A vanishing theorem in positive characteristic and tilting equivalences

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A VANISHING THEOREM IN POSITIVE CHARACTERISTIC AND TILTING EQUIVALENCES

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Abstract. Let $\mathcal{D}_X$ be the sheaf of differential operators ([12]) on a smooth variety X over an algebraically closed field k of characteristic p. We show that if X is a smooth quadric or an incidence variety (a partial flag variety in type $\mathbf{A}_n$) then $\mathrm{H}^i(X, \mathcal{D}_X) = 0$ for $i > 0$. Some applications of this vanishing result to derived categories of coherent sheaves are given.

1. Introduction

Let X be a smooth variety over an algebraically closed field k, and $\mathcal{D}_X$ the sheaf of differential operators on X as defined in [12]. Denote $\mathcal{M}(\mathcal{D}_X)$ the category of left $\mathcal{D}_X$-modules. Objects of this category are quasicoherent sheaves on X equipped with a structure of left $\mathcal{D}_X$-modules. The variety X is said to be D-affine if for any $\mathcal{D}_X$-module $\mathcal{M}$ one has:

- $\mathcal{M}$ is generated by its global sections as a quasicoherent sheaf, that is the natural map $\mathrm{H}^0(X, \mathcal{M}) \otimes \mathcal{O}_X \to \mathcal{M}$ is surjective, and
- higher cohomology of $\mathcal{M}$ vanish, that is $\mathrm{H}^i(X, \mathcal{M}) = 0$ for $i > 0$.

Under these two conditions the functor of global sections Γ provides an equivalence of abelian categories:

$$\Gamma\colon \mathcal{D}_X - \mathrm{mod} \simeq \Gamma(\mathcal{D}_X) - \mathrm{mod}, \tag{1.1}$$

where the category in the right hand side is that of finitely generated modules over $\Gamma(\mathcal{D}_X)$, the ring of global differential operators on X.

The Beilinson–Bernstein theorem ([4]) says that if X is a compact homogeneous space of a complex semisimple Lie group $\mathbf{G}$ (that is $X = \mathbf{G}/\mathbf{P}$, where $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$), then X is D-affine. Hence, for homogeneous spaces one has an equivalence as in (1.1) (localization).

Assume now that the characteristic of the ground field k is non-zero. Consider a semisimple simply connected algebraic group $\mathbf{G}$ over k, and a parabolic subgroup $\mathbf{P} \subset \mathbf{G}$. One can ask whether the homogeneous space $\mathbf{G}/\mathbf{P}$ over k is D-affine in the above sense. Surprisingly enough, very little is known on this subject. In general, as opposed to the characteristic zero case, these spaces fail to be D-affine in positive characteristic, as was shown in [20]. Namely, a counterexample was constructed in *loc.cit.*: starting with the Grassmann variety $\mathrm{Gr}_{2,5}$, an explicit $\mathcal{D}_{\mathrm{Gr}_{2,5}}$-module is produced with a non-zero higher cohomology. We believe that what follows is an exhaustive list of the known results on the D-affinity of homogeneous spaces in positive characteristic. The D-affinity for projective spaces and the flag variety $\mathbf{SL}_3/\mathbf{B}$ was proved in ([13]). For the flag variety of the group in type $\mathbf{B}_2$ a necessary condition for the D-affinity was shown to hold in [2]. For flag varieties $\mathbf{G}/\mathbf{B}$, as was proved in [13], $\mathcal{D}_{\mathbf{G}/\mathbf{B}}$-modules are generated by its global sections. This implies, in particular, that for flag varieties the D-affinity is equivalent to the vanishing of higher

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cohomology of the sheaf $\mathcal{D}$. The above counterexample of [20] shows that for the flag variety of $\mathbf{SL}_5$ the sheaf of differential operators $\mathcal{D}_{\mathbf{SL}_5/\mathbf{B}}$ has a non-zero higher cohomology group.

From now on we fix an algebraically closed field k of characteristic p. All the varieties will be defined over k. The main result of this paper states the vanishing of higher cohomology of the sheaf of differential operators for a class of homogeneous spaces, namely for quadrics of dimension less or equal to four and incidence varieties. Recall that a quadric is a homogeneous space of an orthogonal group, and an incidence variety is a partial flag variety of the group in type $\mathbf{A}_n$ (the variety of partial flags of type $(1, n, n+1)$). The proof of these vanishing results essentially relies on properties of the algebra of "crystalline" differential operators ([7]). Modulo these properties the proof is quite elementary, and does not explicitly use representation theory of Lie algebras in positive characteristic. In fact, vanishing theorems for line bundles are our main tool. Let us sketch the proof. Recall that for a smooth variety X the algebra of crystalline differential operators D_X is an Azumaya algebra on the cotangent bundle to the Frobenius twist X' of X (D_X has a large center that is isomorphic to the algebra of functions on the cotangent bundle to X'); moreover, D_X splits when restricted to the zero section, the splitting bundle being isomorphic to $\mathsf{F}_*\mathcal{O}_X$ ([7]) (where $\mathsf{F}\colon X \to X'$ is the relative Frobenius morphism). Further, the sheaf of differential operators $\mathcal{D}_X$ carries an additional filtration to the usual filtration by degree of an operator, the p-filtration. The sheaf $\mathcal{D}_1$ – the first term of this filtration – is isomorphic to the central reduction of the algebra D_X. This allows to compute cohomology of $\mathcal{D}_1$ using the Koszul resolution of the zero section in the cotangent bundle. The Koszul complex is a filtered complex, and its associated graded complex is quasiisomorphic to the structural sheaf of the Frobenius neighbourhood of the zero section in the cotangent bundle (cf. *loc.cit.*). We prove that in the cases under consideration the latter sheaf has vanishing higher cohomology. This immediately implies the needed vanishing for the first term of the p-filtration. Using results of the Berthelot theory of arithmetic differential operators one shows, by the same argument, that higher terms of the p-filtration has no higher cohomology either. We are done, the sheaf of differential operators being the direct limit of terms of the p-filtration.

As we have seen above, the sheaf $\mathcal{D}_1$ – the first term of the p-filtration – is isomorphic to $\mathcal{E}nd_{\mathcal{O}_X}(\mathsf{F}_*\mathcal{O}_X)$. The vanishing of higher cohomology of $\mathcal{D}_1$ thus implies:

$$\mathrm{E}xt^i(\mathsf{F}_*\mathcal{O}_X, \mathsf{F}_*\mathcal{O}_X) = 0, \tag{1.2}$$

for $i > 0$. Recall that a coherent sheaf $\mathcal{E}$ on a variety X such that $\mathrm{E}xt^i(\mathcal{E}, \mathcal{E}) = 0$ for $i > 0$ is called an almost exceptional sheaf. If, in addition, it generates $\mathrm{D}^b(X)$, the bounded derived category of coherent sheaves on X, then $\mathcal{E}$ is called a tilting sheaf, or a tilting generator ([18]). It is well-known that such a sheaf provides an equivalence between $\mathrm{D}^b(X)$ and the derived category of finitely generated modules over a non-commutative associative algebra A – the algebra of global endomorphisms of the sheaf $\mathcal{E}$ (e.g, *loc.cit.*, Lemma 1.2).

