

# Recursion and growth estimates in renormalizable quantum field theory

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Décembre 2006

IHES/P/06/62

# RECURSION AND GROWTH ESTIMATES IN RENORMALIZABLE QUANTUM FIELD THEORY\*

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ABSTRACT. In this paper we show that there is a Lipatov bound for the radius of convergence for superficially divergent one-particle irreducible Green functions in a renormalizable quantum field theory if there is such a bound for the superficially convergent ones. The radius of convergence turns out to be  $\min\{\rho, 1/b_1\}$ , where  $\rho$  is the bound on the convergent ones, the instanton radius, and  $b_1$  the first coefficient of the  $\beta$ -function.

## 1. THE SET-UP

**1.1. Introduction.** In this paper we explore the recursive structure of the short-distance sector of a renormalizable quantum field theory. Such theories have a finite number of distinct amplitudes  $r \in \mathcal{R} \subset \mathcal{A}$  which need renormalization. We decompose each Green function in accordance with structure functions which need renormalization, the remaining structure functions and superficially convergent Green functions contribute to amplitudes  $a \in \mathcal{A} \setminus \mathcal{R}$  in the set of all amplitudes  $\mathcal{A}$  [14].

For such a theory, the Dyson–Schwinger equations give any amplitude  $s \in \mathcal{A} = \mathcal{A} \setminus \mathcal{R} \cup \mathcal{R}$  in terms of all those amplitudes  $r \in \mathcal{R}$  in the theory which need renormalization, and in terms of an infinite series of integral kernels, the skeleton graphs of the theory. The very fact that we can renormalize by modifying the Lagrangian implies that we have a sub Hopf algebra at our disposal which has the graded elements  $c_k^r$  as generators which correspond to the  $k$ -th order contribution to the amplitude  $r$  [15, 14].

We can relate the integral kernels above to the primitives of the Hopf algebra structure underlying renormalization. The growth of these primitives is hence determined by the growth of integral kernels provided by overall convergent Green functions. A bound on such a growth is hence obtainable from a bound of amplitudes in  $\mathcal{A} \setminus \mathcal{R}$ , where results in constructive field theory are principally available. The typical example is quantum electrodynamics (QED), where the primitives for the vertex function are given by the superficially convergent four fermion  $e^+e^- \rightarrow e^+e^-$  scattering kernel, two-particle irreducible in a suitable channel.

To proceed from there to a bound for amplitudes in  $\mathcal{R}$ , we need a handle on the behaviour of the singular integrations which encompass the short-distance singular sector. Here, we proceed by choosing a suitable set of primitives such that we can reduce the Dyson–Schwinger equations to recursive equations acting on one-variable Mellin transforms. A simple study of the conformal symmetries in the corresponding primitives suffice to determine the form of these Mellin transforms.

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*Date:* December 17, 2006. BU–CMP/06–05.

\*Research supported by grant nsf-dms/0603781. <sup>1</sup>supported by CNRS.

For any  $r \in \mathcal{R}$ , we can define a Green function

$$(1) \quad G^r(a, L) = 1 \pm \sum_k \gamma_k^r(a) L^k,$$

where  $L = \ln -Q^2/\mu^2$  measures the scaling behaviour under the renormalization group flow with respect to a single Euclidean kinematical variable  $Q^2 < 0$ .

Hence, if we have a recursive set of equations giving Green functions  $G^r(a, L)$  in terms of themselves, inserted into integral kernels, we can apply Mellin transforms as defined below upon using

$$(2) \quad G^r(a, \ln -k^2/\mu^2) = G^r(a, \partial_\rho) \left( -\frac{k^2}{\mu^2} \right)^\rho \Big|_{\rho=0},$$

to reduce the evaluation of all integral kernels above to a study of corresponding Mellin transforms.

In such a situation, we hence show how the growth of superficially divergent amplitudes is related to the growth of the superficially convergent ones. To the extent the latter is under control by results in constructive field theory, we get results for the former.

The representation of primitive integral kernels through their Mellin transform then allows us to turn the Dyson–Schwinger equations into a recursive system which determines the  $\gamma_k$  in terms of the  $\gamma_1$ , and determines  $\gamma_1(a)$  recursively through the Taylor coefficients of the relevant Mellin transforms. Modest knowledge of the structure of those transforms:

$$(3) \quad M(p)(\rho) = \frac{\text{res}_p}{\rho(1-\rho)},$$

and a Lipatov bound for them,  $a^{|p|} \text{res}_p \sim |p|^c |c|^{|p|}$  for some suitable  $c$  at large  $|p|$ , allows us to show that solutions to Dyson–Schwinger equations for Green functions  $G^r(a, L)$  have a similar Lipatov bound.

We emphasize that our construction of a basis of primitives with a given Mellin transform resolves overlapping divergences, thanks to the Hochschild cohomology of the relevant Hopf algebras [10].

We have thus a self-similar recursive system determining the formal sums  $\Gamma^r$ ,  $r \in \mathcal{R}$  in terms of themselves and the action of suitable maps  $B_+^{k;r} = \sum_{|p|=k} B_+^{p;r}$  [14]. For all  $r \in \mathcal{R}$ ,

$$(4) \quad X^r(a) = \mathbb{I} \pm \sum_{k \geq 1} a^k B_+^{k;r} (X^r \mathcal{Q}^k),$$

where  $a\mathcal{Q}$  is the invariant charge of the theory, and Green functions are obtained as

$$(5) \quad G(a, L) = \phi_R(X^r(a))(L),$$

for renormalized Feynman rules  $\phi_R$ .

The study of the Hochschild one-cocycles  $B_+^{k,i;r}$  is crucial for a QFT. The Hopf algebra elements  $B_+^{k;r}(\mathbb{I})$  provide the very integral kernels above underlying the DSEs and are the terms which drive the recursion. Their consistent construction automatically takes internal symmetries of the theory into account, [13], by dividing the Hopf algebra by suitable Hopf ideals [23, 24, 13]. A further ideal is defined by our desire to concentrate on amplitudes in  $\mathcal{R}$ .

**1.2. Ideals.** Green functions in field theory decompose into structure functions which have logarithmic short-distance singularities. Locality of field theory guarantees that these short distance singularities are invariant under changes of dimensionfull parameters in the theory, allowing for local counterterms. This is not the only invariance which we can observe in counterterms. A chosen subgraph needs the same counterterm wherever it appear in a larger graph, and in whatever orientation it is inserted, as long as we work in a symmetric renormalization scheme. Furthermore, its counterterm is invariant under modification of its external momenta. We can hence isolate short-distance singularities in subgraphs in a manner such that all subdivergences are functions of a chosen internal momentum of the cograph under consideration.

In so doing we enlarge the Hopf algebra of graphs into a Hopf algebra of colored graphs or decorated rooted trees, which makes the underlying Hochschild cohomology obvious and resolves overlapping divergences [10, 12].

We emphasize that the choices above depend on the theory, renormalization scheme and physical problem one wants to study. Here, we are only concerned with the general set-up, and will only sporadically restrict to a specific theory. Our final result applies to any renormalizable field theory. Specifics of any particular such theory are to be discussed elsewhere.

The freedom we have in the above choices can be summarized by saying we work in an ideal given by

$$(6) \quad \sum_i \Gamma'' \circ_i \text{End}^{-1} \Gamma' = 0,$$

with

$$(7) \quad \text{End}(\Gamma_1 \Gamma_2) = \text{End}(\Gamma_1) \text{End}(\Gamma_2).$$

Here, we use Sweedler notation  $\Delta(\Gamma) = \sum_i \Gamma' \otimes \Gamma''$  and the fact that Feynman graphs have an operadic structure, so that for each term in the sum we have an operadic composition [12] of graphs such that

$$(8) \quad \Gamma = \Gamma'' \circ_i \Gamma'.$$

Finally,  $\text{End}$  is a map which implements such a choice: it chooses a configuration of external momenta for the subgraph  $\Gamma'$ , permutes the orientation of external legs, or changes the insertion place: any modification which will not modify the required counterterm is allowed. By construction, elements in our ideal are superficially convergent, and hence contribute nothing to the short distance singular sector. The above ideal is determined by the choice of the map  $\text{End}$ , which is then determined by the physics one wants to study.

