Dyson Schwinger Equations: From Hopf algebras to Number Theory

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Dyson Schwinger Equations: From Hopf algebras to Number Theory

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Abstract. We consider the structure of renormalizable quantum field theories from the viewpoint of their underlying Hopf algebra structure. We review how to use this Hopf algebra and the ensuing Hochschild cohomology to derive non-perturbative results for the short-distance singular sector of a renormalizable quantum field theory. We focus on the short-distance behaviour and thus discuss renormalized Green functions $G_R(\alpha, L)$ which depend on a single scale $L = \ln q^2/\mu^2$.

1 Introduction

The crucial notion of locality, the structure of Dyson–Schwinger equations and the appearance of mixed motives in the evaluation of Feynman graphs are intimately related. We want to exhibit how these notions come together in quantum field theory. We emphasize the role of Dyson Schwinger equations in this interplay.

Renormalization theory is a time-tested subject put to daily use in many branches of physics. We have seen many of its facets illuminated here at the Fields Institute. In this paper, we focus on its applications in quantum field theory, where a standard perturbative approach is provided through an expansion in Feynman diagrams. In perturbation theory it is mainly a combinatorial problem: determine the needed correction to parameters in the Lagrangian such that the computation allows for finite results in the desired order of perturbation. Whilst the resulting combinatorics of the Bogoliubov recursion, solved by suitable forest formulas, has been known for a long time, the subject regained interest on the conceptual side with the discovery of an underlying Hopf algebra structure behind these recursions.

Non-perturbatively, one faces the equations of motion which the full Green functions have to fulfill. These Dyson–Schwinger equations (DSEs) reflect the self-similarity of amplitudes in quantum field theory, upon studying their skeleton expansion: the computation of propagation or interaction of amplitudes proceeds by

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taking into account that the same propagation or interaction can happen in internal processes. Hence the notion of internal process demands objects which possess internal structure. These turn out to be the celebrated skeleton graphs of a theory. They are crucial in understanding at the same time the algebraic as well as the number-theoretic properties of a field theory.

The main goal of this review is to emphasize one crucial point: How the algebraic structure of the perturbative expansion can lead to non-perturbative solutions, by emphasizing the unifying role of algebraic structures in understanding local renormalization as well as the above self-similar structure of Green functions.

We will also point out along the way how internal symmetries are reflected in this set-up. This is a very recent insight [1], which we can only mention in passing. It goes a long way in establishing a quantum field theory which stands its own ground: instead of deriving quantum gauge invariance from the differential geometry of a classical gauge field theory, we obtain it as a consequence of the algebraic structures of a local quantum theory, and the classical Lagrangian is a derived quantity. It is indeed obtained from the residues of Feynman graphs which are primitive elements in the Hopf algebra of renormalization, an old result in field theory [2].

Finally, in the course of our discussion we will give an idea where one can see signs of universality from this set-up. It is at the time of writing an open research problem to connect the rather systematic account to DSEs exhibited in this talk to other areas where renormalization theory and universality are likewise fundamental notions, as these proceedings hopefully demonstrate.

We first have to review the algebraic structure of quantum field theory, and with it the Hopf algebra structure of a perturbative expansion. From a strictly perturbative viewpoint, this is summarized in this volume in the paper Ebrahimi-Fard and Guo [3], with emphasis given to Rota–Baxter algebras. Our own summary below follows [4]. We hence will be short in our discussion of the perturbative viewpoint, and aim at a qualitative discussion of the above points, exemplified in the much stressed example of the propagator in massless Yukawa theory [5, 6]. Hochschild cohomology then leads the way from perturbative physics to non-perturbative results. We summarize here results detailed in previous works on the structure of DSEs [1, 6, 7, 5, 8, 9] and will end in discussing shortly the number-theory of skeleton graphs, which found a mathematical interpretation as periods of mixed motives recently [10]. We will be content in exhibiting how those primitive graphs mentioned above provide the connection to algebraic geometry and motives.

1.1 A splitting of amplitudes. We assume we work in a renormalizable quantum field theory which provides a finite set $\mathcal{R} \subset \mathcal{A}$ of amplitudes which need renormalization. Here, $\mathcal{A}$ is the set of all amplitudes in a given theory. We hence work with a theory which, from a Lagrangian perspective, provides a finite number of parameters which need renormalization. Note that our notion of an amplitude is such that each monomial in said Lagrangian corresponds to an amplitude. In comparison with the standard terminology, we distinguish the various form-factors provided by a given Green function and regard each corresponding structure function as a Green function. For example the vertex function for the photon decay into an electron-positron pair in quantum electrodynamics (QED) has a form-factor decomposition which provides twelve independent structure functions [11]. Only one of them needs renormalization and corresponds to the term $\bar{\psi}\not{A}\psi$ in the QED

Lagrangian. We denote the projection of the QED vertex function onto this form-factor as the amplitude $\in \mathcal{R}$ corresponding to this term in the Lagrangian.

A general amplitude $a \in \mathcal{A}$ is thus specified by a particular form-factor and the quantum numbers of particles participating in a scattering type experiment contributing to that form-factor. It suffices to consider one-particle irreducible (1PI) Green functions G^a for each amplitude a. The knowledge of these Green function suffices to determine the physics by standard techniques.

We assume that the amplitudes allow to specify an integer $n = n(a)$ which gives the number of external legs. We let $\mathcal{M}_a$ be the set of all 1PI graphs contributing to the amplitude a. We remind the reader that a 1PI graph is a graph which remains connected after removal of any one of its internal edges. By $|\Gamma|$ we denote the number of independent loops in Γ. By $\mathrm{sym}(\Gamma)$ we denote the rank of the automorphism group of the graph.

We then have in general

$$G^a = 1 \pm \sum_{\Gamma \in \mathcal{M}_a} \alpha^{|\Gamma|} \frac{\phi(\Gamma)}{\mathrm{sym}(\Gamma)} = 1 \pm \sum_{k \geq 1} \alpha^k \phi(c_k^a), \tag{1.1}$$

so that

$$c_k^a = \sum_{\Gamma \in \mathcal{M}_a,\, |\Gamma| = k} \frac{\Gamma}{\mathrm{sym}(\Gamma)}, \tag{1.2}$$

is the sum over all 1PI graphs of order k contributing to the amplitude a. In the above, we take the plus sign if $n(a) \geq 3$ and the minus sign for $n(a) = 2$. We discard tadpole amplitudes, $n(a) = 1$, and vacuum amplitudes, $n(a) = 0$. A main point of the subsequent discussion will concern the structure of the sum

$$\Gamma^a = \mathbb{I} \pm \sum_{\Gamma \in \mathcal{M}_a} \alpha^{|\Gamma|} \frac{\Gamma}{\mathrm{sym}(\Gamma)}, \tag{1.3}$$

in the above.

The set of amplitudes $\mathcal{A}$ decomposes for a renormalizable theory into two disjoint subsets

$$\mathcal{A} = \mathcal{A}_+ \cup \mathcal{R}. \tag{1.4}$$

Here, to stress it once more, $\mathcal{R}$ is the set of amplitudes for form-factors of one-particle irreducible graphs which need renormalization. On the other hand, $\mathcal{A}_+$ is the set of amplitudes which are overall finite.

Those amplitudes behave very differently. For an element $r \in \mathcal{R}$ we can write

$$\Gamma^r = \mathbb{I} \pm \sum_{\Gamma \in \mathcal{M}^r} \alpha^{|\Gamma|} \frac{\Gamma}{\mathrm{sym}(\Gamma)} = \mathbb{I} \pm \sum_{k \geq 1} \alpha^k B_+^{k;r}(\Gamma^r Q^{n_r k}), \tag{1.5}$$

where $B_+^{k;r}$ are Hochschild one-cocycles as explained below. The quantity Q is intimately related to the notion of an invariant charge [12]. For momentum space Feynman rules ϕ we have

$$\frac{\partial_L \ln \phi(Q^{n_r})}{\partial L}|_{L=0} = \beta(\alpha), \tag{1.6}$$

where β is the β-function of the theory which describes the running of the charge. Hence Q^{n_r} is a monomial in the Γ^r, $r \in \mathcal{R}$, or their inverses. n_r is an integer such that $Q^{n_r} - \mathbb{I}$ is of order one in α. Finally, $L = \ln q^2/\mu^2$ sets the scale for that running charge.

From (1.5), we have thus a self-similar recursive system determining the formal sums Γ^r, $r \in \mathcal{R}$ in terms of themselves and the action of suitable maps $B_+^{k;r}$. The study of the maps $B_+^{k;r}$ is crucial for a QFT. The Hopf algebra elements $B_+^{k;r}(\mathbb{I})$ provide the skeleton graphs underlying the DSEs and are the terms which drive the recursion which leads to the full theory, and which connects QFT to motives [10].

In contrast, for $a \in \mathcal{A}_+$, we have

$$\Gamma^a = \mathbb{I} \pm \sum_{\Gamma \in \mathcal{M}^a} \alpha^{|\Gamma|} \frac{\Gamma}{\mathrm{sym}(\Gamma)} = \mathbb{I} \pm \sum_k B_+^{k;a}(M_a Q^{n_r k}), \tag{1.7}$$

where M_a is another monomial in Γ^s, $s \in \mathcal{R}$, or their inverses. Hence, amplitudes from the set $\mathcal{A}_+$ are determined from the knowledge of the ones in $\mathcal{R}$.

