rho c_1c_2\ldots c_s} \;\times \\ &\quad \times\, \partial_{d_1}\phi_{\mu a_1a_2\ldots a_{s-1}}\, \frac{1}{\Box^{\frac{s-1}{2}}} \mathcal{K}_{\sigma\tau\,|\,d_2f_2|\ldots|d_sf_s}\,, \end{aligned} \quad (29)$$

for s odd. The action (29) has been obtained from (27) after one integration by part, all the other operations being mere change of labels.

Now, we introduce some tensor-valued differential forms. For instance the Weinberg tensor field $\mathcal{K}$ defines a tensor-valued two-form $\mathcal{R}_1$ *via*

$$(\mathcal{R}_1)_{a_1b_1|\ldots|a_{s-1}b_{s-1}} = \frac{1}{2}\, \mathcal{K}_{\mu\nu|a_1b_1|\ldots|a_{s-1}b_{s-1}}\, dx^\mu \wedge dx^\nu\,, \quad (30)$$

while the symmetric tensor gauge field ϕ_s defines a tensor-valued one-form $e \in \odot^{s-1}(\mathbb{R}^{D*}) \otimes \Omega^1(\mathbb{R}^D)$ by

$$e_{a_1\ldots a_{s-1}} = \phi_{\mu\, a_1\ldots a_{s-1}}\, dx^\mu\,. \quad (31)$$

Also, the background coframe defines a vector-valued one-form

$$(e_0)^a = \delta^a_\mu\, dx^\mu\,. \quad (32)$$

It is tempting to treat the spin-s field one-form $e^{a_1 \ldots a_{s-1}}$ as a sort of "vielbein" for higher-spins perturbing the pure spin-two flat background e_0^a, as suggested by Vasiliev [21]. In this way, the curvature two-form (30) can be thought as the generalization of the linearized Riemann two-form in the moving-frame formulation of gravity [20]. Actually, one may also introduce a "Lorentz connection" one-form

$$(\omega_1)_{a_1 b_1 | \, a_2 \ldots a_{s-1}} = \partial_{[a_1} \phi_{b_1] \, \mu \, a_2 \ldots a_{s-1}} \, dx^\mu \, . \tag{33}$$

(The notations has been chosen in such a way as to easily make contact with the materials reviewed in Section 2 of [34].)

In the even-spin case, the action can be written in the following "Einstein–Cartan–Weyl" form by making use of the former differential forms:

$$\begin{aligned} S[\phi_s] \;&=\; \varepsilon_{a_1 b_1 \ldots c_1 d_1 f_1} \ldots \varepsilon_{a_{s-1} b_{s-1} \ldots c_{s-1} d_{s-1} f_{s-1}} \, \eta^{b_2 \ldots b_{s-1}} \ldots \eta^{c_2 \ldots c_{s-1}} \; \times \\ &\quad \times \int e_0^{b_1} \wedge \ldots \wedge e_0^{c_1} \wedge e^{a_1 \ldots a_{s-1}} \;\wedge\; \frac{1}{\Box^{\frac{s}{2}-1}} \mathcal{R}_1^{d_1 f_1 | \ldots | d_{s-1} f_{s-1}} \, , \end{aligned} \tag{34}$$

while the odd-spin case goes as follows:

$$\begin{aligned} S[\phi_s] \;&=\; \varepsilon_{a_1 b_1 \ldots c_1 d_1 f_1} \, \varepsilon_{a_2 b_2 \ldots c_2 d_2 f_2} \, \ldots \varepsilon_{a_s b_s \ldots c_s d_s f_s} \, \eta^{f_1 a_s} \, \eta^{b_1 b_3 \ldots b_s} \ldots \eta^{c_1 c_3 \ldots c_s} \; \times \\ &\quad \times \int e_0^{b_2} \wedge \ldots \wedge e_0^{c_2} \wedge \omega_1^{a_1 d_1 | \, a_2 a_3 \ldots a_{s-1}} \;\wedge\; \frac{1}{\Box^{\frac{s-1}{2}}} \mathcal{R}_1^{d_2 f_2 | \ldots | d_s f_s} \, . \end{aligned} \tag{35}$$

We implicitly understood everywhere that a symmetrization over all indices labeled by the same Latin letter should be performed.

The writing of the actions (34) and (35) suggests that they might make sense in an arbitrary curved background at the condition that the linearized curvature be replaced with its full non-Abelian counterpart. As a preliminary step in this direction, we show in the next subsection that the above Einstein–Cartan–Weyl actions can be seen as a flat spacetime limit of a MacDowell–Mansouri-like [35] action quadratic in curvatures and torsions taking value in some $(A)dS_D$ higher-spin algebra when $D \geqslant 4$. (For $D = 3$, the action looks more like a Chern–Simons action, in agreement with the fact that the theory is "topological" in the sense that there are no local physical degrees of freedom in three dimensions for $s > 0$.)

2.2.3 Non-local actions à la MacDowell and Mansouri

The isometry algebra of $(A)dS_D$ manifold is presented *via* its translation-like generators P_a and Lorentz generators M_{ab} $(a, b = 0, 1, \ldots, D-1)$ together with their commutation relations

$$[\, M_{ab} \, , \, M_{cd} \,] \;=\; i \, (\eta_{ac} \, M_{db} - \eta_{bc} \, M_{da} - \eta_{ad} \, M_{cb} + \eta_{bd} \, M_{ca}) \, , \tag{36}$$

$$[\, P_a \, , \, M_{bc}] \;=\; i \, (\eta_{ab} \, P_c - \eta_{ac} \, P_b) \, , \tag{37}$$

$$[P_a, P_b] \;=\; i \, \Lambda \, M_{ab} \, . \tag{38}$$

By defining $M_{\hat{D}\,a} := (\Lambda)^{-1/2} \, P_a$, it is possible to collect all generators into the generators M_{AB} where $A = 0, 1, \ldots, D-1, \hat{D}$. These generators M_{AB} span a pseudo-orthogonal algebra since they satisfy the commutation relations

$$[M_{AB}, M_{CD}] \;=\; i\,(\eta_{AC}M_{DB} - \eta_{BC}M_{DA} - \eta_{AD}M_{CB} + \eta_{BD}M_{CA})\,,$$

where η_{AB} is the mostly minus invariant metric of the corresponding pseudo-orthogonal algebra. This is easily understood from the geometrical construction of $(A)dS_D$ as the hyperboloid defined by $X^A X_A = \frac{(d-1)(d-2)}{2\Lambda}$ which is obviously invariant under the pseudo-orthogonal group. It is possible to derive the Poincaré algebra $\mathfrak{io}(D-1,1)$ from the $(A)dS_D$ isometry algebra by performing the Inönü-Wigner contraction $\Lambda \to 0$, in which limit the generators P_a become commuting genuine translation generators. The constant-curvature spacetime algebras can be uniformly realized as follows

$$M_{AB} \;=\; -i\,X_{[A}\frac{\partial}{\partial X^{B]}}\,, \tag{39}$$

if one takes $\frac{\partial}{\partial X^{\hat{D}}} \sim 0$ and $X_{\hat{D}} \sim 0$ in the flat limit $\Lambda \to 0$.

Since the gauge fields and parameters are unconstrained in the non-local formulation, it is natural to make use of the so-called off-shell constant-curvature spacetime higher-spin algebras which were discussed recently in [36, 34] and which we will now review in many details according to the present perspective. These higher-spin algebras can be easily defined as the Lie algebras of polynomials in the operators (39) endowed with the commutator as Lie bracket. In more abstract terms, they are the Lie algebras coming from the realization of the universal enveloping algebra induced by the unitary representation (39) of the constant-curvature spacetime isometry algebra. In more concrete terms, we will consider the Weyl-ordered monomials in the isometry algebra generators defined by (39)

$$T_{a_1b_1|\dots|a_tb_t|a_{t+1}\dots\, a_{s-1}} = M_{a_1b_1}\dots M_{a_tb_t}P_{a_{t+1}}\dots P_{a_{s-1}} + \text{perms} \tag{40}$$

as the most convenient basis of generators for our purpose ($t \in \mathbb{N}$ and $s \in \mathbb{N}_0$), where "perms" stands for the sum of all nontrivial permutations of the generators M and P. The symbol of the differential operators (40) is a tensor irreducible under $GL(D,\mathbb{R})$ with symmetries labeled by the two-row Young tableau

$$\begin{array}{|c|c|c|c|c|c|}\hline a_1 & \cdots & a_t & a_{t+1} & \cdots & a_{s-1} \\ \hline b_1 & \cdots & b_t \\ \cline{1-3}\end{array}\;. \tag{41}$$

In order to mimic MacDowell–Mansouri formulation, one defines a connection one-form ω taking values in the higher-spin algebra

$$\omega(x^\mu, dx^\nu, M_{AB}) \;=\; -i\,dx^\nu\,\omega_\nu^{a_1b_1|\dots|a_tb_t|a_{t+1}\dots\,a_{s-1}}\,T_{a_1b_1|\dots|a_tb_t|a_{t+1}\dots\,a_{s-1}}$$

and whose non-Abelian curvature is the two-form $\mathcal{R} = d\omega + \omega^2$. The component of ω linear in P_a is the moving frame e^a of the spacetime manifold while the component linear in M_{ab} is its Lorentz connection ω^{ab}. In the pure gravity case, the coefficient $\mathcal{R}^{ab}$ of M_{ab} in $\mathcal{R}$ is the sum $\mathcal{R}^{ab} = R^{ab} + \Lambda\, e^a \wedge e^b$ of the Riemann two-form plus cosmological terms while the coefficient T^a of P_a in $\mathcal{R}$ is the torsion. The components $\omega^{a_1b_1|\dots|a_tb_t|a_{t+1}\dots\,a_{s-1}}$ of the connection ω are assumed to be irreducible tensors under $GL(D,\mathbb{R})$ described by the Young diagram (41), as can be done without loss of generality.

In general, if a connection one-form is decomposed as a sum $\omega = \omega_0 + \omega_1$ of a vacuum solution ω_0 plus a small fluctuation ω_1, then its curvature can also be expanded in powers of the fluctuation: at order zero, one has $\mathcal{R}_0 = d\omega_0 + \omega_0^2 = 0$ by assumption, and at order one, the linearized curvature reads $\mathcal{R}_1 = D_0\omega_1 = d\omega_1 + [\omega_0, \omega_1]_+$, where the background covariant derivative $D_0 = d + [\omega_0,\ ]_\pm$ is nilpotent, $D_0^2 = \mathcal{R}_0 = 0$. The linearization of the gauge transformations $\delta_\epsilon\omega = d\epsilon + [\omega, \epsilon]_-$ reads $\delta\omega_1 = D_0\epsilon$ and leaves the linearized curvature invariant. In the present case, the background higher-spin connection is assumed to be purely gravitational in the sense that

$$\omega_0 \,=\, -i\,(e_0^a\,P_a \,+\, \omega_0^{ab}\,M_{ab})\,. \tag{42}$$

Moreover, if the gravitational background is assumed to be a vacuum solution of the constant-curvature spacetime algebra, then the background connection one-form describes the corresponding constant-curvature spacetime manifold, since $\mathcal{R}_0 = 0$ decomposes as $R_0^{ab} = -\Lambda\, e_0^a \wedge e_0^b$ and $T_0^a = 0$. In order to evaluate the action of the covariant derivative with respect to this background, it is sufficient to compute the commutator of P and M with any monomial T. A nice property of the Weyl ordering is that the commutator of a Lie algebra element with a Weyl-ordered element of the universal enveloping algebra preserves the Weyl ordering. Therefore the generators T transform as tensors under the adjoint action of the Lorentz algebra spanned by the M_{ab}'s and it is convenient to split the background covariant derivative into the sum $D_0 = D_0^L + [e_0,\ ]_\pm$ where D_0^L is the covariant derivative with respect to the background Lorentz connection. The commutator between a translation-like generator P and any generator T is easily computed

$$\begin{aligned} \left[\, P_a\,,\, T_{b_1c_1|......|b_tc_t|b_{t+1}\ldots\, b_{s-1}} \,\right] \;&=\; 2i\sum_{i=1}^{t} T_{b_1c_1|......|b_{i-1}c_{i-1}|b_{i+1}c_{i+1}|......|b_tc_t|b_{t+1}\ldots\, b_{s-1}\,[c_i}\ \eta_{b_i]\,a} \\ &\quad + i\,\Lambda\, T_{b_1c_1|......|b_tc_t|ab_{t+1}|b_{t+2}\ldots\, b_{s-1}} \\ &\quad + \,\ldots\, + \, i\,\Lambda\, T_{b_1c_1|......|b_tc_t|ab_{s-1}|b_{t+1}\ldots\, b_{s-2}}\,. \end{aligned} \tag{43}$$

While the background one-form (42) is assumed to contain the spin-two gauge fields (e_0^a, ω_0^{ab}) only, the fluctuation one-form may contain the infinite tower of symmetric tensor gauge fields. In particular, the components along the pure translation-like generators in

$$\omega_1 \,=\, e^{a_1\ldots a_{s-1}}\,P_{a_1}\ldots P_{a_{s-1}} \,+\, \mathcal{O}(M_{ab}) \tag{44}$$

are frame-like one-forms $e^{a_1\ldots a_{s-1}}$ given by (31) in some proper gauge. More precisely, the linearized gauge transformations $\delta\omega_1 = D_0\epsilon$ read in components

$$\delta_\epsilon e^{a_1\ldots a_{s-1}} \,=\, D_0^L \epsilon^{a_1\ldots a_{s-1}} \,+\, (e_0)_c\ \epsilon^{c(a_1|a_2\ldots\, a_{s-1})} \tag{45}$$

and

$$\begin{aligned} \delta_\epsilon\omega_1^{a_1b_1|\ldots|a_tb_t|a_{t+1}\ldots\, a_{s-1}} \;&=\; D_0^L\epsilon^{a_1b_1|\ldots|a_tb_t|a_{t+1}\ldots\, a_{s-1}} \,+\, (e_0)_c\ \epsilon^{a_1b_1|\ldots|a_tb_t|c(a_{t+1}\ldots\, a_{s-1})} \\ &\quad -\Lambda\,\mathbf{Y}_{A}\,\Big(\,\epsilon^{a_2b_2|\ldots|a_tb_t|a_{t+1}\ldots\, a_{s-1}\,[a_1}e_0^{b_1]} \\ &\qquad -\,\ldots\,+\,\epsilon^{a_1b_1|\ldots|a_{t-1}b_{t-1}|a_{t+1}\ldots\, a_{s-1}\,[a_t}e_0^{b_t]}\,\Big) \end{aligned} \tag{46}$$

for $t > 0$, due to the commutation relations (43). The frame-like one forms $e_\mu^{a_1 \ldots a_{s-1}}$ can be seen as rank-s tensors reducible under $GL(D,\mathbb{R})$ which can be decomposed into the sum of two tensors irreducible under $GL(D,\mathbb{R})$ respectively labeled by the Young tableau

$$\begin{array}{|c|c|c|c|}\hline a_1 & \quad\cdots\quad & a_{s-1} & \mu \\ \hline \end{array}$$

and

$$\begin{array}{|c|c|c|}\hline a_1 & \quad\cdots\quad & a_{s-1} \\ \hline \mu \\ \cline{1-1} \end{array} \tag{47}$$

The gauge variation $\eta_{\mu c}\epsilon^{c(a_1|a_2\ldots\, a_{s-1})}$ can be chosen in such a way as to precisely cancel the "hook" part in $e_\mu^{a_1 \ldots a_{s-1}}$ labeled by the Young tableau (47). In this metric-like gauge, the identification (31) holds. Pursuing the analogy with gravity, the other components of ω_1 should be expressed in terms of these dynamical fields e *via* some torsion constraints on the curvature $\mathcal{R}$. These constraints are only known at linearized order where they take the form

$$\mathcal{R}_1^{a_1 b_1|\ldots|a_t b_t|a_{t+1}\ldots\, a_{s-1}} = 0\,, \qquad \text{for} \quad 0 \leqslant t < s-1\,. \tag{48}$$

The commutation relations (43) lead to the following expression for the linearized curvatures,

$$\mathcal{R}_1^{a_1\ldots\, a_{s-1}} = D_0^L e^{a_1\ldots\, a_{s-1}} + (e_0)_c \wedge \omega_1^{c(a_1|a_2\ldots\, a_{s-1})}$$

and

$$\begin{aligned} \mathcal{R}_1^{a_1 b_1|\ldots|a_t b_t|a_{t+1}\ldots\, a_{s-1}} \quad = \quad & D_0^L \omega_1^{a_1 b_1|\ldots|a_t b_t|a_{t+1}\ldots\, a_{s-1}} \\ & + (e_0)_c \wedge \omega_1^{a_1 b_1|\ldots|a_t b_t|c(a_{t+1}\ldots\, a_{s-1})} \\ & + \Lambda\, \mathbf{Y}_{\!A}\, \omega_1^{a_2 b_2|\ldots|a_t b_t|a_{t+1}\ldots\, a_{s-1}\,[a_1} \wedge e_0^{b_1]} \\ & + \ldots + \Lambda\, \mathbf{Y}_{\!A}\, \omega_1^{a_1 b_1|\ldots|a_{t-1} b_{t-1}|a_{t+1}\ldots\, a_{s-1}\,[a_t} \wedge e_0^{b_t]} \end{aligned} \tag{49}$$

for $t \neq 0$.

Therefore, the torsion constraints (48) are solved as

$$(\omega_1)_{[\mu}^{a_1 b_1|\ldots|a_t b_t|}{}_{\nu]}^{(a_{t+1}\ldots\, a_{s-1})} = (D_0^L)_{[\mu}(\omega_1)_{\nu]}^{a_1 b_1|\ldots|a_t b_t|a_{t+1}\ldots\, a_{s-1}} + \mathcal{O}(\Lambda)\,. \tag{50}$$

In the metric-like gauge, these relations may be used recursively to express the auxiliary one forms with mixed symmetries in terms of the frame-like field. For instance, when $t = 0$ and $\Lambda = 0$, Equation (50) reproduces (33). Moreover, then the Riemann-like two-form $(\mathcal{R}_1)_{\mu\nu}^{a_1 b_1|\ldots|a_{s-1} b_{s-1}}$ may be identified with the Weinberg tensor according to (30).