We then decompose graphs as

$$(9) \quad \Gamma = \underbrace{\sum_i \Gamma'' \circ_i \text{End}^{-1}(\Gamma') P(\Gamma'')}_{p(\Gamma)} + \sum_i \Gamma'' \circ_i \text{End}^{-1} \Gamma' P(\Gamma'') P(\Gamma').$$

By construction, the first term in the rhs is a primitive element  $p = p(\Gamma)$  in the Hopf algebra of decorated rooted trees, and  $\text{End}^{-1} = \text{End} \circ S$ .

Note that for a primitive graph  $\gamma$  and an endomorphism  $\text{End}$  which changes its external leg structure to say zero momentum transfer, the above ideal simply says that the short distance singular sector of  $\gamma$  and  $\text{End}(\gamma)$  are the same.

More instructive is the example of the case of say a graph with a single divergent subgraph  $\Gamma = \Gamma'' \circ \Gamma'$ . We then have

$$(10) \quad p(\Gamma) = \Gamma'' \circ [\Gamma' - \text{End}(\Gamma')].$$

This is clearly primitive as  $[\Gamma' - \text{End}(\Gamma')]$  is in the chosen ideal, and  $p(\Gamma)$  in (9) is primitive in general because all its subgraph are in the ideal, by construction. We emphasize that in all our applications we will never discard any terms on the rhs of (9), but will always calculate the full graph  $\Gamma$ , but use (9) as a convenient tool to come to a manageable form for the primitives which drive the recursion, on the expense to have an enlarged set of such integral kernels.

For our purposes, we choose an ideal which sets  $m = 0$  in structure functions and evaluates external momenta at a symmetric point  $-Q^2$  with regard to their external momenta, and iterates graphs into each other in accordance with the definition of the Mellin transform defined below. As a renormalization scheme we choose subtraction at a fixed  $-Q^2 = \mu^2$ .

**1.3. Mellin transform.** Any so constructed primitive  $p = p(\Gamma)$  is a degree homogenous combination of Hopf algebra elements of degree  $|p|$ . We let  $-Q^2$  be the above kinematical variable which we keep. We define the Mellin transform  $M(p)$  of  $p$  as

$$(11) \quad M(p)(\rho) = [-Q^2]^\rho \int \text{Int}_p(-Q^2) \left\{ \frac{1}{|p|} \sum_{i=1}^{|p|} [k_i^2]^{-\rho} \right\} \prod_{i=1}^{|p|} d^4 k_i,$$

where  $\text{Int}_p(-Q^2)$  is the integrand determined by  $p$ . We let

$$(12) \quad \text{Int}_p^-( -Q^2) := \text{Int}_p(-Q^2) - \text{Int}_p(\mu^2).$$

We define renormalized Feynman rules for a symmetric momentum scheme with subtractions at  $-Q^2 = \mu^2$  by

$$(13) \quad \phi_R(B_+^p(X))(-Q^2/\mu^2) = \int \text{Int}_p^-( -Q^2) \left\{ \frac{1}{|p|} \sum_{i=1}^{|p|} \phi_R(X)(-k_i^2/\mu^2) \right\} \prod_{i=1}^{|p|} d^4 k_i.$$

We have

$$(14) \quad \phi_R(B_+^p(X))(-Q^2/\mu^2) = \lim_{\rho \rightarrow 0} \phi_R(X)(\partial_{-\rho}) M(p)(\rho) [(-Q^2/\mu^2)^{-\rho} - 1],$$

where  $\partial_{-\rho} = -\frac{\partial}{\partial \rho}$ .

A Green function is then defined as the image under such Feynman rules applied to a fixpoint of a combinatorial DSE. In the coming sections, we first study the asymptotics of a combinatorial DSE, then the growth after applying  $\phi_R$ .

Let us now look at an example in  $\phi_4^4$  theory. We define the vertex Green function by setting  $-Q^2 = s = t = u$ , and set  $m = 0$  in all propagators in accordance with an ideal which isolates the short distance singularities in massless Green functions.

Taking the symmetry in external legs into account, the Hopf algebra series reads to order  $g^2$

$$(15) \quad 1 + g^2 \text{ (circle) } + 3g^2 \left( \frac{1}{4} \text{ (two circles) } + \frac{1}{2} [ \text{ (crossed circles) } + \text{ (other crossed circles) } ] \right).$$

We define a primitive

$$\begin{aligned}
 p_2 &= \frac{1}{4} \text{Diagram 1} + \frac{1}{2} \left[ \text{Diagram 2} + \text{Diagram 3} \right] - \frac{3}{2} \text{Diagram 4} \circ \text{End} \left( \text{Diagram 5} \right) \\
 (16) \quad &= \frac{1}{4} \text{Diagram 1} - \frac{1}{2} \text{Diagram 2}.
 \end{aligned}$$

We define a combinatorial DSE

$$(17) \quad X(g) = 1 + g \frac{3}{2} B_+^{p_1}(X(g)^2) + 3g^2 B_+^{p_2}(X(g)^3),$$

where  $p_1 = \text{Diagram 6}$ .

It has a fixpoint which reproduces to order  $g^2$  the above expansion.

Let us turn this into an integral equation:

$$\begin{aligned}
 (18) \quad G(g, \ln -Q^2/\mu^2) &= 1 + \frac{3}{2} g \int \left\{ \frac{G(g, \ln -k^2/\mu^2)^2}{k^2(k+Q)^2} \right\}_- d^4 k \\
 &+ 3g^2 \int \frac{1}{8} [G(g, \ln -k^2/\mu^2) + G(g, \ln -l^2/\mu^2)]^3 \\
 &\quad \left\{ \frac{2k \cdot l - 2(k+l) \cdot s - 2s^2}{k^2 l^2 (k+Q)^2 (l+Q)^2 (l-k)^2} \right\}_- d^4 k d^4 l,
 \end{aligned}$$

where  $\}_-$  indicates subtraction at  $-Q^2 = \mu^2$ . One verifies that the solution to this integral equation agrees to order  $g^2$  with the perturbative renormalization in a symmetric momentum scheme for the vertex function in  $\phi_4^4$  theory, as it should. Its solution also agrees to any order in  $g^2$  with the leading order in  $L = \ln -Q^2/\mu^2$  in that order of  $g^2$ , and also with the next to leading order, as we have taken the two loop primitive into account.

Addition of further primitives delivers the lower powers in  $L$ . The Mellin transform  $M_{p_1} = \frac{1}{\rho(1-\rho)}$  is trivial, the Mellin transform  $M_{p_2}$  much less so and can be found in [2].

**1.4. Mellin transforms as a geometric series.** The above Mellin transforms  $M_p(\rho)$  have the form

$$(19) \quad M_p(\rho) = \frac{f_p(\rho)}{\rho(1-\rho)},$$

in a natural manner, where the pole at  $\rho = 0$  reflects the short-distance singularity, and the pole at  $\rho = 1$  reflects the fact that in our ideal we have massless internal propagation.

The series  $f_p(\rho) = \text{res}_p + \mathcal{O}(\rho)$  determines the residue  $\text{res}_p$  of the transform. We keep the factor  $1/(1-\rho)$  explicit though, to maintain the form of the Mellin transform similar, in accordance with its conformal symmetries at  $-Q^2 = 1$ .

By the above definition of the Feynman rules it is a purely algebraic exercise [2, 11] to define a new series of primitives  $p_1 = p$ ,  $p_2 = B_+^p(B_+^p(\mathbb{I})) - \frac{1}{2}B_+^p(\mathbb{I})B_+^p(\mathbb{I})$ , and so on, such that

$$(20) \quad M_p(\rho) = \sum_n q_n \frac{\text{res}_{p_n}}{\rho(1-\rho)}.$$

We henceforth assume a basis of primitives  $\{p\}$  such that

$$(21) \quad M_p(\rho) = \frac{\text{res}_p}{\rho(1-\rho)},$$

which is convenient in the following. This finishes the discussion of our analytic set-up and we next remind ourselves of some basic properties of recursive systems relevant to Dyson–Schwinger equations.