Hence, we focus on amplitudes $r \in \mathcal{R}$ where it turns out that the self-similarity and the properties of the Hochschild one-cocycles $B_+^{k;r}$ are paramount to their understanding, as well with respect to perturbative renormalization as with respect to nonperturbative physics.

1.2 The example of massless QED. Massless quantum electrodynamics starts from the Lagrangian

$$\mathcal{L}_{\mathrm{QED}} = \bar{\psi}\left[\partial\!\!\!/ + A\!\!\!/\right]\psi + \frac{1}{4} F \cdot F. \tag{1.8}$$

It has three monomials and as a renormalizable theory we hence can work with a set

$$\mathcal{R}_{\mathrm{QED}} = \{ \; \text{} \; , \; \text{  } \; , \; \text{} \; \}, \tag{1.9}$$

in standard notation.

The Green function for the vertex has an elaborate form-factor expansion (with $n($  $) = 3$),

$$G_\mu \text{} = \phi(\Gamma \text{}) = \gamma_\mu F_1(p_1^2, p_2^2, p_3^3, \alpha, \mu) + \cdots, \tag{1.10}$$

where it is only F_1 which needs renormalization. We let P_1 be the projector onto this form factor.

The Green functions for these monomials are then given as

$$G^{\bar{\psi} A\!\!\!/ \psi} = 1 + \sum_{\Gamma \in \mathcal{M}} \text{  } \alpha^{|\Gamma|} \mathrm{P}_1 \phi(\Gamma), \tag{1.11}$$

$$G^{\bar{\psi} \partial\!\!\!/ \psi} = 1 - \sum_{\Gamma \in \mathcal{M}} \text{  } \alpha^{|\Gamma|} \mathrm{P}_2 \phi(\Gamma), \tag{1.12}$$

$$G^{F \cdot F} = 1 - \sum_{\Gamma \in \mathcal{M}} \text{} \alpha^{|\Gamma|} \mathrm{P}_3 \phi(\Gamma). \tag{1.13}$$

The projectors P_2 onto the kinetic term of the inverse fermion propagator is redundant in a massless theory and only needed in a massive theory, and the projector P_3 onto the transversal part of the inverse photon propagator is likewise redundant in the Landau gauge where the tree level term remains transversal. We often write $G^r = \phi(\Gamma^r)$ as a shorthand for such equations, with suitable projectors understood in the application of the Feynman rules ϕ.

Note that the Green function for the full vertex, containing eleven further amplitudes of the set $\mathcal{A}_+$, is obtained by omitting the projector P_1 in the above.

We stick to the notion that each monomial in the Lagrangian has its own Green function, while all other amplitudes belong to the set $\mathcal{A}_+$.

The goal is to calculate the corresponding 1PI Green functions order by order in the fine-structure α of the theory, by applying Feynman rules to these 1PI graphs of a renormalizable theory under consideration. There are two problems here: each single graph is mapped by the Feynman rules to an ill-defined quantity, and furthermore, after labourosly eliminating these divergences, the resulting series is not of the convergent type.

Progress with both problems is possible thanks to the algebraic structures underlying Feynman graphs using the Lie and Hopf algebras discussed below.

As we said before, 1PI Green functions are parameterized by the quantum numbers, -masses, momenta, spin and such-, of the particles participating in the scattering process under consideration. Physicists denote propagating particles by lines, and the perturbative expansion in terms of graphs is organized such that external half-lines denote the particles parameterizing the Green function under consideration, while internal edges and vertices describe internal propagations and vertices.

Note that the Lagrangian L of massless quantum electrodynamics is obtained accordingly as

$$L = \hat{\phi}(\ \longrightarrow\ )^{-1} + \hat{\phi}(\ \overset{\S}{\longrightarrow}\ ) + \hat{\phi}(\ \sim\ )^{-1} = \bar{\psi}\partial\!\!\!/\psi + \bar{\psi}A\!\!\!/\psi + \frac{1}{4}F\cdot F, \tag{1.14}$$

where $\hat{\phi}$ are coordinate space Feynman rules. Here, inversion like $\hat{\phi}(\ \longrightarrow\ )^{-1}$ takes account of the fact that monomials quadratic in the fields refer to inverse propagators, while $\longrightarrow$ and $\sim$ refer to the free propagators of QED.

This is not to say that there are no other Green function in quantum electrodynamics. But we focus here on the Green functions which need renormalization, and this, for a renormalizable field theory, gives us a finite set of terms to be considered. It is indeed the recursive self-similar nature of amplitudes from the set $\mathcal{R}$ which drives the need for renormalization. That the unrenormalized amplitudes suffer from short-distance divergences is a mere accident of perturbation theory which disappears once the non-perturbative fixpoint equations of motion, the DSEs, have been taken into account.

The unrenormalized momentum space Feynman rules ϕ assign to a graph a function ($\Gamma^{[0]}$ and $\Gamma^{[1]} = \Gamma^{[1]}_{\rm int} \cup \Gamma^{[1]}_{\rm ext}$ being the set of vertices v and internal and external edges e of Γ) of the form

$$\phi(\Gamma)(\{p_f\}) = \int \prod_{v\in\Gamma^{[0]}} \phi(v)\delta^{(4)}\left(\sum_{f\ \text{incident}\ v} k_f\right) \prod_{e\in\Gamma^{[1]}_{\rm int}} \text{Prop}(k_e)\frac{d^4k_e}{4\pi^2}. \tag{1.15}$$

They are determined from the knowledge of the Feynman rules for interaction vertices v and the knowledge of free covariances $\text{Prop}(k_e)$ for each internal edge e. In the above form (1.15) a Dirac mass

$$\delta^{(4)}\left(\sum_{f\in\Gamma^{[1]}_{\rm ext}} p_f\right), \tag{1.16}$$

will factor out of the expression for momentum conservation for $|\Gamma^{[1]}_{\rm ext}| = n(a)$, $\Gamma \in \mathcal{M}_a$, external momenta p_f.

As a result, formally the unrenormalized Green function is obtained as

$$G^r_{\rm u}(\alpha;\{p_f\};z) = \phi(\Gamma^r)\,(\alpha;\{p_f\};z)\,, \tag{1.17}$$

where we have introduced a suitably chosen regulator z, needed in perturbation theory but not non-perturbatively.

Note that in (1.15) the four-dimensional Dirac mass for each internal vertex guarantees momentum conservation at each such vertex and restricts the number of four-dimensional integrations to the number of independent cycles in the graph. These integrals suffer from UV singularities which render the integration over the momenta in internal cycles ill-defined. We remind the reader that the problem persists in coordinate space, where one confronts the continuation of products of distributions to regions of coinciding support. We restrict ourselves here to a discussion of the situation in momentum space and refer the reader to the literature for the situation in coordinate space [13].

The problem of perturbative renormalization is to make sense out of this situation term by term: We have to determine invertible series $Z^r(\alpha,z)$ for all $r\in\mathcal{R}$, hence in the parameters of the Lagrangian

$$L = \sum_{r\in\mathcal{R}} \hat{\phi}^{\pm 1}(r), \tag{1.18}$$

such that the modified Lagrangian

$$\tilde{L} = \sum_{r\in\mathcal{R}} Z^r(\alpha,z)\,\hat{\phi}^{\pm 1}(r), \tag{1.19}$$

produces a perturbation series in graphs which allows for the removal of the regulator z. Again, $\hat{\phi}$ are the Feynman rules in coordinate space, with the understanding that they evaluate an amplitude r to the corresponding tree level term.

Let us first describe how this transition is achieved using the Lie- and Hopf algebra structure of the perturbative expansion which we summarize below:

- Decide on the free fields and local interactions of the theory, appropriately specifying quantum numbers (spin, mass, flavor, color and such) of fields, restricting interactions so as to obtain a renormalizable theory.
- Determine the Feynman rules for free propagators from free field theory. The Feynman rules for interaction vertices then follow from locality: the ability to compensate by local counterterms actually fixes the structure of interaction vertices modulo the absolute values of masses and charges which parameterize the chosen theory.
- Consider the set of all 1PI graphs with edges corresponding to those free-field propagators. Together with the vertices this allows to construct a pre-Lie algebra of graph insertions. Anti-symmetrize this pre-Lie product to get a Lie algebra $\mathcal{L}$ of graph insertions and define the Hopf algebra $\mathcal{H}$ which is dual to the enveloping algebra $\mathcal{U}(\mathcal{L})$ of this Lie algebra.
- Realize that the coproduct and antipode of this Hopf algebra give rise to the forest formula which generates local counterterms upon introducing a suitable Rota–Baxter map, a renormalization scheme in physicists' parlance.
- Use the Hochschild cohomology of this Hopf algebra to show that you can absorb singularities in local counterterms, hence in the form described in (1.19) above.

This gives a rather satisfying account of renormalization theory. As an added bonus, if you work with an an intermediate complex regulator these steps can be summarized as to construct the Birkhoff decomposition of the unrenormalized Feynman rules, regarded as an element in $\mathrm{Spec}(\mathcal{G})$, the character group of this Hopf algebra [15, 16, 17]. This settles perturbative renormalization.

The situation is even better when we look at the structure of the full non-perturbative Green functions and concentrate on the short-distance behaviour. Here, a regulator is not at all needed. Indeed, in recent years, we have learned how to make progress with non-perturbative physics [1, 5, 7, 8], and continue as follows:

- Show that the elements c_k^r form a sub Hopf algebra.
- Determine the Hochschild one-cocycles $B_+^{k;a}$ for this sub Hopf algebra from the primitive elements of the Hopf algebra.
- Construct the DSEs as fixpoint equations with the elements $\phi(B_+^{k;a}(\mathbb{I}))$ as kernels corresponding to the Dyson skeleton expansion.
- For the set $\mathcal{R}$, determine the recursion relations which follow from the renormalization group, applying the consequences of the representations of one-parameter groups of automorphisms of the above Hopf algebras [17] in this set-up.
- Solve the DSE for the remaining unknown anomalous dimensions.
- Aim to establish functional equations which connect an expansion in α to an expansion in $1/\alpha$.