Let $\mathbf{G}$ be a semisimple simply connected algebraic group over k. Assume that $p > h$, where h is the Coxeter number of $\mathbf{G}$. The main results of [7] and [8] imply immediately that for a homogeneous space $\mathbf{G}/\mathbf{P}$ the bundle $\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{P}}$ generates the derived category of coherent sheaves $\mathrm{D}^b(\mathbf{G}/\mathbf{P})$ (see Section 6). The vanishing of higher cohomology of the sheaf $\mathcal{D}_1$ on $\mathbf{G}/\mathbf{P}$, therefore, can be rephrased as saying that the sheaf $\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{P}}$ is tilting. Putting together our knowledge of cohomology of the sheaf $\mathcal{D}_1$ on homogeneous spaces, we are able to state:

Theorem 1.1. Let X be a homogeneous space of one of the following types:

- Projective space $\mathbb{P}^n$
- The flag variety $\mathbf{G}/\mathbf{B}$ of the group $\mathbf{G}$ in type $\mathbf{B}_2$
- Quadric of dimension ≤ 4
- Incidence variety of type $(1, n, n+1)$

and assume $p > h$. Then $\mathsf{F}_*\mathcal{O}_X$ is a tilting bundle in $\mathrm{D}^b(X)$.

Proof. The fact that $\mathsf{F}_*\mathcal{O}_X$ is a generator for all homogeneous spaces is proved in Lemma 6.2 below. The vanishing of higher cohomology of the sheaf $\mathcal{D}_1$ for projective spaces and the flag variety in type $\mathbf{B}_2$ follows from [13] and [2], respectively. Theorem 4.1 and Theorem 5.1 of the present paper give the necessary vanishing for quadrics and incidence varieties. □

Corollary 1.1. *Let X be as in Theorem 1.1. Then one has an equivalence of derived categories:*

$$\mathrm{D}^b(X) \simeq \mathrm{D}^b(\mathrm{End}(\mathsf{F}_*\mathcal{O}_X) - \mathrm{mod}). \tag{1.3}$$

Proof. Follows from Theorem 1.1 and Lemma 1.2 of [18]. □

It should be noted that derived categories of coherent sheaves of varieties from Theorem 1.1 have been known explicitly for a long time. Namely, in all these cases complete exceptional collections are known to exist ([3], [19]). However, there are a great many exceptional collections in these derived categories: indeed, the braid group is known to act on the set of exceptional collections ([9]). The sheaf $\mathsf{F}_*\mathcal{O}_X$, however, is associated to a variety X via an intrinsic procedure. The equivalence furnished by the sheaf $\mathsf{F}_*\mathcal{O}_X$ (when it holds, of course) is, in some sense, a distinguished one. In a forthcoming paper ([26]) we show that there are examples of non-homogeneous (Fano) varieties when $\mathsf{F}_*\mathcal{O}_X$ gives as well a tilting object.

It is an old result of Hartshorne saying that for projective spaces $\mathbb{P}^n$ the sheaf $\mathsf{F}_*\mathcal{O}_{\mathbb{P}^n}$ splits into the direct sum of line bundles:

$$\mathsf{F}_*\mathcal{O}_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{p_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(-n)^{p_n}, \tag{1.4}$$

where p_i's are certain multiplicities that depend only on p (these are polynomials on p). Obviously, this implies Theorem 1.1 for $\mathbb{P}^n$ (by the Beilinson theorem for projective spaces ([3])). Note also that the converse implication is also true, i.e. the tilting property of the bundle $\mathsf{F}_*\mathcal{O}_{\mathbb{P}^n}$ implies decomposition (1.4). For quadrics Theorem 1.1 implies a similar decomposition of $\mathsf{F}_*\mathcal{O}$ into the direct sum of line bundles and spinor bundle(s) (cf. [19]). It would be tempting to conjecture that the sheaf $\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{P}}$ is tilting for homogeneous spaces $\mathbf{G}/\mathbf{P}$. At present, however, there are too few examples in favour of this guess. Hopefully, future research will reveal further ones. Finally, let us remark that the vanishing result for differential operators that we prove in the present paper was known previously ([13]) only for projective spaces $\mathbb{P}^n$ and for the flag variety of the group $\mathbf{SL}_3$ (the latter variety being an incidence variety for $n = 3$).

The paper is organized as follows. In Section 2 we recall some facts from linear algebra and state vanishing theorems for line bundles that we will need. In Section 3 we recall the definition and main properties of differential operators. We then pass to crystalline differential operators and show a link between cohomology vanishing for differential operators on homogeneous spaces and that for line bundles on cotangent bundles to these spaces. The proofs of vanishing theorems for quadrics and incidence varieties occupy Sections 4 and 5, respectively. In the last section we discuss applications of these vanishing results to derived categories of coherent sheaves.

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2. Preliminaries

Throughout we fix an algebraically closed field k of characteristic p.

2.1. **Frobenius morphism.** The *absolute Frobenius morphism* on a scheme is the identity on point spaces and raising to the p-th power locally on functions. The absolute Frobenius morphism is not a morphism of k-schemes. Let $\pi : X \to \mathrm{Spec}(k)$ be a scheme. Let X' be the scheme obtained from X by base change with the absolute Frobenius morphism on $\mathrm{Spec}(k)$, i.e., the underlying topological space of X' is that of X with the same structure sheaf $\mathcal{O}_X$ of rings, only the underlying k-algebra structure on $\mathcal{O}_{X'}$ is twisted as $\lambda \odot f = \lambda^{1/p} f$, for $\lambda \in k$ and $f \in \mathcal{O}_{X'}$. Using this description of X', *the relative Frobenius morphism* $\mathsf{F} : X \to X'$ is defined in the same way as the absolute Frobenius morphism and it is a morphism of k-schemes.

2.2. **Koszul resolutions.** Let V be a finite dimensional vector space over k with a basis $\{e_1, \dots, e_n\}$. Recall that the *r-th exterior power* $\wedge^r V$ of V is defined to be the r-th tensor power $V^{\otimes r}$ of V divided by the submodule generated by the elements:

$$u_1 \otimes \dots \otimes u_r - (-1)^{\mathrm{sgn}\sigma} u_{\sigma_1} \otimes \dots \otimes u_{\sigma(r)}$$

for all the permutations $\sigma \in \Sigma_r$ and $u_1, \dots, u_r \in V$. Similarly, the *r-th symmetric power* $\mathsf{S}^r V$ of V is defined to be the r-th tensor power $V^{\otimes r}$ of V divided by the submodule generated by the elements

$$u_1 \otimes \dots \otimes u_r - u_{\sigma_1} \otimes \dots \otimes u_{\sigma(r)}$$

for all the permutations $\sigma \in \Sigma_r$ and $u_1, \dots, u_r \in V$.

Let

$$0 \to V' \to V \to V'' \to 0 \tag{2.1}$$

be a short exact sequence of vector spaces. For any $n > 0$ there is a functorial exact sequence (the Koszul resolution, ([16], II.12.12))

$$\dots \to \mathsf{S}^{n-i} V \otimes \wedge^i V' \to \dots \to \mathsf{S}^{n-1} V \otimes V' \to \mathsf{S}^n V \to \mathsf{S}^n V'' \to 0. \tag{2.2}$$

Another fact about symmetric and exterior powers is the following ([15], Exercise 5.16). For a short exact sequence as (2.1) one has for each n the filtrations

$$\mathsf{S}^n V = F_n \supset F_{n-1} \supset \dots \quad \text{and} \quad \bigwedge^n V = F'_n \supset F'_{n-1} \supset \dots \tag{2.3}$$

such that

$$F_i/F_{i-1} \simeq \mathsf{S}^{n-i}V' \otimes \mathsf{S}^i V'' \tag{2.4}$$

and

$$F'_i/F'_{i-1} \simeq \bigwedge^{n-i} V' \otimes \bigwedge^{i} V'' \tag{2.5}$$

When either V' or V'' is a one-dimensional vector space, these filtrations on exterior powers of V degenerate into short exact sequences. If V'' is one-dimensional then one obtains:

$$0 \to \wedge^r V' \to \wedge^r V \to \wedge^{r-1} V' \otimes V'' \to 0. \tag{2.6}$$