By using the expression (49) together with the former remarks, one can check that the MacDowell–Mansouri-like action

$$\begin{aligned} S[\phi_s] \quad = \quad & \frac{1}{\Lambda}\, \varepsilon_{a_1 b_1 \ldots c_1 d_1 f_1} \cdots \varepsilon_{a_{s-1} b_{s-1} \ldots c_{s-1} d_{s-1} f_{s-1}}\, \eta^{b_2 \ldots b_{s-1}} \cdots \eta^{c_2 \ldots c_{s-1}} \times \\ & \times \int e_0^{b_1} \wedge \ldots \wedge \mathcal{R}_1^{c_1 a_1|a_2 \ldots a_{s-1}} \wedge \frac{1}{\Box_{(A)dS}^{\frac{s}{2}-1}} \mathcal{R}_1^{d_1 f_1|\ldots|d_{s-1} f_{s-1}}\,, \end{aligned} \tag{51}$$

reproduces the Einstein–Cartan–Weyl-like action (34) at order zero in Λ, in the metric-like gauge. More precisely, one should first take the $\Lambda \to 0$ limit in the action (51) and then one uses the zero-torsion constraints to express the auxiliary one-forms in terms of the frame-like field. In the pure gravity case $s = 2$, one recovers MacDowell–Mansouri action [35]. In the odd spin case, it is the action

$$S[\phi_s] = \frac{1}{2\Lambda}\,\varepsilon_{a_1 b_1 \ldots c_1 d_1 f_1}\,\varepsilon_{a_2 b_2 \ldots c_2 d_2 f_2}\,\cdots\varepsilon_{a_s b_s \ldots c_s d_s f_s}\,\eta^{f_1 a_s}\,\eta^{b_1 b_3 \ldots b_s}\cdots\eta^{c_1 c_3 \ldots c_s}\,\times$$
$$\times \int e_0^{b_2}\wedge\ldots\wedge\mathcal{R}_1^{a_1 d_1|c_2 a_2|\,a_3\ldots a_{s-1}}\wedge\frac{1}{\Box_{(A)dS}^{\frac{s-1}{2}}}\mathcal{R}_1^{d_2 f_2|\ldots|d_s f_s}\,. \quad (52)$$

which can reproduce the action (35). We implicitly understood everywhere that a symmetrization over all indices labeled by the same Latin letter should be performed. The "d'Alembertian" in (anti) de Sitter is *not* determined uniquely from its flat spacetime limit. In general,

$$\Box_{(A)dS} = \nabla^2 + \mathcal{O}(\Lambda)\,,$$

where the term $\mathcal{O}(\Lambda)$ is an operator acting on the spin degrees of freedom. A convenient requirement in order to remove this ambiguity could be that $\Box_{(A)dS}$ should commute with the $(A)dS$ covariant derivative, hence it is tempting to define $\Box_{(A)dS}$ as the anticommutator $[\,D_0\,,\,D_0^\dagger\,]_+$ because it commutes with the differential D_0.

The MacDowell–Mansouri-like actions (51)-(52) are automatically gauge invariant since the Lagrangian is quadratic in the linearized curvatures. Notice that these MacDowell–Mansouri-like actions may provide quadratic actions in constant-curvature spacetime within the unconstrained approach. This issue should be investigated further. We should also point out that these quadratic actions are of the same MacDowell-Mansouri form as the Lopatin–Vasiliev action [20] but the latter is local and has a different structure for the contraction of indices. This is possible because the tangent indices are not constrained to be traceless here and so more freedom is allowed in the contraction of indices.

Let us conclude this subsection with some speculative observations. The appealing feature of the quadratic actions (51)-(52) is that the starting point of Vasiliev *et al.* in their construction [37] of cubic vertices, invariant under non-Abelian gauge transformations associated with the constrained ("on-shell") higher-spin algebra, was the formulation of symmetric tensor gauge fields à la MacDowell–Mansouri *via* a local constrained frame-like formulation [20]. Therefore, by analogy, our result suggests that a non-linear Lagrangian for the non-Abelian higher-spin gauge theory with unconstrained ("off-shell") higher-spin algebra – if any – could be of the non-local MacDowell–Mansouri-like form presented here. Although elusive, such a non-local expression quadratic in the curvatures has some precedents. Indeed, the expressions (51)-(52) are reminiscent of the two-dimensional non-local action $S[g]$, quadratic in the worldsheet scalar curvature, which is obtained from the Polyakov action $S^P[g,X]$ by integrating out the D massless Klein–Gordon scalars $X^\mu(\sigma)$ describing the position of the bosonic string in the target space [23]. The harmless non-locality of this action and of the free higher-spin actions fall into the same category. An analogous picture for the full MacDowell–Mansouri-like actions would be in agreement with the folklore stating that a non-Abelian gauge theory of higher-spin fields might be interpreted as the effective theory of some more fundamental theory describing extended objects. In any case, we believe that the frame-like actions presented here deserve to be explored further.

3 Mixed-symmetry tensor gauge fields

In the present section we generalize the gauge theory of free rank-s symmetric tensor fields to the case of massless gauge fields with components transforming in an arbitrary irrep. of the general linear group, labeled by a Young diagram Y made of s columns. The reader is now assumed to have read the appendix A because the fundamental definitions are not repeated here. Following the terminology introduced in Section A.2.2, we say that the gauge field $\phi_{_Y}$ is a (differential) hyperform of $\Omega^Y_{(s)}(\mathbb{R}^D)$.

3.1 Bargmann–Wigner programme

3.1.1 Local, constrained approach of Labastida

It is natural to try to generalize the work of Fronsdal (briefly reviewed in Subsection 2.1.1) to arbitrary mixed-symmetry tensor gauge fields. In [38], Labastida conjectured some gauge invariances and determined a local gauge-invariant wave operator which was supposed to describe the proper degrees of freedom, but he was not able to prove that one may reach a gauge where the on-shell physical degrees of freedom provide the appropriate UIR of $O(D-2)$.

Labastida used a set of commuting oscillators [38] and thereby chose the symmetric convention for Young tableaux. Nevertheless, it turns out to be convenient for our later purposes to deal with fields in the antisymmetric convention. So, throughout the present section 3.1, the gauge field $\phi_{_Y}$ is understood to be a (differential) multiform of $\Omega^{\ell_1,\dots,\ell_s}_{[s]}(\mathbb{R}^D)$ whose components are in the irrep. of $GL(D,\mathbb{R})$ labeled by the Young diagram $Y=(\ell_1,\dots,\ell_s)$. Each basis element $d_i x^\mu$ of each exterior algebra $\wedge(\mathbb{R}^{D*})$ plays the role of a graded oscillator. We introduce the Labastida operator defined by

$$\mathrm{F} := \Box - d_i\, d_i^\dagger + \frac{1}{2}\, d_i\, d_j \mathrm{Tr}_{ij}\,, \tag{53}$$

where there is always an implicit summation from 1 to s over all repeated Latin indices. Each term on the right-hand-side commutes with the operator $\mathrm{Tr}_{ij}*_i$, hence the Labastida operator F preserves the $GL(D,\mathbb{R})$-irreducibility conditions (105). In other words, the Young symmetrizer $\mathbf{Y}_A$ commutes with the operator F, so that if $\phi_{_Y}\in\Omega^Y_{(s)}(\mathbb{R}^D)$, then the mixed-symmetry Labastida tensor $\mathcal{F}_{_Y}:=\mathrm{F}\phi_{_Y}$ also belongs to $\Omega^Y_{(s)}(\mathbb{R}^D)$.

It is natural to postulate that the field equation is

$$\mathcal{F}_{_Y}\approx 0\,, \tag{54}$$

and that the gauge transformations take the form

$$\delta_\epsilon\phi_{_Y} \;=\; \mathbf{Y}_A\, d_i\epsilon_i\,, \tag{55}$$

where ϵ_i are differential multiforms belonging to $\Omega^{\ell_1,\dots,\ell_i-1,\dots,\ell_s}_{[s]}(\mathbb{R}^D)$. The gauge transformation of the Labastida tensor under (55) is given by

$$\delta_\epsilon \mathcal{F}_Y = \frac{1}{2}\,\mathbf{Y}_A\, d_i\, d_j\, d_k\,(\mathrm{Tr}_{ij}\epsilon_k)\,, \tag{56}$$

due to the identity (115). The equation (56) is the analogue of (4). The commutation relations (116) suggest to require that $\mathrm{Tr}_{(ij}\epsilon_{k)} = 0$. Notice that this condition is weaker than the tracelessness of every parameter independently. The gauge invariance of the wave equation (54) was one of the requirements of Labastida in order to determine uniquely his relativistic wave operator in the symmetric convention [38]. One may easily check that the translation of Labastida's requirements in the antisymmetric convention also fix uniquely the wave operator. Hence the Labastida tensor in the symmetric convention of [38] must be equal to a linear combination of the Labastida tensor $\mathcal{F}_Y$ in the antisymmetric convention.

The main technical problems in the local approach are of course the trace conditions to be imposed on the gauge field and the gauge parameters. In the general mixed-symmetry case ϕ_Y, it is very difficult to determine them from first principle, contrary to the completely symmetric case, because there is now a wide variety of inequivalent ways to take traces. Moreover, a troublesome aspect in the construction of Labastida is that the double-trace constraints that he imposes on the gauge field ϕ_Y are in general *not* invariant under his gauge transformations.[9] For instance the analogue of the double-trace constraint of Labastida reads $\mathrm{Tr}_{(ij}\mathrm{Tr}_{kl)}\phi_Y = 0$ in the antisymmetric convention. But the identity

$$\mathrm{Tr}_{(ij}\mathrm{Tr}_{kl)}\big(d_m\epsilon_m\big) = 4\,d^\dagger_{(i}\big(\mathrm{Tr}_{jk}\epsilon_{l)}\big) + d_m\big(\mathrm{Tr}_{(ij}\mathrm{Tr}_{kl)}\epsilon_m\big)$$

shows that the former double-trace constraint is in general not preserved by gauge transformations (55) where the parameters are only subject to the trace constraint $\mathrm{Tr}_{(ij}\epsilon_{k)} = 0$. It is then fair to say that the problem of constructing a local action principle for arbitrary gauge fields ϕ_Y is still open.

3.1.2 Higher derivative, unconstrained approach

The curvature tensor of Weinberg was appropriately generalized in [9] by extending the cohomological results of [8] to arbitrary mixed-symmetry tensor fields. The definitions and main properties of the curvature tensors in the general case under consideration are reviewed in Section A.2.2. The curvature tensor field $\mathcal{K}_{\overline{Y}} \in \Omega^{\overline{Y}}_{(s)}(\mathbb{R}^D)$ for the mixed-symmetry tensor gauge field $\phi_Y \in \Omega^{Y}_{(s)}(\mathbb{R}^D)$ is obtained by taking s curls, $\mathcal{K}_{\overline{Y}} = d_1 \dots d_s \phi_Y$ and $\overline{Y}$ is the Young diagram obtained by adding a row of length s on top of the Young diagram Y. The curvature tensor is invariant under the gauge transformations (55) without any trace constraint on the gauge parameters ϵ_i. The Bianchi-like identities are the set of equations $d_i\mathcal{K}_{\overline{Y}} = 0$ $(i = 1, \dots, s)$.

The commutation relation

$$[\,\mathrm{Tr}_{ij}\,,\, d_i d_j\,]_- = \Box - d_i d_i^\dagger - d_j d_j^\dagger\,, \tag{57}$$

[9]We are grateful to A. Pashnev and M. Tsulaia for calling this fact to our attention.

where *no* sum on the indices i and j is understood, follows from (115) and implies in turn the operatorial identity $\mathrm{Tr}_{12}\, d_1 \ldots d_s \;=\; d_3 \ldots d_s \mathrm{F}$. Applied on the gauge field ϕ_{Y} , this last identity leads to the generalization of the Damour–Deser identity for arbitrary mixed-symmetry fields

$$\mathrm{Tr}_{12}\, \mathcal{K}_{\overline{Y}} \;=\; d_3\, d_4\, \ldots\, d_s\, \mathcal{F}_{Y}\, . \tag{58}$$

Therefore, the Labastida equation (54) implies the Ricci-flat-like equation

$$\mathrm{Tr}\, \mathcal{K}_{\overline{Y}} \;\approx\; 0\,, \tag{59}$$

stating that the curvature tensor is traceless on-shell, in agreement with (106). In analogy with the situation reviewed in Subsection 2.1.4, the Ricci-flat-like equation (59) implies the compensator equation

$$\mathcal{F}_{Y} \;\approx\; \frac{1}{2}\, \mathbf{Y}_A\, d_i\, d_j\, d_k\, \alpha_{ijk}\,, \tag{60}$$

where $\alpha_{ijk} \;=\; \alpha_{(ijk)}$ are some (differential) hyperforms associated with the Young diagrams obtained by removing three boxes in distinct columns of Y. The compensator fields α_{ijk} are pure-gauge fields expected to vary according to

$$\delta_{\epsilon}\alpha_{ijk} \;=\; \mathrm{Tr}_{(ij}\epsilon_{k)} \tag{61}$$

in order to compensate the variation (56) of the Fronsdal tensor in the third-order field equation (60). As one can see, the Labastida equation (54) arises as a partial gauge-fixing of the compensator equation.

The results explained in the previous paragraph were announced in [17] but the complete proof was not presented there because of the lack of space. For the sake of completeness, we now sketch the subtle use of Poincaré lemmas that enables to relate the Ricci-flat-like equation (59) with the compensator equation (60) *via* the Damour–Deser identity (58). The argument is deeply rooted in the following lemma, the proof of which is given in Appendix B.1

Lemma 1. *Let $\mathcal{P}$ be a differential hyperform of $\Omega_{(s)}(\mathbb{R}^D)$. Then,*

$$d_s\mathcal{P} = 0 \quad \Longrightarrow \quad d_i\mathcal{P} = 0\,, \quad \forall\, i \in \{1, \ldots, s\}\,. \tag{62}$$

As a corollary of the lemma 1, we have the implication

$$\Big(\prod_{i=1}^{k} d_{s-k+i}\Big)\, \mathcal{P} = 0 \quad \stackrel{\text{Lemma 1}}{\Longrightarrow} \quad \Big(\prod_{i\in I} d_i\;\Big)\, \mathcal{P} = 0\,, \quad \forall I \subset \{1,2,\ldots,s\} \mid \#I = k\,, \tag{63}$$

for any integer $k \in \{1, \ldots, s\}$, which can easily been proved by induction. The properties (63) and (121) combined together prove the following

Proposition 1. *Let $\mathcal{P}$ be a differential hyperform of $\Omega_{(s)}(\mathbb{R}^D)$. Then,*

$$\Big(\prod_{i=1}^{k} d_{s-k+i}\Big)\, \mathcal{P} = 0 \quad \Longrightarrow \quad \Big(\prod_{i\in I}\; d^{\{i\}}\Big)\, \mathcal{P} \;=\; 0\,, \qquad \forall\, I \subset \{1,2,\ldots,s\} \mid \#I = k\,, \tag{64}$$

In other words, the proposition 1 provides a sufficient condition for the cocycle condition $d^k\mathcal{P}=0$ of the generalized cohomology group ${}^{(k)}H^{(i_1,\ldots,i_s)}(d)$ associated with the operator $d=d^{\{1\}}+\ldots+d^{\{s\}}$ acting on the space of hyperforms $\Omega_{(s)}(\mathbb{R}^D)$. The generalized Poincaré lemma of [9] proves the triviality of the generalized cohomology groups ${}^{(k)}H^{(\ell_1,\ldots,\ell_s)}(d)$ for $1\leqslant k\leqslant s$, $0<\ell_s$ and $\ell_1<D$. The Ricci-flat-like equation (59) combined with the Damour–Deser identity (58) states that the Fronsdal tensor obeys the equation $d_3d_4\ldots d_s\mathcal{F}_{_Y}\approx 0$. The proposition 1 for $k=s-2$ implies that $d^{s-2}\mathcal{F}_{_Y}\approx 0$. The triviality of ${}^{(2)}H^{Y}(d)$ implies the exactness of the on-shell Fronsdal tensor, $\mathcal{F}_{_Y}\approx d^3\alpha$, as expressed by the compensator equation (60).

3.1.3 Non-local, unconstrained approach of de Medeiros and Hull

As was pointed out in [16], the equations (16) of Francia and Sagnotti were generalized by Hull and de Medeiros in [15] as follows

$$\mathrm{Tr}_{(12}\mathrm{Tr}_{34}\ldots\mathrm{Tr}_{s-1\,s)}\,\mathcal{K}_{\overline{Y}}\approx 0\qquad, \tag{65}$$

for s even. The sum of products of all possible traces over indices all belonging to distinct columns in (65) correspond in (16) to the contraction with the symmetrized powers $\eta_{(\mu_1\mu_2}\ldots\eta_{\mu_{s-1}\mu_s)}$ of the metric tensor. For s odd, the equation may be written in two ways

$$\mathrm{Tr}_{(12}\ldots\mathrm{Tr}_{s-2\,s-1}\mathrm{Tr}_{s\,s+1)}\,d_{s+1}\mathcal{K}_{\overline{Y}}=\mathrm{Tr}_{(12}\ldots\mathrm{Tr}_{s-2\,s-1}\,d^{\dagger}_{s)}\mathcal{K}_{\overline{Y}}\approx 0\,, \tag{66}$$

because of the fact that $\mathcal{K}_{\overline{Y}}$ is of degree zero in the $s+1$th set of antisymmetric indices. One can check explicitly that the operators $\mathrm{Tr}_{ij}{*_i}$ commute with the operator $\mathrm{Tr}_{(12}\ldots\mathrm{Tr}_{2n-1\,2n)}$ when i and j belong to the set $\{1,\ldots,2n\}$ [19]. Therefore, the equations (65)-(66) have the same symmetry properties as the corresponding tensor gauge field $\phi_{_Y}$. As they are, it is not obvious that they describe the proper physical degrees of freedom because the light-cone gauge is hard to reach since the gauge transformations (55) involve many parameters and are highly reducible in general. As a preliminary, we show in the next paragraph that the equations (65)-(66) are equivalent to the following compensator-like equation

$$\mathcal{F}_{_Y}\;\approx\;\frac{1}{2}\,\mathbf{Y}_A\,d_i\,d_j\,d_k\,\mathcal{H}_{ijk}\,, \tag{67}$$

generalizing the equation (14). The essential difference between (67) and the compensator equation (60) is that the tensor fields $\mathcal{H}_{ijk}$ are non-local functions of the gauge field $\phi_{_Y}$ and its partial derivatives. Nevertheless, their gauge transformations are proportional to $\mathrm{Tr}_{(ij}\epsilon_{k)}$ so that the gauge-fixing condition $\mathcal{H}_{ijk}=0$ leads to the Labastida equation (54).

To prove the on-shell equivalence between the deMedeiros–Hull equations (65)-(66) and (67) we need a crucial identity.

Lemma 2. *For any given natural number $n\in\mathbb{N}$,*

$$Tr_{(12}\ldots Tr_{2n-1\,2n)}\,d_1d_2\ldots d_{2n-1}d_{2n}=\Box^{n-1}\,\mathrm{F}\;-\;\frac{n-1}{2n-1}\,\Box^{n-2}\,d_jd_k\;Tr_{jk}\,\mathrm{F}\;+\;d_id_jd_k\mathrm{O}_{ijk}\,,$$

where there is an implicit sum from 1 *to* $2n$ *over every repeated index and* O_{ijk} *denotes a set of differential operators* ($1 \leqslant i,j,k \leqslant 2n$).