Very recently we gained insight how to carry such a program through [6], and will exhibit below one example where it is carried to the end.

What is encouraging are the structural features one can establish for any renormalizable field theory which this program exhibits.

A first observation is that non-perturbatively, no regulator is needed. The anomalous dimensions of propagators and vertices at zero momentum transfer self-regulate the theory. Similar ideas for the conformal case of a vanishing β-function have been employed in the past, and are reviewed in this volume [18].

A second feature is that upon organizing the DSEs in terms of Hochschild one-cocycles, the algebraic structure of the forest formulas remain invariant upon addition of more one-cocycles. This allows for a much more sensible expansion in terms of Hochschild cohomology than the usual truncations done in DSEs, and goes hand in hand with a decomposition of field theory into periods of more and more complex motives. In particular, the decomposition into the one-cocycles $B_+^{k;r}(\mathbb{I})$ and the restriction to a subset of these is a factorization of field theory reminiscent of a decomposition of a zeta function into its Euler factors. This rather far-fetched analogy [19] will be elaborated elsewhere.

From the viewpoint of DSEs, the Birkhoff decomposition becomes a decomposition into a homogenous part and an inhomogenity determined by the chosen renormalization condition. It thus survives the transition to non-perturbative physics.

If the charge $\phi(Q^{n_r})$ gives rise to a vanishing β function, the corresponding Hopf algebra governing the DSEs becomes cocommutative, the dual Lie algebra is abelian and we can solve nonperturbatively by a scaling solution and a simple Mellin transform of the one-cocycles $B_+^{k;r}$ [7, 9].

In the general case, starting from this Mellin transform, one realizes that the renormalization group allows to work with a very simplified form of the coproduct:

we can project on both sides to terms in the span of linear generators. Hence we find two extremely useful simplifications: the restriction to sub Hopf algebras which are much easier to define than the full graph algebras, and the linearization in the use of the coproduct. Together with the fact that the primitives $B_+^{k;r}(\mathbb{I})$ exhibit conformal invariance at the renormalization point, which severely restricts the form of their Mellin transform, this goes a long way, I set my hopes for the future, in understanding better the phenomenon of universality, finding common anomalous dimensions in very different physical systems. But before we can exhibit these features in the promised example, we have to start with the structure of Feynman graphs.

2 Lie- and Hopf algebras of graphs

To capture the structure of a renormalizable quantum field theory, we organize it in terms of graphs. More formally, these graphs Γ will index generators δ_Γ of a Hopf algebra, so that as an algebra it is the free commutative algebra on generators indexed by the 1PI graphs of the theory. We will also consider pre-Lie and Lie algebras, with their generators Z_Γ indexed by the very same graphs. We will often simply write Γ for δ_Γ or Z_Γ when the context is clear.

All algebras are supposed to be over some field $\mathbb{K}$ of characteristic zero, associative and unital, and similarly for co-algebras. The unit (and by abuse of notation also the unit map) will be denoted by $\mathbb{I}$, the co-unit map by $\bar{e}$. Algebra homomorphisms are supposed to be unital. A bialgebra $(A = \bigoplus_{i=0}^\infty A_i, m, \mathbb{I}, \Delta, \bar{e})$ is called graded connected if $A_i A_j \subset A_{i+j}$ and $\Delta(A_i) \subset \bigoplus_{j+k=i} A_j \otimes A_k$, and if $\Delta(\mathbb{I}) = \mathbb{I}\otimes\mathbb{I}$ and $A_0 = k\mathbb{I}$, $\bar{e}(\mathbb{I}) = 1 \in \mathbb{K}$ and $\bar{e} = 0$ on $\bigoplus_{i=1}^\infty A_i$. We call $\ker \bar{e}$ the augmentation ideal of A and denote by P the projection $A \to \ker \bar{e}$ onto the augmentation ideal, $P = id - \mathbb{I}\bar{e}$.

Note that the augmentation ideal contains the quantum world: all graphs containing loops belong to the augmentation ideal. The classical world is captured by $A_0 = k\mathbb{I}$.

Furthermore, we use Sweedler's notation $\Delta(h) = \sum h' \otimes h''$ for the coproduct. We define

$$\mathrm{Aug}^{(k)} = \left(\underbrace{P \otimes \cdots \otimes P}_{k \text{ times}} \right) \Delta^{k-1},\ A \to \{\ker \bar{e}\}^{\otimes k}, \tag{2.1}$$

as a map into the k-fold tensorproduct of the augmentation ideal. We let $A^{(k)} = \ker \mathrm{Aug}^{(k+1)} / \ker \mathrm{Aug}^{(k)}$, $\forall k \geq 1$. All bialgebras considered here are bigraded in the sense that

$$A = \bigoplus_{i=0}^\infty A_i = \bigoplus_{k=0}^\infty A^{(k)}, \tag{2.2}$$

where $A_k \subset \oplus_{j=1}^k A^{(j)}$ for all $k \geq 1$. $A_0 \simeq A^{(0)} \simeq \mathbb{K}$.

The first construction we have to study is the pre-Lie algebra structure of 1PI graphs.

2.1 The Pre-Lie Structure. We are considering 1PI Feynman graphs. 1PI graphs are naturally graded by their number of independent loops, the rank of their first homology group $H_{[1]}(\Gamma, \mathbb{Z})$. We already wrote $|\Gamma|$ for this degree of a graph Γ.

As we stressed several times, a crucial notion is the external leg structure of a graph. It determines the relevant amplitude to which the graph contributes. The

relevant contribution to the counterterm in the Lagrangian is then obtained by evaluating the Feynman rules on the tree level graph which corresponds to that amplitude. It is thus profitable to have a map which assigns to a given graph that tree level graph. This map $\mathbf{res}(\Gamma)$ is easily described: for any graph, we shrink all its internal edges to a point, so that a single vertex, with the external edges attached as half edges, remains. This is the desired tree level graph. For a graph with two external edges, the result is simply the two-point vertex of a form-factor of a single inverse edge, otherwise, if $n(r) > 2$, we get an interaction term in the Lagrangian.

When we evaluate a graph by the Feynman rules, we hence obtain a result in the form

$$\phi(\Gamma) = \phi(\mathbf{res}(\Gamma))X + Y, \tag{2.3}$$

where X is a superficially divergent Green function and Y contributes to amplitudes in $\mathcal{A}_+$.

We will see below how to construct a QFT from primitive graphs Γ. Primitive graphs have no divergent subgraphs which need renormalization. For such graphs Γ, the above decomposition reads

$$\phi(\Gamma) = \phi(\mathbf{res}(\Gamma))\left[\frac{r}{z}\right] + \text{ finite terms} + Y, \tag{2.4}$$

for some regulator z. Here, r is the numerical residue of the graph. Thus,

$$\lim_{z\to 0} z\phi(\Gamma) = r\phi(\mathbf{res}(\Gamma)), \tag{2.5}$$

and hence the name residue also for the map which shrinks all internal edges. Note that $|\mathbf{res}(\Gamma)| = 0$. We emphasize that the set $\mathcal{M}_r$ contains only 1PI graphs Γ such that $|\Gamma| > 0$. The residue $\mathbf{res}(\Gamma)$ is not an element of this set, and hence is no generator in our Hopf algebra. The elements of degree zero are strictly given by the scalars so that we have a connected Hopf algebra, which is justified by the very fact that each Green function is a mere structure function corresponding to an amplitude in the Lagrangian.

Having specified free quantum fields and local interaction terms between them, one immediately obtains the set of 1PI graphs, and can consider for a given external leg structure r the set of graphs with that external leg structure. For a renormalizable theory, we can define a superficial degree of divergence

$$\omega = \sum_{r\in\Gamma^{[1]}_{\text{int}}\cup\Gamma^{[0]}} \omega_r - 4|H_{[1]}(\Gamma,\mathbb{Z})|, \tag{2.6}$$

for each such external leg structure: $\omega(\Gamma) = \omega(\Gamma')$ if $\mathbf{res}(\Gamma) = \mathbf{res}(\Gamma')$, all graphs with the same external leg structure have the same superficial degree of divergence -the hallmark of a renormalizable theory-, and only for a finite number of distinct external leg structures $r \in \mathcal{R}$ will this degree indeed signify a divergence.

For the pre-Lie structure we define a bilinear operation

$$\Gamma_1 * \Gamma_2 = \sum_{\Gamma} n(\Gamma_1, \Gamma_2; \Gamma)\Gamma, \tag{2.7}$$

where the sum is over all 1PI graphs Γ. Here, $n(\Gamma_1, \Gamma_2; \Gamma)$ is a section coefficient which counts the number of ways a subgraph Γ_2 can be reduced (by shrinking it to its residue in the above sense) to a point in Γ such that Γ_1 is obtained. The above sum is evidently finite as long as Γ_1 and Γ_2 are finite graphs, and the graphs

which contribute necessarily fulfill $|\Gamma| = |\Gamma_1| + |\Gamma_2|$ and $\mathbf{res}(\Gamma) = \mathbf{res}(\Gamma_1)$, as $\mathbf{res}(\Gamma_1 \star \Gamma_2) = \mathbf{res}(\Gamma_1)$ by construction.