## 2. THE UNIVERSAL LAW

**2.1. The classical case.** A classical result of combinatorics is Pólya’s asymptotic formula for the number,  $t(n)$ , of rooted, unlabelled trees with  $n$  vertices [20] (translated in [21]):

$$(22) \quad t(n) \sim C\rho^{-n}n^{-3/2}$$

where  $C$  is an explicit constant and  $\rho$  is the radius of convergence of the generating function  $\mathbf{T}(x) = \sum_{n \geq 1} t(n)x^n$  of the class of all rooted, unlabelled trees. Thus  $\rho$  is the reciprocal of Otter’s tree constant [19] (sequence A051491 in [22]). The key to Pólya’s proof is to convert the recursive equation

$$\mathbf{T}(x) = x \exp\left(\sum_{m \geq 1} \mathbf{T}(x^m)/m\right)$$

for rooted, unlabelled trees to a bivariate function

$$\mathbf{E}(x, y) = xe^y \exp\left(\sum_{m \geq 2} \mathbf{T}(x^m)/m\right)$$

at which point the recursive equation becomes  $\mathbf{T}(x) = \mathbf{E}(x, \mathbf{T}(x))$ . Since  $\rho < 1$  this rewriting uses the new variable  $y$  to isolate the portion of the recursive equation which controls the radius of convergence.

Consequently,  $y - \mathbf{E}(x, y)$  is amenable to Weierstrass preparation, which tells us that around  $(\rho, \mathbf{T}(\rho))$ ,  $y - \mathbf{E}(x, y)$  is a product of a holomorphic function which is nonzero around  $(\rho, \mathbf{T}(\rho))$  and a monic polynomial in  $y$  with coefficients which are analytic in  $x$ . The degree of the polynomial is the order of the first nonzero derivative. The failure of the implicit function theorem at  $\rho$  gives that

$$\begin{aligned} \mathbf{T}(\rho) - \mathbf{E}(\rho, \mathbf{T}(\rho)) &= 0 \\ 1 - (\partial_y \mathbf{E})(\rho, \mathbf{T}(\rho)) &= 0 \\ -(\partial_{yy} \mathbf{E})(\rho, \mathbf{T}(\rho)) &\neq 0. \end{aligned}$$

So the Weierstrass polynomial is quadratic which gives that  $\mathbf{T}$  has a square root singularity at  $\rho$ :

$$\mathbf{T}(x) = f(x) + g(x)\sqrt{\rho - x}$$

with  $f$  and  $g$  analytic at  $\rho$ . Then the Cauchy integral theorem gives the desired asymptotics along with a formula for  $C$ .

Asymptotics of the form  $C\rho^{-n}n^{-3/2}$  have since been widely found for classes of rooted trees with recursive definitions and hence (22) is sometimes known as the universal law. Some notable examples generalizing the reach of Pólya’s analysis include [3], [8], and [17]. Surveys include [5] and [18]. See also chapter VII of Flajolet and Sedgewick’s forthcoming book *Analytic Combinatorics* currently available as an online draft [7]<sup>1</sup>.

For all but the last of the above the focus is on improved analytic conditions which still must be checked for any particular case of interest. One approach for

<sup>1</sup>References are drawn from the draft of October 23, 2006.

further generalization is to specify certain combinatorial constructions and then use these constructions to build recursive equations with solutions which automatically satisfy the universal law. This is the approach of [1]. Flajolet and Sedgewick also use a framework of combinatorial constructions to build their recursive equations, getting the universal law for various schemas, such as the exp-log schema [7, VII.1].

**2.2. Recursive systems.** Another approach to generalizing Pólya’s analysis is to move to recursive systems of equations while restricting the complexity of each equation. This is of particular relevance to applications to counting Feynman diagrams since generally only polynomials and geometric series are needed, but systems are difficult to avoid.

First note that any geometric series can be converted to a polynomial at the expense of a new variable. Namely replace  $1/(1 - X)$  with a new variable  $F$  and add the equation

$$F = 1 + F \cdot X.$$

In view of this reduction we can focus on polynomial systems.

Nonnegative polynomial recursive systems give the universal law under some reasonable conditions, as was shown independently by Drmota[6], Lalley[16], and Woods[25]. For our purposes the full generality of the above are not necessary and we’ll follow the presentation of Flajolet and Sedgewick [7, VII.6.3].

Suppose

$$\begin{aligned} y_1 &= \Phi_1(x, y_1, \dots, y_m) \\ &\vdots \\ y_m &= \Phi_m(x, y_1, \dots, y_m) \end{aligned}$$

with the  $\Phi_i$  polynomials with real coefficients.

There are five conditions which together guarantee that each component solution to this system satisfies the universal law. First we say the system is *nonlinear* if at least one of the  $\Phi_i$  is nonlinear in  $y_1, \dots, y_m$ . The system is *nonnegative* if each  $\Phi_i$  has nonnegative coefficients.

The next condition guarantees that the system does in fact behave recursively. For  $\bar{y} = (y_1, \dots, y_m) \in \mathbb{R}[[x]]^m$  define the  $x$ -valuation by  $\text{val}(\bar{y}) = \min_i(\text{val}(y_i))$  where the valuation of a series picks out the index of the first nonzero coefficient, i.e.  $\text{val}(\sum_{n=k}^{\infty} a_n x^n) = k$  when  $a_k \neq 0$ , and with the convention that  $\text{val}(0) = \infty$ . Also define  $d(\bar{y}, \bar{y}') = 2^{-\text{val}(\bar{y} - \bar{y}')}$ . Then the system is *proper* if

$$d(\Phi(\bar{y}), \Phi(\bar{y}')) < K d(\bar{y}, \bar{y}') \quad \text{for some } K < 1.$$

To guarantee that the system behaves as one system rather than many we need to guarantee that all variables play a role in each equation. Specifically, define the dependency graph of the system to be the directed graph on  $\{1, \dots, m\}$  with an edge from  $k$  to  $j$  if  $y_j$  figures in a monomial of  $\Phi_k(x, \bar{y})$ . The system is *irreducible* if the dependency graph is strongly connected, that is there is a path between any two vertices following edges only in their forward direction.

Finally, to remove spurious zero coefficients in the solution series, and to avoid extra singularities on the circle of convergence, define a power series  $\mathbf{T}(x)$  to be *aperiodic* if it is not the case that there is a power series  $\mathbf{U}(x)$  and integers  $a \geq 0$  and  $d \geq 2$  such that  $\mathbf{T}(x) = x^a \mathbf{U}(x^d)$ . If the solutions for each  $y_i$  of the system are aperiodic then the system itself is said to be *aperiodic*.



Then

**Theorem 1** ([7] Theorem VII.6). *Suppose  $\bar{y} = \Phi(\bar{y})$  is a polynomial system that is nonlinear, proper, nonnegative, and irreducible. Then all component solutions  $y_j$  have the same radius of convergence  $\rho < \infty$  and have a square root singularity at  $\rho$ . If furthermore the system is aperiodic then all  $y_j$  satisfy the universal law.*

An outline of the proof is as follows. For more details see [7, Theorem VII.6]. Properness gives that the component solutions are unique, because the solutions can be generated recursively from the zero vector, and along with nonnegativity we get further that the component solutions have nonnegative coefficients. Irreducibility, along with nonnegativity to prevent cancellations, gives that the radius of each  $y_i$  is the same value  $\rho$ . Aperiodicity forces  $\rho$  to be the only singularity of each component solution on the circle of convergence.

The key to the proof is the square root singularity at  $\rho$ . Each of the three independent proofs takes a different approach, though in all cases, as in the single equation situation, it comes down to using the failure of the implicit function theorem at  $\rho$ . Nonlinearity is necessary to avoid singularities which themselves are linear rather than of square root type.

Drmotá [6] proceeds by solving a subset of the equations for the remaining variables and then substituting back into the remaining equations to reduce the number of equations in the system. Iteratively he is able to reduce to one equation at which point the system can be treated classically.

Lalley [16], summarized in [7, Theorem VII.6], considers the linearized Jacobian

$$\left( \partial_{y_j} \Phi_i(x, y_1, \dots, y_m) \right)$$

and uses Perron-Frobenius theory to show that the largest eigenvalue in absolute value at  $x = \rho$  of the Jacobian is precisely 1 and that there is an eigenvector  $v$  with positive coefficients. Then multiplying the system by  $v$  and expanding around  $\rho$  gives the desired asymptotics.

Woods [25] also uses the Jacobian and the largest eigenvalue 1. He continues the analysis on block upper triangular matrices in order to deal with certain non-irreducible cases.

### 3. THE UNIVERSAL LAW FOR FEYNMAN DIAGRAMS

**3.1. QED with one primitive per loop order.** Massless quantum electrodynamics provides three monomials in the Lagrangian which need renormalization, corresponding as one-particle irreducible Green functions to the inverse fermion and photon propagators, and the vertex. We do not yet impose the Ward identity and proceed by setting up the combinatorial structure as it rather typically holds for a theory with three-valent vertices and two types of edges (the degenerate case of a single type of edge is easily interfered).