Similarly, if V' is one-dimensional, the filtration above degenerates to give a short exact sequence:

$$0 \to \wedge^{r-1} V'' \otimes V' \to \wedge^r V \to \wedge^r V'' \to 0. \tag{2.7}$$

Remark 2.1. There is a general characteristic-free definition of the so-called Schur complexes ([1]), of which the Koszul resolution (2.2) is a particular example. One should be careful when dealing with these complexes in positive characteristic as divided and symmetric powers of a module differ in this case. In particular, Lemma 2.1. from [17], which the cited paper relies on, holds for divided powers, and not for symmetric powers as it was stated in *loc.cit.*

2.3. **Vanishing theorems for line bundles.** We recall here some vanishing theorems for line bundles that we will need. Let us first introduce some notations. Let $\mathbf{G}$ be a connected, simply connected, semisimple algebraic group, $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$, and $\mathbf{T}$ a maximal torus. Let $R(\mathbf{T}, \mathbf{G})$ be the root system of $\mathbf{G}$ with respect to $\mathbf{T}$, R^+ the subset of poisitive roots, $S \subset R^+$ the simple roots, and h the Coxeter number of $\mathbf{G}$. By $\langle \cdot, \cdot \rangle$ we denote the natural pairing $X(\mathbf{T}) \times Y(\mathbf{T}) \to \mathbb{Z}$, where $X(\mathbf{T})$ is the group of characters (also identified with the weight lattice) and $Y(\mathbf{T})$ the group of one parameter subgroups of $\mathbf{T}$ (also identified with the coroot lattice). For a subset $I \subset S$ let $\mathbf{P} = \mathbf{P}_I$ denote the associated parabolic subgroup. Recall that the group of characters $X(\mathbf{P})$ of $\mathbf{P}$ can be identified with $\{\lambda \in X(\mathbf{T}) | \langle \lambda, \alpha^\vee \rangle = 0, \text{ for all } \alpha \in I\}$. In particular, $X(\mathbf{B}) = X(\mathbf{T})$. A weight $\lambda \in X(\mathbf{B})$ is called *dominant* if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $s \in S$. A dominant weight $\lambda \in X(\mathbf{P})$ is called $\mathbf{P}$-regular if $\langle \lambda, \alpha^\vee \rangle > 0$ for all $s \notin I$, where $\mathbf{P} = \mathbf{P}_I$ is a parabolic subgroup.

Recall that the prime p is *a good prime* for $\mathbf{G}$ if p is coprime to all the coefficients of the highest root of $\mathbf{G}$ written in terms of the simple roots. For simple $\mathbf{G}$, p is a good prime if $p \geq 2$ for type $\mathbf{A}$ and $p \geq 3$ for type $\mathbf{B}$.

Here is a vanishing theorem by Kumar et al. ([21], Theorem 5) that we will rely on in proving our statements:

Theorem 2.1. *Let X be a homogeneous space $\mathbf{G}/\mathbf{P}$, $\mathrm{T}^*(X)$ the total space of the cotangent bundle of X, and $\pi\colon \mathrm{T}^*(X) \to X$ the projection. Assume that char k is a good prime for $\mathbf{G}$. Let $\lambda \in X(\mathbf{P})$ be a $\mathbf{P}$-regular weight. Then*

$$\mathrm{H}^i(\mathrm{T}^*(X), \pi^*\mathcal{L}(\lambda)) = \mathrm{H}^i(X, \mathcal{L}_\lambda \otimes \pi_*\mathcal{O}_{\mathrm{T}^*(X)}) = 0, \tag{2.8}$$

for $i > 0$.

Here $\mathcal{L}_\lambda$ denotes a line bundle that corresponds to the weight λ. In particular, if $X = \mathbf{G}/\mathbf{B}$ then one has:

$$\mathrm{H}^i(\mathrm{T}^*(X), \mathcal{O}_{\mathrm{T}^*(X)}) = 0, \tag{2.9}$$

for $i > 0$.

Remark 2.2. If $\mathbf{P} \supset \mathbf{B}$ is a proper parabolic sugroup then the vanishing of higher cohomology of the sheaf $\mathcal{O}_{\mathrm{T}^*(\mathbf{G}/\mathbf{P})}$ on $\mathrm{T}^*(\mathbf{G}/\mathbf{P})$ is not known in general. Recall that this is known in characteristic zero ([10]). A cruical tool to prove such a vanishing is the Grauert–Riemenschneider theorem which is not valid in positive characteristic. There are some cases, however, when the sheaf $\mathcal{O}_{\mathrm{T}^*(\mathbf{G}/\mathbf{P})}$ can be shown to have no higher cohomology. First, this holds in type $\mathbf{A}_n$ as nilpotent orbits are normal in type $\mathbf{A}_n$ in positive characteristic ([24], Propositions 4.6 and 4.9). Secondly, if a parabolic subgroup $\mathbf{P}$ corresponds to a simple short root then higher cohomology of $\mathcal{O}_{\mathrm{T}^*(\mathbf{G}/\mathbf{P})}$ vanish as well ([21], Theorem 6). It is believed, however, that the vanishing of higher cohomology of $\mathcal{O}_{\mathrm{T}^*(\mathbf{G}/\mathbf{P})}$ must hold for any parabolic subgroup $\mathbf{P}$ ([11], Comments to Chapter 5). More generally, the vanishing as in (2.8) should hold for any dominant line bundle $\mathcal{L}_\lambda$.

3. Differential operators

3.1. **Generalities.** This subsection is taken from [22]. Let R be a commutative algebra over k.

Definition 3.1. The ring of k-linear differential operators $D_k(R)$ on R is an $R \otimes_k R$-subalgebra of $\mathrm{End}_k(R)$ defined by

$$D_k(R) = \{\phi \in \mathrm{End}_k(R) : I^n \cdot \phi = 0,\ n \gg 0\},$$

where I denotes the kernel of the product map $R \otimes_k R \to R$.

The $R \otimes_k R$-submodules

$$D_k^n(R) = \{\phi \in \mathrm{End}_k(R) : I^{n+1} \cdot \phi = 0\},$$

defines a filtration of $D_k(R)$. Elements in $D_k^n(R)$ are called differential operators of degree $\leq n$. When I is a finitely generated ideal there is a second filtration of $D_k(R)$ given by the $R \otimes_k R$-submodules

$$D_k^{(n)}(R) = \{\phi \in \mathrm{End}_k(R) : I^{(n+1)} \cdot \phi = 0\},$$

where $I^{(n)}$ denotes the ideal in $R \otimes_k R$ generated by elements of the form a^n, $a \in I$. This filtration is particularly nice when the characteristic p of k is positive. In this case $I^{(p^n)}$ is generated by elements of the form $a^{p^n} \otimes 1 - 1 \otimes a^{p^n}$, and hence $D_k^{(p^n-1)}(R) = \mathrm{End}_{R^{p^n}}(R)$, where R^{p^n} denotes the subring of R of p^n-powers (here we use that k is algebraically closed and hence perfect). In particular,

$$D_k(R) = \bigcup_n \mathrm{End}_{R^{p^n}}(R).$$

The right side of this equation shows that $D_k(R)$ is independent of k, and we therefore suppress k from the notation and write $D(R)$ instead of $D_k(R)$.

Lemma 3.1. *Assume that k has positive characteristic p and that R is a finitely generated k-algebra. For every multiplicative subset S of R there exists a natural isomorphism of left R_S-modules*

$$(D(R))_S \simeq D(R_S),$$

where the localization on the left is performed as a left R-module.