The proof is given in Appendix B.2. Applying the operator appearing in Lemma 2 for $n = [\frac{s+1}{2}]$ on the gauge field $\phi_{_Y}$, one gets the on-shell equality

$$\Box^{n-1}\,\mathcal{F}_{_Y}\;-\;\frac{n-1}{2n-1}\,\Box^{n-2}\,d_j d_k\,\mathrm{Tr}_{jk}\,\mathcal{F}_{_Y}\;+\;d_i d_j d_k \Sigma_{ijk} \approx 0\,, \tag{68}$$

for the multiforms $\Sigma_{ijk} := \mathrm{O}_{ijk}\phi_{_Y}$, by virtue of the equations (65)-(66). Taking a trace of both sides of the equation (68), leads to

$$\Box^{n-1}\,\mathrm{Tr}_{ij}\mathcal{F}_{_Y}\;\approx d_k\sigma_k\,, \tag{69}$$

for some multiforms σ_k. Inserting (69) into (68) gives (67).

3.1.4 Bargmann–Wigner equations

Following the discussion in the subsection 2.1.4, we stress that the sth-order Ricci-flat-like equation (59) is equivalent to a set of first-order field equations for $\mathcal{K}_{\overline{Y}}$. Indeed, the vanishing of the Ricci-like tensor means that the on-shell Weinberg tensor field $\mathcal{K}_{\overline{Y}}$ takes values in an irrep. of $O(D-1,1)$. The Bargmann–Wigner equations are somehow the converse statement. Let $\mathcal{K}_{\overline{Y}}$ be a differential hyperform with components in a tensorial irrep. of the Lorentz group $O(D-1,1)$ whose symmetries are labeled by the Young diagram $\overline{Y}$ (in the antisymmetric convention). As explained in the appendix A.2.2, the Bianchi-like identities (122) imply that the hyperform $\mathcal{K}_{\overline{Y}}$ is exact, which means that it is precisely the curvature tensor of a gauge field $\phi_{_Y}$ taking values in an irreducible representation of $GL(D,\mathbb{R})$ labeled by the Young diagram Y. This proves the equivalence between the Ricci-flat-like equation (59) obeyed by the Weinberg tensor field, and the Bianchi-like equations (122) obeyed by an $O(D-1,1)$-irreducible tensor fields with the same symmetries as the Weinberg tensor. Moreover, due to the commutation relation (115) the compatibility condition between the Bianchi-like identities (122) and the tracelessness property (59) are the transversality conditions

$$d_i^\dagger \mathcal{K}_{\overline{Y}} \approx 0 \quad (i=1,\dots,s)\,. \tag{70}$$

The equations (70) and (122) are called the Bargmann–Wigner equations since they generalize (21). They were proposed in [9, 16] as field equations for mixed-symmetry tensor gauge fields. By definition, the Bargmann–Wigner equations state that the differential hyperform $\mathcal{K}_{\overline{Y}}$ is harmonic on-shell.

Up to now, we have achieved to prove the equivalence of the Labastida equation (54), the Ricci-flat equations (59), compensator (60), the deMedeiros–Hull equations (65)-(66) and the Bargmann–Wigner equations (70) and (122). In order to prove that they describe the proper physical degrees of freedom, it is sufficient to do so for one of these equations: this is done in the appendix C for the Bargmann–Wigner equations. As a corollary, this completes the Bargmann–Wigner programme for arbitrary finite-component fields in any dimension, as summarized in the following theorem.

Theorem 1. *(Bargmann–Wigner's programme) [17]*

Let $\overline{Y}$ be an allowed Young diagram $(\bar{\ell}_1, \ldots, \bar{\ell}_s)$ with at least two rows of equal length s and $Y := (\bar{\ell}_1 - 1, \ldots, \bar{\ell}_s - 1)$ be the Young diagram $(\ell_1, \ldots, \ell_s)$ obtained by removing the first row of $\overline{Y}$.
Any tensorial irreducible representation of the group $O(D-1,1)$ with finite-dimensional representation space $V_{\overline{Y}}^{O(D-1,1)}$ where $V = \mathbb{R}^D$, provides a massless unitary irreducible representation of the group $IO(D-1,1)$ associated with the Young diagram Y: Its infinite-dimensional representation space is the space of harmonic differential multiforms $\mathcal{K}_{\overline{Y}}$ of spin s taking values in $V_{\overline{Y}}^{O(D-1,1)}$. The latter space is isomorphic to the Hilbert space $\mathcal{H}_Y$ of physical states $\varphi_Y \in L^2(\mathbb{R}^D) \otimes V_Y^{O(D-2)}$ that are solutions of $\Box\varphi_Y \approx 0$.
Any single-valued massless unitary irreducible representation of $IO(D-1,1)$ induced from a finite-dimensional irreducible representation of $O(D-2)$ is equivalent to a representation obtained in this way.

3.2 Fierz–Pauli programme

In the first subsection, we discuss the state of the art in order to clarify what is new in the present work with respect to the extensive literature on the subject. In the second subsection, a non-local Lagrangian for any mixed-symmetry tensor gauge field is written in compact form, two particular cases of which are exhibited in the third subsection.

3.2.1 Local actions

Local covariant Lagrangians have already been obtained for gauge fields labeled by the most general "hook" diagrams $(\ell_1, 1, \ldots, 1)$ [39], "two-row" diagrams $(2, \ldots, 2, 1, \ldots, 1)$ [40] and "two-column" diagrams (ℓ_1, ℓ_2) [41, 42] in approaches where trace constraints are imposed on the higher-spin fields. On the one hand, a decisive step towards the explicit completion of the Fierz–Pauli programme has been performed in the $OSp(1,1|2)$ formalism [18]. The drawback of this formalism is that it requires some technically involved computations in order to write the quadratic action only in terms of the $Sp(2)$ singlet variables (*i.e.* the constrained mixed-symmetry gauge field). This last step has never been performed explicitly for the mixed-symmetry case to our knowledge. On the other hand, in [43] Labastida introduced an explicit self-adjoint Einstein-like tensor corresponding to his field equation and conjectured that this Einstein-like tensor would provide the local constrained quadratic action for a tensor gauge field labeled by an arbitrary Young diagram. The problem of his approach is that he could not prove in full generality that his choice of trace constraints would lead to the proper physical degrees of freedom.

More recently, an algorithm for the construction of quadratic actions for mixed-symmetry tensor gauge fields was given in the BRST approach [44]. Finally, de Medeiros and Hull conjectured in [19] the rough form of a non-local Einstein-like tensor but they did not give the precise coefficients of its expansion in powers of traces, neither did they prove that their Einstein-like equation describes the proper physical degrees of freedom.

In this sense, the non-local second-order action that we write in Theorem 2 provides the first explicit realization of the Fierz–Pauli programme in full generality. More accurately, our analysis is restricted to

Minkowski spacetime and to fields with a finite number of components. Incidentally, we should mention that for "massless" mixed-symmetry tensor gauge fields, the Bargmann–Wigner programme for the anti de Sitter group $SO(D-1,2)$ has already been examined in many details [45] and the Fierz–Pauli programme has recently experienced considerable progresses [46]. Also, the completion of the Bargmann–Wigner programme has recently been extended to all massless irreps (including infinite-component ones) of the Poincaré group $ISO(D-1,1)$ [47]. The non-locality property of the action proposed here remains elusive and it would be pleasant to explicitly derive its local counterparts. Actually, the BRST algorithm of [44] indirectly ensures the existence of a local action invariant under unconstrained gauge transformations, but with many auxiliary fields. In the same way, the work of [18] may be interpreted as a proof of the existence of a local second-order action invariant under constrained gauge transformations.

3.2.2 Non-local actions

The main idea is that the use of the Levi-Civita tensors enables a straightforward generalization of the results of Subsection 2.2.1 to the mixed-symmetry case. Still, one should make sure to take the appropriate traces and that the result is projected on the proper symmetry.

Our main results are summarized in compact form in the following theorem. Subsequently, we provide two examples and then describe in more details the construction of the non-local Lagrangian for arbitrary mixed-symmetry tensor gauge fields.

Theorem 2. *(Fierz–Pauli's programme)*

Let s be a positive integer. The smallest even integer that is not smaller than s is denoted by $\overline{s}:=2[\frac{s+1}{2}]=s+\varepsilon(s)$. Let $Y:=(\ell_1,\ldots,\ell_{\overline{s}})$ be a Young diagram with first row of length s (that is to say, $\ell_{\overline{s}}=0$ when s is odd) and such that $\ell_1+\ell_2\leqslant D-2$. Let $\phi_{_Y}$ be a gauge field with components in the tensorial irreducible representation of the group $GL(D,\mathbb{R})$ with (finite-dimensional) representation space $V_Y^{GL(D,\mathbb{R})}$ where $V=\mathbb{R}^D$.

The second-order quadratic action

$$S\,[\,\phi_{_Y}]\;=\;\langle\;\phi_{_Y}\mid \mathrm{K}\mid\phi_{_Y}\,\rangle$$

defined by the self-adjoint kinetic operator

$$\mathrm{K}\;=\;\mathrm{Tr}^{\,(\mathrm{D}-1)\frac{\overline{s}}{2}-|\mathrm{Y}|}\circ *^{\overline{s}}\circ\frac{1}{\Box^{\frac{\overline{s}}{2}}}\circ\Big(\prod_{i=1}^{\overline{s}}d_i\Big)\qquad, \tag{71}$$

with

$$\mathrm{Tr}^{\,(\mathrm{D}-1)\frac{\overline{s}}{2}-|\mathrm{Y}|}\;:=\;\prod_{j=1}^{\frac{\overline{s}}{2}}\big(\mathrm{Tr}_{j\;\overline{s}-j+1}\big)^{D-1-\ell_j-\ell_{\overline{s}-j+1}}\,,$$

is manifestly gauge-invariant under the transformations

$$\delta_\epsilon\mid\phi_{_Y}\,\rangle\;=\;\sum_{i=1}^{s}d_i\mid\epsilon_i\,\rangle\,, \tag{72}$$

where ϵ_i are differential multiforms belonging to $\Omega^{\ell_1\,,\,\dots\,,\,\ell_i-1\,,\,\dots\,,\,\ell_s}_{[s]}(\mathbb{R}^D)$.

Let $\overline{Y}$ be the Young diagram obtained by adding one row of length s to the Young diagram Y. The equation of motion derived from this action may be cast into the form

$$\frac{\delta S\,[\,\phi_{_Y}]}{\delta\,\langle\,\phi_{_Y}\,|}\approx 0\quad\iff\quad|\,\mathcal{G}_{_Y}\rangle\approx 0\,, \tag{73}$$

where the Einstein tensor $\mathcal{G}_{_Y}$ is defined by

$$\mathcal{G}_{_Y}\;:=\left\{\begin{array}{ll}\mathbf{Y}_S\;\mathrm{Tr}^{\,(\mathrm{D}-1)\frac{\mathrm{s}}{2}-|\mathrm{Y}|}\,\widetilde{\mathcal{K}}_{\widetilde{\overline{Y}}}\approx 0 & \textit{for } s \textit{ even}\,,\\ \mathbf{Y}_S\;\mathrm{Tr}^{\,(\mathrm{D}-1)\frac{\mathrm{s}+1}{2}-|\mathrm{Y}|}\,d^{\dagger}_{s+1}\widetilde{\mathcal{K}}_{\widetilde{\overline{Y}}}\approx 0 & \textit{for } s \textit{ odd}\,,\end{array}\right. \tag{74}$$

with $\widetilde{\mathcal{K}}_{\widetilde{\overline{Y}}}$ the dual of the curvature tensor $\mathcal{K}_{\overline{Y}}$ and $\widetilde{\overline{Y}}$ the dual of the Young diagram $\overline{Y}$.

The (infinite-dimensional) space of field configurations extremizing the action $S[\,\phi_{_Y}]$ carries the massless unitary irreducible representation of the group $IO(D-1,1)$ associated with the Young diagram Y: it is isomorphic to the Hilbert space $\mathcal{H}_Y$ of physical states $\varphi_{_Y}\in L^2(\mathbb{R}^D)\otimes V_Y^{O(D-2)}$ that are solutions of $\Box\varphi_{_Y}\approx 0$.

In order to help the reader to get used to the notations involved in the theorem 2 and to provide some flavor of the general proof, we present two particular examples with mixed symmetry gauge fields (one for each parity of the spin s).

An odd-spin example

We first consider the gauge field $\phi_{_Y}\in V_Y^{GL(D,\mathbb{R})}$ with the associated Young diagram $Y=(\ell_1,\ell_2,\ell_3,0)=(2,1,1,0)$. The spin is $s=3$, hence $\overline{3}=2[\frac{3+1}{2}]=3+\varepsilon(3)=4$. The tensor gauge field components read $\phi_{\mu_1^1\mu_1^2\mu_1^3\,;\,\mu_2^1}$. The ket $|\,\phi_{_Y}\,\rangle$ is assumed to be expressed in the symmetric convention, which means that it is totally symmetric in $(\mu_1^1\,,\mu_1^2\,,\mu_1^3)$ and obeys $\phi_{(\mu_1^1\mu_1^2\mu_1^3\,;\,\mu_2^1)}=0$. The Young tableau associated to $\phi_{\mu_1^1\mu_1^2\mu_1^3\,;\,\mu_2^1}$ is depicted as follows

$$Y\quad=\quad\begin{array}{|c|c|c|}\hline \mu_1^1 & \mu_1^2 & \mu_1^3\\ \hline \mu_2^1 \\ \cline{1-1}\end{array}\;. \tag{75}$$

The curvature tensor $\mathcal{K}_{\overline{Y}}$ will have components $\mathcal{K}_{\mu_1^1\mu_2^1\mu_3^1\,|\,\mu_1^2\mu_2^2\,|\,\mu_1^3\mu_2^3}$ described by the Young tableau

$$\overline{Y}\quad=\quad\begin{array}{|c|c|c|}\hline \mu_1^1 & \mu_1^2 & \mu_1^3\\ \hline \mu_2^1 & \mu_2^2 & \mu_2^3\\ \hline \mu_3^1 \\ \cline{1-1}\end{array} \tag{76}$$

that is, as a Young diagram, $\overline{Y}=(3,2,2,0)=(2,1,1,0)+(1,1,1,0)$. The curvature tensor is expressed in the antisymmetric convention because of the presence in the Lagrangian of the $\overline{s}=4$ Levi-Civita tensors

$$\varepsilon^{\mu_1^1\mu_2^1\dots\,\mu_D^1}\,\varepsilon^{\mu_1^2\mu_2^2\dots\,\mu_D^2}\,\varepsilon^{\mu_1^3\mu_2^3\dots\,\mu_D^3}\,\varepsilon^{\mu_1^4\mu_2^4\dots\,\mu_D^4} \tag{77}$$

contracted with the $s=3$ derivatives of the gauge field components

$$\partial_{\mu_3^1}\partial_{\mu_2^2}\partial_{\mu_2^3}\phi_{\mu_1^1\mu_1^2\mu_1^3\,;\,\mu_2^1}\,.$$

In order for the Ricci-flat-like equation $\mathrm{Tr}\mathcal{K}_{\overline{Y}} \approx 0$ to define a nontrivial theory, we must have $D \geqslant \ell_1 + \ell_2 + 2 = 5$. We choose here $D = 5$.

Continuing the construction of the Lagrangian, we have to act with $\varepsilon(3) = 1$ extra derivative $\partial_{\mu_1^4}$ on the gauge field $\phi_{\mu_1^1\mu_1^2\mu_1^3\,;\,\mu_2^1}$. The components of the bra $\langle\, \phi_Y \,|$ are written, in the symmetric convention, as $\phi_{\mu_5^4\mu_5^3\mu_5^2\,;\,\mu_4^4}$ and they correspond to the Young tableau

$$\begin{array}{|c|c|c|}\hline \mu_5^4 & \mu_5^3 & \mu_5^2 \\ \hline \mu_4^4 \\ \cline{1-1} \end{array} \quad . \tag{78}$$

Finally, the trace operator $\mathrm{Tr}^{\,(D-1)\frac{\overline{s}}{2}-|Y|} = (\mathrm{Tr}_{14})^2(\mathrm{Tr}_{23})^2$ reads, in components,

$$(\eta_{\mu_5^1\mu_2^4}\eta_{\mu_4^1\mu_3^4})\,(\eta_{\mu_3^2\mu_3^3}\eta_{\mu_4^2\mu_4^3})\,. \tag{79}$$

Summarizing, the action is explicitly written as

$$S[\phi_Y] = \frac{1}{2}\int d^5x \left[\phi_{\mu_5^4\mu_5^3\mu_5^2\,;\,\mu_4^4}(\eta_{\mu_5^1\mu_2^4}\eta_{\mu_4^1\mu_3^4}\eta_{\mu_3^2\mu_3^3}\eta_{\mu_4^2\mu_4^3})(\varepsilon^{\mu_1^1\cdots\,\mu_5^1}\,\ldots\,\varepsilon^{\mu_1^4\cdots\,\mu_5^4})\,\frac{1}{\Box}\,\partial_{\mu_1^4}\partial_{\mu_3^1}\partial_{\mu_2^2}\partial_{\mu_2^3}\,\phi_{\mu_1^1\mu_1^2\mu_1^3\,;\,\mu_2^1}\right].$$

At this stage, it is instructive to draw the $GL(5,\mathbb{R})$ Young diagram Z corresponding to the product (77), in which we mark by a "$\times$" the cells corresponding to the components of the bra $\langle\, \phi_Y \,|$ (and ket $|\, \phi_Y \,\rangle$) and by a "$-$" the cells corresponding to the partial derivatives. The components of the metric tensors are marked by a "$\circ$". It gives

$$Z \;=\; \begin{array}{|c|c|c|c|}\hline \mu_1^1 & \mu_1^2 & \mu_1^3 & \mu_1^4 \\ \hline \mu_2^1 & \mu_2^2 & \mu_2^3 & \mu_2^4 \\ \hline \mu_3^1 & \mu_3^2 & \mu_3^3 & \mu_3^4 \\ \hline \mu_4^1 & \mu_4^2 & \mu_4^3 & \mu_4^4 \\ \hline \mu_5^1 & \mu_5^2 & \mu_5^3 & \mu_5^4 \\ \hline \end{array} \;=\; \begin{array}{|c|c|c|c|}\hline \times & \times & \times & - \\ \hline \times & - & - & \circ \\ \hline - & \circ & \circ & \circ \\ \hline \circ & \circ & \circ & \times \\ \hline \circ & \times & \times & \times \\ \hline \end{array} \quad . \tag{80}$$

The differential multiform $d_{\overline{s}}\mathcal{K}_{\overline{Y}} = d_{{}_4}\mathcal{K}_{\overline{Y}}$ is labeled by the Young diagram $\overline{Y}_+ := Y + (1,1,1,1) = (3,2,2,1)$. The ket $*^{\overline{s}}\, d_{\overline{s}} \,|\, \mathcal{K}_{\overline{Y}} \,\rangle = (*_1 *_2 *_3 *_4)\; d_{{}_4} \,|\, \mathcal{K}_{\overline{Y}} \,\rangle$ enters in the Lagrangian with the following tensorial components, in the antisymmetric convention,

$$(\,*_1\, *_2\, *_3\, *_4\;\; d_{{}_4}\, \mathcal{K}_{\overline{Y}}\,)^{\mu_4^1\mu_5^1\,|\,\mu_3^2\mu_4^2\mu_5^2\,|\,\mu_3^3\mu_4^3\mu_5^3\,|\,\mu_2^4\mu_3^4\mu_4^4\mu_5^4}\,.$$

Only one $GL(5,\mathbb{R})$-irreducible component of the above tensor, also denoted by $(\widetilde{d_{{}_4}\mathcal{K}})_{\widetilde{\overline{Y}_+}}$, survives inside the action. It is labeled by the Young diagram $\widetilde{\overline{Y}_+} = (5,5,5,5,5) - (1,2,2,3) = (4,3,3,2)$ and the corresponding Young tableau reads

$$\widetilde{\overline{Y}_+} \;=\; \begin{array}{|c|c|c|c|}\hline \mu_5^4 & \mu_5^3 & \mu_5^2 & \mu_5^1 \\ \hline \mu_4^4 & \mu_4^3 & \mu_4^2 & \mu_4^1 \\ \hline \mu_3^4 & \mu_3^3 & \mu_3^2 \\ \cline{1-3} \mu_2^4 \\ \cline{1-1} \end{array} \;=\; \begin{array}{|c|c|c|c|}\hline \times & \times & \times & \circ \\ \hline \times & \circ & \circ & \circ \\ \hline \circ & \circ & \circ \\ \cline{1-3} \circ \\ \cline{1-1} \end{array} \quad . \tag{81}$$

It may be obtained by rotating $Z \curvearrowright$ by 180 degrees and removing the cells of the Young tableau $\overline{Y}_+$ corresponding to $d_{{}_4}\mathcal{K}_{\overline{Y}}$. In terms of $SL(5,\mathbb{R})$-irreducible representations, this tensor is equivalent to $d_{{}_4}\,\mathcal{K}_{\overline{Y}}$.