The smallest Hopf algebra which allows for a correct renormalization of the full theory treats the sum of all primitive graphs at a given loop order as a single Hochschild cocycle, and we start from there. This situation is described by the

following system.

$$\begin{aligned} X_1 &= 1 + \sum_{k \geq 1} x^k \frac{X_1^{2k+1}}{(1-X_2)^{2k}(1-X_3)^k} \\ X_2 &= x \frac{X_1}{(1-X_2)(1-X_3)} \\ X_3 &= x \frac{X_1}{(1-X_2)^2} \end{aligned}$$

After converting the geometric series we get

$$\Phi = \begin{cases} X_1 &= 1 + X_1 F_1 \\ F_1 &= x X_1^2 F_2^2 F_3 + x X_1^2 F_2^2 F_3 F \\ X_2 &= x X_1 F_2 F_3 \\ F_2 &= 1 + F_2 X_2 \\ X_3 &= x X_1 F_2^2 \\ F_3 &= 1 + F_3 X_3 \end{cases}$$

$\Phi$  is nonnegative, polynomial, nonlinear, and irreducible.  $\Phi$  can be seen to be aperiodic simply by calculating the first few terms.  $\Phi$  itself is not proper. However  $\Phi^2$  describes the same solutions and the other properties, nonnegative, polynomial, nonlinear, irreducible, and aperiodic, remain true for powers.

To see that  $\Phi^2$  is proper suppose we have 2 vectors

$$v = (x_1, f_1, x_2, f_2, x_3, f_3)$$

and

$$v' = (x'_1, f'_1, x'_2, f'_2, x'_3, f'_3)$$

at distance  $2^{-n}$ ; so write  $x'_1 = x_1 + x''_1$  with  $x''_1$  having no term of degree less than  $n$  and similarly for the other coordinates. Consider the difference in  $X_1$  coordinates after applying  $\Phi$ :

$$x_1 f_1 - x'_1 f'_1 = -x''_1 f''_1 - x''_1 f_1 - x_1 f''_1$$

which has no terms of degree less than  $n$ . Further if  $f_1$  has no constant term and  $f''_1$  has no term of degree  $n$  then the difference in the  $X_1$  coordinate has no terms of degree less than  $n+1$ . Argue similarly for  $F_2$  and  $F_3$ . For the  $X_2$  coordinate after applying  $\Phi$  we get a difference of:

$$x x_1 f_2 f_3 - x x'_1 f'_2 f'_3 = -x x''_1 f''_2 f''_3 - x x''_1 f''_2 f_3 - x x''_1 f_2 f''_3 - x x_1 f''_2 f''_3 - x x_1 f''_2 f_3 - x x_1 f_2 f''_3$$

which has no terms of degree less than  $n+1$ . Notice also that the new  $X_2$  coordinate has no constant term for both initial vectors. Argue similarly for  $X_3$  and  $F_1$ .

Now consider applying  $\Phi$  a second time. Apply the above arguments again but notice that we are in the ‘‘further’’ case for  $X_1$ ,  $F_2$  and  $F_3$ , so all coordinates now have a difference with no terms of degree less than  $n+1$ , that is the distance has decreased by at least  $1/2$ . Thus any  $1 > K > 1/2$  will give that  $\Phi^2$  is proper.

Consequently by Theorem 1 we know that all 6 solution series to  $\Phi^2$  and hence all 3 solution series to the original system have the same radius of convergence  $\rho$  and have coefficients with the asymptotic form

$$C \rho^{-n} n^{-3/2}$$

In order to understand the asymptotic growth rate for the QED system with primitives summed at each loop order it remains to understand the radius  $\rho$ . Substituting  $X_2$  and  $X_3$  into  $F_2$  and  $F_3$  respectively we get  $F_2 = 1 + xX_1F_2^2F_3 = F_3$ . Thus the original system can be rewritten

$$\begin{aligned} X_1 &= 1 + \frac{xX_1^3F_2^3}{1 - xX_1^2F_2^3} \\ F_2 &= 1 + xX_1F_2^3 \end{aligned}$$

Rearrange

$$X_1 = 1 + \frac{X_1^2(F_2 - 1)}{1 - X_1(F_2 - 1)}$$

then solve to get

$$F_2 = \frac{1 - 2X_1^2}{X_1(1 - 2X_1)}$$

Substitute back into the equation for  $F_2$  and expand to get

$$-x + X_1 + (6x - 5)X_1^2 + 8X_1^3 + (-12x - 4)X_1^4 + 8xX_1^6 = 0$$

As a polynomial in  $X_1$  this has discriminant

$$4096x^2(32x^2 - 8x + 1)(-2 + 27x)^2$$

So the radius of the system is

$$\frac{2}{27}$$

In view of the fact that the radius is an important value associated to the system and that this system is canonically associated to QED we're led to the following question as to what is the physical meaning of  $2/27$  in QED? The relevance of this number extends to any system which provides a recursive system similar to QED. We can proceed similarly for other theories; the results of some examples are summarized in Appendix A.

**3.2. Polynomially many primitives per loop order.** Combining all primitives at a given loop order into one Hochschild cocycle which drives the Dyson–Schwinger equation defines the smallest sub Hopf algebra which still renormalizes the full theory correctly. It is often instructive to disentangle the primitives in different ways, for example in accordance with the transcendental nature of  $\text{res}_p$ . This motivates to consider a slightly more general condition on the number of primitives, enough to apply the polynomial systems result. we next assume there to be  $p(k)$  primitives at  $k$  loops where  $p$  is a polynomial.

To see that this circumstance reduces to a nonnegative polynomial system it suffices to show that

$$\sum_{k \geq \ell} p(k)B^k$$

can be written as a sum of powers of geometric series. This follows from two facts.

First, the falling factorials

$$\{k(k-1) \cdots (k-n+1) : n \geq 0\}$$

form a basis for polynomials in  $k$ , and

$$k^n = \sum_{j=1}^n S_2(n, j)k(k-1) \cdots (k-j+1)$$

where  $S_2(n, j)$  are the Stirling numbers of the second kind, A008277 in [22], so in particular are nonnegative, and hence for  $p(k)$  with nonnegative coefficients we only need nonnegative coefficients of the falling factorials.

Second, notice that for  $\ell \geq n$

$$\begin{aligned} & \sum_{k \geq \ell} k(k-1) \cdots (k-n+1) B^k \\ &= B^n \frac{d^n}{dB^n} \sum_{k \geq \ell} B^k \\ &= B^n \sum_{j=0}^n \binom{n}{j} \left( \frac{d^j}{dB^j} \frac{1}{1-B} \right) \left( \frac{d^{n-j}}{dB^{n-j}} B^\ell \right) \\ &= \sum_{j=0}^n \ell(\ell-1) \cdots (\ell-n+j+1) \binom{n}{j} \left( \frac{1}{1-B} \right)^{j+1} B^{\ell+j} \end{aligned}$$

where all the coefficients are nonnegative.

One case where there is a natural interpretation is QED with a linear number of generators, namely

$$X_1 = 1 + \sum_{k \geq 1} p(k) x^k \frac{X_1^{2k+1}}{(1-X_2)^{2k} (1-X_3)^k}$$

with  $X_2$  and  $X_3$  as before and with  $p(k)$  linear, which corresponds to counting with Cvitanović's gauge invariant sectors [4].

**3.3. Other systems.** Johnson, Baker, and Willey [9] use gauge invariance to reduce the QED system to

$$X = x \sum_{k \geq 0} \left( \frac{x}{1-X} \right)^k = \frac{x(1-X)}{1-X-x}$$

While amenable to the universal law analysis, this recursive equation can be solved exactly by the quadratic formula. We get

$$X = \frac{1 + \sqrt{1-4x}}{2}$$

giving the Catalan numbers, A000108 in [22], as coefficients. The radius is  $1/4$  which is considerably larger than  $2/27$ , showing how powerful gauge invariance is. Note that it is only the inverse photon propagator  $1-X$  which needs renormalization, and that it appears in the denominator.

#### 4. THE GROWTH OF $\gamma_1$

After these considerations of the combinatorial side, we discuss analytic aspects.