One then has:

Theorem 2.1 *The operation $*$ is pre-Lie:*

$$[\Gamma_1 * \Gamma_2] * \Gamma_3 - \Gamma_1 * [\Gamma_2 * \Gamma_3] = [\Gamma_1 * \Gamma_3] * \Gamma_2 - \Gamma_1 * [\Gamma_3 * \Gamma_2]. \tag{2.8}$$

Together with the corresponding Lie algebra $\mathcal{L}$ one is led to consider the dual of its universal enveloping algebra $\mathcal{U}(\mathcal{L})$ using the theorem of Milnor and Moore and the above grading by the loop number. This graded dual, obtained from the usual Kronecker pairing, is a Hopf algebra $\mathcal{H}(m, \mathbb{I}, \Delta, \bar{e})$ which is commutative but not co-commutative. $\mathcal{H}$ is a graded commutative Hopf algebra which suffices to describe perturbative renormalization theory [14, 15, 16].

2.2 Multiplicative subtraction. The above algebra structures are available once one has decided on the set of 1PI graphs of interest, delivering the renormalization of any such chosen local quantum field theory. As to be expected, gauge theories provide particular properties with respect to the appearance of sub Hopf algebras which explain the Slavnov Taylor identities for the couplings [1], while the identities related to kinematics of Green functions expressing transversality of physical degrees of freedom for the gauge boson, are reflected in the presence of Hopf ideals [20] and a semi-direct product structure between radiation and matter [21].

From the above, one-particle irreducible graphs Γ provide the linear generators δ_Γ, with span $\mathcal{H}_{\rm lin} = {\rm span}(\delta_\Gamma)$, of the Hopf algebra $\mathcal{H} = \oplus_{i=0}^{\infty}\mathcal{H}_i$. Disjoint union of graphs provides the commutative product. We let P be the projector into the augmentation ideal $\mathcal{H}_{\rm aug} = \oplus_{i=1}^{\infty}\mathcal{H}_i$ and $P_{\rm lin}$ be the projector into $\mathcal{H}_{\rm lin} \subset \mathcal{H}_{\rm aug}$.

Let now Γ be a 1PI graph. We find the Hopf algebra $\mathcal{H}$ as described above to have a co-product explicitely given as $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$:

$$\Delta(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma, \tag{2.9}$$

where the sum is over all unions of 1PI superficially divergent proper subgraphs (hence subgraphs with residue in $\mathcal{R}$ each), and we extend this definition to products of graphs so that we get a bi-algebra.

Having a co-product, two further structure maps of $\mathcal{H}$ are immediate, the co-unit and the antipode. The co-unit $\bar{e}$ vanishes on any non-trivial Hopf algebra element, $\bar{e}(\mathbb{I}) = 1$, $\bar{e}(X) = 0$. The antipode is

$$S(\Gamma) = -\Gamma - \sum_{\gamma \subset \Gamma} S(\gamma)\Gamma/\gamma. \tag{2.10}$$

Note that for each term in the restricted sum $[P \otimes P]\Delta(\Gamma) = \sum_i \Gamma'_{(i)} \otimes \Gamma''_{(i)}$ we have unique gluing data G_i such that

$$\Gamma = \Gamma''_{(i)} \leftarrow_{G_i} \Gamma'_{(i)}, \ \forall i. \tag{2.11}$$

These gluing date describe the necessary bijections to glue the components $\Gamma'_{(i)}$ back into $\Gamma''_{(i)}$ so as to obtain Γ: using them, we can reassemble the whole from its parts. Each possible gluing can be interpreted as a composition in the insertion operad of Feynman graphs [22, 23]. This gluing has a faithful representation as an iteration of Feynman integrals, upon replacing the δ-function for momentum conservation at a given insertion point by the expression for the inserted graph,

and identifying the continuous quantum numbers at external legs of the inserted graphs with the momentum labels of the edges attached to the removed vertex, in accordance with the gluing data. It is one of the challenges with regard to analytic progress with Feynman graphs to understand this operation as a generalization of the notion of iterated integrals and their algebraic structure.

Disjoint scattering processes give rise to independent amplitudes, so one is led to the study of characters of the Hopf algebra, maps $\phi : \mathcal{H} \to V$ such that $\phi \circ m = m_V(\phi \otimes \phi)$.

Such maps assign to any element in the Hopf algebra an element in a suitable target space V. The study of tree-level amplitudes in lowest order perturbation theory justifies to assign to each edge a propagator and to each elementary scattering process a vertex which define the Feynman rules $\phi(\mathbf{res}(\Gamma))$ and the underlying Lagrangian, on the level of residues (in the graphical sense above) of these very graphs. As graphs themselves are constructed from edges and vertices, such residues, one is led to assign to each Feynman graph an evaluation in terms of an integral over the continues quantum numbers assigned to edges or vertices, which leads to the familiar integrals over momenta in closed loops mentioned before, and hence leads to the Feynman rules (1.15), $\phi \in \mathrm{Spec}(\mathcal{G})$, as before.

Next, we choose a map $R : V \to V$, from which we obviously demand that is does not modify the UV-singular structure, and furthermore that it obeys

$$R(xy) + R(x)R(y) = R(R(x)y) + R(xR(y)), \tag{2.12}$$

an equation which guarantees the multiplicativity of renormalization and is at the heart of the Birkhoff decomposition which emerges below: it tells us that elements in V split into two parallel subalgebras given by the image and kernel of R. Such Rota–Baxter algebras play a role for associative algebras which is similar to the role Yang–Baxter algebras play for Lie algebras. The structure of these algebras allows to connect renormalization theory to integrable systems [3]. The situation is then remarkably similar to the factorization and Birkhoff decomposition in many studies of dynamical systems or problems in condensed matter theory, see for example Korepin et.al. in this volume [24]. Also, most of the results obtained initially for a specific renormalization scheme like minimal subtraction can be obtained in general upon a structural analysis of the corresponding Rota–Baxter algebras.

In renormalization theory we define a further character S_R^ϕ which deforms $\phi \circ S$ slightly and delivers the counterterm for Γ in the renormalization scheme R:

$$S_R^\phi(\Gamma) = -Rm_V(S_R^\phi \otimes \phi \circ P)\Delta = -R[\phi(\Gamma)] - R\left[\sum_{\gamma \subset \Gamma} S_R^\phi(\gamma)\phi(\Gamma/\gamma)\right], \tag{2.13}$$

a slight modification of the inverted Feynman rules

$$\phi \circ S = m_V(S \circ \phi \otimes \phi \circ P)\Delta = -\phi(\Gamma) - \sum_{\gamma \subset \Gamma} \phi \circ S(\gamma)\phi(\Gamma/\gamma). \tag{2.14}$$

Note that $S_R^\phi \in \mathrm{Spec}(\mathcal{G})$ thanks to (2.12).

The classical results of renormalization theory follow using this group structure: We obtain the renormalization of Γ by the application of a renormalized character

$$S_R^\phi \star \phi(\Gamma) = m_V(S_R^\phi \otimes \phi)\Delta \tag{2.15}$$

and Bogoliubov's $\bar{R}$ operation as

$$\bar{R}(\Gamma) = m_V(S_R^\phi \otimes \phi)(\mathrm{id} \otimes P)\Delta(\Gamma) = \phi(\Gamma) + \sum_{\gamma \subset \Gamma} S_R^\phi(\gamma)\phi(\Gamma/\gamma), \tag{2.16}$$

so that we have

$$S_R^\phi \star \phi(\Gamma) = \bar{R}(\Gamma) + S_R^\phi(\Gamma). \tag{2.17}$$

Here, $S_R^\phi \star \phi$ is an element in the group of characters $\mathcal{G}$ of the Hopf algebra, with the group law given by the convolution

$$\phi_1 \star \phi_2 = m_V \circ (\phi_1 \otimes \phi_2) \circ \Delta, \tag{2.18}$$

so that the co-product, co-unit and co-inverse (the antipode) give the product, unit and inverse of this group, as befits a Hopf algebra. This Lie group has the previous Lie algebra $\mathcal{L}$ of graph insertions as its Lie algebra: $\mathcal{L}$ exponentiates to $\mathcal{G}$. This finishes perturbative renormalization theory. Further results in particular with regard to the vicinity of $\phi \in \mathrm{Spec}(\mathcal{G})$ as tested by one-parameter groups of automorphisms of the Hopf algebra were obtained in [17]. This illuminates in particular the renormalization group, whose ubiquitous applications are discussed for example in this volume [25].

3 The role of Hochschild cohomology and nonperturbative physics

The Hochschild cohomology of the combinatorial Hopf algebras which we discuss here plays three major roles in quantum field theory: it allows to prove locality from the accompanying filtration by the augmentation degree coming from the kernels $\ker \mathrm{Aug}^{(k)}$, it allows to write the quantum equations of motion in terms of the Hopf algebra primitives, elements in $\mathcal{H}_{\mathrm{lin}} \cap \{\ker \mathrm{Aug}^{(2)} / \ker \mathrm{Aug}^{(1)}\}$, and identifies the relevant sub-Hopf algebras formed by time-ordered products. Before we discuss these properties, let us first introduce the relevant Hochschild cohomology [15].