The Euler-Lagrange derivatives are proportional to

$$\mathbf{Y}_{S}\left[(\mathrm{Tr}_{14})^2(\mathrm{Tr}_{23})^2\,(\widetilde{d_4\,\mathcal{K}})_{\widetilde{\overline{Y}_+}}\right]\approx 0 \tag{82}$$

or, in components,

$$\frac{\delta S}{\delta\phi_{\mu_5^4\mu_5^3\mu_5^2\,;\,\mu_4^4}}\propto \mathbf{Y}_{S}\left[(\eta_{\mu_5^1\mu_2^4}\eta_{\mu_4^1\mu_3^4}\eta_{\mu_3^2\mu_3^3}\eta_{\mu_4^2\mu_4^3})(\widetilde{d_4\mathcal{K}})_{\mu_2^4\mu_3^4\mu_4^4\mu_5^4\,|\,\mu_3^3\mu_4^3\mu_5^3\,|\,\mu_3^2\mu_4^2\mu_5^2\,|\,\mu_4^1\mu_5^1}\right]$$

where $\mathbf{Y}_{S}$ is the Young projector associated with the Young tableau (78). In the field equations (and also in the Lagrangian), only one $GL(5,\mathbb{R})$-irreducible component of the tensor product $\eta_{\mu_5^1\mu_2^4}\eta_{\mu_4^1\mu_3^4}\eta_{\mu_3^2\mu_3^3}\eta_{\mu_4^2\mu_4^3}$ will contribute. It is the irreducible component characterized by the Young tableau X such that the product $X\cdot Y$ contains $\widetilde{\overline{Y}_+}$ in its decomposition. We find that $X=(2,2,2,2)$. Drawing the tableau,

$$X \quad = \quad \begin{array}{|c|c|c|c|}\hline \circ & \circ & \circ & \circ \\ \hline \circ & \circ & \circ & \circ \\ \hline \end{array}\quad . \tag{83}$$

Indeed, it is easy to check, using the Littlewood-Richardson rules, that

$$\begin{array}{|c|c|c|c|}\hline \circ & \circ & \circ & \circ \\ \hline \circ & \circ & \circ & \circ \\ \hline \end{array}\ \cdot\ \begin{array}{|c|c|c|}\hline \times & \times & \times \\ \hline \times \\ \cline{1-1} \end{array}\quad\supset\quad \begin{array}{|c|c|c|c|}\hline \times & \times & \times & \circ \\ \hline \times & \circ & \circ & \circ \\ \hline \circ & \circ & \circ \\ \cline{1-3} \circ \\ \cline{1-1} \end{array}\quad .$$

According to the definitions introduced in the appendix A.1.1, one may say that, on-shell, the field $(\widetilde{d_4\,\mathcal{K}})_{\widetilde{\overline{Y}_+}}$ takes values in a tensorial representation of $SL(5,\mathbb{R})$ labeled by the difference $\widetilde{\overline{Y}_+}-Y$ of $\mathcal{Y}$ where the substraction of the Young diagram Y corresponds to the trace constraints (82) imposed by the equations of motion. Due to the isomorphism $V^{SL(5,\mathbb{R})}_{\widetilde{\overline{Y}_+}}\cong V^{SL(5,\mathbb{R})}_{\overline{Y}_+}$, the former tensorial representation is equivalent to a tensorial representation labeled by the difference $\overline{Y}_+-Y$ corresponding to the field $(d_4\,\mathcal{K})_{\overline{Y}_+}$ on which are imposed the trace constraints

$$\mathbf{Y}_{S}\left[\mathrm{Tr}_{14}\mathrm{Tr}_{23}\,d_4\,\mathcal{K}_{\overline{Y}}\right]\approx 0\,, \tag{84}$$

labeled by Y. This provides a group-theoretical proof of the fact that the Einstein-like equations (82) are equivalent to the deMedeiros–Hull equations (84). They respectively are particular instances of (73) and (66).

An even-spin example

We next consider the gauge field $\phi_Y\in V_Y^{GL(D,\mathbb{R})}$ with the associated Young diagram $Y=(\ell_1,\ell_2,\ell_3,\ell_4)=(3,2,2,2)$. We choose the dimension $D=7$. The spin is $s=4$, hence $\overline{4}=2[\frac{4+1}{2}]=4+\varepsilon(4)=4$. The tensor gauge field components read $\phi_{\mu_1^1\mu_1^2\mu_1^3\mu_1^4\,;\,\mu_2^1\mu_2^2\mu_2^3\mu_2^4\,;\,\mu_3^1}$. The associated Young tableau is depicted as follows

$$\begin{array}{|c|c|c|c|}\hline \mu_1^1 & \mu_1^2 & \mu_1^3 & \mu_1^4 \\ \hline \mu_2^1 & \mu_2^2 & \mu_2^3 & \mu_2^4 \\ \hline \mu_3^1 \\ \cline{1-1} \end{array}\quad . \tag{85}$$

The curvature tensor $\mathcal{K}_{\overline{Y}}$ has components $\mathcal{K}_{\mu_1^1\mu_1^2\mu_1^3\mu_1^4\,;\,\mu_2^1\mu_2^2\mu_2^3\mu_2^4\,;\,\mu_3^1\mu_3^2\mu_3^3\mu_3^4\,;\,\mu_4^1}$ described by the Young tableau

$$\begin{array}{|c|c|c|c|}\hline \mu_1^1 & \mu_1^2 & \mu_1^3 & \mu_1^4 \\ \hline \mu_2^1 & \mu_2^2 & \mu_2^3 & \mu_2^4 \\ \hline \mu_3^1 & \mu_3^2 & \mu_3^3 & \mu_3^4 \\ \hline \mu_4^1 \\ \cline{1-1} \end{array}\quad . \tag{86}$$

where $\overline{Y} = (4,3,3,3) = (3,2,2,2) + (1,1,1,1)$. The curvature tensor is expressed in the antisymmetric convention because of the presence in the Lagrangian of the $\overline{s} = 4$ Levi-Civita tensors

$$\varepsilon^{\mu_1^1\mu_2^1\cdots\,\mu_7^1}\,\varepsilon^{\mu_1^2\mu_2^2\cdots\,\mu_7^2}\,\varepsilon^{\mu_1^3\mu_2^3\cdots\,\mu_7^3}\,\varepsilon^{\mu_1^4\mu_2^4\cdots\,\mu_7^4} \tag{87}$$

contracted with the $s = 4$ derivatives of the gauge field components

$$\partial_{\mu_3^2}\partial_{\mu_3^3}\partial_{\mu_3^4}\partial_{\mu_4^1}\phi_{\mu_1^1\mu_1^2\mu_1^3\mu_1^4\,;\,\mu_2^1\mu_2^2\mu_2^3\mu_2^4\,;\,\mu_3^1}\,.$$

With $D = 7$, the Ricci-flat-like equation $\mathrm{Tr}\,\mathcal{K}_{\overline{Y}} \approx 0$ defines a nontrivial theory, since $\ell_1 + \ell_2 + 2 \leqslant D$.

The components of the bra $\langle\ \phi_Y\ |$ are written, in the symmetric convention, as $\phi_{\mu_7^1\mu_7^2\mu_7^3\mu_7^4\,;\,\mu_6^1\mu_6^2\mu_6^3\mu_6^4\,;\,\mu_5^4}$. Finally, the trace operator $\mathrm{Tr}^{\,(D-1)\frac{\overline{s}}{2}-|Y|} = \mathrm{Tr}_{14}(\mathrm{Tr}_{23})^2$ reads, in components,

$$\eta_{\mu_5^1\mu_4^4}\,\eta_{\mu_4^2\mu_4^3}\,\eta_{\mu_5^2\mu_5^3}\,.$$

Summarizing, the action is explicitly written as

$$\begin{aligned} S[\phi_Y] \;=\; \frac{1}{2}\int d^7x\,\Big[\,&\phi_{\mu_7^1\mu_7^2\mu_7^3\mu_7^4\,;\,\mu_6^1\mu_6^2\mu_6^3\mu_6^4\,;\,\mu_5^4}\,(\eta_{\mu_5^1\mu_4^4}\,\eta_{\mu_4^2\mu_4^3}\,\eta_{\mu_5^2\mu_5^3})\,(\varepsilon^{\mu_1^1\cdots\,\mu_7^1}\,\ldots\,\varepsilon^{\mu_1^4\cdots\,\mu_7^4}) \\ &\frac{1}{\Box}\,\mathcal{K}_{\mu_1^1\mu_1^2\mu_1^3\mu_1^4\,;\,\mu_2^1\mu_2^2\mu_2^3\mu_2^4\,;\,\mu_3^1\mu_3^2\mu_3^3\mu_3^4\,;\,\mu_4^1}\,\Big]\,. \end{aligned}$$

This construction is more transparent when drawing the $GL(7,\mathbb{R})$ Young diagram Z corresponding to the product (87), in which we mark by a "$\times$" the cells corresponding to the components of the bra $\langle\ \phi_Y\ |$ (and ket $|\ \phi_Y\ \rangle$) and by a "$-$" the cells corresponding to the partial derivatives. The components of the metric tensors are marked by a "$\circ$". It gives

$$Z \;=\; \begin{array}{|c|c|c|c|}\hline \mu_1^1 & \mu_1^2 & \mu_1^3 & \mu_1^4 \\ \hline \mu_2^1 & \mu_2^2 & \mu_2^3 & \mu_2^4 \\ \hline \mu_3^1 & \mu_3^2 & \mu_3^3 & \mu_3^4 \\ \hline \mu_4^1 & \mu_4^2 & \mu_4^3 & \mu_4^4 \\ \hline \mu_5^1 & \mu_5^2 & \mu_5^3 & \mu_5^4 \\ \hline \mu_6^1 & \mu_6^2 & \mu_6^3 & \mu_6^4 \\ \hline \mu_7^1 & \mu_7^2 & \mu_7^3 & \mu_7^4 \\ \hline \end{array} \;=\; \begin{array}{|c|c|c|c|}\hline \times & \times & \times & \times \\ \hline \times & \times & \times & \times \\ \hline \times & - & - & - \\ \hline - & \circ & \circ & \circ \\ \hline \circ & \circ & \circ & \times \\ \hline \times & \times & \times & \times \\ \hline \times & \times & \times & \times \\ \hline \end{array}\quad .$$

Only one $GL(7,\mathbb{R})$-irreducible component of the differential multiform $*_1 *_2 *_3 *_4 \mathcal{K}_{\overline{Y}}$ survives inside the action. The corresponding differential hyperform is denoted by $\widetilde{\mathcal{K}}_{\overline{\widetilde{\overline{Y}}}}$ and is labeled by the Young diagram $\overline{\widetilde{\overline{Y}}} = (7,7,7,7) - (3,3,3,4) = (4,4,4,3)$. The associated Young tableau reads

$$\begin{array}{|c|c|c|c|}\hline \mu_7^4 & \mu_7^3 & \mu_7^2 & \mu_7^1 \\ \hline \mu_6^4 & \mu_6^3 & \mu_6^2 & \mu_6^1 \\ \hline \mu_5^4 & \mu_5^3 & \mu_5^2 & \mu_5^1 \\ \hline \mu_4^4 & \mu_4^3 & \mu_4^2 \\ \cline{1-3}\end{array} \quad = \quad \begin{array}{|c|c|c|c|}\hline \times & \times & \times & \times \\ \hline \times & \times & \times & \times \\ \hline \times & \circ & \circ & \circ \\ \hline \circ & \circ & \circ \\ \cline{1-3}\end{array} \quad . \tag{88}$$

It has been obtained by rotating $Z \curvearrowright$ by 180 degrees and removing the cells corresponding to the Young diagram $\overline{Y}$. In terms of $SL(7,\mathbb{R})$-irreducible representations, the tensor $\widetilde{\mathcal{K}}_{\overline{\widetilde{\overline{Y}}}}$ is equivalent to $\mathcal{K}_{\overline{Y}}$.

The Euler-Lagrange equations are

$$0 \approx \frac{\delta S}{\delta \phi_{\mu_7^1 \mu_7^2 \mu_7^3 \mu_7^4 \,;\, \mu_6^1 \mu_6^2 \mu_6^3 \mu_6^4 \,;\, \mu_5^4}} \propto \mathbf{Y}_S \left[(\eta_{\mu_5^1 \mu_4^4}\, \eta_{\mu_4^2 \mu_4^3}\, \eta_{\mu_5^2 \mu_5^3})\, \widetilde{\mathcal{K}}_{\mu_7^4 \mu_6^4 \mu_5^4 \mu_4^4 \,|\, \mu_7^3 \mu_6^3 \mu_5^3 \mu_4^3 \,|\, \mu_7^2 \mu_6^2 \mu_5^2 \mu_4^2 \,|\, \mu_7^1 \mu_6^1 \mu_5^1} \right] \tag{89}$$

where $\mathbf{Y}_S$ is the projector on the symmetries of $\phi_{\mu_7^1 \mu_7^2 \mu_7^3 \mu_7^4 \,;\, \mu_6^1 \mu_6^2 \mu_6^3 \mu_6^4 \,;\, \mu_5^4}$. In the field equations (and thus in the Lagrangian), only one $GL(7,\mathbb{R})$-irreducible component of the tensor product $\eta_{\mu_5^1 \mu_4^4}\, \eta_{\mu_4^2 \mu_4^3}\, \eta_{\mu_5^2 \mu_5^3}$ will contribute. It is the irreducible component characterized by the Young tableau X such that the tensor product $X \cdot Y$ contains $\overline{\widetilde{\overline{Y}}}$ in its decomposition. We find $X = (2,2,1,1)$. Drawing the diagram,

$$X \quad = \quad \begin{array}{|c|c|c|c|}\hline \circ & \circ & \circ & \circ \\ \hline \circ & \circ \\ \cline{1-2}\end{array} \quad .$$

It is easy to check, using the Littlewood-Richardson rule, that

$$\begin{array}{|c|c|c|c|}\hline \circ & \circ & \circ & \circ \\ \hline \circ & \circ \\ \cline{1-2}\end{array} \ \cdot \ \begin{array}{|c|c|c|c|}\hline \times & \times & \times & \times \\ \hline \times & \times & \times & \times \\ \hline \times \\ \cline{1-1}\end{array} \quad \supset \quad \begin{array}{|c|c|c|c|}\hline \times & \times & \times & \times \\ \hline \times & \times & \times & \times \\ \hline \times & \circ & \circ & \circ \\ \hline \circ & \circ & \circ \\ \cline{1-3}\end{array} \quad .$$

Due to the isomorphism $V^{SL(7,\mathbb{R})}_{\overline{\widetilde{\overline{Y}}}} \cong V^{SL(7,\mathbb{R})}_{\overline{Y}}$, the Einstein-like equations (89) are equivalent to the deMedeiros–Hull equations

$$\mathbf{Y}_S \left[\mathrm{Tr}_{1\,4} \mathrm{Tr}_{2\,3}\, \mathcal{K}_{\overline{Y}} \right] \approx 0\,. \tag{90}$$

The equations (89) and (90) respectively provide a particular example of (73) and (65).

3.2.3 Proof of Theorem 2

The proof may be divided in three distinct parts. Firstly, we show that our definition of the action produces a result different from zero, which is a non-trivial statement due to the numerous contractions of various irreducible tensors. Secondly, the kinetic operator (71) is proven to be self-adjoint, which implies that the equations of motion indeed are (73). Thirdly, the Euler–Lagrange equations (73) are shown to be equivalent to the equations of Hull and de Medeiros. At the light of the results of Section 3.1, this step ends the proof of Theorem 2. The simpler way to start the proof of Theorem 2 is to explicit the construction of the Lagrangian step by step and exhibit the Young tableaux corresponding to the diverse objects involved, because the procedure is very simple even though the multiplicity of indices somehow casts a shadow on this quality.