**4.1. The recursions.** Consider the Dyson–Schwinger equation

$$X(x) = \mathbb{I} - \sum_{k \geq 1} \sum_{i=0}^{s_k} x^k p_i(k) B_+^{k,i} (XQ^k)$$

where  $Q = X^r$  with  $r < 0$  an integer, and  $p_i(k)$  coefficients, not necessarily polynomial. We'll use the notation  $F_{k,i}(\rho)$  for the Mellin transform of the integral kernel.

Starting with  $r < 0$  is justified in light of the last example of the previous section, we will generalize this soon enough.

Specializing [15, (26)] to this case we get the recursion

$$(23) \quad \gamma_k(x) = \frac{1}{k} \gamma_1(x) (1 + rx \partial_x) \gamma_{k-1}(x)$$

independently of the  $p_i(k)$ . Using the dot notation,  $\gamma \cdot U = \sum \gamma_k U^k$ , of [15] we have

$$(24) \quad \gamma \cdot L = \sum_k \sum_i x^k p_i(k) (1 + \gamma \cdot \partial_{-\rho})^{-rk+1} (1 - e^{-L\rho}) F_{k,i}(\rho)|_{\rho=0}$$

Taking one  $L$  derivative and setting  $L$  to 0 we get

$$(25) \quad \gamma_1 = \sum_k \sum_i x^k p_i(k) (1 + \gamma \cdot \partial_{-\rho})^{-rk+1} \rho F_{k,i}(\rho)|_{\rho=0}$$

Restricting to  $\rho F_{k,i}(\rho) = r_{k,i}/(1-\rho)$  allows us to write a much tidier recursion for  $\gamma_1$ . Taking two  $L$  derivatives of (24) and setting  $L = 0$  we get

$$\begin{aligned} 2\gamma_2 &= - \sum_k \sum_i x^k p_i(k) (1 + \gamma \cdot \partial_{-\rho})^{-rk+1} \rho^2 F_{k,i}(\rho)|_{\rho=0} \\ &= - \sum_k \sum_i x^k p_i(k) (1 + \gamma \cdot \partial_{-\rho})^{-rk+1} \rho F_{k,i}(\rho)|_{\rho=0} + \sum_k x^k r_{k,i} p_i(k) \\ &= -\gamma_1 + \sum_{k \geq 1} \sum_i r_{k,i} p_i(k) x^k \quad \text{from (25)} \end{aligned}$$

Thus we can from now on ignore the sum over  $i$  and let  $p(k) = \sum_i r_{k,i} p_i(k)$ . Then from (23)

$$\gamma_1 = \sum_{k \geq 1} p(k) x^k - 2\gamma_2 = \sum_{k \geq 1} p(k) x^k - \gamma_1 (1 + rx \partial_x) \gamma_1$$

giving

**Proposition 2.**

$$\gamma_{1,n} = p(n) + \sum_{j=1}^{n-1} (-rj - 1) \gamma_{1,j} \gamma_{1,n-j}$$

**4.2. Finding the radius.** We see from the proposition that if  $\sum p(k) x^k$  is Gevrey- $n$ , that is  $\sum x^k p(k)/(k!)^n$  converges, but not Gevrey- $(n-1)$ , then  $\gamma_1$  is at best Gevrey- $n$ .

Of most interest for our applications are the cases where only finitely many  $p(k)$  are nonzero but all are nonnegative and where  $p(k) = c^k k!$  giving the Lipatov bound; so  $\sum p(k) x^k$  is Gevrey-1. Assume that  $p(k) \geq 0$  and

$$\sum_{k \geq 1} x^k \frac{p(k)}{k!} = f(x)$$

has radius  $0 < \rho \leq \infty$  and  $f(x) > 0$  for  $|x| \leq \rho$ . The above two cases are included as  $f(x)$  a polynomial and  $f(x) = cx/(1-cx)$  respectively. In such circumstances  $\gamma_1$  is also Gevrey-1 and the radius is the minimum of  $\rho$  and  $-1/(ra_1)$  (where we view  $-1/(ra_1)$  as  $+\infty$  in the case  $a_1 = 0$ ) which we can see as follows<sup>2</sup>.

<sup>2</sup>Similarly  $\sum p(k) x^k$  Gevrey- $n$  for  $n > 1$  leads to  $\gamma_n$  Gevrey- $n$

Let  $a_n = \gamma_{1,n}/n!$ . Then  $a_1 = \gamma_{1,1} = p(1)$  and

$$\begin{aligned} a_n &= \frac{p(n)}{n!} + \sum_{j=1}^{n-1} (-rj - 1) \binom{n}{j}^{-1} a_j a_{n-j} \\ &= \frac{p(n)}{n!} + \frac{1}{2} \sum_{j=1}^{n-1} (-rj - 1 - r(n-j) - 1) \binom{n}{j}^{-1} a_j a_{n-j} \\ &= \frac{p(n)}{n!} + \left(-r\frac{n}{2} - 1\right) \sum_{j=1}^{n-1} \binom{n}{j}^{-1} a_j a_{n-j} \end{aligned}$$

To achieve an upper bound on the radius of convergence of  $\sum a_n x^n$  take the first and last terms of the sum to get

$$a_n \geq \frac{p(n)}{n!} - r \frac{n-2}{n} a_1 a_{n-1}$$

for  $n \geq 2$ . So the radius of  $\sum a_n x^n$  is no more than the radius of the recursively defined series with equality above, say

$$b_n = \frac{p(n)}{n!} - r \frac{n-2}{n} b_1 b_{n-1}$$

for  $n \geq 2$  with  $b_1 = a_1$ . Immediately we see that if  $a_1 = 0$  the radius of  $\sum b(n)x^n$  is  $\rho$ . Otherwise consider

$$n(n-1)b_n = \frac{p(n)n(n-1)}{n!} - r(n-1)(n-2)b_1 b_{n-1}$$

Equivalently with  $\mathbf{B}(x) = \sum b(n)x^n$  we get

$$\mathbf{B}''(x) = f''(x) - r b_1 x \mathbf{B}''(x)$$

Solving for  $\mathbf{B}''(x)$

$$\mathbf{B}''(x) = \frac{f''(x)}{1 + r a_1 x}$$

which, since differentiation does not change the radius of a series, has radius  $\min\{\rho, -1/(r a_1)\}$ , and thus so does  $\mathbf{B}(x)$ .

For the lower bound on the radius we need a few preliminary results. First a simple combinatorial fact.

**Lemma 3.** *Given  $0 < \theta < 1$*

$$\frac{1}{n} \binom{n}{j} \geq \frac{\theta^{-j+1}}{j}$$

for  $1 \leq j \leq \theta n$  and  $n \geq 2$ .

*Proof.* Fix  $n$ . Write  $j = \lambda n$ ,  $0 < \lambda \leq \theta$ . Then

$$\frac{1}{n} \binom{n}{j} = \frac{1}{n} \binom{n}{\lambda n} \geq \frac{n^{\lambda n - 1}}{(\lambda n)^{\lambda n}} = \frac{\lambda^{-\lambda n + 1}}{\lambda n} \geq \frac{\theta^{-j+1}}{j}$$

□

Second we need to understand the behaviour of  $\sum a_n x^n$  at the radius of convergence.

**Lemma 4.** *Using notation as above and with  $\mathbf{A}(x) = \sum a_n x^n$  with radius of convergence  $\rho_a$  we have that  $\limsup_{x \rightarrow \rho_a} \mathbf{A}(x)(1 + x r a_1)/f(x) \leq 1$*

*Proof.* Take any  $0 < \theta < 1/2$ . Using the previous lemma

$$\begin{aligned}
a_n &= \frac{p(n)}{n!} + \left(-r\frac{n}{2} - 1\right) \sum_{j=1}^{n-1} \binom{n}{j}^{-1} a_j a_{n-j} \\
&\leq \frac{p(n)}{n!} - rn \sum_{1 \leq j \leq \theta n} \binom{n}{j}^{-1} a_j a_{n-j} - r\frac{n}{2} \sum_{\theta n \leq j \leq n-\theta n} \binom{n}{j}^{-1} a_j a_{n-j} \\
&\leq \frac{p(n)}{n!} - r \sum_{1 \leq j \leq \theta n} j\theta^{j-1} a_j a_{n-j} - r \binom{n}{\theta n}^{-1} \frac{n}{2} \sum_{\theta n \leq j \leq n-\theta n} a_j a_{n-j} \\
&\leq \frac{p(n)}{n!} - r \sum_{1 \leq j \leq \theta n} j\theta^{j-1} a_j a_{n-j} - \frac{r}{2} n\theta^{\theta n} \sum_{\theta n \leq j \leq n-\theta n} a_j a_{n-j}
\end{aligned}$$

so the coefficients of  $\mathbf{A}(x)$  are bounded above by the coefficients of

$$f(x) - xr\mathbf{A}'(\theta x)\mathbf{A}(x) - \frac{r}{2} \left(x \frac{d}{dx} \mathbf{A}^2\right)(\theta^\theta x)$$