3.1 Hochschild cohomology of bialgebras. Let $(A, m, \mathbb{I}, \Delta, \epsilon)$ be a bialgebra, as before. We regard linear maps $L : A \to A^{\otimes n}$ as n-cochains and define a coboundary map b, $b^2 = 0$, by

$$bL := (\mathrm{id} \otimes L) \circ \Delta + \sum_{i=1}^{n} (-1)^i \Delta_i \circ L + (-1)^{n+1} L \otimes \mathbb{I}, \tag{3.1}$$

where Δ_i denotes the coproduct applied to the i-th factor in $A^{\otimes n}$, which defines the Hochschild cohomology of A.

For the case $n = 1$, (3.1) reduces to, for $L : A \to A$,

$$bL = (\mathrm{id} \otimes L) \circ \Delta - \Delta \circ L + L \otimes \mathbb{I}. \tag{3.2}$$

Note that for $h \in A^{(k)}$, we have $L(h) \in A^{k+1}$.

The category of objects (A, C) which consists of a commutative bialgebra A and a Hochschild one-cocycle C on A has an initial object $(\mathcal{H}_{\mathrm{rt}}, B_+)$, where $\mathcal{H}_{\mathrm{rt}}$ is the Hopf algebra of (non-planar) rooted trees and the closed but non-exact one-cocycle B_+ grafts a product of rooted trees together at a new root [15].

In Feynman graph Hopf algebras we will consider many one-cocycles $B_+^{k;r}$. The closedness of these $B_+^{k;r}$ in the context of Feynman graph Hopf algebras will turn out to be crucial for what follows. In passing, we note that such one-cocycles transpose

to the universal enveloping algebra on the dual side. With $B_+ : \mathcal{H} \to \mathcal{H}_{\rm lin}$ a one-cocycle, it turns out that by the standard Kronecker pairing [15, 16], the dual map $B_+^\vee : \mathcal{L} \to U(\mathcal{L})$ is a one-cocycle in Lie algebra cohomology.

3.2 The roles of Hochschild cohomology. The Hochschild cohomology of the Hopf algebras of 1PI graphs sheds light on the structure of 1PI Green functions. To determine the relevant Hochschild one-cocycles of a Feynman graph Hopf algebra $\mathcal{H}$, one determines first the primitives graphs γ of the Hopf algebra, which by definition fulfill

$$\Delta(\gamma) = \gamma \otimes \mathbb{I} + \mathbb{I} \otimes \gamma. \tag{3.3}$$

Using the pre-Lie product above, one then determines maps (possibly first on suitably chosen sub Hopf algebras based on graph which have γ as a cograph)

$$B_+^\gamma : \mathcal{H} \to \mathcal{H}_{\rm lin} \tag{3.4}$$

such that

$$B_+^\gamma(h) = B_+^\gamma(h) \otimes \mathbb{I} + \left(\mathrm{id} \otimes B_+^\gamma\right) \Delta(h), \tag{3.5}$$

where $B_+^\gamma(h) = \sum_\Gamma n_+(\gamma, h, \Gamma)\Gamma$. The new section coefficients $n_+(\gamma, h, \Gamma)$ are closely related [1] to the section coefficients (2.7) we had before, but are normalized so that (3.5) holds.

Using the definition of the Bogoliubov map $\bar{R}$ this immediately shows that

$$\bar{R}(B_+^\gamma(h)) = \int \mathrm{D}_\gamma \leftarrow_{G_i} \left[S_R^\phi * \phi(h)\right], \tag{3.6}$$

which proves locality of counterterms inductively upon recognizing that B_+^γ increases the augmentation degree. By D_γ we denote the measure assigned by the Feynman rules to the primitive γ,

$$\phi(B_+^\gamma(\mathbb{I})) =: \int D_\gamma. \tag{3.7}$$

The insertion of the functions for the subgraph is achieved using the relevant gluing data of (2.11), and the start of the induction on primitive graphs, which belong to $\mathcal{H}^{(1)}$, is immediate by Weinberg's theorem [29]. We hence obtain, by applying the Rota–Baxter map R to $\bar{R}$ in (3.6), the crucial statement of renormalization theory: the counterterm in perturbation theory is obtained by replacing all subgraphs by their renormalized expressions.

If we want to understand now why a field theory can be renormalized by local counterterms, we have to understand that it can be organized such that at each order all its contributions are in the image of a Hochschild closed one-cocycle.

It turns out that this can be achieved already at the combinatorial level. The sum over all graphs Γ^r contributing to a given amplitude can indeed be obtained as the solution to a fixpoint equation which puts these sums (for all $r \in \mathcal{R}$) into the image of those Hochschild one-cocycles $B_+^{k;r}$.

Indeed, to recover the quantum equation of motions, the DSEs, from Hochschild cohomology, one proves that

$$\Gamma^r = \mathbb{I} \pm \sum_{\gamma,\, \mathbf{res}(\gamma)=r} \frac{\alpha^{|\gamma|}}{\mathrm{sym}(\gamma)} B_+^\gamma(X_\gamma), \tag{3.8}$$

where

$$X_\gamma = \prod_{e \in \gamma_{\rm int}^{[1]}} \prod_{v \in \gamma^{[0]}} \Gamma^v / \Gamma^e = \Gamma^r Q^{n_r|\gamma|}, \tag{3.9}$$

in a renormalizable theory. Upon application of the Feynman rules the Hopf algebra elements $B_+^\gamma(\mathbb{I})$ turn to the integral kernels of the usual Dyson–Schwinger equations. We obtain the required recursive form (1.5) of the DSEs for $r \in \mathcal{R}$. This allows for new non-perturbative approaches as exhibited in a moment.

The one-cocycles introduced above allow to determine sub Hopf algebras of the form

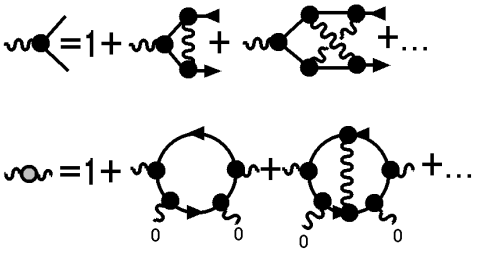
$$\Delta(c_k^r) = \sum_{j=0}^{k} P_{k,j}^r \otimes c_{k-j}^r, \tag{3.10}$$

where the c_j^r are defined in Eq.(1.2) and the $P_{k,j}^r$ are polynomials in these variables given below. These sub Hopf algebras do not necessitate the considerations of single Feynman graphs any longer, but allow to establish renormalization directly for the sum of all graphs at a given loop order.

They hence establish a Hopf algebra structure on time-ordered products in momentum space. For theories with internal symmetries one expects and indeed finds that the existence of these sub-algebras establishes relations between graphs which are the Slavnov–Taylor identities between the couplings in the Lagrangian [1].

3.3 QED as an example. Let us present the DSEs for massless QED. Thanks to the Ward–Takahashi identity we only need to give them for two elements in $\mathcal{R}_{\rm QED}$, the vertex  and the photon propagator  .

We give them first in graphical form using the results of [26].



. (3.11)

All internal edges are full propagators. Note that in the second equation we took a double derivative with respect to the external momentum at the inverse photon propagator. The corresponding Green function is indicated on the lhs. That makes it effectively a four-point function with two zero-momentum external photons. Our Feynman rules, incorporating our definition of an amplitude, are normalized to evaluate the tree-level term to unity, and we get an expansion in terms of 1PI four-photon amplitudes, organized in a skeleton expansion in this equation.

The combinatorial DSEs are thus simply

$$\Gamma^r = \mathbb{I} \pm \sum_{\gamma \in \mathcal{H}^{(1)} \cap \mathcal{H}_{\rm lin},\, \mathbf{res}(\gamma)=r} \alpha^{|\gamma|} B_+^\gamma(\Gamma^r Q^{2|\gamma|}) = \mathbb{I} \pm \sum_{k \geq 1} \alpha^k B_+^{k;r}(\Gamma^r Q^{2k}), \tag{3.12}$$

for $r \in \{$,  $\}$. The sum is over all 1PI primitive graphs with the desired residue r. Also,

$$Q = \frac{\Gamma^{\;\;\;}}{\Gamma^{\;\;\;}\sqrt{\Gamma^{\;\;\;}}}. \tag{3.13}$$

Furthermore,

$$B_+^{k;r} = \sum_{\gamma \in \mathcal{H}^{(1)} \cap \mathcal{H}_{\rm lin}, \mathbf{res}(\gamma)=r, |\gamma|=k} \frac{1}{\mathrm{sym}(\gamma)} B_+^{\gamma}, \tag{3.14}$$

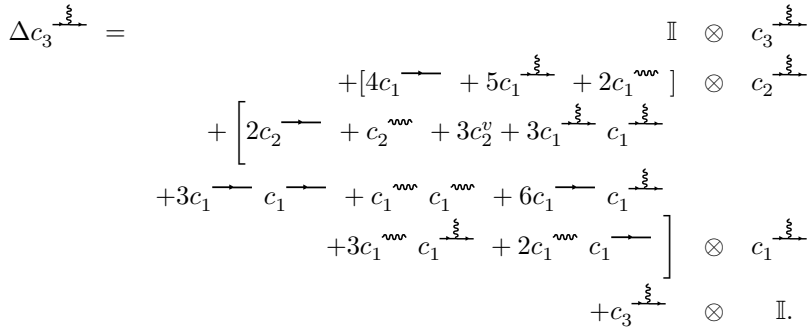
and B_+^{γ} glues vertex functions and propagator functions into edges of γ in accordance with Hochschild cohomology.