(1°) The starting point is the product of the $\overline{s}$ Levi-Civita tensors corresponding to the operator $*_1 \ldots *_{\overline{s}}$ in (71). In components, this product reads

$$\varepsilon^{\mu^1_1\mu^1_2\cdots\,\mu^1_D}\,\varepsilon^{\mu^2_1\mu^2_2\cdots\,\mu^2_D}\,\ldots\,\varepsilon^{\mu^{\overline{s}}_1\mu^{\overline{s}}_2\cdots\,\mu^{\overline{s}}_D}\,. \qquad (91)$$

Obviously, this product defines an irreducible representation of $GL(D,\mathbb{R})$ labeled by the following Young tableau

$$\mathrm{Z} \;=\; \begin{array}{|c|c|c|c|}\hline \mu^1_1 & \mu^2_1 & \cdots & \mu^{\overline{s}}_1 \\ \hline \mu^1_2 & \mu^2_2 & \cdots & \mu^{\overline{s}}_2 \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline \mu^1_D & \mu^2_D & \cdots & \mu^{\overline{s}}_D \\ \hline \end{array} \qquad (92)$$

All other indices present in the Lagrangian have to be contracted with the contravariant indices of the Levi-Civita tensors in (91), therefore we will have to “store” into the tableau (92) the indices of the components of the gauge fields, partial derivatives and metric tensors.

Since the components of the tensor gauge field in the bra and in the ket are contracted with the Levi-Civita tensors in the action, the antisymmetrization is automatic so that one may assume without loss of generality that the only algebraic constraints on the gauge field is that it is totally symmetric in the indices appearing in the rows of Y. Only the $GL(D,\mathbb{R})$-irreducible components of $\phi_{_Y}$ will appear in the Lagrangian. For the ket $|\,\phi_{_Y}\rangle$, the tensor gauge field components read

$$\phi_{\mu^1_1\mu^2_1\cdots\,\mu^s_1\,;\,\mu^1_2\mu^2_2\,\cdots\,\mu^{r_2}_2\,;\,\ldots\ldots\,;\,\mu^1_{\ell_1}\mu^2_{\ell_1}\cdots\,\mu^{r_{\ell_1}}_{\ell_1}}$$

where r_a is the length of the ath row in Y. The Young tableau (102) corresponding to the gauge field can be obtained by looking at the Young tableau Y included in the left upper corner of (92).

The $\overline{s}$ partial derivatives in the operator $d_1 \ldots d_{\overline{s}}$ read in components

$$\partial_{\mu^1_{\ell_1+1}}\partial_{\mu^2_{\ell_2+1}}\ldots\partial_{\mu^s_{\ell_s+1}}(\partial_{\mu^{s+1}_1})^{\varepsilon(s)}\,.$$

The contraction of the components

$$\partial_{\mu^1_{\ell_1+1}}\partial_{\mu^2_{\ell_2+1}}\ldots\partial_{\mu^s_{\ell_s+1}}(\partial_{\mu^{s+1}_1})^{\varepsilon(s)}\phi_{\mu^1_1\mu^2_1\cdots\,\mu^s_1\,;\,\mu^1_2\mu^2_2\,\cdots\,\mu^{r_2}_2\,;\,\ldots\ldots\,;\,\mu^1_{\ell_1}\mu^2_{\ell_1}\cdots\,\mu^{r_{\ell_1}}_{\ell_1}} \qquad (93)$$

with the Levi-Civita tensors (91) in the Lagrangian projects the derivatives of the gauge field on the components of the curvature tensor whose symmetry properties are characterized by the Young diagram $\overline{Y} := (\ell_1+1, \ell_2+1, \ldots, \ell_s+1)$ and Young tableau (124). This explains the appearance of the curvature tensor in the ket of the Euler-Lagrange equations (73). In the odd-spin case where $\varepsilon(s)=1$ and $\overline{s}=s+1$, an extra partial derivative $\partial_{\mu_1^{s+1}}$ is applied on the curvature tensor. The index of this extra partial derivative is not antisymmetrized with the index of any other partial derivative, as can be seen in (93) by the fact that no other partial derivative index $\mu^i_{\ell_i+1}$ possesses the same column index: $i \neq \overline{s}$. Therefore the contraction of (93) with (91) is nonzero. The first derivative of the odd-spin curvature tensor is characterized by the Young diagram $\overline{Y}_+ := (\ell_1+1, \ell_2+1, \ldots, \ell_s+1, 1)$.

An important point to understand next is that the components corresponding to the bra $\langle\, \phi_{Y}\mid$ can be chosen as

$$\phi_{\mu_D^{\overline{s}}\mu_D^{\overline{s}-1}\ldots\mu_D^{1+\varepsilon(s)}\,;\,\mu_{D-1}^{\overline{s}}\mu_{D-1}^{\overline{s}-1}\ldots\mu_D^{\overline{s}-r_2+1}\,;\,\ldots\,;\,\mu_{D-\ell_1+1}^{\overline{s}}\ldots\mu_{D-\ell_1+1}^{\overline{s}-r_{\ell_1}+1}}\,. \tag{94}$$

It is easier to state the preceding point in terms of Young tableaux and diagrams. The previous ordering of the indices of the bra ϕ_Y can be read off from (92): One rotates the Young diagram Y corresponding to ϕ_Y by 180 degrees ($\curvearrowright$ in the plan of the sheet of paper) and places it at the right-bottom corner of (92). The indices appearing in the cells of the rotated Y Young diagram coincide with the components of $\langle\, \phi_Y\mid$.

The indices that remain uncontracted in (92) are traced by the operator $\text{Tr}^{\,(D-1)\frac{\overline{s}}{2}-|Y|}$, as indicated in (71). The resulting action is nonvanishing because no two indices μ^i_j and $\mu^{i'}_j$ with the same row index j are contracted by the same epsilon tensor. In the Lagrangian, all the indices with the same row label i could be totally symmetrized without giving a vanishing result. In fact, by construction of the Lagrangian, such an operation would be redundant. We have explained how the curvature tensor $\mathcal{K}_{\overline{Y}}$ appeared in the Lagrangian, as well as the action of an extra derivative $\partial_{\mu_1^{s+1}}$ when $\varepsilon(s)=1$. By contraction with the epsilon-tensors (91), the curvature tensor $\mathcal{K}_{\overline{Y}}$ is dualized on every column, giving $*^s\,\mathcal{K}_{\overline{Y}}$ for s even and $*^{s+1}\,d_{s+1}\,\mathcal{K}_{\overline{Y}}$ for s odd.

By construction, for s even, only the $GL(D,\mathbb{R})$-irreducible component of the differential multiform $*^s\,\mathcal{K}_{\overline{Y}}$ which is labeled by the Young diagram $\widetilde{\overline{Y}} \in \mathbb{Y}^s$ will survive in the action. [See Appendix A.1.2 for the general definition of the dual Young diagram $\widetilde{\overline{Y}}$ and tensor.] The corresponding differential hyperform is denoted by $\widetilde{\mathcal{K}}_{\widetilde{\overline{Y}}} \in V^{GL_D}_{\widetilde{\overline{Y}}}$ and $V=\mathbb{R}^D$. The coordinates of $\widetilde{\overline{Y}}$ are $(D-\ell_s-1, D-\ell_{s-1}-1, \ldots, D-\ell_1-1)$. One can read the components of $\widetilde{\mathcal{K}}_{\widetilde{\overline{Y}}}$ and understand its appearance in the Lagrangian by inspecting the Young tableau (92): Mark with a "$\bullet$" the cells of (92) which correspond to $\mathcal{K}_{\overline{Y}}$. Then rotate (92) $\curvearrowright$ by 180 degrees. The empty cells now sit at the top of the rotated tableau and give the Young tableau $\widetilde{\overline{Y}}$ associated with the components of $\widetilde{\mathcal{K}}_{\widetilde{\overline{Y}}}$. Now consider the Young tableau Y included in the left upper-corner of $\widetilde{\overline{Y}}$. From (94) and the paragraph below (94), it corresponds to the Young tableau associated with the components of the bra $\langle\, \phi_Y\mid$. The remaining indices of $\widetilde{\overline{Y}}$ correspond to the components of the operator $\text{Tr}^{\,(D-1)\frac{s}{2}-|Y|}$. Note that the cells in which these remaining indices appear do not constitute a Young diagram.

In the odd-spin case, the $GL(D,\mathbb{R})$-irreducible component of the differential multiform $*^{s+1}\,d_{s+1}\,\mathcal{K}_{\overline{Y}}$ which survives in the Lagrangian is labeled by $\widetilde{\overline{Y}_+}$. The corresponding differential hyperform is denoted

$(\widetilde{d_{s+1}\mathcal{K}})_{\widetilde{\overline{Y}_+}}$ and transforms in $V^{GL_D}_{\widetilde{\overline{Y}_+}}$ with $V=\mathbb{R}^D$. The coordinates of $\widetilde{\overline{Y}_+}\in\mathbb{Y}^{s+1}$ are $(D-1, D-\ell_s-1, D-\ell_{s-1}-1,\ldots,D-\ell_1-1)$. Similarly as in the even-spin case, the Young tableau associated with the components of $(\widetilde{d_{s+1}\mathcal{K}})_{\widetilde{\overline{Y}_+}}$ is obtained from (92) by marking with a "$\bullet$" the cells of (92) which correspond to $(d_{s+1}\mathcal{K})_{\overline{Y}_+}$ and rotating (92) $\curvearrowright$ by 180 degrees. The empty cells which sit now at the top of the rotated tableau give the Young tableau $\widetilde{\overline{Y}_+}$ associated with the components of $(\widetilde{d_{s+1}\mathcal{K}})_{\widetilde{\overline{Y}_+}}$. Again, the Young tableau Y included in the left upper-corner of $\widetilde{\overline{Y}_+}$ corresponds to the components of the bra $\langle\ \phi_Y\ |$ in (94). The remaining indices which sit below and at the right of the Young tableau $Y\subset\widetilde{\overline{Y}_+}$ correspond to the components of the operator $\mathrm{Tr}^{\,(D-1)\frac{s+1}{2}-|Y|}$. The cells in which these remaining indices appear do not constitute a Young diagram.

(2°) The detailed construction of the Lagrangian explained above enables us to provide a Young-diagramatic proof of the self-adjoint property of the kinetic operator, $\mathrm{K}^\dagger=\mathrm{K}$. Take the rectangular Young diagram with D rows and $\overline{s}$ columns which underlies (92). Mark with a "$\times$" the cells corresponding to $|\ \phi_Y\ \rangle$ and $\langle\ \phi_Y\ |$ and fill with the symbol "$-$" the cells corresponding to the partial derivatives. Finally, mark with the symbol "$\circ$" the cells that remain, which correspond to the trace operators $\mathrm{Tr}^{\,(D-1)\frac{\overline{s}}{2}-|Y|}$. Denote the resulting rectangular Young tableau by the symbol Z. Now rotate Z by 180 degrees $\curvearrowright$. What appears is not yet Z, since each symbol "$-$" has to jump upward over the symbols "$\circ$". Because there is an even number $\overline{s}$ of "$-$" and because to each "$\circ$" in the ith column there is a corresponding "$\circ$" in the $(\overline{s}-i+1)$th column, there is an even number of jumps. The rotation of Z corresponds to taking the adjoint of $\langle\ \phi_Y\ |\ K\ |\ \phi_Y\ \rangle$. A jump in a column corresponds to a transposition in the indices of a Levi-Civita tensor, therefore an even number of transposition brings a factor $+1$. Finally, there is an even number of integrations by part because $\overline{s}$ is even. We have thus showed that $\mathrm{K}^\dagger=\mathrm{K}$. Moreover, it is now obvious that the action is invariant under the gauge transformations (72) because it depends on the ket $|\ \phi_Y\ \rangle$ only through the curvature.

(3°) The Euler-Lagrange equations $\frac{\delta S[\,\phi_Y]}{\delta\langle\,\phi_Y|}\approx 0$ are obtained by varying the action with respect to the bra $\langle\ \phi_Y\ |$, so by definition they have the symmetries of $|\ \phi_Y\ \rangle$. In the odd-spin case, it means that one sets (on-shell) to zero the component of $\left[\mathrm{Tr}^{\,(D-1)\frac{s+1}{2}-|Y|}\right](\widetilde{d_{s+1}\mathcal{K}})_{\widetilde{\overline{Y}_+}}$ which belongs to $V^{GL_D}_Y$. In the even-spin case, one sets (on-shell) to zero the components of $\left[\mathrm{Tr}^{\,(D-1)\frac{s}{2}-|Y|}\right]\widetilde{\mathcal{K}}_{\widetilde{\overline{Y}}}$ which belong to $V^{GL_D}_Y$. The operator between square brackets is in general reducible, it decomposes under $GL(D,\mathbb{R})$ into a sum of irreducible powers of the metric tensor. However, only a certain $GL(D,\mathbb{R})$-irreducible component labeled X^{even} for s even (X^{odd} for s odd) will survive in the field equation, the component for which the division $\widetilde{\overline{Y}}/X^{even}$ contains Y and the component for which the division $\widetilde{\overline{Y}_+}/X^{odd}$ contains Y (see Appendix A.1.1 for the division rule with Young diagrams). As recalled in Appendix A.1.2, with respect to $SL(D,\mathbb{R})$, the irreducible representations $\overline{Y}$ and $\widetilde{\overline{Y}}$ are equivalent: $V^{SL_D}_{\widetilde{\overline{Y}}}\cong V^{SL_D}_{\overline{Y}}$ ($V=\mathbb{R}^D$). Similarly, $\widetilde{\overline{Y}_+}$ and $\overline{Y}_+$ are equivalent irreps of $SL(D,\mathbb{R})$. The dual $SL(D,\mathbb{R})$-irreducible representation $\widetilde{\overline{Y}}$ (respectively $\widetilde{\overline{Y}_+}$) is called the contragredient $SL(D,\mathbb{R})$-irreducible representation of $\overline{Y}$ (of $\overline{Y}_+$), see e.g. the third reference of [48]. Consequently, the field equations for s even imply that the component of the trace $\mathrm{Tr}^{\frac{s}{2}}\,\mathcal{K}_{\overline{Y}}$ which belongs to $V^{SL(D,\mathbb{R})}_Y$ is set to zero. In the odd-spin case, it means that the field equations set to zero the component of the trace $\mathrm{Tr}^{\frac{s+1}{2}}\, d_{s+1}\mathcal{K}_{\overline{Y}}$ which belongs to $V_Y^{SL(D,\mathbb{R})}$. The latter two field equations are therefore equivalent to the equations (65) and (66), which in turn are equivalent to the Ricci-flat-like equations $\mathrm{Tr}\,\mathcal{K}_{\overline{Y}} \approx 0$.

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A Notation and conventions

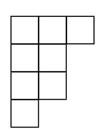
In this section, we review former results, introduce the fundamental definitions and take the opportunity to fix the notation.

A.1 Young diagrams and tensorial representations

We essentially extracted the standard definitions and properties on irreps and Young diagrams from various "textbook" references such as [48] (see also [49] and the appendix of the second reference of [8]).

A.1.1 Young diagrams and irreducible representations

A **Young diagram** Y is a diagram which consists of a finite number $s > 0$ of columns of identical squares (referred to as the **cells**) of finite non-increasing lengths $\ell_1 \geqslant \ell_2 \geqslant \ldots \geqslant \ell_s \geqslant 0$. The total number of cells of the Young diagram Y is denoted by $|Y| = \sum_{j=1}^{s} \ell_j$. The set of Young diagrams with at most s columns is denoted by $\mathbb{Y}^s$. We identify any Young diagram Y with its "coordinates" $(\ell_1, \ldots, \ell_s)$. For instance,

$Y \equiv$  $\in \mathbb{Y}^3$

is identified with the triple $(4, 3, 1) \in \mathbb{N}^3$. A **Young tableau** is a Young diagram where each cell contains an index.

Let $\mathcal{Y}$ be the Abelian group made of all formal finite sums of Young diagrams with integer coefficients. This group is $\mathbb{N}$-graded by the number $|Y|$ of boxes: $\mathcal{Y} = \sum_{n\in\mathbb{N}} \mathcal{Y}_n$. The famous "Littlewood–Richardson rule" defines a multiplication law which endows $\mathcal{Y}$ with a structure of graded commutative ring. The product of two Young diagrams X and Y is defined as

$$X \cdot Y \,=\, \sum_Z m_{X\,Y\,|\,Z}\; Z\,,$$

where the coefficients $m_{X\,Y\,|\,Z} = m_{Y\,X\,|\,Z}$ are the number of distinct labeling of the Young diagram Z obtained from the Littlewood–Richardson rule. As one can see, $|X \cdot Y| = |X| + |Y|$. A related operation in

$\mathcal{Y}$ is the "division" of Z by Y defined as

$$Z/Y = \sum_X m_{X\,Y\,|\,Z}\, X\,,$$

where the sum is over Young diagrams X such that the product $X \cdot Y$ contains the term Z (with coefficient $m_{X\,Y\,|\,Z}$).

Multilinear applications with a definite symmetry are associated with a definite Young tableau, while the symmetry in itself is specified by the Young diagram. Let V be a finite-dimensional vector space of dimension D over a field $\mathbb{K}$ and V^* its dual. The dual of the nth tensor power $V^{\otimes n}$ of V is canonically identified with the space of multilinear forms of rank n: $(V^{\otimes n})^* \cong (V^*)^{\otimes n}$. Let Y be a Young diagram whose first column has length $\ell_1 < D$ and let us consider that each of the $|Y|$ copies of V^* in the tensor product $(V^*)^{\otimes |Y|}$ is labeled by one cell of Y. The **Schur module** $V_Y^{GL_D}$ is defined as the vector space of all multilinear forms T in $(V^*)^{\otimes |Y|}$ such that :

(i) T is completely antisymmetric in the entries of each column of Y,

(ii) complete antisymmetrization of T in the entries of a column of Y and another entry of Y that is on the right-hand side of the column vanishes.

The space $V_Y^{GL_D}$ is an irreducible subspace invariant for the natural action of GL_D on $(V^*)^{\otimes |Y|}$. Its elements were called **hyperforms** by P. J. Olver [7].

Let Y be a Young diagram and T an arbitrary multilinear form in $(V^*)^{\otimes |Y|}$, one defines the multilinear form $\mathcal{Y}_A(T) \in (V^*)^{\otimes |Y|}$ by

$$\mathcal{Y}_A(T) = T \circ \mathcal{A}_Y \circ \mathcal{S}_Y$$

with

$$\mathcal{A}_Y = \sum_{c \in C} (-)^{\varepsilon(c)} c\,, \quad \mathcal{S}_Y = \sum_{r \in R} r\,,$$

where C is the group of permutations which permute the entries of each column, $\varepsilon(c)$ is the parity of the permutation c, and R is the group of permutations which permute the entries of each row of Y. It can be proved that any $\mathcal{Y}_A(T)$ belongs to $V_Y^{GL_D}$ and that the application $\mathcal{Y}_A$ of $End\big((V^*)^{\otimes |Y|}\big)$ satisfies the condition $\mathcal{Y}_A^2 = \lambda \mathcal{Y}_A$ for some number $\lambda \neq 0$. Thus $\mathbf{Y}_A = \lambda^{-1} \mathcal{Y}_A$ is a projection of $(V^*)^{\otimes |Y|}$ onto itself, i.e. $\mathbf{Y}_A^2 = \mathbf{Y}_A$, with image $\mathrm{Im}(\mathbf{Y}_A) = V_Y^{GL_D}$. The projector $\mathbf{Y}_A$ is referred to as the **Young symmetrizer** in the **antisymmetric convention for the Young diagram** Y.