Since all coefficients are nonnegative, for any  $0 < x < \rho_a$  we have

$$\mathbf{A}(x) \leq f(x) - xr\mathbf{A}'(\theta x)\mathbf{A}(x) - \frac{r}{2} \left(x \frac{d}{dx} \mathbf{A}^2\right)(\theta^\theta x)$$

which is continuous in  $\theta$ , so for fixed  $0 < x < \rho_a$  we can let  $\theta \rightarrow 0$  giving

$$\mathbf{A}(x) \leq f(x) - xra_1\mathbf{A}(x)$$

so

$$\frac{\mathbf{A}(x)(1 + xra_1)}{f(x)} \leq 1$$

for  $0 < x < \rho_a$ . The result follows.  $\square$

Let  $\rho_a$  be the radius of  $\sum a_n x^n$ . If  $\rho_a = \rho$  then we're done, so suppose  $\rho_a < \rho$ . To get the lower bound on  $\rho_a$  it remains then to prove that  $\rho_a \geq -1/(ra_1)$  when  $a_1 \neq 0$  and to prove a contradiction in the case that  $a_1 = 0$ . Take any  $\epsilon > 0$ . Then there exists an  $N > 0$  such that for  $n > N$

$$\begin{aligned}
a_n &\leq \frac{p(n)}{n!} - ra_1 a_{n-1} - r \frac{1}{n-1} \sum_{j=2}^{n-2} a_j a_{n-j} \\
&\leq \frac{p(n)}{n!} - ra_1 a_{n-1} + \epsilon \sum_{j=2}^{n-2} a_j a_{n-j} \\
&\leq \frac{p(n)}{n!} - ra_1 a_{n-1} + \epsilon \sum_{j=1}^{n-1} a_j a_{n-j}
\end{aligned}$$

Define

$$c_n = \begin{cases} a_n & \text{if } a_n > \frac{p(n)}{n!} - rc_1 c_{n-1} + \epsilon \sum_{j=1}^{n-1} c_j c_{n-j} \\ \frac{p(n)}{n!} - rc_1 c_{n-1} + \epsilon \sum_{j=1}^{n-1} c_j c_{n-j} & \text{otherwise (in particular when } n > N) \end{cases}$$

In particular  $c_1 = a_1$ . The radius of  $\sum a_n x^n$  is at least as large as the radius of  $\mathbf{C}(x) = \sum c_n x^n$ . Rewriting with generating series

$$\mathbf{C}(x) = f(x) - ra_1 x \mathbf{C}(x) + \epsilon \mathbf{C}^2(x) + P_\epsilon(x)$$

where  $P_\epsilon(x)$  is some polynomial. This equation can be solved by the quadratic formula. The discriminant is

$$(26) \quad (ra_1x + 1)^2 - 4\epsilon(f(x) + P_\epsilon(x))$$

By assumption we are interested in  $|x| < \rho$  where (26) has no singularities, so the radius of  $\mathbf{C}(x)$  is the closest root to 0 of (26); call it  $\rho_\epsilon$ . Consider  $\epsilon = 0$  giving

$$(ra_1x + 1)^2$$

which has  $-1/(ra_1)$  as its closest root to 0 when  $a_1 \neq 0$ ; call this value  $\rho_0$ .

Then to get the lower bound on the radius of  $\sum a_n x^n$  it remains only to prove the following lemma.

**Lemma 5.** *With notation and assumptions as above if  $a_1 \neq 0$ ,  $\lim_{\epsilon \rightarrow 0} |\rho_\epsilon| \geq \rho_0$ , while if  $a_1 = 0$  we have a contradiction.*

*Proof.* In view of Lemma 4 this is a short exercise in analysis.

By construction the coefficient of  $x^n$  in  $P_\epsilon(x)$  is bounded by  $a_n + rc_1c_{n-1} \leq a_n + ra_1a_{n-1}$  since  $c_n \geq a_n$  for all  $n \geq 1$ . So  $P_\epsilon(x)$  has coefficients which are nonnegative and bounded by those of  $\mathbf{A}(x)(1 + rxa_1)$ . Thus by Lemma 4, the continuity of  $P_\epsilon(x)$  at  $\rho_a$ , and the assumption that  $\rho_a < \rho$ , we see that  $f(\rho_a) + P_\epsilon(\rho_a) \leq f(\rho_a) + \liminf_{x \rightarrow \rho_a} (\mathbf{A}(x)(1 + rxa_1)) < \infty$ . By the nonnegativity of the coefficients of  $f$  and  $P_\epsilon$  we can choose  $M > 0$  such that  $|f(x) + P_\epsilon(x)| < M$  independently of  $\epsilon$  for  $|x| \leq \rho_a$ .

Suppose  $a_1 \neq 0$ . Take any  $\eta > 0$ . Consider  $|x| \leq \rho_a$ . Choose  $\delta > 0$  such that  $(ra_1x + 1)^2 < \delta$  implies  $|x - \rho_0| < \eta$ . Pick  $\epsilon < \delta/(4M)$ . Then  $(ra_1\rho_\epsilon + 1)^2 = 4\epsilon(f(\rho_\epsilon) + P_\epsilon(\rho_\epsilon)) < \delta$  so  $|\rho_\epsilon - \rho_0| < \eta$

Suppose on the other hand that  $a_1 = 0$ . Take  $0 < \delta < 1$ . Then, since  $|\rho_\epsilon| \leq \rho_a$ , we get that for  $\epsilon < \delta/(4M)$ ,  $1 = 4\epsilon(f(\rho_\epsilon) + P_\epsilon(\rho_\epsilon)) < \delta$  which is a contradiction.  $\square$

Taking the two bounds together we get the final result

**Theorem 6.** *Assume  $\sum_{k \geq 1} x^k p(k)/k!$  has radius  $\rho$ . Then  $\sum x^n \gamma_{1,n}/n!$  converges with radius of convergence  $\min\{\rho, -1/(r\gamma_{1,1})\}$ , where  $-1/(r\gamma_{1,1})$  is interpreted to mean  $+\infty$  in the case  $\gamma_{1,1} = 0$ .*

**4.3. Nonnegative systems.** Now suppose we have a system of Dyson-Schwinger equations

$$X^r(x) = \mathbb{I} - \sum_{k \geq 1} \sum_{i=0}^{s_k} x^k p_i^r(k) B_+^{k,i;r} (X^r Q^k)$$

for  $r \in \mathcal{R}$  with  $\mathcal{R}$  a finite set,  $p_i^r(k) \geq 0$ , and where

$$Q = \prod_{r \in \mathcal{R}} X^r(x)^{s_r}$$

with integers  $s_r < 0$  for all  $r \in \mathcal{R}$ .

Then as before from [15, (26)] we have

$$(27) \quad \gamma_k^r(x) = \frac{1}{k} \left( \gamma_1^r(x) + \sum_{j \in \mathcal{R}} s_j \gamma_1^j(x) x \partial_x \right) \gamma_{k-1}^r(x)$$

again independent of the  $p^r(k)$ .

Assume that there is one insertion place, and so one variable  $\rho$ , and that the Mellin transform of the integral kernel is a geometric series  $\rho F_{k,i}^r(\rho) = r_{k,i;r}/(1-\rho)$ .