The sub Hopf algebra is given as follows [1]. Let us look at the set $\mathcal{R}_{\rm QED}$. Each of the series $\Gamma^r = \Gamma^r(\alpha)$ is a formal series in α, with coefficients c_k^r. The sought-after polynomial $P_{k,j}^r$ in the variables c_m^s is simply the Taylor coefficient of order j in the Taylor expansion in α of the expression

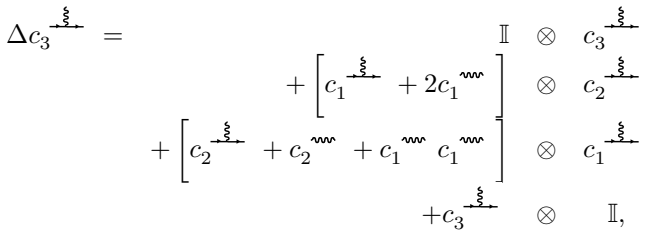
$$\Gamma^r Q^{2(k-j)}, \tag{3.15}$$

using (3.13). This is nothing but the argument of the one-cocycle in (3.12) and this is true in general. There is a wonderful and completely general phenomenon appearing here [1, 7]: if we restrict the sum over one-cocycles in (3.12) at some loop order, we obtain the same sub Hopf algebra. Those sub Hopf algebras are completely universal and independent of the order of the skeleton expansion we take into account. This is in my opinion a first clue as to why phenomena like universality can appear in recursive dynamical systems, to which quantum field theories evidently belong, at least in their short-distance sector.

To see this result in action, let us a look at a three loop example for the QED vertex.



If we now impose the Hopf ideal generated by the Ward identities [20] the above simplifies to



and the Hopf algebra structure were cocommutative were it not for the non-vanishing beta function, which ensures non-vanishing $c_k^{\sim\sim}$.

Actually, taken into account the classical results of Baker, Johnson and Willey [27], the Hopf algebra would be trivial were the β-function to vanish. A thorough discussion of such structures in QED and their connection to the representation theory of the QED Hopf algebra will be given elsewhere [21].

Taking the Ward identity into account, the combinatorial DSEs are even simpler as then

$$Q = \frac{1}{\sqrt{\Gamma\ \wwwvert}}. \tag{3.16}$$

In particular, for the photon we get

$$\Gamma\ \wwwvert\ = \mathbb{I} - \sum_{\gamma\in\mathcal{H}^{(1)}\cap\mathcal{H}_{\rm lin},\,\mathbf{res}(\gamma)=\ \wwwvert} \alpha^{|\gamma|} B_+^\gamma\left(\left[\Gamma\ \wwwvert\ \right]^{1-|\gamma|}\right). \tag{3.17}$$

Note that for the one-loop skeleton we have $|\gamma| = 1$, the argument of the one-cocycle is $\mathbb{I}$ trivially, and hence this term becomes an inhomogenous part in the DSE, and provides the fermion determinant for the QED Lagrangian.

4 The structure of Green functions

It is now time to summarize the structure of Green functions. We closely follow [1, 5, 6, 7].

4.1 Green functions for $\mathcal{R}$. For a given superficially divergent amplitude $r \in \mathcal{R}$ we let Γ^r be the sum

$$\Gamma^r = \mathbb{I} \pm \sum_{\mathbf{res}(\Gamma)=r} \alpha^{|\Gamma|}\frac{\Gamma}{\mathrm{sym}(\Gamma)}, \tag{4.1}$$

over all 1PI graphs Γ contributing to that amplitude, with α a loop-counting small parameter. Projection onto suitable form factors $\phi(r)$ allows the sum to start with unity, so that by application of the Feynman rules, $\phi(\Gamma^r)$ is the corresponding structure function and the Lagrangian L is given by

$$L = \sum_{r\in\mathcal{R}} \hat{\phi}^{\pm 1}(r), \tag{4.2}$$

where we take the plus sign for $n(r) \geq 3$ and the minus sign for $n(r) = 2$.

We write

$$\Gamma^r = \mathbb{I} \pm B_+^r(\Gamma^r, Q(\{\Gamma^i\})), \tag{4.3}$$

where the Hochschild one-cocycle

$$B_+^r(\Gamma^r, Q) = \sum_{k\geq 1} \alpha^k B_+^{k;r}(\Gamma^r Q^{n_r k}), \tag{4.4}$$

is a sum of one-cocycles $B_+^{k,r}$ and Q^{n_r} is a monomial in the Γ^r:

$$Q^{n_r} = \frac{1}{\Gamma^{\mathbf{res}(\gamma)}}\frac{\prod_{v\in\gamma^{[0]}}\Gamma^v}{\prod_{e\in\gamma^{[1]}_{\rm int}}\Gamma^e}, \tag{4.5}$$

for any $\gamma \in mathcalH^{(1)}$ and $|\gamma| = 1$. In general, different such γ have different internal edges and vertices, and hence we obtain different $Q = Q(\gamma)$. For Q to be well-defined, we divide by the relations which equate them all. This is the origin of the Slavnov–Taylor identities for the couplings [1].

The $B_+^{k,r}$ themselves are obtained from the skeleton graphs γ of the theory:

$$B_+^{k,r} = \sum_{\gamma\in\mathcal{H}^{(1)}\cap\mathcal{H}_{\rm lin},\mathbf{res}(\gamma)=r} \frac{1}{\mathrm{sym}(\gamma)} B_+^\gamma, \tag{4.6}$$

where the sum is over all Hopf algebra primitives γ contributing to the amplitude r at k loops. These maps are defined to be closed Hochschild one-cocycles on the sub Hopf algebra generated by their concatentations and products [1, 7].

Effectively, (4.3) reduces the sum (4.1) over all graphs to a sum over primitive ones, making use of the recursive structure of this fixpoint equation which determines the sum of graphs which contribute to a chosen amplitude. The sums involved typically reflect the universal law of [28] and will be discussed in detail in upcoming work.

The c_j^r are the linear generators of a sub-Hopf algebra:

Theorem 4.1 *i) There exists a charge Q as above, maps $B_+^{k;r}$ and polynomials $P_{k,j}^r$ such that*

$$\Gamma^r = \mathbb{I} \pm \sum_k \alpha^k B_+^{k;r}\left(\Gamma^r Q^{n_r k}\right), \tag{4.7}$$

$$\Delta B_+^{k;r} = B_+^{k;r} \otimes \mathbb{I} + \left(\mathrm{id} \otimes B_+^{k;r}\right)\Delta, \tag{4.8}$$

$$\Delta c_k^r = \sum_{j=0}^{k} P_{k,j}^r \otimes c_{k-j}^r, \tag{4.9}$$

which make the free commutative algebra in generators $\{c_k^r\}$ into a sub Hopf algebra $\mathcal{H}_{\mathrm{c}}(\Delta, m, S, \epsilon)$ of the Feynman graph Hopf algebra.
ii) The polynomials $P_{k,j}^r$ are the Taylor coefficients of the expansion of the argument of the k-th one-cocycle to order j in α.

We call the sub Hopf algebra $\mathcal{H}_{\mathrm{c}}$ the Hopf algebra of time-ordered products, as the sum of all graphs with the same residue delivers precisely that. Feynman rules are then defined in accordance with the Hochschild cohomology as iterated integrals on subgraphs.

For a study of the short-distance behaviour of QFT we have to look at the amplitudes in $\mathcal{R}$. For such amplitudes, we can make use of the scaling properties of our Hopf algebras wrt one-parameter groups of automorphisms, and combine the sub Hopf algebra structure above with the general results on complex graded commutative Hopf algebras of [16, 17].

We have to make sure that any Green function which is superficially divergent and appears in the integrand generated by $B_+^{k;r}(\mathbb{I})$ only depends on a single scale. To achieve this in accordance with our approach is a non-trivial demand for amplitudes with $n(r) > 2$. The corresponding vertex functions $\phi_R(\Gamma^r)$, $\phi_R := S_R^\phi * \phi$, depend a priori on $n(r)-1$ linear independent external momenta p_f. Those external momenta will be internal momenta inside the DSEs, where they are to be integrated out thanks to the underlying iterated structure of those equations. Thanks to the fact that short-distance singularities are local we can isolate them into vertex functions at zero momentum transfer, and decompose a divergent amplitude further into an amplitude which depends on a single scale, and an amplitude in $\mathcal{A}_+$. The choice of such a decomposition corresponds to a choice of a basis of primitive elements in the Hopf algebra to work with, and an easy recursion argument [22] shows that we can choose an appropriate basis of such primitives so that we can set up the DSEs such that all divergent subgraphs depend only on a single scale.

Hence there exists a basis of graphs and external structures in the Hopf algebra such that

$$\phi_R(Q) = \phi_R(Q)(\alpha, L), \tag{4.10}$$

where $L = \ln q^2/\mu^2$ is the single scale which determines the running of the invariant charge. In this chosen basis, short distance singularities are captured by Green functions which are functions of two dimensionless variables α, L, with a remarkable duality between these two variables first observed in [5].