Proof. Fix a positive integer n. As R is a finitely generated k-algebra it is finitely generated as a module over the subring R^{p^n}. This implies that the exists a natural isomorphism

$$\mathrm{End}_{R^{p^n}}(R)_S \simeq \mathrm{End}_{R_S^{p^n}}(R_S),$$

Now conclude the argument by using the description of $D(R)$ above (in positive characteristic). □

Let X be a variety over k. The sheaf of k-linear differential operators $\mathcal{D}_X$ on X ([12]) is an $\mathcal{O}_X$-subalgebra of $\mathcal{E}nd_k(\mathcal{O}_X)$ which is quasicoherent for both $\mathcal{O}_X$-module structures. If $X = \mathrm{Spec}(R)$ is affine, the sheaf $\mathcal{D}_X$ coincides with the quasicoherent $\mathcal{O}_X$-bialgebra associated to the R-bialgebra $D_k(R)$ defined above.

Recall that for a variety X over k the relative Frobenius morphism $\mathsf{F}\colon X \to X'$ is defined (here X' is the Frobenius twist of X). It follows from the above that

$$\mathcal{D}_X = \bigcup_n \mathcal{E}nd_{\mathcal{O}_X}(\mathsf{F}_{n*}\mathcal{O}_X), \tag{3.1}$$

where $\mathsf{F}_n = \mathsf{F} \circ \cdots \circ \mathsf{F}\colon X \to X^{(n)}$, ($n$ times) and $X^{(n)}$ is the n-th Frobenius twist of X.

Denote $\mathcal{D}_r = \mathcal{E}nd_{\mathcal{O}_X}(\mathsf{F}_{r*}\mathcal{O}_X)$. Recall that a variety X is said to be Frobenius split if the homomorphism $\mathcal{O}_{X'} \to \mathsf{F}_*\mathcal{O}_X$ of $\mathcal{O}_{X'}$-modules is split. Hence $\mathcal{O}_{X'}$ is a direct summand in $\mathsf{F}_*\mathcal{O}_X$. The following statement was shown in [2] (Proposition, Sec.1):

Lemma 3.2. *Let X be a Frobenius split variety. Then*

$$H^i(X, \mathcal{D}_X) = 0 \Leftrightarrow H^i(X, \mathcal{D}_r) = 0 \textit{ for any } r \in \mathbb{N}. \tag{3.2}$$

Homogeneous spaces of semisimple algebraic groups are Frobenius split ([23]).

Remark 3.1. Lemma 3.2 and the above counterexample to the D-affinity ([20]) imply that for the flag variety $\mathbf{SL}_5/\mathbf{B}$ there is a non-zero higher cohomology group:

$$\mathrm{H}^1(\mathbf{SL}_5/\mathbf{B}, \mathcal{D}_r) \neq 0, \tag{3.3}$$

for some $r \geq 1$. A priori, one can say nothing about these r's. Hence, higher cohomology of the sheaf $\mathcal{D}_1$ may still vanish.

3.2. **Crystalline differential operators.** The material of this subsection is taken from [7]. We recall, following *loc.cit.* (see also [6] and [8]), some properties of crystalline differential operators (differential operators without divided powers, or PD-differential operators in the terminology of Berthelot and Ogus).

Let X be a smooth variety, $\mathcal{T}_X^*$ the cotangent bundle, and $\mathrm{T}^*(X)$ the total space of $\mathcal{T}_X^*$.

Definition 3.2. The sheaf D_X of *crystalline differential operators* on X is defined as the enveloping algebra of the tangent Lie algebroid, i.e., for an affine open $U \subset X$ the algebra $\mathrm{D}(U)$ contains the subalgebra $\mathcal{O}$ of functions, has an $\mathcal{O}$-submodule identified with the Lie algebra of vector fields

$Vect(U)$ on U, and these subspaces generate $\mathrm{D}(U)$ subject to relations $\xi_1\xi_2 - \xi_2\xi_1 = [\xi_1, \xi_2] \in Vect(U)$ for $\xi_1, \xi_2 \in Vect(U)$, and $\xi \cdot f - f \cdot \xi = \xi(f)$ for $\xi \in Vect(U)$ and $f \in \mathcal{O}(U)$.

If $\mathrm{char}(k) = 0$ then D_X is isomorphic to the sheaf of differential operators $\mathcal{D}_X$ defined in the previous subsection. If $\mathrm{char}(k) = p$ then D_X shares some features with the characteristic zero case; for example, D_X carries an increasing filtration "by order of a differential operator", and the associated graded $gr(\mathrm{D}_X) \cong \mathcal{O}_{\mathrm{T}^*X}$ canonically. On the other hand, some phenomena are special to the characteristic p setting. We have an action map $\mathrm{D}_X \to \mathcal{E}nd_k(\mathcal{O}_X)$, which is not injective, unlike the characteristic zero case. For example, if $X = \mathbb{A}^1 = Spec(k[x])$, the section $\partial_x^p \neq 0$ of D_X acts by zero on $\mathcal{O}$. Further, the algebra $\mathsf{F}_*\mathrm{D}_X$ is known to have a large center, which is isomorphic to $\mathsf{F}_*\pi_*\mathcal{O}_{\mathrm{T}^*(X)} = \mathrm{gr}(\mathsf{F}_*\mathrm{D}_X)$ (here $\pi\colon \mathrm{T}^*(X) \to X$ denotes the projection). Thus, there exists a sheaf of algebras $\mathbb{D}_X$ on $\mathrm{T}^*(X')$ such that $\pi_*\,\mathbb{D}_X = \mathsf{F}_*\mathrm{D}_X$ (by abuse of notation we denote the projection $\mathrm{T}^*(X') \to X'$ by the same letter π). Moreover, $\mathbb{D}_X$ is an Azumaya algebra on $\mathrm{T}^*(X')$. Let $i\colon X' \hookrightarrow \mathrm{T}^*(X')$ be the zero section embedding. Then $i^*\,\mathbb{D}_X$ splits as an Azumaya algebra, the splitting bundle being $\mathsf{F}_*\mathcal{O}_X$. In other words, $i^*\,\mathbb{D}_X = \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_X)$.

3.3. **Cohomology of the Frobenius neighbourhood.** Our goal is to study the cohomology of the sheaf $\mathcal{D}_X$ for homogeneous spaces. Consider the first term of the p-filtration, that is the bundle $\mathcal{E}nd(\mathsf{F}_*\mathcal{O}_X)$. Given the above, one has:

$$\mathrm{H}^k(X', \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_X)) = \mathrm{H}^k(X', i^*\,\mathbb{D}_X) = \mathrm{H}^k(\mathrm{T}^*(X'), i_*i^*\,\mathbb{D}_X) = \mathrm{H}^k(\mathrm{T}^*(X'), \mathbb{D}_X \otimes i_*\mathcal{O}_{X'}), \tag{3.4}$$

the last isomorphism in (3.4) follows from the projection formula. Consider the bundle $\pi^*\mathcal{T}^*_{X'}$. There is a tautological section s of this bundle such that the zero locus of s coincides with X'. Hence, one obtains the Koszul resolution:

$$0 \to \det\,(\pi^*\mathcal{T}_{X'}) \to \cdots \to \wedge^k(\pi^*\mathcal{T}_{X'}) \to \wedge^{k-1}(\pi^*\mathcal{T}_{X'}) \to \cdots \to \mathcal{O}_{\mathrm{T}^*(X')} \to i_*\mathcal{O}_{X'} \to 0. \tag{3.5}$$