Actually the construction of the Young symmetrizer introduced above by first symmetrizing the intries of the rows and then antisymmetrizing the entries of the columns of a given Young tableau could as well have been defined with antisymmetrization first followed by symmetrization. The corresponding irreducible GL_D-modules are isomorphic and the corresponding projector is called the Young symmetrizer in the **symmetric convention** for the Young diagram Y and is denoted by $\mathbf{Y}_S$. The changes of convention $\mathbf{Y}_S \circ \mathbf{Y}_A$ and $\mathbf{Y}_A \circ \mathbf{Y}_S$ are mere changes of basis in the Schur module $V_Y^{GL_D}$. Notice that for Young diagrams Y made of one row (or one column), it is not necessary to specify the choice of convention because both symmetrizers produce the same result; and the corresponding hyperforms of the Schur module $V_Y^{GL_D}$ are usually said to be **completely (anti)symmetric tensors**. In all other cases, the hyperforms are also called **mixed-symmetry tensors** in the litterature.

Example: The simplest instance of a mixed-symmetry tensor is the tensor $\overset{A}{T}_{\mu\nu|\rho}$ of rank three associated with the "hook" tableau $\begin{ytableau} \mu & \rho \\ \nu \end{ytableau}$ identified with the couple $(2,1) \in \mathbb{N}^2$. We chose the antisymmetric convention so that $\overset{A}{T}_{\mu\nu|\rho} = \overset{A}{T}_{[\mu\nu]|\rho}$ and $\overset{A}{T}_{[\mu\nu|\rho]} = 0$, where square brackets always denote complete antisymmetrization over all indices with strength one. In the symmetric convention, we would have a tensor $\overset{S}{T}_{\mu\rho\,;\,\nu}$ such that $\overset{S}{T}_{\mu\rho\,;\,\nu} = \overset{S}{T}_{(\mu\rho)\,;\,\nu}$ and $\overset{S}{T}_{(\mu\rho\,;\,\nu)} = 0$, where curved brackets always denote complete symmetrization over all indices with strength one. We can switch from one convention to the other by the following changes of basis $\overset{S}{T}_{\mu\rho\,;\,\nu} = -\overset{A}{T}_{\nu(\mu\,;\,\rho)}$ and $\overset{A}{T}_{\mu\nu\,;\,\rho} = \overset{S}{T}_{\rho[\mu\,;\,\nu]}$.

If the vector space V is endowed with a non-degenerate symmetric bilinear form (*i.e.* a metric) with signature (p,q) where $p+q=D$, then the subspace $V_Y^{O(p,q)}$ of traceless hyperforms in the Schur module $V_Y^{GL_D}$ is irreducible under the group $O(p,q)$. Whenever the sum of the lengths of the first two columns of Y is greater than D, $\ell_1+\ell_2>D$, then the irreducible space is identically zero: $V_Y^{O(p,q)}=\{0\}$. So Young diagrams such that $\ell_1+\ell_2 \leqslant D$ are said to be **allowed**. All non-zero finite-dimensional irreps of $O(p,q)$ are uniquely characterized by the datum of an allowed Young diagram.

Let $\mathcal{Y}_{>0}$ be the Abelian monoid made of all formal finite sums of Young diagrams with non-negative integer coefficients. Finite direct sums of irreps of GL_D may therefore be labeled by elements of $\mathcal{Y}_{>0}$ *via* the rule

$$V^{GL_D}_{m\,X+n\,Y} = m\,V^{GL_D}_X \oplus n\,V^{GL_D}_Y\,,$$

where the positive integer coefficients $m,n\in\mathbb{N}$ must be interpreted as the multiplicity of the corresponding representations. The same is true for the groups $O(p,q)$. The evaluation of the Kronecker product of two irreps of GL_D can be done by means of the Littlewood–Richardson rule which gives

$$V^{GL_D}_X \otimes V^{GL_D}_Y = V^{GL_D}_{X\cdot Y} = \bigoplus_Z m_{X\,Y\,|\,Z}\,V^{GL_D}_Z\,. \tag{95}$$

A related operation is that of contraction of one set of contravariant indices of symmetry Z with a subset of a set of covariant tensor indices of symmetry Y to yield a sum of covariant tensors with indices of symmetry X given by the division rule

$$V^{GL_D}_Z \,/\, V^{GL_D}_Y = V^{GL_D}_{Z/Y} = \bigoplus_X m_{X\,Y\,|\,Z}\,V^{GL_D}_X\,.$$

The irreps of GL_D may be reduced to direct sums of irreps of $O(p,q)$ by extracting all possible trace terms formed by contraction with products of the metric tensor and its inverse. The reduction is given by the branching rule

$$GL_D \downarrow O(p,q) \quad : \quad V^{GL_D}_Y \downarrow V^{O(p,q)}_{Y/\Delta}\,, \tag{96}$$

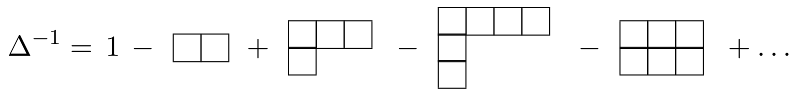
where Δ is the formal infinite sum

$$\Delta = 1 + \ydiagram{2} + \ydiagram{4} + \ydiagram{2,2} + \ydiagram{6} + \ydiagram{4,2} + \ydiagram{2,2,2} + \ldots$$

corresponding to the sum of all possible powers of the metric tensor. The decomposition (96) actually has a useful converse

$$O(p,q) \uparrow GL_D \quad : \quad V_Y^{O(p,q)} \uparrow V_{Y/\Delta^{-1}}^{GL_D}\,, \tag{97}$$

because the series Δ has an inverse



The operation (97) leads to a formal finite sum of irreps, some of which with strictly negative integer coefficients that have to be interpreted as constraints on some trace of the corresponding tensor basis. (Remark: These constraints are not preserved by the full GL_D group.)

A.1.2 Multiform and hyperform algebras

The elements of the algebra $\odot\big(\wedge(V^*)\big)$ of symmetric tensor products of antisymmetric forms $\in \wedge(V^*)$ are called **multiforms**. The subspace $\odot^s\big(\wedge(V^*)\big)$ of sums of symmetric products of s antisymetric forms is denoted by

$$\wedge_{[s]}(V) \equiv \underbrace{\wedge(V^*) \odot \ldots \odot \wedge(V^*)}_{s \text{ factors}}\,. \tag{98}$$

The D generators of the ith factor $\wedge(V^*)$ are written $d_i x^\mu$ ($i = 1, 2, \ldots, s$). By definition, the multiform algebra $\wedge_{[s]}(\mathbb{R}^D)$ is presented by the commutation relations

$$d_i x^\mu \, d_j x^\nu \;=\; (-)^{\delta_{ij}}\, d_j x^\nu \, d_i x^\mu\,, \tag{99}$$

where the wedge product is not written explicitly.

Let G be an Abelian group. The direct sum $V_* = \oplus_g V_g$ is called the G-**graded space** associated with the family of vector spaces $\{V_g\}_{g\in G}$. Moreover, if V is an algebra such that for any two elements $\alpha \in V_g$ and $\beta \in V_h$ the product $\alpha\,\beta \in V_{g\cdot h}$, then V is said to be a G-graded algebra. As an example, the algebra $\wedge_{[s]}(V)$ is $\mathbb{N}^s$-graded

$$\wedge_{[s]}(V) = \bigoplus_{(\ell_1,\ldots,\ell_s)\in\mathbb{N}^s} \wedge_{[s]}^{\ell_1,\ell_2,\ldots,\ell_s}(V)\,, \tag{100}$$

where an element α of $\wedge_{[s]}^{\ell_1,\ell_2,\ldots,\ell_s}(V)$ reads

$$\alpha \;=\; \frac{1}{\ell_1!\ldots\ell_s!}\;\alpha_{[\mu_1^1\ldots\mu_{\ell_1}^1]\,|\,\ldots\ldots\,|\,[\mu_1^s\ldots\mu_{\ell_s}^s]}\; d_1x^{\mu_1^1}\wedge\ldots\wedge d_1x^{\mu_{\ell_1}^1}\;\odot\ldots\ldots\odot\; d_sx^{\mu_1^s}\wedge\ldots\wedge d_sx^{\mu_{\ell_s}^s}\,. \tag{101}$$

Each exterior algebra is $\mathbb{Z}_2$-graded by the parity of the antisymmetric form. This induces a $\mathbb{Z}_2$-grading of the algebra $\wedge_{[s]}(V)$ given by the parity $\varepsilon(\ell_1 + \ldots + \ell_s)$ of the multiform $\alpha \in \wedge_{[s]}^{\ell_1,\ell_2,\ldots,\ell_s}(V)$. The algebra of multiforms is therefore graded commutative [see Equation (99)].

If $(\ell_1, \ldots, \ell_s)$ defines a Young diagram Y, then one can form a Young tableau by placing all the μ^i_j indices in (101) corresponding to the ith exterior algebra $\wedge(V^*)$ in the ith column of Y :

$$\begin{array}{|c|c|c|c|c|c|}
\hline
\mu^1_1 & \mu^2_1 & \cdots & \mu^{r_2}_1 & \cdots & \mu^s_1 \\
\hline
\mu^1_2 & \mu^2_2 & \cdots & \mu^{r_2}_2 & \multicolumn{2}{c}{} \\
\cline{1-4}
\vdots & \vdots & \multicolumn{1}{c}{\vdots \;\; \cdots} & \multicolumn{3}{c}{} \\
\cline{1-2}
\mu^1_{\ell_2} & \mu^2_{\ell_2} & \multicolumn{4}{c}{} \\
\cline{1-2}
\vdots & \multicolumn{5}{c}{} \\
\cline{1-1}
\mu^1_{\ell_1} & \multicolumn{5}{c}{} \\
\cline{1-1}
\end{array} \tag{102}$$

So the space $\wedge^{\ell_1,\ell_2,\ldots,\ell_s}_{[s]}(V)$ of multiforms is an eigenspace of the operator $\mathcal{A}_Y$ antisymmetrizing over the indices placed in the same column. Conversely, any hyperform in the antisymmetric convention can be seen as a multiform. This induces a natural product on the space of hyperforms.

From now on, we will assume that V is equipped with a metric. Then the Hodge dual operations

$$*_i : \wedge^{\ell_1,\ldots,\ell_i,\ldots,\ell_s}_{[s]}(V) \to \wedge^{\ell_1,\ldots,D-\ell_i,\ldots,\ell_s}_{[s]}(V)\,, \quad 1 \leqslant i \leqslant s \tag{103}$$

in each subspace $\wedge^{\ell_i}(V^*)$ may be defined. In practice, the operator $*_i$ acts as the Hodge operator on the ith antisymmetric form in the tensor product. To remain in the space $\otimes(V^*)$ of covariant tensors requires the use of the metric in order to lower contravariant indices.

Using the metric, another simple operation that can be defined is the trace. The convention is that we always take the trace over indices in two different columns, say the ith and jth. We denote this operation by

$$\mathrm{Tr}_{ij} : \wedge^{\ell_1,\ldots,\ell_i,\ldots,\ell_j,\ldots,\ell_s}_{[s]}(V) \to \wedge^{\ell_1,\ldots,\ell_i-1,\ldots,\ell_j-1,\ldots,\ell_s}_{[s]}(V)\,, \quad i \neq j\,. \tag{104}$$

Using the previous definitions of multiforms, Hodge dual and trace operators, we may reformulate the definition of the Schur module as follows: Let α be a multiform in $\wedge^{\ell_1,\ldots,\ell_s}_{[s]}(V)$. If

$$\ell_j \leqslant \ell_i < D\,, \quad \forall\, i,j \in \{1,\ldots,s\} :\; i \leqslant j\,,$$

then one obtains the equivalence

$$\mathrm{Tr}_{ij}\,\{\, *_i\, \alpha \,\} \;=\; 0 \quad \forall\, i,j :\; 1 \leqslant i < j \leqslant s \qquad \Longleftrightarrow \qquad \alpha \in V^{GL_D}_{(\ell_1,\ldots,\ell_s)}\,. \tag{105}$$

Indeed, the condition (i) is satisfied since α is a multiform and the condition (ii) is simply rewritten in terms of tracelessness conditions.

Let Y be an allowed Young diagram, $\ell_1 + \ell_2 \leqslant D$. The further irreducibility condition obeyed by a multiform $\alpha \in V_Y^{O(D-1,1)} \subset V_Y^{GL_D}$, is the vanishing of all possible traces. Using the irreducibility conditions (105) under GL_D one may show that the vanishing of the trace over the indices placed in the first two columns implies the vanishing of all other possible traces:

$$\text{If} \quad \alpha \in V_Y^{GL_D}\,, \text{ then:} \qquad \mathrm{Tr}\,\alpha \,=\, 0 \quad \Longleftrightarrow \quad \mathrm{Tr}_{ij}\,\alpha \,=\, 0 \quad \forall\, i,j \in \{1,\ldots,s\} \quad \Longleftrightarrow \quad \alpha \in V_Y^{O(p,q)}\,, \tag{106}$$

where we defined $\mathrm{Tr} \equiv \mathrm{Tr}_{12}$.

Let $Y = (\ell_1, \ldots, \ell_s)$ be any Young diagram in $\mathbb{Y}^s$. We define the **dual Young diagram** $\widetilde{Y} := (\tilde{\ell}_1, \ldots, \tilde{\ell}_s)$ by the following lengths of its columns: $\tilde{\ell}_i := D - \ell_{s+1-i}$ for $i \in \{1, \ldots, s\}$. Let α be a multiform of $\wedge_{[s]}^{\ell_1,\ldots,\ell_s}(V)$. One denotes by $\widetilde{\alpha} \in \wedge_{[s]}^{\tilde{\ell}_1,\ldots,\tilde{\ell}_s}(V)$ the **dual multiform** defined by

$$\widetilde{\alpha} := *^s\,\alpha\,, \qquad \text{where} \qquad *^s \,\equiv\, \prod_{i=1}^{s} *_i\ .$$

The dual multiform $\widetilde{\alpha}$ belongs to the same representation space of SL_D as α. If $\alpha_Y \in V_Y^{GL_D}$ is a hyperform labeled by the Young diagram Y, then the dual multiform $\widetilde{\alpha}_{\widetilde{Y}}$ is in the irrep. of GL_D associated with the dual Young diagram $\widetilde{Y}$, *i.e.* $\widetilde{\alpha}_{\widetilde{Y}} \in V_{\widetilde{Y}}^{GL_D}$, called the **contragradient representation** of $V_Y^{GL_D}$. Actually, the representations are equivalent under SL_D.

If $Y = (\ell_1, \ell_2, \ldots, \ell_s)$ is an allowed Young diagram, $\ell_1 + \ell_2 \leqslant D$, then the Young diagram $Y^* = (D - \ell_1, \ell_2, \ldots, \ell_s)$ is also an allowed Young diagram called **associated Young diagram**. In such case, if $\alpha_Y \in V_Y^{O(p,q)}$ is a hyperform in the irrep. of $O(p,q)$ corresponding to the Young diagram Y, then the multiform $*_1\alpha_Y$ is in the irrep. of $O(p,q)$ labeled by the associated Young diagram Y^*, *i.e.* $*_1\alpha_Y \in V_{Y^*}^{O(p,q)}$. The two irreps of $O(p,q)$ become equivalent when they are restricted to $SO(p,q)$. Notice that, for an allowed Young diagram, all columns but the first one have length $\ell_i < D/2$ $(2 \leqslant i \leqslant s)$. Therefore each inequivalent finite-dimensional irreps of $SO(p,q)$ is uniquely characterized by a Young diagram with columns of length smaller than $D/2$.

The metric on V allows to endow the space $\wedge_{[s]}(V)$ of multiforms with a non-degenerate symmetric bilinear form

$$(\ ,\ )\,:\,\wedge_{[s]}(V) \odot \wedge_{[s]}(V) \,\longrightarrow\, \mathbb{K} \tag{107}$$

called **scalar product** defined by taking the scalar product in each of the s exterior algebras $\wedge(V^*)$. More explicitly,

$$(\,\alpha\,,\beta\,) \,=\, \frac{1}{\ell_1!\ldots\ell_s!}\ \alpha_{\mu^1_1\ldots\mu^1_{\ell_1}\,|\,\ldots\,|\,\mu^s_1\ldots\mu^s_{\ell_s}}\ \beta^{\mu^1_1\ldots\mu^1_{\ell_1}\,|\,\ldots\,|\,\mu^s_1\ldots\mu^s_{\ell_s}}\ .$$

for two multiforms α and β which read in components as in (101). The scalar product is positive definite if and only if the metric on V is. *Via* the left multiplication in $\wedge_{[s]}(V)$ the generators $d_i x^\mu$ can be interpreted as operators. Their adjoint $(d_i x^\mu)^\dagger$ for the scalar product reproduces the interior product in each exterior algebra because the operators $d_i x^\mu$ and $(d_j x^\nu)^\dagger$ satisfy the canonical graded commutation relations

$$[d_i x^\mu, (d_j x^\nu)^\dagger]_\pm \,=\, \delta_{ij}\,\eta^{\mu\nu}\,, \tag{108}$$

where $[\ ,\ ]_\pm$ stands for the $\mathbb{Z}_2$-graded commutator, $\eta_{\mu\nu}$ are the components of the (pseudo-Riemannian) metric on V and $\eta^{\mu\lambda}\eta_{\lambda\nu} = \delta^\mu_\nu$. The anticommutation relations (108) also imply that $\wedge_{[s]}(V)$ is isomorphic to a Fock space whose creation operators would be the $d_i x^\mu$'s and the destruction operators the $(d_i x^\mu)^\dagger$'s. In terms of the latter operators, the trace operators Tr_{ij} defined in (104) can be written as $\mathrm{Tr}_{ij} = \eta_{\mu\nu}(d_i x^\mu)^\dagger (d_j x^\nu)^\dagger$.