Rewriting the system of Dyson-Schwinger equations with dot notation we have

$$(28) \quad \gamma^r \cdot L = \sum_k \sum_i x^k p_i^r(k) \prod_{j \in \mathcal{R}} (1 + \gamma^j \cdot \partial_{-\rho})^{-s_j k + 1} (1 - e^{-L\rho}) F_{k,i}^r(\rho)|_{\rho=0}$$

As before we can find tidier recursions for the  $\gamma_1^r$  by comparing the first and second  $L$  derivatives of (28). We get

$$\gamma_1^r = \sum_k \sum_i x^k p_i^r(k) \prod_{j \in \mathcal{R}} (1 + \gamma^j \cdot \partial_{-\rho})^{-s_j k + 1} \rho F_{k,i}^r(\rho)|_{\rho=0}$$

and

$$\begin{aligned} 2\gamma_2^r &= - \sum_k \sum_i x^k p_i^r(k) \prod_{j \in \mathcal{R}} (1 + \gamma^j \cdot \partial_{-\rho})^{-s_j k + 1} \rho^2 F_{k,i}^r(\rho)|_{\rho=0} \\ &= -\gamma_1^r + \sum_{k \geq 1} \sum_i r_{k,i;r} p_i^r(k) x^k \quad \text{since } \rho F_{k,i}^r(\rho) = \frac{r_{k,i;r}}{1-\rho} \end{aligned}$$

Thus letting  $p(k) = \sum_i r_{k,i;r} p_i(k)$  and using (27)

$$\gamma_1^r = \sum_{k \geq 1} p^k(k) x^k - 2\gamma_2^r = \sum_{k \geq 1} p^r(k) x^k - \gamma_1^r(x)^2 - \sum_{j \in \mathcal{R}} s_j \gamma_1^j(x) x \partial_x \gamma_1^r(x)$$

giving

$$\gamma_{1,n}^r = p^r(n) + \sum_{i=1}^{n-1} (-s_r i - 1) \gamma_{1,i}^r \gamma_{1,n-i}^r + \sum_{\substack{j \in \mathcal{R} \\ j \neq r}} \sum_{i=1}^{n-1} (-s_j i) \gamma_{1,n-i}^j \gamma_{1,i}^r$$

To attack the growth of the  $\gamma_1^r$  we will again assume that

$$\sum_{k \geq 1} x^k \frac{p^r(k)}{k!} = f^r(x)$$

has radius  $0 < \rho_r \leq \infty$  and  $f^r(x) > 0$  for  $|x| \leq \rho_r$ . We'll proceed by similar bounds to before.

Let  $a_n^r = \gamma_{1,n}^r/n!$ . Then

$$a_n^r = \frac{p^r(n)}{n!} + \sum_{i=1}^{n-1} (-s_r i - 1) a_i^r a_{n-i}^r \binom{n}{i}^{-1} + \sum_{\substack{j \in \mathcal{R} \\ j \neq r}} \sum_{i=1}^{n-1} (-s_j i) a_{n-i}^j a_i^r \binom{n}{i}^{-1}$$

Taking the last term in each sum we have

$$a_n^r \geq \frac{p^r(n)}{n!} - \left( \sum_{j \in \mathcal{R}} s_j a_1^j \right) \frac{n-2}{n} a_{n-1}^r$$

Let  $b_n^r$  be the series defined by  $b_1^r = a_1^r$  and equality in the above recursion. Let  $\mathbf{B}^r(x) = \sum b_n^r x^n$ . Then as before if  $\sum_{j \in \mathcal{R}} s_j a_1^j = 0$  the radius of  $\mathbf{B}(x)$  is  $\rho_r$  and otherwise consider

$$\mathbf{B}^r(x)'' = f^r(x)'' - \left( \sum_{j \in \mathcal{R}} s_j a_1^j \right) x \mathbf{B}^r(x)''$$

Solving for  $\mathbf{B}^r(x)''$  we get that the radius of  $\sum a_n^r x^n$  is at most

$$\min \left\{ \rho_r, \frac{-1}{\sum_{j \in \mathcal{R}} s_j a_1^j} \right\}$$

again interpreting the second possibility to be  $\infty$  when  $\sum_{j \in \mathcal{R}} s_j a_1^j = 0$ .

In the other direction take any  $\epsilon > 0$  then there exists an  $N > 0$  such that for  $n > N$  we get

$$a_n^r \leq \frac{p^r(n)}{n!} - \left( \sum_{j \in \mathcal{R}} s_j a_1^j \right) a_{n-1}^r + \epsilon \sum_{i=1}^{n-1} \sum_{j \in \mathcal{R}} a_i^r a_{n-i}^j$$

Taking  $\mathbf{C}^r(x)$  to be the series whose coefficients satisfy the above recursion with equality for when this gives a result  $\geq a_n^r$  and equal to  $a_n^r$  otherwise we get

$$\mathbf{C}^r(x) = f^r(x) - \left( \sum_{j \in \mathcal{R}} s_j a_1^j \right) x \mathbf{C}^r(x) + \epsilon \sum_{j \in \mathcal{R}} \mathbf{C}^r(x) \mathbf{C}^j(x) + P_\epsilon^r(x)$$

where  $P_\epsilon^r$  is a polynomial.

Summing over  $r$  we get a recursive equation for  $\sum_{r \in \mathcal{R}} \mathbf{C}^r(x)$  of the same form as in the single equation case. Note that since each  $\mathbf{C}^r$  is a series with nonnegative coefficients there can be no cancellation of singularities and hence the radius of convergence of each  $\mathbf{C}^r$  is at least that of the sum. The equivalent of Lemma 4 for this case follows from

$$\begin{aligned} \sum_{r \in \mathcal{R}} a_n^r &\leq \sum_{r \in \mathcal{R}} \frac{p^r(n)}{n!} - \sum_{j \in \mathcal{R}} s_j a_1^j \sum_{r \in \mathcal{R}} a_{n-1}^r + \sum_{r, j \in \mathcal{R}} \sum_{i=1}^{n-1} (-s_j i) a_{n-i}^j a_i^r \binom{n}{i}^{-1} \\ &\leq \sum_{r \in \mathcal{R}} \frac{p^r(n)}{n!} - \sum_{j \in \mathcal{R}} s_j a_1^j \sum_{r \in \mathcal{R}} a_{n-1}^r \\ &\quad + \max_j (-s_j) \sum_{i=2}^{n-2} i \binom{n}{i}^{-1} \left( \sum_{r \in \mathcal{R}} a_{n-i}^r \right) \left( \sum_{r \in \mathcal{R}} a_i^r \right) \\ &= \sum_{r \in \mathcal{R}} \frac{p^r(n)}{n!} - \sum_{j \in \mathcal{R}} s_j a_1^j \sum_{r \in \mathcal{R}} a_{n-1}^r \\ &\quad + \max_j (-s_j) n \sum_{2 \leq i \leq \theta n} \binom{n}{i}^{-1} \left( \sum_{r \in \mathcal{R}} a_{n-i}^r \right) \left( \sum_{r \in \mathcal{R}} a_i^r \right) \\ &\quad + \max_j (-s_j) \frac{n}{2} \sum_{\theta n \leq i \leq n - \theta n} \binom{n}{i}^{-1} \left( \sum_{r \in \mathcal{R}} a_{n-i}^r \right) \left( \sum_{r \in \mathcal{R}} a_i^r \right) \end{aligned}$$

for  $\theta$  as in Lemma 4 with  $\sum_{r \in \mathcal{R}} \mathbf{A}^r(x)$  in place of  $\mathbf{A}(x)$ , where  $\mathbf{A}^r(x) = \sum a^r(n) x^n$ . Then continue the argument as in Lemma 4 with  $\sum_{r \in \mathcal{R}} f^r(x)$  in place of  $f(x)$  and  $\max_j (-s_j)$  in place of  $-r$ , and using the second term to get the correct linear part as  $\theta \rightarrow 0$ .

Thus by the analysis of the single equation case we get a lower bound on the radius of  $\sum a_n^r x^n$  of  $\min_{s \in \mathcal{R}} \{\rho_s, -1/\sum_{j \in \mathcal{R}} s_j a_1^j\}$ . In particular if  $r \in \mathcal{R}$  is such that  $\rho_r$  is minimal we see that the radius of  $\sum a_n^r x^n$  is exactly  $\min\{\rho_r, -1/\sum_{j \in \mathcal{R}} s_j a_1^j\}$ .

Suppose the radius of  $\sum a_n^s x^n$  was strictly greater than that of  $\sum a_n^r x^n$ . Then we can find  $\beta > \delta > 0$  such that

$$a_n^r > \beta^n > \delta^n > a_n^s$$

for  $n$  sufficiently large. Pick a  $k \geq 1$  such that  $a_k^s > 0$ . Then

$$\delta^n > a_n^s \geq \frac{-s_r k! a_k^s}{n \cdots (n-k+1)} a_{n-k}^r > \frac{-s_r k! a_k^s}{n \cdots (n-k+1)} \beta^{n-k}$$

so

$$\frac{\delta^k}{-s_r a_k^s} \left( \frac{\delta}{\beta} \right)^{n-k} > \frac{k!}{n \cdots (n-k+1)}$$

which is false for  $n$  sufficiently large, giving a contradiction. Thus all the  $\sum a_n^s x^n$  have the same radius  $\min_{r \in \mathcal{R}} \{\rho_r, -1/\sum_{j \in \mathcal{R}} s_j a_1^j\}$

From this we can conclude that each  $\sum x^n \gamma_{1,n}^r / n!$  also converges with radius  $\min_{r \in \mathcal{R}} \{\rho_r, -1/\sum_{j \in \mathcal{R}} s_j \gamma_{1,1}^j\}$ , where the second possibility is interpreted as  $\infty$  when  $\sum_{j \in \mathcal{R}} s_j \gamma_{1,1}^j = 0$ .