Let us tensor the resolution (3.5) by the sheaf $\mathbb{D}_X$. The cohomology group on the right in (3.4) can be computed via the above Koszul resolution. The following lemma, which reduces cohomology of a filtered complex to those of its associated graded one, is standard:

Lemma 3.3. *For $j \geq 0$ one has:*

$$\mathrm{H}^j(\mathrm{T}^*(X), \mathsf{F}^* \wedge^k (\pi^*\mathcal{T}_{X'})) = 0 \quad \Rightarrow \quad \mathrm{H}^j(\mathrm{T}^*(X'), \mathbb{D}_X \otimes \wedge^k (\pi^*\mathcal{T}_{X'})) = 0. \tag{3.6}$$

Proof. We need to compute the hypercohomology of the complex $C^k\colon = \wedge^k(\pi^*\mathcal{T}_{X'}) \otimes \mathbb{D}_X, k = -n, \dots, 0$ (where $n = \dim X$). Take the direct image of the complex $C^\bullet$ with respect to π:

$$\mathrm{R}^\bullet\pi_*C^k = \mathrm{R}^\bullet\pi_*(\wedge^k(\pi^*\mathcal{T}_{X'}) \otimes \mathbb{D}_X) = \pi_*(\wedge^k(\pi^*\mathcal{T}_{X'}) \otimes \mathbb{D}_X) = \mathsf{F}_*\mathrm{D}_X \otimes \wedge^k(\mathcal{T}_{X'}), \tag{3.7}$$

the morphism π being affine. The complex $\mathrm{R}^\bullet\pi_*C^k = \mathsf{F}_*\mathrm{D}_X \otimes \wedge^k(\mathcal{T}_{X'})$ is a filtered complex, the associated complex being isomorphic to $\mathrm{gr}(\mathsf{F}_*\mathrm{D}_X) \otimes \wedge^k(\mathcal{T}_{X'}) = \mathsf{F}_*\pi_*\mathcal{O}_{\mathrm{T}^*(X)} \otimes \wedge^k(\mathcal{T}_{X'})$. Clearly, for $i \geq 0$

$$\mathrm{H}^i(X', \mathrm{gr}(\mathsf{F}_*\mathrm{D}_X) \otimes \wedge^k(\mathcal{T}_{X'})) = 0 \quad \Rightarrow \quad \mathrm{H}^i(X', \mathsf{F}_*\mathrm{D}_X \otimes \wedge^k(\mathcal{T}_{X'})) = 0. \tag{3.8}$$

There are isomorphisms:

$$\mathrm{H}^i(X', \mathsf{F}_*\pi_*\mathcal{O}_{\mathrm{T}^*(X)} \otimes \wedge^k(\mathcal{T}_{X'})) = \mathrm{H}^i(X, \pi_*\mathcal{O}_{\mathrm{T}^*(X)} \otimes \mathsf{F}^* \wedge^k (\mathcal{T}_{X'})) = \mathrm{H}^i(\mathrm{T}^*_X, \pi^*\mathsf{F}^* \wedge^k (\mathcal{T}_{X'})). \tag{3.9}$$

The last group is isomorphic to $\mathrm{H}^i(\mathrm{T}^*_X, \mathsf{F}^*\pi^* \wedge^k (\mathcal{T}_{X'}))$, hence the statement of the lemma. $\square$

Remark 3.2. The complex $\tilde{C}^k \colon = \mathsf{F}^*\pi^* \wedge^k (\mathcal{T}_{X'})$ is quasiisomorphic to $\mathsf{F}^* i_* \mathcal{O}_{X'}$, the structural sheaf of the Frobenius neighbourhood of the zero section. Below we show that if X is a smooth quadric of dimension ≤ 4 or an incidence variety then

$$\mathrm{H}^j(\mathrm{T}^*(X), \mathsf{F}^* i_* \mathcal{O}_{X'}) = 0 \tag{3.10}$$

for $j > 0$.

4. Quadrics

Fix a notation first. In the next two sections we denote $\pi \colon \mathrm{T}^*(X) \to X$ the projection, X being a scheme and $\mathrm{T}^*(X)$ the cotangent bundle to X.

Theorem 4.1. *Let Q_n be a smooth quadric of dimension $n \leq 4$. Assume that p is an odd prime. Then $\mathrm{H}^i(\mathsf{Q}_n, \mathcal{D}_{\mathsf{Q}_n}) = 0$ for $i > 0$.*

Proof. A smooth quadric $\mathsf{Q}_n \subset \mathbb{P}(V)$ is a homogeneous space, and a hypersurface of degree two in $\mathbb{P}(V)$. We first prove that $\mathrm{H}^i(\mathsf{Q}_n, \mathcal{D}_1) = 0$ for $i > 0$. Consider an adjunction sequence

$$0 \to \mathcal{T}_{\mathsf{Q}_n} \to \mathcal{T}_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathsf{Q}_n} \to \mathcal{O}_{\mathsf{Q}_n}(2) \to 0, \tag{4.1}$$

and the Euler sequence on $\mathbb{P}(V)$ restricted to Q_n:

$$0 \to \mathcal{O}_{\mathsf{Q}_n} \to V \otimes \mathcal{O}_{\mathsf{Q}_n}(1) \to \mathcal{T}_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathsf{Q}_n} \to 0. \tag{4.2}$$

Show first that $\mathrm{H}^i({\mathsf{Q}_n}', \mathsf{F}_*\pi_*\mathcal{O}_{\mathrm{T}^*(\mathsf{Q}_n)}) = 0$ for $i > 0$. Clearly, $\mathrm{Q}_1 = \mathbb{P}^1$ and $\mathrm{Q}_2 = \mathbb{P}^1 \times \mathbb{P}^1$. For projective spaces the vanishing of higher cohomology of differential operators was proved in ([13]). If $n = 3$ then the quadric Q_3 is a homogeneous space of the group in type $\mathbf{B}_2$ such that the parabolic subgroup $\mathbf{P}$, defining Q_3, corresponds to the short simple root in the root system of type $\mathbf{B}_2$. As was discussed in Remark 2.2, for a parabolic subgroup $\mathbf{P}$ that correspond to a short simple root in the given root system the sheaf $\mathcal{O}_{\mathrm{T}^*(\mathbf{G}/\mathbf{P})}$ has vanishing higher cohomology ([21], Theorem 6). For $n = 4$ the quadric $\mathbf{Q}_4$ is isomorphic to the Grassmann variety $\mathrm{Gr}_{2,4}$, that is to a homogeneous space of the group $\mathbf{SL}_4$. Recall that for a parabolic subgroup $\mathbf{P}$ of the group $\mathbf{SL}_m$ the sheaf $\mathcal{O}_{\mathrm{T}^*(\mathbf{SL}_\mathrm{m}/\mathbf{P})}$ has vanishing higher cohomology ([24], Propositions 4.6 and 4.9, cf. also Remark 2.2). Hence, for $n \leq 4$ one obtains:

$$\mathrm{H}^i({\mathsf{Q}_n}', \mathsf{F}_*\pi_*\mathcal{O}_{\mathrm{T}^*(\mathsf{Q}_n)}) = \mathrm{H}^i(\mathsf{Q}_n, \pi_*\mathcal{O}_{\mathrm{T}^*(\mathsf{Q}_n)}) = \mathrm{H}^i(\mathsf{Q}_n, \mathsf{S}^\bullet \mathcal{T}_{\mathsf{Q}_n}) = 0 \quad \text{for} \quad i > 0. \tag{4.3}$$