If $(\ell_1, \ell_2, \ldots, \ell_s)$ defines a Young diagram Y, then the operators $\mathcal{A}_Y$ and $\mathcal{S}_Y$ are well-defined on $\wedge_{[s]}^{\ell_1,\ell_2,\ldots,\ell_s}(V)$. Moreover, they are self-adjoint, therefore the two Young symmetrizers are the adjoint of each other: $\mathbf{Y}^\dagger_{_A} = \mathbf{Y}_{_S}$. There is no ambiguity once a Young tableau is specified. This implies that one may define the non-degenerate product on the space of hyperforms

$$(\ ,\ )_Y \,:\, V_Y^{GL_D} \odot V_Y^{GL_D} \to \mathbb{K} \tag{109}$$

defined by

$$(\,\alpha\,,\,\beta\,)_Y := (\,\alpha\,,\,\mathbf{Y}_{_A}\beta\,)\,, \tag{110}$$

where α and β may be taken to be multiforms of $\wedge_{[s]}^{\ell_1,\ell_2,\ldots,\ell_s}(V)$ but the result depends only on their irreducible component in $V_Y^{GL_D}$. One observes that β may naturally be assumed to be in the antisymmetric convention, and α in the symmetric convention. Indeed, $(\,\alpha\,,\,\beta\,)_Y = (\,\alpha\,,\,\mathbf{Y}_{_A}\beta\,)_Y = (\,\mathbf{Y}_{_S}\alpha\,,\,\beta\,)_Y$ because the symmetrizers are projectors and adjoint with respect to each other. In Dirac's terminology, one may take the "bras" to be hyperforms in the symmetric convention and the "kets" in the antisymmetric convention.[10]

A.2 Differential complexes

The objective of the works presented in [7, 8, 9, 10] was to construct complexes for irreducible tensor fields of mixed symmetries, thereby generalizing to some extent the calculus of differential forms.

A.2.1 Multicomplex of differential multiforms

We start with basic definitions from homological algebra. A **differential complex** is defined to be an $\mathbb{N}$-graded space $V_* = \oplus_{i\in\mathbb{N}} V_i$ with a nilpotent endomorphism d of degree one, i.e. there is a chain of linear transformations

$$\ldots \xrightarrow{d} V_{i-1} \xrightarrow{d} V_i \xrightarrow{d} V_{i+1} \xrightarrow{d} \ldots$$

such that $d^2 = 0$. A well-known example of such structure is the de Rham complex for which the vector space is the set $\Omega^*(\mathbb{R}^d)$ of differential forms graded by the form degree. The role of the nilpotent operator is played by the exterior derivative $d = dx^\mu\partial_\mu$. One can now define the quotient $H^*(d) := \frac{\mathrm{Ker}_d}{\mathrm{Im}_d}$ called the **cohomology** of d. This space inherits the grading of V_*. The elements of $H(d)$ are called (**co-**)**cycles**. Elements of Im_d are said to be trivial or **exact** (co)-cycles.

A straightforward generalization of the previous definitions is to consider a more complicated grading. More specifically, one takes $\mathbb{N}^s$ as Abelian group ($s \geqslant 2$). A **multicomplex** of order $s \in \mathbb{N}$ is defined to be

[10] We are grateful to Marc Henneaux for calling these observations to our attention.

an $\mathbb{N}^s$-graded space $V_{(*,\ldots,*)} = \bigoplus_{(i_1,\ldots,i_s)\in\mathbb{N}^s} V_{(i_1,\ldots,i_s)}$ with s nilpotent endomorphisms d_j $(1 \leqslant j \leqslant s)$ such that

$$d_j V_{(i_1,\ldots,i_j,\ldots,n_s)} \subset V_{(i_1,\ldots,i_j+1,\ldots,i_s)}.$$

A multicomplex of order one is a usual differential complex. A concrete realization of this definition is the space of **differential multiforms** whose elements are sums of products of the generators $d_j x^\mu$ with smooth functions as coefficients.

More precisely, the space of differential multiforms is the graded tensor product of $C^\infty(\mathbb{R}^D)$ with s symmetrized copies of the exterior algebra $\wedge(\mathbb{R}^{D*})$ where $\mathbb{R}^{D*}$ is the dual space with basis $d_i x^\mu$ $(1 \leqslant i \leqslant s$, thus there are s times D of them). We denote this multigraded space $C^\infty(\mathbb{R}^D) \otimes \wedge_{[s]}(\mathbb{R}^D)$ as

$$\Omega_{[s]}(\mathbb{R}^D) = \bigoplus_{(\ell_1,\ldots,\ell_s)\in\mathbb{N}^s} \Omega^{\ell_1,\ell_2,\ldots,\ell_s}_{[s]}(\mathbb{R}^D)\,, \tag{111}$$

by analogy with the de Rham complex $\Omega^*(\mathbb{R}^D) = \Omega_{[1]}(\mathbb{R}^D)$. The tensor field $\alpha_{\mu^1_1\ldots\mu^1_{\ell_1}|\ldots|\mu^s_1\ldots\mu^s_{\ell_s}}(x)$ defines a multiform $\alpha \in \Omega^{\ell_1,\ldots,\ell_s}_{[s]}(\mathbb{R}^D)$ which explicitly reads

$$\alpha \;=\; \frac{1}{\ell_1!\ldots\ell_s!}\;\alpha_{\mu^1_1\ldots\mu^1_{\ell_1}\,|\,\ldots\,|\,\mu^s_1\ldots\mu^s_{\ell_s}}(x)\;\; d_1x^{\mu^1_1}\wedge\ldots\wedge d_1x^{\mu^1_{\ell_1}}\;\odot\ldots\odot\; d_sx^{\mu^s_1}\wedge\ldots\wedge d_sx^{\mu^s_{\ell_s}}\,. \tag{112}$$

In the sequel, when we refer to the differential multiform α we speak either of (112) or of its components. More generally, we call (smooth covariant) **tensor field** any element of the space $\bigotimes(\mathbb{R}^{D*})\otimes C^\infty(\mathbb{R}^D)$.

We endow $\Omega_{[s]}(\mathbb{R}^D)$ with the structure of a multicomplex by defining s **exterior derivatives**

$$d_i : \Omega^{\ell_1,\ldots,\ell_i,\ldots,\ell_s}_{[s]}(\mathbb{R}^D) \to \Omega^{\ell_1,\ldots,\ell_i+1,\ldots,\ell_s}_{[s]}(\mathbb{R}^D)\,,\quad 1\leqslant i\leqslant s\,, \tag{113}$$

defined by taking the exterior derivative with respect to the ith set of antisymmetric indices. Naturally, for each label i $(1\leqslant i\leqslant s)$ one can define the cohomology group $H^*(d_i) \equiv \dfrac{\text{Ker}_{d_i}}{\text{Im}_{d_i}}$. The nilpotent operators $d_j \equiv d_j x^\mu \partial_\mu$ generalize the exterior differential of the de Rham complex.

If the manifold $\mathbb{R}^D$ is endowed with a metric then, by using the Hodge operators $*_i$ introduced previously, one may also define the **coderivatives**

$$d_i^\dagger \;:=\; (-)^{q+1+\ell_i(D-\ell_i+1)}\;*_i\;d_i\,*_i\;:\;\Omega^{\ell_1,\ldots,\ell_i,\ldots,\ell_s}_{[s]}(\mathbb{R}^D) \to \Omega^{\ell_1,\ldots,\ell_i-1,\ldots,\ell_s}_{[s]}(\mathbb{R}^D)\,,\quad 1\leqslant i\leqslant s\,. \tag{114}$$

As usual, the Laplacian or d'Alembertian may be defined by the anticommutator $\Box = [d_i, d_i^\dagger]_+$. A multiform α in $\Omega_{[s]}(\mathbb{R}^D)$ is said to be **harmonic** if it is closed $(d_i\alpha = 0)$ and coclosed $(d_i^\dagger\alpha = 0)$ for all $i\in\{1,\ldots,s\}$. Notice the very useful identities

$$[\,\text{Tr}_{ij}\,,\,d_k\,]_\pm \;=\; 2\,\delta_{k(i}\,d^\dagger_{j)}\,, \tag{115}$$

and

$$[\,d_i\,,\,d_j\,]_\pm \;=\; 0\,,\qquad [\,d_i\,,\,d_j^\dagger\,]_\pm \;=\; \delta_{ij}\,\Box\,. \tag{116}$$

A.2.2 Generalized complex of differential hyperforms

Let N be a natural number not smaller than 2. An N**-complex** is defined as a graded space $V_* = \oplus_i V_i$ equipped with an endomorphism d of degree 1 that is nilpotent of order $N \geqslant 2$: $d^N = 0$. The **generalized cohomology** of the N-complex V_* is the family of $N-1$ graded spaces ${}^{(k)}H(d)$ with $1 \leqslant k \leqslant N-1$ defined by ${}^{(k)}H(d) = \mathrm{Ker}(d^k)/\mathrm{Im}(d^{N-k})$, i.e. ${}^{(k)}H^*(d) = \oplus_i {}^{(k)}H^i(d)$ where

$$ {}^{(k)}H^i(d) = \Big\{ \alpha \in V_i \mid d^k\alpha = 0,\ \alpha \sim \alpha + d^{N-k}\beta,\ \beta \in V_{i+k-N} \Big\} . $$

Proposition 2. *[10] Any multicomplex structure of order $N-1$ possesses a canonical N-complex structure.*

This fact plays a crucial role in the gauge structure of mixed-symmetry tensor gauge fields. The proof is rather simple.

Proof: In order to connect the two definitions one has to build an $\mathbb{N}$-grading from the $\mathbb{N}^s$-grading of the multicomplex $V_{(*,\ldots,*)} = \bigoplus_{(i_1,\ldots,i_s)\in\mathbb{N}^s} V_{(i_1,\ldots,i_s)}$ endowed with the s nilpotent endomorphisms d_j. A simple choice is to consider the **total grading** defined by the sum $i \equiv \sum_{j=1}^s i_j$. We introduce the operator

$$ d_T \equiv \sum_{j=1}^{s} d_j $$

which possesses the nice property of being of total degree one. Two convenient cases arise:

- $[\, d_i \,, d_j \,]_+ \;=\; 0$: Usually the nilpotent operators d_j are taken to be anticommuting and therefore d is nilpotent. This case is rather standard in homological perturbation theory.

- $[\, d_i \,, d_j \,]_- \;=\; 0$ when $i \neq j$ and ${d_i}^2 \;=\; 0$: From our present perspective, commuting d_j's are indeed quite interesting because, in that case, d_T is in general nilpotent of order $s+1$ and the space V is endowed with a $(s+1)$-complex structure. Indeed, every term in the expansion of d_T^{s+1} contains at least one of the d_j twice. □

Due to the (anti)commutation relations (116), the second case in the proof is illustrated by the multicomplex $\Omega_{[s]}(\mathbb{R}^D)$ of differential multiforms.

The **total cohomology group** [10] is the generalized cohomology group ${}^{(k)}H^{(i_1,\ldots,i_s)}(d_T)$ associated with the operator d_T and the $\mathbb{N}^s$-grading, whose elements $\alpha \in V_{(i_1,\ldots,i_s)}$ satisfy the set of cocycle conditions

$$ \prod_{i\in I} d_i \;\alpha \;=\; 0\,, \qquad \forall\, I \subset \{1,2,\ldots,s\} \mid \#I = k\,, \tag{117} $$

with the equivalence relation

$$ \alpha \;\sim\; \alpha \;+ \sum_{\substack{J \subset \{1,2,\ldots,s\} \\ \#J = s-k+1}} \prod_{j\in J} d_j\, \beta_J\,, \tag{118} $$

where β_J belongs to $V_{(j_1,\ldots,j_s)}$ with

$$j_k \equiv \begin{cases} i_k & \text{if } k \notin J\,, \\ i_k - 1 & \text{if } k \in J\,. \end{cases}$$

This can be easily seen by decomposing the cocycle condition $d_T^k\alpha = 0$ and the equivalence relation $\alpha \sim \alpha + d_T^{s-k+1}\beta$ in $\mathbb{N}^s$ degree.

A **differential hyperform** [7] is a $GL(D,\mathbb{R})$-irreducible tensor field, that is, an element of $C^\infty(\mathbb{R}^D) \otimes V_Y^{GL(D,\mathbb{R})}$. We denote by $\Omega^Y_{(s)}(\mathbb{R}^D)$ the space of differential hyperforms associated with the Young diagram Y made of s columns. We also introduce the $\mathbb{Y}^s$-graded space

$$\Omega_{(s)}(\mathbb{R}^D) = \sum_{Y \in \mathbb{Y}^s} \Omega^Y_{(s)}(\mathbb{R}^D)\,. \tag{119}$$

In order to endow the space $\Omega_{(s)}(\mathbb{R}^D)$ with a structure of multicomplex, one may introduce the maps [9, 10]

$$d^{\{i\}} : \Omega_{(s)}^{(\ell_1,\ldots,\ell_i,\ldots,\ell_s)}(\mathbb{R}^D) \to \Omega_{(s)}^{(\ell_1,\ldots,\ell_i+1,\ldots,\ell_s)}(\mathbb{R}^D)\,, \tag{120}$$

for $1 \leqslant i \leqslant s$ and $\ell_{i+1} > \ell_i$. This operator is defined as follows: take the derivative of a differential hyperform of $\Omega^Y_{(s)}$ and consider the image in $\Omega^{Y^{\{i\}}}_{(s)}$ where $Y^{\{i\}}$ is the Young diagram obtained from Y by adding one more cell in the ith column. In other words, $d^{\{i\}} \equiv \mathbf{Y}_A^{\{i\}} \circ \partial$. Since hyperforms in the antisymmetric convention may also be seen as multiforms, the action of an operator $d^{\{i\}}$ may be expressed as a linear combination of the action of the exterior derivatives d_j. So we have the obvious property that, for any differential hyperform α of $\Omega_{(s)}(\mathbb{R}^D)$,

$$\begin{aligned} &\Big(\prod_{i\in I} d_i\Big)\,\alpha = 0\,, \qquad \forall I \subset \{1,2,\ldots,s\} \mid \#I = k \\ &\qquad\qquad \Longrightarrow \quad \Big(\prod_{i\in I} d^{\{i\}}\Big)\alpha = 0\,, \quad \forall I \subset \{1,2,\ldots,s\} \mid \#I = k\,. \end{aligned} \tag{121}$$

We proved in [9] the triviality of the generalized cohomology groups ${}^{(k)}H^{(\ell_1,\ldots,\ell_s)}(d)$ for $1 \leqslant k \leqslant s$, $0 < \ell_s$ and $\ell_1 < D$, in the space of differential hyperforms $\Omega_{(s)}(\mathbb{R}^D)$ with $d = d^{\{1\}} + \ldots + d^{\{s\}}$, thereby extending the results of [7, 8]. In particular, for ${}^{(1)}H^{\overline{Y}}(d)$ where $\overline{Y} \in \mathbb{Y}^s$ is a Young diagram made of s columns, one may show[11] that the closure conditions of a hyperform $\mathcal{K}_{\overline{Y}} \in \Omega^{\overline{Y}}_{(s)}(\mathbb{R}^D)$ are equivalent to

$$d_i \mathcal{K}_{\overline{Y}} = 0\,, \quad (i = 1,\ldots,s) \tag{122}$$

and that imply the following exactness of the differential hyperform

$$\mathcal{K}_{\overline{Y}} = d_1 \ldots d_s \phi_Y\,, \tag{123}$$

where ϕ_Y is a differential hyperform belonging to $\Omega^Y_{(s)}(\mathbb{R}^D)$ with Y the Young diagram obtained by removing the first row of $\overline{Y}$. Such an exact hyperform $\mathcal{K}_{\overline{Y}}$ is called the **curvature tensor** of the **gauge field** ϕ_Y.

[11] See Corollary 1 of [9] for more details.

If the components $\phi_{\mu^1_1 \ldots \mu^1_{\ell_1} \,|\, \ldots \,|\, \mu^s_1 \ldots \mu^s_{\ell_s}}$ of the gauge field are characterized by the Young tableau (102), then the components $\mathcal{K}_{\mu^1_1 \ldots \mu^1_{\ell_1+1} \,|\, \ldots \,|\, \mu^s_1 \ldots \mu^s_{\ell_s+1}}$ are described by the Young tableau

μ^1_1	μ^2_1	$\cdots$	$\mu^{r_2}_1$	$\cdots$	μ^s_1
μ^1_2	μ^2_2	$\cdots$	$\mu^{r_2}_2$	$\cdots$	μ^s_2
μ^1_3	μ^2_3	$\cdots$	$\mu^{r_2}_3$		
$\vdots$	$\vdots$	$\vdots$ $\cdots$			
$\mu^1_{\ell_2+1}$	$\mu^2_{\ell_2+1}$				
$\vdots$					
$\mu^1_{\ell_1+1}$					

(124)

Analogously, for ${}^{(s)}H^Y(d)$ where Y is a Young diagram made of s columns, one may show that the closure condition of a hyperform $\phi_{_Y} \in \Omega^Y_{(s)}(\mathbb{R}^D)$ is equivalent to

$$d_1 \ldots d_s \phi_{_Y} = 0\,, \tag{125}$$

and they imply the following exactness of the differential hyperform

$$\phi_{_Y} = \mathcal{S}_Y \sum_{i=1}^{s} d_i \epsilon_i = \sum_{i=1}^{s} d^{\{i\}} \epsilon_{\{i\}}\,, \tag{126}$$

where the ϵ_i are differential multiforms belonging to $\Omega^{\ell_1,\ldots,\ell_i-1,\ldots,\ell_s}_{[s]}(\mathbb{R}^D)$ while the $\epsilon_{\{i\}}$ are differential hyperforms (or zero if they are not well-defined) belonging to $\Omega^{(\ell_1,\ldots,\ell_i-1,\ldots,\ell_s)}_{(s)}(\mathbb{R}^D)$. Such an exact hyperform $\phi_{_Y}$ is called the a **pure gauge field**.

The norm of the functions in $L^2(\mathbb{R}^D)$ together with the scalar product on $\wedge_{[s]}(\mathbb{R}^D)$ define a natural non-degenerate symmetric bilinear form on the space $\Omega_{[s]}(\mathbb{R}^D)$ of differential multiforms, so that the codifferential $d_i^\dagger$ in (114) becomes the adjoint of the exterior derivative d_i. This implies that one may define the following **scalar product** on the space of differential hyperforms

$$\langle\ ,\ \rangle\,:\,\Omega^Y_{(s)}(\mathbb{R}^D) \odot \Omega^Y_{(s)}(\mathbb{R}^D) \,\rightarrow\, \mathbb{R} \tag{127}$$

defined by

$$\langle\, \alpha \mid \beta \,\rangle := \int d^D x\ (\,\alpha\,,\,\beta\,)_Y\,, \tag{128}$$

where $(\,\alpha\,,\,\beta\,)_Y$ is the scalar product (110) on $V_Y^{GL(D,\mathbb{R})}$. We remind the reader that, without loss of generality, one may take the bras $\langle \alpha \mid$ to be differential hyperforms in the symmetric convention and the kets $\mid \beta\,\rangle$ in the antisymmetric convention.