In perturbation theory the Feynman rules now allow us to write the renormalized Green functions (compare to the unrenormalized ones in (1.1)), in the momentum scheme so that $G_R^r(\alpha, 0) = 1$, as

$$G_R^r(\alpha, L) = \phi_R(\Gamma^r) = 1 + \sum_k \alpha^k \phi_R(c_k^r)(L). \tag{4.11}$$

We can expand in a different manner

$$G_R^r(\alpha, L) = 1 + \sum_{k\geq 1} \gamma_k^r(\alpha) L^k, \tag{4.12}$$

and the renormalization group dictates relations between the γ_k^r. We work them out in a moment and note the renormalization group equations first:

$$\frac{\partial G_R^r(\alpha, L)}{\partial L} - \left[n_e(r)\gamma_1^e(\alpha) + \beta(\alpha)\frac{\partial}{\partial \alpha} \right] G_R^r(\alpha, L) = 0. \tag{4.13}$$

Here, n_e is the number of external legs of type e in the residue r, and γ_1^e is the corresponding anomalous dimension, obtained by taking a derivative in (4.12),

$$\gamma_1^e := \frac{\partial G^e(\alpha, L)}{\partial L}|_{L=0}. \tag{4.14}$$

$\beta(\alpha)$ is the β-function corresponding to the charge $\phi_R(Q)$. It is given through the anomalous dimensions in a simple form. If

$$Q^{n_r} = \frac{\prod_v [\Gamma^v]^{n_v}}{\prod_e [\Gamma^e]^{n_e/2}}, \tag{4.15}$$

then

$$\beta(\alpha) = \sum_v n_v \gamma_1^v - \frac{1}{2}\sum_e n_e \gamma_1^e. \tag{4.16}$$

We let

$$\beta_{\rm comb} := \sum_v n_v \Gamma^v - \frac{1}{2}\sum_e n_e \Gamma^e, \tag{4.17}$$

be the corresponding series in α with coefficients in the Hopf algebra.

Next, we note that in the case of a linear DSE [7, 9], we get

$$\partial_L \phi(Q)(L) = 0. \tag{4.18}$$

As a result scaling

$$G(\alpha, L) = e^{-\gamma(\alpha)L}, \tag{4.19}$$

solves the linear DSE so that

$$\gamma_k(\alpha) = \frac{\gamma_1(\alpha)^k}{k!}. \tag{4.20}$$

In such a case, the corresponding Hopf algebra structure of the c_k^r is cocommutative and the dual Lie algebra structure abelian. The violation of conformal invariance

captured by a non-vanishing β-function reflects itself in the non-vanishing commutators of the Feynman graph Lie algebra.

A beautiful result of [17] is the scattering formula, which expresses higher poles in the regulator through iterated residues. This translates into a statement on the coefficients of higher logs in the leading log expansion of a quantum field theory. We are assured that to get to these higher log contributions from a Hopf algebra element, we simply have to apply the coproduct sufficiently often so as to decompose it into the components in $\mathcal{H}^{(1)}$. We then find the contribution to the k-th power of log by the product of the residues of all those primitives.

Hence, to proceed in general we consider the map

$$P_{\text{lin}}^{(n)} = \underbrace{P_{\text{lin}} \otimes \cdots \otimes P_{\text{lin}}}_{n \text{ times}} \Delta^{n-1} \tag{4.21}$$

where P_{lin} is the projector into the linear span of generators of the Hopf algebra. From [1, 7] we have:

Theorem 4.2 *The linearized coproduct is obtained as*

$$P_{\text{lin}}^{(2)}\Gamma^r = P_{\text{lin}}\Gamma^r \otimes P_{\text{lin}}\Gamma^r + P_{\text{lin}}Q \otimes \alpha\partial_\alpha\Gamma^r,$$

where

$$P_{\text{lin}}Q = \beta_{\text{comb}}. \tag{4.22}$$

This allows us to understand the iterative structure of the next-to... leading log expansion (4.12).

We define for $n > 1$

$$\sigma_n := \frac{1}{n!}\, m^{n-1}\, \underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{n \text{ times}}\, \Delta^{n-1}, \tag{4.23}$$

and σ_1 is the residue defined by

$$\sigma_1 = \partial_L \phi_R \left(S \star Y_{\text{aug}}\right)(L)|_{L=0}. \tag{4.24}$$

Actually, σ_n evaluates to the coefficient of the L^n term in the evaluation of a Hopf algebra element by the renormalized Feynman rules, by the scattering type formula [17].

We have

$$h \notin H_{\text{lin}} \Rightarrow \sigma_1(h) = 0, \tag{4.25}$$

so we can use Theorem 2 and, by the above definition (4.12) of $\gamma_k^r(\alpha)$,

$$\gamma_k^r(\alpha) = \sigma_k(\Gamma^r(\alpha)). \tag{4.26}$$

Projection onto the linear generators delivers the desired formula for the expansion in L, $\forall k \geq 2$:

$$\gamma_k^r(\alpha) = \frac{1}{k}\left[\gamma_1^r(\alpha)\gamma_{k-1}^r(\alpha) + \sum_j s^j \gamma_1^j(\alpha)\alpha\partial_\alpha\gamma_{k-1}^r(\alpha)\right]. \tag{4.27}$$

This gives us a second clue towards universality in field theory: not only simplify the rather complicated graph Hopf algebras to the rather simple Hopf algebras

of time-ordered products, but from these Hopf algebras we only need to remind ourselves of the simplest linear part in them: the underlying complications of diffeomorphisms of physical parameters only very mildly interfere with the short-distance sector of a theory.

With the the above choice of basis we can now introduce the Mellin transform by raising internal propagators $\mathrm{Prop}(k_e)$ with momentum k_e to non-integer powers ρ. A derivative wrt ρ then amounts to the insertion of logarithmic corrections $\ln k_e^2$ into the Feynman integrals $\phi_R(B_+^{k;r}(\mathbb{I}))$, which is all what is needed to proceed in view of the expansion (4.12) for internal Green functions.

The DSEs turn into equations which determine γ_1^r as we will see in a moment, while the further terms in the L expansion are determined from (4.27) above. Green functions also have the usual expansion in α which is triangular wrt γ_k:

$$\gamma_k^r(\alpha) = \sum_{j\geq k} \gamma_{k,j}^r \alpha^j. \tag{4.28}$$

We can hence proceed to work out the recursion relations which express the functions γ_k^r through the functions γ_1^r for $k > 1$, and turn the Dyson–Schwinger equations into an implicit equation which allows to determine the sole unknown coefficients $\gamma_{1,j}^r$ from the knowledge of the above Mellin transforms.

Before we finally discuss an example, let us mention a third clue towards universality: the fact that by construction the above primitives are invariant under conformal inversion severely restricts the form of the Mellin transform and analytic differences can to a large extent be compensated by a redefinition of the relevant couplings. We hope to have some more concrete results along those lines in the future.

4.2 A simple example. For concreteness, we consider massless Yukawa theory and consider all self-iterations of the one-loop massless fermion propagator, with subtractions in the momentum scheme at $q^2 = \mu^2$. Our Green function is an inverse propagator with momentum q and a function of two variables a and $L = \ln q^2/\mu^2$. We ignore radiative corrections at the bosonic line and also at vertices, so the set $\mathcal{R}$ has a single element and the superscript r is suppressed henceforth. We restrict ourselves to a single element in $\mathcal{H}^{(1)} \cap \mathcal{H}_{\mathrm{lin}}$ and hence to a single one-cocycle B_+. We rederive the results of [5] for this case.

We write the perturbative series for the Dyson–Schwinger equation as

$$X(a) = \mathbb{I} - aB_+\left(\frac{1}{X(a)}\right), \tag{4.29}$$

where $\phi(B_+(\mathbb{I}))$ provides the one-loop self-energy integral to be iterated. Note that upon setting $X(a) = \mathbb{I} - \underline{X}(a)$, this is the equation for the self-energy $\underline{X}(a) = -P_{\mathrm{lin}}X(a)$ studied in [5].

We have

$$Q = 1/X^2 \to P_{\mathrm{lin}}(Q) = -2\underline{X}(a), \tag{4.30}$$

and find the linearized coproduct

$$P_{\mathrm{lin}}^{(2)}X(a) = P_{\mathrm{lin}}X(a) \otimes (P_{\mathrm{lin}} - 2a\partial_a)X(a). \tag{4.31}$$

This is Proposition 1 of [5] and we also get

Theorem 4.3 *The next-to next-to... leading log expansion in L is given through the anomalous dimension $\gamma_1(a)$ as*

$$\gamma_k(a) = \frac{1}{k}\gamma_1(a)(1 - 2a\partial_a)\gamma_{k-1}(a). \tag{4.32}$$

This is Proposition 2 of [5]. As a result, can work out with ease the recursions which express γ_k, $k > 1$ through the Taylor coefficients of γ_1.

Such recursions are obtained for any non-linear DSE by iterating Theorem 4.2. We observe that we only need the cocommutative part in the determination of the coproduct as is evident from the definition (4.23) of σ_k, $k > 1$. The non-cocommutative part is always of lower degree in L in the obvious filtration by L.

It remains to understand how to compute $\gamma_1(\alpha)$. We proceed here by the Mellin transform. It reads

$$F(\rho) = \frac{1}{q^2}\int d^4k \frac{k \cdot q}{[k^2]^{1+\rho}(k+q)^2}_{|_{q^2=1}} = \frac{1}{\rho(\rho-2)} =: \frac{r}{\rho} + \sum_{i\geq 0} f_i \rho^i. \tag{4.33}$$

Let us introduce a short hand notation as

$$\gamma \cdot U := \sum_{k=1}^{\infty} \gamma_k(\alpha) U^k. \tag{4.34}$$

The Dyson–Schwinger equation becomes

$$\gamma \cdot L = \alpha(1 + \gamma \cdot \partial_{-\rho})^{-1}[e^{-L\rho} - 1]F(\rho)|_{\rho=0}, \tag{4.35}$$

where we evaluate the rhs at $\rho = 0$ after taking derivatives. The functional dependence of the non-linear DSE on $XQ = X^{-1}$ reflects itself on the rhs.