The sequence (4.1) gives rise to a short exact sequence (see Section 2):

$$0 \to \wedge^r \mathcal{T}_{\mathsf{Q}_n} \to \wedge^r \mathcal{T}_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathsf{Q}_n} \to \wedge^{r-1} \mathcal{T}_{\mathsf{Q}_n} \otimes \mathcal{O}_{\mathsf{Q}_n}(2) \to 0. \tag{4.4}$$

Let us show that $\mathrm{H}^i({\mathsf{Q}_n}', \wedge^k(\mathcal{T}_{{\mathsf{Q}_n}'}) \otimes \mathsf{F}_*\pi_*\mathcal{O}_{\mathrm{T}^*(\mathsf{Q}_n)}) = 0$ for $i > r$. We argue by induction on r, the base of induction being $r = 1$. Tensoring the sequence (4.1) (on ${\mathsf{Q}_n}'$) with $\mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n}$, one gets:

$$0 \to \mathcal{T}_{{\mathsf{Q}_n}'} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n} \to \mathcal{T}_{\mathbb{P}(V)} \otimes \mathcal{O}_{{\mathsf{Q}_n}'} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n} \to \mathcal{O}_{{\mathsf{Q}_n}'}(2) \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n} \to 0. \tag{4.5}$$

By Theorem 2.1, one has for $i > 0$:

$$\mathrm{H}^i({\mathsf{Q}_n}', \mathcal{O}_{{\mathsf{Q}_n}'}(2) \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n}) = \mathrm{H}^i(\mathsf{Q}_n, \mathcal{O}_{\mathsf{Q}_n}(2p) \otimes \mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n}) = 0, \tag{4.6}$$

Tensoring sequence (4.2) (on ${\mathsf Q_n}'$) with $\mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n}$ gives:

$$0 \to \mathcal{O}_{{\mathsf Q_n}'} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n} \to V \otimes \mathcal{O}_{{\mathsf Q_n}'}(1) \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n} \to \mathcal{T}_{\mathbb{P}(V)} \otimes \mathcal{O}_{{\mathsf Q_n}'} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n} \to 0. \tag{4.7}$$

We see that $\mathrm{H}^i(\mathsf{Q}_n', V \otimes \mathcal{O}_{{\mathsf Q_n}'}(1) \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n}) = 0$ for $i > 0$, the bundle $\mathcal{O}_{\mathsf{Q}_n}(1)$ being ample. We saw above that $\mathrm{H}^i(\mathsf{Q}_n, \mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n}) = 0$ for $i > 0$. Hence, $\mathrm{H}^i({\mathsf Q_n}', \mathcal{T}_{\mathbb{P}(V)} \otimes \mathcal{O}_{{\mathsf Q_n}'} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n}) = 0$ for $i > 0$. From sequence (5.7) we conclude that $\mathrm{H}^i({\mathsf Q_n}', \mathcal{T}_{{\mathsf Q_n}'} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n}) = 0$ for $i > 1$. The same arguments show that $\mathrm{H}^i({\mathsf Q_n}', \mathcal{T}_{{\mathsf Q_n}'} \otimes \mathcal{O}_{{\mathsf Q_n}'}(2) \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n}) = 0$ for $i > 1$. Assume that for $r \leq m$ one has $\mathrm{H}^i({\mathsf Q_n}', \wedge^k(\mathcal{T}_{{\mathsf Q_n}'}) \otimes \mathsf{F}_*\pi_*\mathcal{O}_{\mathrm{T}^*(\mathsf{Q}_n)}) = 0$ for $i > r$. Let us prove that $\mathrm{H}^i({\mathsf Q_n}', \wedge^{r+1}(\mathcal{T}_{{\mathsf Q_n}'}) \otimes \mathsf{F}_*\pi_*\mathcal{O}_{\mathrm{T}^*(\mathsf{Q}_n)}) = 0$ for $i > r+1$. Tensoring sequence (5.6) (on ${\mathsf Q_n}'$) with $\mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n}$, one gets:

$$0 \to \wedge^{r+1}\mathcal{T}_{{\mathsf Q_n}'} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n} \to \wedge^{r+1}\mathcal{T}_{\mathbb{P}(V)} \otimes \mathcal{O}_{{\mathsf Q_n}'} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n} \to \wedge^r\mathcal{T}_{{\mathsf Q_n}'} \otimes \mathcal{O}_{{\mathsf Q_n}'}(2) \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n} \to 0. \tag{4.8}$$

By the inductive assumption we can also assume that $\mathrm{H}^i({\mathsf Q_n}', \wedge^r(\mathcal{T}_{{\mathsf Q_n}'}) \otimes \mathcal{O}_{{\mathsf Q_n}'}(2) \otimes \mathsf{F}_*\pi_*\mathcal{O}_{\mathrm{T}^*(\mathsf{Q}_n)}) = 0$ for $i > r$. Let us show that $\mathrm{H}^i({\mathsf Q_n}', \wedge^{r+1}\mathcal{T}_{\mathbb{P}(V)} \otimes \mathcal{O}_{{\mathsf Q_n}'} \otimes \mathsf{F}_*\mathsf{S}^l\mathcal{T}_{\mathsf{Q}_n}) = 0$ for $i > 0$. The long exact sequence associated to sequence (4.8) will imply $\mathrm{H}^i({\mathsf Q_n}', \wedge^{r+1}(\mathcal{T}_{{\mathsf Q_n}'}) \otimes \mathsf{F}_*\pi_*\mathcal{O}_{\mathrm{T}^*(\mathsf{Q}_n)}) = 0$ for $i > r+1$. From sequence (4.2) one obtains a short exact sequence:

$$0 \to \wedge^{j-1}\mathcal{T}_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathsf{Q}_n} \to \wedge^j V \otimes \mathcal{O}_{\mathsf{Q}_n}(j) \to \wedge^j\mathcal{T}_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathsf{Q}_n} \to 0. \tag{4.9}$$

Glueing together these short exact sequences for $j = 1, \dots r+1$, one gets an exact sequence:

$$0 \to \mathcal{O}_{\mathsf{Q}_n} \to V \otimes \mathcal{O}_{\mathsf{Q}_n}(1) \to \wedge^2 V \otimes \mathcal{O}_{\mathsf{Q}_n}(2) \to \cdots \to \wedge^{r+1}\mathcal{T}_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathsf{Q}_n} \to 0. \tag{4.10}$$

Tensoring the above sequence (on ${\mathsf Q_n}'$) with $\mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n}$, we conclude that $\mathrm{H}^i({\mathsf Q_n}', \wedge^{r+1}\mathcal{T}_{\mathbb{P}(V)} \otimes \mathcal{O}_{{\mathsf Q_n}'} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_{\mathsf{Q}_n}) = 0$ for $i > 0$. Using Lemma 3.3, we get $\mathrm{H}^i(\mathsf{Q}_n, \mathcal{D}_1) = 0$ for $i > 0$.

To finish the proof of Theorem 4.1, recall that in the work [5] Berthelot constructed sheaves of arithmetic differential operators $\bar{\mathrm{D}}_k$ of level k. The construction yields that $\bar{\mathrm{D}}^k$ is an Azumaya algebra on $\mathrm{T}^*(X^k)$, where X^k is the k-th Frobenius twist of X. Further, the Azumaya algebra $\bar{\mathrm{D}}_k$ splits when restricted to the zero section, the splitting bundle being $\mathsf{F}^k_*\mathcal{O}_X$. In other words, the central reduction of $\bar{\mathrm{D}}_k$ is isomorphic to $\mathcal{D}_k$. The sheaf $\bar{\mathrm{D}}_1$ is isomorphic to $\mathbb{D}$, the sheaf of crystalline differential operators from Section 3. The arguments used above for the sheaf $\mathcal{D}_1$ on a quadric can be applied as well to the sheaves $\mathcal{D}_k$. We thus get $\mathrm{H}^i(\mathsf{Q}_n, \mathcal{D}_k) = 0$ for $i > 0$. Remembering (3.1), one gets the statement. □

Remark 4.1. The vanishing of higher cohomology of $\mathcal{D}_1$ for quadrics of dimesnion $n \leq 4$ and for all p was established earlier, by a different methdod, in [25].