Given a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ on a functional space, a **quadratic action** for the field ϕ is a bilinear functional $S[\phi] = \langle\, \phi \mid \mathrm{K} \mid \phi\,\rangle$ entirely determined by the datum of a self-adjoint (pseudo)differential operator K called **kinetic operator**. Because of the non-degeneracy of the bilinear form, the action $S[\phi]$ is extremized for configurations obeying the **field equation** $\mathrm{K} \mid \phi\,\rangle = 0$. Translation invariance requires the kinetic operator K to be independent of the coordinates x, hence the field equation is a linear partial differential equation (PDE) with constant coefficients. Boundary conditions and regularity requirements should be specified when solving PDEs.[12] For instance, in order to convert linear PDEs into algebraic equations by going to the momentum representation, we consider the gauge field $\phi_{_Y}$ either as a rapidly decreasing smooth function or as a tempered distribution, that is the ket $\mid \phi_{_Y}\rangle \in \mathcal{S}(\mathbb{R}^D) \otimes V_Y^{GL(D,\mathbb{R})}$ and the bra $\langle\, \phi_{_Y} \mid \in \mathcal{S}'(\mathbb{R}^D) \otimes V_Y^{GL(D,\mathbb{R})}$. The action $S\,[\,\phi_{_Y}]$ is said to be **gauge invariant** under (72) if $\langle d_i \epsilon_i \mid \mathrm{K} \mid \phi_{_Y}\rangle = 0$ for all ϵ_i and $\phi_{_Y}$. This gauge invariance property is equivalent to the **Noether identity** $d_i^\dagger \mathrm{K} = 0$ since the bilinear form is non-degenerate.

B Technical lemmas

B.1 Proof of Lemma 1

We consider any two adjacent columns of the differential hyperform $\mathcal{P}_{\ldots|\mu_1\ldots\mu_r|\nu_1\ldots\nu_q|\ldots}$, and we want to show that the following implication holds (without expliciting the other columns this time; they play no role in the proof)

$$\mathcal{P}_{\mu_1\ldots\mu_r|[\nu_1\ldots\nu_q,\rho]} = 0 \quad \Longrightarrow \quad \partial_{[\rho}\mathcal{P}_{\mu_1\ldots\mu_r]|\nu_1\ldots\nu_q} = 0\,, \tag{129}$$

where a coma stands for a derivative. In the case where $q = r$, the above implication is trivial ($\mathcal{P}$ is then symmetric under the exchange of the two columns), so we assume $q < r$ from now on.

(A) Since $\mathcal{P} \in \Omega_{(s)}(\mathbb{R}^D)$, one has $\mathcal{P}_{[\mu_1\ldots\mu_r|\nu_1]\nu_2\ldots\nu_q} \equiv 0$ which gives $\mathcal{P}_{\mu_1\ldots\mu_r|\nu_1\ldots\nu_q} \equiv r(-)^r \mathcal{P}_{\nu_1[\mu_1\ldots\mu_{r-1}|\mu_r]\nu_2\ldots\nu_q}$. Without bothering about coefficients, we write

$$\mathcal{P}_{\nu_1[\mu_1\ldots\mu_{r-1}|\mu_r]\nu_2\ldots\nu_q} \propto \mathcal{P}_{\mu_1\ldots\mu_r|\nu_1\ldots\nu_q}\,. \tag{130}$$

(B) We antisymmetrize on the first $(r{+}2)$ indices of the differential hyperform $\mathcal{P}$, yielding $\mathcal{K}_{[\mu_1\ldots\mu_r|\nu_1\nu_2]\nu_3\ldots\nu_q} \equiv 0$. Decomposing this identity, we see three classes of terms appearing, where ν_1 and ν_2 are

1. both in the first column,

2. one in the first column, the second in the other,

3. both in the second column.

[12] Throughout this article, we are sometimes sloppy concerning such technical issues of functional analysis because our main concern is algebraic. Practically, this means that we always implicitly assume that the functional space we work with is such that the objects we talk about and the operations we perform on them, are well defined. There is no lack of rigor in such assumption because they may be legitimated and we refer to textbooks such as [50] for details.

Explicitly one finds

$$0 \equiv a\Big(\mathcal{P}_{\nu_1[\mu_1\dots\mu_{r-1}|\mu_r]\nu_2\nu_3\dots\nu_q} - \mathcal{P}_{\nu_2[\mu_1\dots\mu_{r-1}|\mu_r]\nu_1\nu_3\dots\nu_q}\Big) + b\,\mathcal{P}_{\nu_1\nu_2[\mu_1\dots\mu_{r-2}|\mu_{r-1}\mu_r]\nu_3\dots\nu_q} + c\,\mathcal{P}_{\mu_1\dots\mu_r|\nu_1\dots\nu_q]}\,, \tag{131}$$

for some non-vanishing coefficients $a, b, c \in \mathbb{N}_0$. This allows us to write $\mathcal{P}_{\nu_1\nu_2[\mu_1\dots\mu_{r-2}|\mu_{r-1}\mu_r]\nu_3\dots\nu_q}$ as a linear combinaison of $\mathcal{P}_{\mu_1\dots\mu_r|\nu_1\dots\nu_q}$ and $\Big(\mathcal{P}_{\nu_1[\mu_1\dots\mu_{r-1}|\mu_r]\nu_2\nu_3\dots\nu_q} - \mathcal{P}_{\nu_2[\mu_1\dots\mu_{r-1}|\mu_r]\nu_1\nu_3\dots\nu_q}\Big)$. Using (130) one obtains

$$\mathcal{P}_{\nu_1\nu_2[\mu_1\dots\mu_{r-2}|\mu_{r-1}\mu_r]\nu_3\dots\nu_q} \propto \mathcal{P}_{\mu_1\dots\mu_r|\nu_1\dots\nu_q}\,. \tag{132}$$

(C) Starting this time from $\mathcal{P}_{[\mu_1\dots\mu_r|\nu_1\nu_2\nu_3]\nu_4\dots\nu_q} \equiv 0$ and using the relations (130) and (132), one obtains similarly $\mathcal{P}_{\nu_1\nu_2\nu_3[\mu_1\dots\mu_{r-3}|\mu_{r-2}\mu_{r-1}\mu_r]\nu_4\dots\nu_q} \propto \mathcal{P}_{\mu_1\dots\mu_r|\nu_1\dots\nu_q}$. At the end of the day one gets

$$\mathcal{P}_{\nu_1\dots\nu_q[\mu_{q+1}\dots\mu_r|\mu_1\dots\mu_q]} \propto \mathcal{P}_{\mu_1\dots\mu_r|\nu_1\dots\nu_q}\,. \tag{133}$$

As a consequence of our starting hypothesis Equation (129), we have $\mathcal{P}_{\nu_1\dots\nu_q[\mu_{q+1}\dots\mu_r|\mu_1\dots\mu_q\,,\rho]} = 0$, and finally, using Relation (133), $\partial_{[\rho}\mathcal{P}_{\mu_1\dots\mu_r]|\nu_1\dots\nu_q} = 0$.

B.2 Proof of Lemma 2

The proof is somewhat tedious because it requires some care with the combinatorial gymnastic.

By definition,

$$\mathrm{Tr}_{(12}\dots\mathrm{Tr}_{2n-1\,2n)} = \frac{1}{(2n)!}\sum_{\pi\in\mathfrak{S}_{2n}}\Big(\prod_{i\in\{1,\dots,n\}}\mathrm{Tr}_{\pi(2i-1)\pi(2i)}\Big)\,.$$

To start with, one makes use of (115) for $i \neq j$ in order to rearrange the factors in the following sum over all permutations π of the set $\{1,\dots,2n\}$

$$\mathrm{Tr}_{(12}\dots\mathrm{Tr}_{2n-1\,2n)}\,d_1d_2\dots d_{2n-1}d_{2n} = = \frac{1}{(2n)!}\sum_{\pi\in\mathfrak{S}_{2n}}\Big(\prod_{i=1}^{n}\mathrm{Tr}_{\pi(2i-1)\pi(2i)}d_{\pi(2i-1)}d_{\pi(2i)}\Big)\,. \tag{134}$$

Then, one evaluates each factor

$$\mathrm{Tr}_{\pi(2i-1)\pi(2i)}d_{\pi(2i-1)}d_{\pi(2i)} = \Box - d_{\pi(2i-1)}d^\dagger_{\pi(2i-1)} - d_{\pi(2i)}d^\dagger_{\pi(2i)} + d_{\pi(2i-1)}d_{\pi(2i)}\mathrm{Tr}_{\pi(2i-1)\pi(2i)}\,, \tag{135}$$

by using (57). Now, one inserts (135) into the products

$$\prod_{i=1}^{n}\mathrm{Tr}_{\pi(2i-1)\pi(2i)}d_{\pi(2i-1)}d_{\pi(2i)} = \Box^n + \Box^{n-1}\Big(-\sum_{j=1}^{2n}d_jd_j^\dagger + \sum_{i=1}^{n}d_{\pi(2i-1)}d_{\pi(2i)}\mathrm{Tr}_{\pi(2i-1)\pi(2i)}\Big) + \Box^{n-2}\sum_{j=1}^{2n-2}\sum_{k=j+1+\varepsilon(j)}^{2n}d_{\pi(j)}d_{\pi(k)}d^\dagger_{\pi(j)}d^\dagger_{\pi(k)} + \sum_{i,j,k=1}^{2n}d_id_jd_k(\dots)\,, \tag{136}$$

We evaluated and grouped the terms in (136) according to the number of d'Alembertians and curls, by making use of the commutation relation (116). More precisely, the decomposition in powers of the d'Alembertian goes as follows.

$\Box^n$: The leading term comes from picking the d'Alembertian in each of the n factor in the product.

$\Box^{n-1}$: The terms come from choosing a factor and taking d'Alembertian in the $n-1$ remaining factors. Still, one may either choose in the right-hand-side of (135) one of the term of the form $d_j d_j^\dagger$ or the last term with the trace.

$\Box^{n-2}$: In degree $n-2$, the terms are of two types: either they contain two curls and are of the form $d_j d_k d_j^\dagger d_k^\dagger$ or they contain at least three curls. The first type of terms comes from choosing two factors in the product and one term of the form $d\,d^\dagger$ in each of them. All other choices give rise to terms of the second type.

$\Box^{n-3}$: All terms of degree $n-3$ or lower in the d'Alembertian include at least three curls. All such terms have been put together in the last term of (136).

Eventually, one should perform the sum over all permutations of the $2n$ elements in the set $\{1,\dots,2n\}$. The result is

$$\begin{aligned}
&\frac{1}{(2n)!}\sum_{\pi\in\mathfrak{S}_{2n}}\Big(\prod_{i=1}^{n}\mathrm{Tr}_{\pi(2i-1)\pi(2i)}d_{\pi(2i-1)}d_{\pi(2i)}\Big)\\
&=\Box^n+\Box^{n-1}\Big(-\sum_{j=1}^{2n}d_jd_j^\dagger+\frac{1}{2(2n-1)}\sum_{j,k=1}^{2n}d_jd_k\mathrm{Tr}_{jk}\Big)\\
&+\frac{n-1}{2n-1}\Box^{n-2}\sum_{j,k=1}^{2n}d_jd_kd_j^\dagger d_k^\dagger+\sum_{i,j,k=1}^{2n}d_id_jd_k(\dots)\,,
\end{aligned}\tag{137}$$

because of the two identities

$$\sum_{\pi\in\mathfrak{S}_{2n}}\sum_{i=1}^{n}d_{\pi(2i-1)}d_{\pi(2i)}\mathrm{Tr}_{\pi(2i-1)\pi(2i)}=n\sum_{\pi\in\mathfrak{S}_{2n}}d_{\pi(1)}d_{\pi(2)}\mathrm{Tr}_{\pi(1)\pi(2)}\,,$$

and

$$\sum_{\pi\in\mathfrak{S}_{2n}}\sum_{j=1}^{2n-2}\sum_{k=j+1+\varepsilon(j)}^{2n}d_{\pi(j)}d_{\pi(k)}d^\dagger_{\pi(j)}d^\dagger_{\pi(k)}=2n(n-1)\sum_{\pi\in\mathfrak{S}_{2n}}d_{\pi(1)}d_{\pi(2)}d^\dagger_{\pi(1)}d^\dagger_{\pi(2)}\,,$$

supplemented by the fact that for any object s_{jk} symmetric in its indices j and k,

$$\sum_{\pi\in\mathfrak{S}_{2n}}s_{\pi(1)\pi(2)}=(2n-2)!\sum_{j,k=1}^{2n}s_{jk}\,.\tag{138}$$

Finally, by making use of the definition (53) in (137) and going back to the departure equation (134), one obtains by straightforward algebra

$$\begin{aligned}
\mathrm{Tr}_{(12}\dots\mathrm{Tr}_{2n-1\,2n)}\,d_1\dots d_{2n}&=\Box^{n-1}\,\mathrm{F}+\frac{n-1}{2n-1}\Box^{n-2}\sum_{j,k=1}^{2n}d_jd_k\big(-\Box\,\mathrm{Tr}_{jk}+d_j^\dagger d_k^\dagger\big)\\
&+\sum_{i,j,k=1}^{2n}d_id_jd_k(\dots)\,,
\end{aligned}$$

The commutation relation (115) ends the proof the lemma 2.

C Light-cone

The proof of the theorem was already sketched in the appendix A of [42] but we present it here in full details in order to be self-contained. In physical terms, the proof amounts to show that, on-shell fieldstrengths are essentially gauge fields in the light-cone gauge [18, 51].

Indeed, in order to prove the theorem 1 it is convenient to introduce a **light-cone basis** associated with any light-like vector p^μ: we define it as a basis of $\mathbb{R}^{D-1,1}$ such that the light-like direction $+$ is normalized along the vector while the light-like direction $-$ is orthogonal and the remaining space-like directions define the transverse hyperplane $\mathbb{R}^{D-2}$. Hence, $p^+ = 1$ is the only non-vanishing component of the vector p^μ in this basis.

Lemma 3. *Let p^μ be a given vector on the lightcone (defined by $p^2 = 0$) in Minkowski space $\mathbb{R}^{D-1,1}$.*

This vector defines the operators $\overline{p}_i = p_\mu d_i x^\mu$ and their adjoint $\overline{p}_i^\dagger = p_\mu (d_i x^\mu)^\dagger$. Any multiform $|\,\alpha\,\rangle$ of the Fock space $\wedge_{[s]}^{\ell_1,\ell_2,\dots,\ell_s}(\mathbb{R}^D)$ with $\ell_s > 0$ such that

$$\overline{p}_i \mid \alpha \rangle = 0\,, \quad \overline{p}_i^\dagger \mid \alpha \rangle = 0\,, \qquad \forall\, i \in \{1,\dots,s\}$$

reads in the light-cone basis

$$\mid \alpha \rangle \;=\; \overline{p}_1 \overline{p}_2 \dots \overline{p}_s \mid \beta \rangle\,,$$

where $|\,\beta\,\rangle \in \wedge_{[s]}^{\ell_1-1,\ell_2-1,\dots,\ell_s-1}(\mathbb{R}^{D-2})$ is a multiform on the transverse hyperplane $\mathbb{R}^{D-2}$.

Proof of Lemma 3 : As explained in Appendix A.1.2, the space $\wedge_{[s]}(\mathbb{R}^D)$ is isomorphic to a Fock space whose creation operators are the $d_i x^\mu$ and the destruction operators the $(d_i x^\mu)^\dagger$. In the light-cone basis, the condition $\overline{p}_i^\dagger \mid \alpha \rangle = (d_i x^-)^\dagger \mid \alpha\rangle = 0$ states that the ith Fock space $\cong \wedge(\mathbb{R}^D)$ is in the vacuum for the creation operator $d_i x^+$.[13] Thus the occupation number of $d_i x^+$ is zero for all integers i from 1 to s.

On the one hand, the condition $\overline{p}_i \mid \alpha \rangle = d_i x^- \mid \alpha \rangle = 0$ states that the ith Fock space $\cong \wedge(\mathbb{R}^D)$ has maximal occupation number for the creation operator $d_i x^-$. For any fixed i, this operator is Grassmann-odd, thus its maximal occupation number is equal to one. This is true for all integers i, hence $|\,\alpha\,\rangle = d_1 x^- d_2 x^- \dots d_s x^- \mid \beta \rangle$ for some multiform $|\,\beta\,\rangle \in \wedge_{[s]}^{\ell_1-1,\ell_2-1,\dots,\ell_s-1}(\mathbb{R}^D)$. On the other hand, we have also shown that the occupation number is zero for all creation operators $d_i x^+$, thus $|\,\beta\,\rangle$ is transverse and belongs to $\wedge_{[s]}(\mathbb{R}^{D-2})$. □

Proof of Theorem 1 : The on-shell harmonicity of the differential hyperform $\mathcal{K}_{\overline{Y}}$ implies that the massless Klein–Gordon equation $\Box \mathcal{K}_{\overline{Y}} \approx 0$ is obeyed. Let us Fourier transform the tensor field components $\mathcal{K}_{\overline{Y}}(x)$ in such a way that the harmonicity conditions become algebraic.[14] The d'Alembert equation implies that

[13] Because the metric is off-diagonal in the light-cone directions.

[14] Boundary conditions and regularity requirements should be specified when solving PDEs. In Theorem 1, we implicitly assumed that the "ket" $|\,\varphi_Y\,\rangle \in L^2(\mathbb{R}^D) \otimes V_Y^{O(D-2)}$. This choice is convenient because (a) it provides an obvious norm for $\mathcal{H}_Y$, (b) it selects solutions such that $|\,\varphi_Y(x)\,| \stackrel{|x|\to\infty}{\longrightarrow} 0$, and (c) if we consider φ_Y as a temperate distribution (since the "bra" $\langle\,\varphi_Y \mid\, \in \mathcal{S}'(\mathbb{R}^D) \otimes V_Y^{O(D-2)}$) then we are always allowed to convert linear PDEs into algebraic equations by going to the momentum representation.

the support of the Fourier transform $\mathcal{K}_{\overline{Y}}(p)$ is on the mass-shell $p^2 \approx 0$, so that the momentum vector p^μ is light-like on-shell. For each Fourier mode of the tensor field $\mathcal{K}_{\overline{Y}}(p)$ associated with a momentum vector p^μ, let us introduce a light-cone basis. As follows from Lemma 3, the harmonicity conditions impose that the components of each Fourier mode are on-shell equal to

$$\mathcal{K}_{\overline{Y}}(p) \approx \overline{p}_1 \ldots \overline{p}_s \phi_Y(p)$$

for some transverse multiform $\phi_Y(p)$ labeled by the Young diagram Y. It is now easy to prove that the on-shell $O(D-1,1)$-irreducibility conditions of the components $\mathcal{K}_{\overline{Y}}(p)$ imply the $O(D-2)$-irreducibility condition of the components of $\phi_Y(p)$. Therefore the harmonicity conditions restrict the hyperform $\mathcal{K}_{\overline{Y}}(p)$ to carry an UIR of $O(D-2)$ labeled by the Young diagram Y. This conclusion is true for any Fourier mode, therefore it applies to the complete Fourier transform as well. □

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