**4.4. Systems with some  $s_r > 0$ .** Let us relax the restriction that  $s_r < 0$  and that  $p^r(n) \geq 0$ . It is now difficult to make general statements concerning the radius of convergence of the  $\sum a_n^r x^n$ . For example consider the system

$$\begin{aligned} a_n^1 &= \frac{p^1(n)}{n!} + \sum_{j=1}^{n-1} (2j-1) a_j^1 a_{n-j}^1 \binom{n}{j}^{-1} - \sum_{j=1}^{n-1} j a_j^1 a_{n-j}^2 \binom{n}{j}^{-1} \\ a_n^2 &= \frac{p^2(n)}{n!} - \sum_{j=1}^{n-1} (j+1) a_j^2 a_{n-j}^2 \binom{n}{j}^{-1} + \sum_{j=1}^{n-1} 2j a_j^2 a_{n-j}^1 \binom{n}{j}^{-1} \end{aligned}$$

so  $s_1 = -2$  and  $s_2 = 1$ . Suppose also that

$$\begin{aligned} p^2(2) &= 0 \\ a_1^1 &= a_1^2 \\ p^2(n) &= -2(n-1)! a_1^2 a_{n-1}^1 \end{aligned}$$

Then  $a_2^2 = 0$  and inductively  $a_n^2 = 0$  for  $n \geq 2$  so the system degenerates to

$$\begin{aligned} a_n^1 &= \frac{p^1(n)}{n!} + \sum_{j=1}^{n-1} (2j-1) a_j^1 a_{n-j}^1 \binom{n}{j}^{-1} - \frac{n-1}{n} a_1^1 a_{n-1}^1 \\ a_n^2 &= \begin{cases} a_1^1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We still have a free choice of  $p^1(n)$ , and hence control of the radius of the  $a^1$  series. On the other hand the  $a^2$  series trivially has infinite radius of convergence.

Generally, finding a lower bound on the radii of the solution series, remains approachable by the preceding methods while control of the radii from above is no longer apparent.

Precisely, for any  $\epsilon > 0$

$$\begin{aligned} |a_n^r| &\leq \frac{|p^r(n)|}{n!} + \left| - \sum_{j \in \mathcal{R}} s_j a_1^j \right| |a_{n-1}^r| + \sum_{i=1}^{n-2} |(-s_r i - 1)| |a_i^r| |a_{n-i}^r| \binom{n}{i}^{-1} \\ &\quad + \sum_{\substack{j \in \mathcal{R} \\ j \neq r}} \sum_{i=1}^{n-2} |(-s_j i)| |a_{n-i}^j| |a_i^r| \binom{n}{i}^{-1} \\ &\leq \frac{|p^r(n)|}{n!} + \left| - \sum_{j \in \mathcal{R}} s_j a_1^j \right| |a_{n-1}^r| + \epsilon \sum_{i=1}^{n-1} \sum_{j \in \mathcal{R}} |a_i^r| |a_{n-i}^j| \end{aligned}$$

So, for a lower bound on the radius we may proceed as in the nonnegative case using the absolute value of the coefficients and achieving that the radius of the  $\sum a_n^r x^n$ , and hence that of  $\sum x^n \gamma_{1,n}^r / n!$ , is at least

$$\min_{r \in \mathcal{R}} \left\{ \rho_r, \left| \frac{-1}{\sum_{j \in \mathcal{R}} s_j \gamma_{1,1}^j} \right| \right\}$$

where the second possibility is interpreted as  $\infty$  when  $\sum_{j \in \mathcal{R}} s_j \gamma_{1,1}^j = 0$ .

Note that this gives the lower bound on the radius of convergence as the minimum of the first instanton singularity (which one expects to be the radius for  $p(k)/k!$ ) and the inverse of the first term in the  $\beta$  function of the theory. Furthermore, we emphasize that Ward identities typically allow a restriction to systems where all  $s_r < 0$ . A more detailed discussion will be given in future work where the general approach described here will be discussed with regard to the specific details of the relevant renormalizable theories of interest. Finally, we note that the appearance of the inverse of the first term in the  $\beta$ -function makes sense: in the conformal case of a vanishing  $\beta$ -function we would not expect a constraint on the minimum of the radius to come from perturbation theory.

## 5. APPLICATIONS OF THE GROWTH OF $\gamma_1$

While expectations for the growth of  $p(k)/k!$  in terms of instanton singularities are routine in the context of path integral estimates, the path integral is merely a successful heuristic to parameterize our lack of understanding of quantum field theory. Rigorous estimates for the growth of superficially convergent Green functions, and hence the  $p(k)$ , can sometimes be obtained using constructive field theory, at least as bounds for the radius [26]. We emphasize that such results can be turned by our methods into similarly rigorous results for superficially divergent Green functions.

A more complete discussion, dedicated to the renormalizable quantum field theories in four dimensions, will be given elsewhere.

### APPENDIX A. OTHER THEORIES COMBINATORIALLY WITH ONE PRIMITIVE PER LOOP ORDER

For each of the following systems the solution series have coefficients satisfying the universal law. In the mixed  $\phi^3, \phi^4$  case there is one primitive per vertex per loop order. Unfortunately the full power of symmetry factors is not available in this simple combinatorial set-up, leading to the different variants.

Theory	System	Radius
$\phi^3$	$X_1 = 1 + X_1 F_1$ $F_1 = x X_1^2 F_2^3 (1 + F_1)$ $X_2 = \frac{x X_1 F_2^2}{2}$ $F_2 = 1 + F_2 X_2$	Smallest positive real root of $3581577x^4 - 4443984x^3 + 2332368x^2 - 539136x + 32768$ . Numerically 0.09061681898407704...
$\phi^4$	$X_1 = 1 + x^2 X_1^3 F_2^4 F_1 + \frac{3}{2} x X_1 F_2^2$ $F_1 = x X_1 F_2^2 (1 + F_1)$ $X_2 = x F_2 + x^2 X_1 F_2^3$ $F_2 = 1 + F_2 X_2$	Root of a degree 10 polynomial. Numerically 0.12968592295019730...
$\phi^4$ variant	$X_1 = 1 + x^2 X_1^3 F_2^4 F_1 + x X_1 F_2^2$ rest as in previous case	Root of a degree 9 polynomial. Numerically 0.13856076790723086...
mixed $\phi^3, \phi^4$	$X_1 = 1 + x X_1 X_2 F_3^2 + x X_1^2 F_3^3$ $+ 2x F_3^2 X_1$ $+ X_2 X_1^2 F_3 (F_a F_b + F_a + F_b$ $+ F_a x X_2^2 F_3^3 + F_b x X_2 F_3^2$ $+ x^2 X_1^2 X_2 F_3^5 + x X_1^2 F_3^3$ $+ x X_2 F_3^2)$ $X_2 = 1 + X_2 (F_a F_b + F_a + F_b$ $+ F_a x X_1^2 F_3^3 +$ $F_b x X_2 F_3^2 + x^2 X_1^2 X_2 F_3^5)$ $+ X_1^3 F_3 (F_a F_b + F_a + F_b$ $+ F_a x X_1^2 F_3^3 + F_b x X_2 F_3^2$ $+ x^2 X_1^2 X_2 F_3^5$ $+ x X_1^2 F_3^3 + x X_2 F_3^2)$ $+ \frac{3}{2} x X_2 F_3^2$ $F_a = x^2 X_2^2 F_3^4 + x X_2 F_3^2 F_a$ $F_b = x^2 X_1^4 F_3^6 + x X_1^2 F_3^3 F_b$ $X_3 = x X_1 F_3^2 + x F_3 + x^2 X_2 F_3^3$ $F_3 = 1 + X_3 F_3$	0.02145...

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