Remark 4.2. Undoubetdly, Theorem 4.1 should hold for quadrics of any dimension. If we knew the vanishing (4.3) for arbitrary n then all the other arguments would apply verbatim. Unfortunately, at present, we are not able to show that (4.3) holds for any n, though it is very likely that there exists an elementary proof to this fact. This question will be addressed in a subsequent paper.

5. Incidence varieties

Let V be a vector space over k of dimension n, and X the flag variety $\mathbb{F}_{1,n-1,n}$, a smooth divisor in $\mathbb{P}(V) \times \mathbb{P}(V^*)$ of bidegree $(1,1)$.

Theorem 5.1. $\mathrm{H}^i(X, \mathcal{D}_X) = 0$ *for* $i > 0$.

Proof. We argue as in the proof of Theorem 4.1. Consider the line bundle $\mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1)$ over $\mathbb{P}(V) \times \mathbb{P}(V^*)$. Then X is isomorphic to the zero locus of a section of $\mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1)$. One has an adjunction sequence:

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} \otimes \mathcal{O}_X \to \mathcal{O}_X(1) \boxtimes \mathcal{O}_X(1) \to 0. \tag{5.1}$$

Note that $\mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} = \mathcal{T}_{\mathbb{P}(V)} \boxplus \mathcal{T}_{\mathbb{P}(V^*)}$. One has the Euler sequences:

$$0 \to \mathcal{O}_{\mathbb{P}(V)} \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to \mathcal{T}_{\mathbb{P}(V)} \to 0, \tag{5.2}$$

and

$$0 \to \mathcal{O}_{\mathbb{P}(V^*)} \to V^* \otimes \mathcal{O}_{\mathbb{P}(V^*)}(1) \to \mathcal{T}_{\mathbb{P}(V^*)} \to 0. \tag{5.3}$$

Hence a short exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} \oplus \mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} \to V^* \otimes \mathcal{O}_{\mathbb{P}(V^*)}(1) \boxplus \mathcal{O}_{\mathbb{P}(V)}(1) \otimes V \to \mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} \to 0 \tag{5.4}$$

We have already seen that in type **A** one has the cohomology vanishing:

$$\mathrm{H}^i(X', \mathsf{F}_*\pi_*\mathcal{O}_{\mathrm{T}^*(X)}) = \mathrm{H}^i(X, \pi_*\mathcal{O}_{\mathrm{T}^*(X)}) = \mathrm{H}^i(X, \mathsf{S}^\bullet\mathcal{T}_X) = 0 \tag{5.5}$$

for $i > 0$. Similarly, the sequence (5.1) gives rise to a short exact sequence:

$$0 \to \wedge^r\mathcal{T}_X \to \wedge^r\mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V)} \otimes \mathcal{O}_X \to \wedge^{r-1}\mathcal{T}_X \otimes \mathcal{O}_X(1) \boxtimes \mathcal{O}_X(1) \to 0. \tag{5.6}$$

As above, let us show that $\mathrm{H}^i(X'), \wedge^k(\mathcal{T}_{X'}) \otimes \mathsf{F}_*\pi_*\mathcal{O}_{\mathrm{T}^*(X)}) = 0$ for $i > r$. We argue by induction on r, the base of induction being $r = 1$. Tensoring the sequence (5.1) (on X') with $\mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X$, one gets:

$$0 \to \mathcal{T}_{X'} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X \to \mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X \otimes \mathcal{O}_{X'} \to \mathcal{O}_{X'}(1) \boxtimes \mathcal{O}_X(1) \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X \to 0. \tag{5.7}$$

By Theorem 2.1, one has for $i > 0$:

$$\mathrm{H}^i(X', \mathcal{O}_{X'}(1) \boxtimes \mathcal{O}_X(1) \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X) = \mathrm{H}^i(X, \mathcal{O}_{X'}(p) \boxtimes \mathcal{O}_X(p) \otimes \mathsf{S}^\bullet\mathcal{T}_X) = 0, \tag{5.8}$$

On the other hand, tensoring sequence (5.9) (considered first on $\mathbb{P}(V) \times \mathbb{P}(V^*)'$ and then restricted to X') with $\mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X$ gives:

$$0 \to \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X^{\oplus 2} \to (V^* \otimes \mathcal{O}_{\mathbb{P}(V^*)}(1) \boxplus \mathcal{O}_{\mathbb{P}(V)}(1) \otimes V) \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X \to \mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)'} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X \to 0 \tag{5.9}$$

The leftmost term in the above sequence has vanishing higher cohomology by (5.5). The middle term is the bundle $\mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X$ tensored with a semiample sheaf on X (i.e. the sheaf isomorphic to either $\mathcal{O}_X(k) \boxtimes \mathcal{O}_X$ or $\mathcal{O}_X \boxtimes \mathcal{O}_X(k)$. Let $\mathcal{L}$ be a semiample line bundle on X. For any $l \geq 1$ there is a filtration on $\mathsf{S}^l(\mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} \otimes \mathcal{O}_X)$, the graded factors of this filtration being isomorphic to $\mathsf{S}^i(\mathcal{T}_X) \otimes (\mathcal{O}_X(l-i) \boxtimes \mathcal{O}_X(l-i))$. Tensoring $\mathsf{S}^l(\mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} \otimes \mathcal{O}_X)$ with $\mathcal{L}$, the graded factors $\mathsf{S}^i\mathcal{T}_X \otimes (\mathcal{O}_X(l-i) \boxtimes \mathcal{O}_X(l-i))$ get twisted by $\mathcal{L}$. For $i < l$ the higher cohomology of the corresponding graded factor vanish by Theorem 2.1. On the other hand, the higher cohomology of $\mathsf{S}^l(\mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} \otimes \mathcal{O}_X) \otimes \mathcal{L}$ are easily seen to vanish (use the Koszul resolutions associated to the Euler sequences (5.2) and (5.3), and the Kempf theorem ([14])). One obtains that $\mathrm{H}^i(X, \mathsf{S}^\bullet\mathcal{T}_X \otimes \mathcal{L}) = 0$ for $i > 1$. Hence, the cohomology of the middle term in (5.5) vanishes for $i > 1$, and $\mathrm{H}^i(X', \mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)'} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X) = 0$

for $i > 1$. Coming back to sequence (5.7), one gets $\mathrm{H}^i(X', \mathcal{T}_{X'} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X) = 0$ for $i > 1$. The inductive step is completely analogous to that in the proof of Theorem 4.1 (one uses sequence (5.9) to ensure that $\mathrm{H}^i(X', \wedge^{r+1}\mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)'} \otimes \mathcal{O}_{X'} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X) = 0$ for $i > 1$). This allows to complete the induction argument.

□

6. Tilting equivalences

6.1. **Tilting sheaves.** Recall some facts about tilting sheaves. The definition and lemma below are taken from ([18]).