The only unknown quantities in this equation are the Taylor coefficients $\gamma_{1,j}$ which are implicitly defined through the Taylor coefficients of the Mellin transform (4.33) above.

Taking a derivative of (4.35) wrt L and setting L to zero allows us to read them off:

$$\gamma_1 = \alpha(1 + \gamma \cdot \partial_{-\rho})^{-1}\rho F(\rho)|_{\rho=0} \tag{4.36}$$

$$= \alpha r + \alpha \left(\sum_{k\geq 1}[\gamma \cdot \partial_{-\rho}]^k\right)\left[\sum_{k=1}^{\infty}\rho^k f_{k-1}\right]|_{\rho=0}, \tag{4.37}$$

so $\gamma_{1,1} = r$ universally. The resulting recursions determine the non-perturbative solution given in [5].

As a final remark, we mention that a functional equation can be assigned to the solution of the DSE, relating a weak to a strong coupling expansion making use of the analytic structure behind the recursion above. It is reminiscent of a functional equation for a ζ-function in two variables for the function field case [6].

Furthermore, in the general case one needs multivariate Mellin transforms, which is an immediate source for the appearance of transcendentals in solutions to DSEs [6]. The same holds actually for the related study of higher one-cocycles. A few short remarks finish our exploration of DSEs.

5 From Primitives to Motives

At long last we arrive in understanding the role of primitives $\gamma \in \mathcal{H}^{(1)}$ in field theory. Their Mellin transforms provide the basic constituents of a field theory,

with each Taylor coefficient of such a transform an interesting period in its own right.

Knowledge of these Mellin transforms is at the time of writing sparse and restricted to low degrees in α, even if we are only to look at the the residue, the first Taylor coefficient in such a Mellin transform. Indeed, in accordance with (2.4) we find

$$\phi(B_+^\gamma)(\mathbb{I}) = \frac{r_\gamma}{\rho} + \text{finite terms}, \tag{5.1}$$

and our first task is to find the residue r_γ for primitive graphs γ.

Still, such numbers provide much to explore for a mathematician, and lead deep into the territory of algebraic geometry and motivic theory. Data on such numbers have been amply provided in collaboration with David Broadhurst [30, 31] and John Gracey [32, 33], and hide in many computations in particle physics [34].

While it is nice that those Feynman graphs could be given a motivic interpretation in [10], we are apparently (rather depressingly, the evaluation of the wheel with n-spokes studied by us in that paper is already a formidable task in this respect) a far cry from a full motivic classification of Feynman graphs, though it almost certainly exists. The results of [10] where a first step in this direction. Whilst incomplete, they give hope for the future. One proceeds by assigning to a graph Γ a graph polynomial and finds the relevant pairings which are responsible for the period assigned to the desired residue to be determined from the interplay between the zeros of the graph polynomial (the graph hypersurface) and the simplex of integration. The existence of subgraphs containing closed loops (albeit convergent) ensures that this interplay has non-trivial cohomology, and gives the resulting periods motivic interpretation in those simple cases we could study.

The message to be kept is that upon organizing the short-distance singular sector in a self-consistent manner, we find that we can solve the DSEs upon understanding the number-theoretic and motivic nature of the elements $B_+^{k;r}(\mathbb{I})$.

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References

[1] D. Kreimer, *Anatomy of a gauge theory*, Annals of Physics (2006), online first, in press, hep-th/0509135.

[2] A. I. Vainshtein, V. I. Zakharov, V. A. Novikov, M. A. Shifman, *The ABC of Instantons*, Sov. Phys. Ushpekhi **25** (1982), 195-215.

[3] K. Ebrahimi-Fard, L. Guo, *Rota–Baxter algebras in renormalization of perturbative quantum field theory*, this volume; hep-th/0604116.

[4] D. Kreimer, *The Hopf algebra structure of renormalizable quantum field theory*, in Encyclopedia of Mathematical Physics, eds. J.-P. Francoise, G. L. Naber, S. T. Tsou. Oxford: Elsevier 2006 (ISBN 978-0-1251-2666-3).

[5] D. J. Broadhurst and D. Kreimer, *Exact solutions of Dyson-Schwinger equations for iterated one-loop integrals and propagator-coupling duality,* Nucl. Phys. B **600** (2001) 403, hep-th/0012146.

[6] D. Kreimer, K. Yeats, *An etude in non-linear Dyson-Schwinger equations,* arXiv:hep-th/0605096, Nucl. Phys. **B** Proc. Suppl., in press.

[7] C. Bergbauer, D. Kreimer, *Hopf algebras in renormalization theory: Locality and Dyson–Schwinger equations from Hochschild cohomology*, Proc. 75 Renc. Phys. & Math. , L. Nyssen, ed., Eur. Math. Soc., in press, hep-th/0506190.
[8] D. Kreimer, *Factorization in quantum field theory: An exercise in Hopf algebras and local singularities,* arXiv:hep-th/0306020, contributed to Les Houches School of Physics: Frontiers in Number Theory, Physics and Geometry, Les Houches, France, 9-21 Mar 2003, in press.
[9] D. Kreimer, *An étude in linear Dyson–Schwinger equations,* preprint, IHES/P-06-23. MPI-Math, Bonn, Proceedings, to appear.
[10] S. Bloch, H. Esnault, D. Kreimer, *On Motives Associated to Graph Polynomials,* Comm. Math. Phys. **267** (2006) 181-225; arXiv:math.ag/0510011.
[11] A. Kizilersu, M. Reenders and M. R. Pennington, *One loop QED vertex in any covariant gauge: Its complete analytic form,* Phys. Rev. D **52** (1995) 1242 [arXiv:hep-ph/9503238].
[12] D. J. Gross, *Application of the renormalization group to high energy physics,* in *Les Houches 1975: Methods in field theory,* edited by R. Balian and J. Zinn-Justin (North-Holland/World Scientific, Amsterdam/Singapore, 1976/1981).
[13] D. Kreimer, C. Bergbauer, *The Hopf algebra of rooted trees in Epstein-Glaser renormalization,* Annales Henri Poincaré **6** (2005) 343 [arXiv:hep-th/0403207].
[14] D. Kreimer, *On the Hopf algebra structure of perturbative quantum field theories,* Adv. Theor. Math. Phys. **2** (1998) 303 [arXiv:q-alg/9707029].
[15] A. Connes, D. Kreimer, *Hopf algebras, renormalization and noncommutative geometry,* Commun. Math. Phys. **199** (1998) 203 [arXiv:hep-th/9808042].
[16] A. Connes, D. Kreimer *Renormalization in quantum field theory and the Riemann-Hilbert problem. I: The Hopf algebra structure of graphs and the main theorem,* Commun. Math. Phys. **210** (2000) 249 [arXiv:hep-th/9912092].
[17] A. Connes, D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. II: The beta-function, diffeomorphisms and the renormalization group,* Commun. Math. Phys. **216** (2001) 215, hep-th/0003188.
[18] I. Todorov, *Constructing conformal field theory models*, this volume.
[19] D. Kreimer, *What is the trouble with Dyson-Schwinger equations?*, Nucl. Phys. Proc. Suppl. **135** (2004) 238, hep-th/0407016.
[20] W. van Suijlekom, *The Hopf algebra of Feynman graphs in QED,* arXiv:hep-th/0602126, Lett. Math. Phys. in press.
[21] D. Kreimer, *On QED*, in preparation.
[22] D. Kreimer, *Structures in Feynman graphs: Hopf algebras and symmetries,* Proc. Symp. Pure Math. **73** (2005) 43 [arXiv:hep-th/0202110].
[23] M. Markl, S. Shnider, J. Stasheff, *Operads in Algebra, Topology and Physics*, Oxford Univ. Press 2002.
[24] A. R. Its, B.-Q. Jin, V. E. Korepin, *Entropy of XY spin chain and block Toeplitz determinants*, this volume.
[25] D. G. C. McKeon, *Using the renormalization group*, this volume.
[26] D. J. Broadhurst, R. Delbourgo and D. Kreimer, *Unknotting the polarized vacuum of quenched QED,* Phys. Lett. B **366** (1996) 421 [arXiv:hep-ph/9509296].
[27] M. Baker, K. Johnson, *Quantum electrodynamics at small distances,* Phys. Rev. **183** (1969) 1292.
[28] J. Bell, S. Burris, and K. Yeats, *Counting Rooted Trees: The Universal Law* $t(n)C\rho^{-n}n^{-3/2}$, Elect. J. Comb. **13** (2006) # R63, math.CO/0512432.
[29] S. Weinberg, *High-energy behavior in quantum field theory,* Phys. Rev. **118** (1960) 838.
[30] D. Broadhurst, D. Kreimer, *Knots and numbers in* Φ^4 *theory to 7 loops and beyond,* Int. J. Mod. Phys. C **6** (1995) 519 [arXiv:hep-ph/9504352].
[31] D. Broadhurst, D. Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops,* Phys. Lett. B **393** (1997) 403 [arXiv:hep-th/9609128].
[32] J. Gracey, *Practicalities of renormalizing quantum field theories*, this volume; hep-th/0605037.
[33] D. Broadhurst, J. Gracey, D. Kreimer, *Beyond the triangle and uniqueness relations: Non-zeta counterterms at large N from positive knots,* Z. Phys. C **75** (1997) 559 [arXiv:hep-th/9607174].
[34] S. Weinzierl, *The art of computing loop integrals*, this volume; hep-th/0604068.