Definition 6.1. A coherent sheaf $\mathcal{E}$ on X is called a *tilting generator* of the bounded derived category $\mathcal{D}^b(X)$ of coherent sheaves on X if the following holds:

(1) The sheaf $\mathcal{E}$ is a tilting object in $\mathcal{D}^b(X)$ – that is, for any $i \geq 1$ we have $\mathrm{E}xt^i(\mathcal{E}, \mathcal{E}) = 0$
(2) The sheaf $\mathcal{E}$ generates the derived category $\mathcal{D}^b(X)$ of complexes bounded from above – that is, if for some object $\mathcal{F} \in \mathcal{D}^b(X)$ we have $\mathrm{R}Hom^\bullet(\mathcal{E}, \mathcal{F}) = 0$, then $\mathcal{F} = 0$.

Tilting sheaves are a tool to construct derived equivalences. One has:

Lemma 6.1. *Let X be a smooth scheme, $\mathcal{E}$ a tilting generator of the derived category $\mathcal{D}^b(X)$, and denote $R = \mathcal{E}nd(\mathcal{E})$. Then the algebra R is left-Noetherian, and the correspondence $\mathcal{F} \mapsto \mathrm{R}Hom^\bullet(\mathcal{E}, \mathcal{F})$ extends to an equivalence*

$$\mathcal{D}^b(X) \to \mathcal{D}^b(R\text{-mod}^{\mathrm{fg}}) \tag{6.1}$$

between the bounded derived category $\mathcal{D}^b(X)$ of coherent sheaves on X and the bounded derived category $\mathcal{D}^b(R\text{-mod}^{\mathrm{fg}})$ of finitely generated left R-modules.

One of the main results of [7] is the derived Beilinson–Bernstein localization theorem (*loc.cit.*, Theorem 3.2) for crystalline differential operators. It asserts that for $p > h$, where h is the Coxeter number of $\mathbf{G}$, there is an equivalence of derived categories:

$$\mathrm{D}^b(X', \mathrm{D}_{X'}-\mathrm{mod}_c) \simeq \mathrm{D}^b(U_0(\mathfrak{g})\text{-mod}^{\mathrm{fg}}). \tag{6.2}$$

Here $X = \mathbf{G}/\mathbf{B}$, and $U_0(\mathfrak{g})$ is the central reduction (modulo the "Harish-Chandra part" of the center) of the universal enveloping algebra of $\mathfrak{g}$ ([7]). We refer the reader to *loc.cit.* for the definition of both categories in (6.2). A similar equivalence (singular localization) is proved for $\mathbf{G}/\mathbf{P}$ in [8]. An immediate corollary to these equivalences is the following:

Lemma 6.2. *Let $X = \mathbf{G}/\mathbf{P}$ be a homogeneous spaces of a semisimple algebraic group $\mathbf{G}$ over k. Assume that $p > h$. Then the bundle $\mathsf{F}_*\mathcal{O}_X$ is a generator in $\mathrm{D}^b(X)$.*

Proof. One has to show that if an object $\mathcal{M}$ of $\mathrm{D}^b(X)$ is orthogonal to $\mathsf{F}_*\mathcal{O}_X$, i.e. $\mathrm{R}Hom_X(\mathcal{M}, \mathsf{F}_*\mathcal{O}_X) = 0$, then $\mathcal{M} = 0$. From the orthogonality condition and by the adjunction of direct and inverse image functors one obtains that such an $\mathcal{M}$ satisfies (up to passing to the dual)

$$\mathbb{H}^*(X, \mathsf{F}^*\mathcal{M}) = 0. \tag{6.3}$$

The Frobenius morphism is an affine morphism, thus from (6.3) we get the equality $\mathbb{H}^*(X, \mathsf{F}_*\mathsf{F}^*\mathcal{M}) = 0$. Now the object $\mathsf{F}_*\mathsf{F}^*\mathcal{M}$ is a complex of $\mathsf{F}_*\mathcal{D}_X$-modules; indeed, the object $\mathsf{F}_*\mathsf{F}^*\mathcal{M}$ is a complex of modules over $\mathcal{E}nd_{\mathcal{O}_X}(\mathsf{F}_*\mathcal{O}_X)$ (the Cartier descent), and the latter algebra is the central reduction of the algebra $\mathsf{F}_*\mathcal{D}_X$. Using the equivalence (6.2) and the condition (6.3), we see that the object $\mathsf{F}_*\mathsf{F}^*\mathcal{M}$ of $\mathrm{D}^b(X', \mathcal{D}_{X'} - \mathrm{mod})$ is zero (since the functor establishing the equivalence (6.2) is the derived global section functor). By the projection formula one has $\mathsf{F}_*\mathsf{F}^*\mathcal{M} = \mathsf{F}_*\mathcal{O}_X \otimes \mathcal{M}$, hence $\mathcal{M} = 0$.

$\square$

Theorem 1.1 now follows.

References

[1] K. Akin, D. Buchsbaum, J. Weyman, “Schur functors and Schur complexes”, Adv. in Math. **44** (1982), no. 3, pp. 207–278

[2] H. H. Andersen, M. Kaneda, “On the D-affinity of the flag variety in type B_2”, Mansucripta Math., **103** (2000), pp. 393–399

[3] A. Beilinson, “Coherent sheaves on $\mathbb{P}^n$ and problems in linear algebra”, (Russian) Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, pp. 68–69

[4] A. Beilinson, J. Bernstein, “Localisation de $\mathfrak{g}$-modules”, C.R.Acad. Sci. Paris, Ser. I **292** (1981), pp. 15–18

[5] P. Berthelot, “$\mathcal{D}$-modules arithmétiques I. Opérateurs différentiels de niveau fini” , Ann. Sci. Ecole Norm. Sup. **29** (1996), pp. 185–272

[6] R. Bezrukavnikov, “Noncommutative counterparts of the Springer resolution”, arXiv:math.RT/0604445

[7] R. Bezrukavnikov, I. Mirković, D. Rumynin, “Localization of modules over semisimple Lie algebras in prime characteristic”, arXiv:math.RT/0205144

[8] R. Bezrukavnikov, I. Mirković, D. Rumynin, “Singular localization and intertwining functors for semisimple Lie algebras in prime characteristic”, arXiv:math.RT/0602075

[9] A. Bondal, “Representations of associative algebras and coherent sheaves”, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, pp. 25–44

[10] W. Borho, H. Kraft, “Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen”, Comment. Math. Helv. **54** (1979), no. 1, pp. 61–104.

[11] M. Brion, S. Kumar, “Frobenius splitting methods in geometry and representation theory”, Progress in Mathematics, v. 231, Birkhauser

[12] A. Grothendieck, EGA IV, Etude locale des schémas et des morphismes de schémas, Inst. Hautes Etudes Sci. Publ. Math. **32** (1967)

[13] B. Haastert, “Über Differentialoperatoren und $\mathbb{D}$-Moduln in positiver Charakterik”, Manuscripta Math., **58** (1987), no. 4, pp. 385–415

[14] W. J. Haboush, “A short proof of the Kempf vanishing theorem”, Invent. Math. **56** (1980), no. 2, pp. 109–112

[15] R. Hartshorne, “Algebraic geometry”, Springer Verlag, Heidelberg, New York (1977), GTM 52

[16] J. C. Jantzen, “Representations of algebraic groups”, 2nd ed. (2003), AMS

[17] K. Joshi, “Kodaira-Akizuki-Nakano vanishing: a variant”, Bull. London Math. Soc. **32** (2000), no. 2, pp. 171–176

[18] D. Kaledin, “Derived equivalences by quantization”, arXiv:math.AG/0504584

[19] M. Kapranov, “On the derived categories of coherent sheaves on some homogeneous spaces”, Invent. Math. **92** (1988), no. 3, pp. 479–508

[20] M. Kashiwara, N. Lauritzen, “Local cohomology and $\mathcal{D}$-affinity in positive characteristic”, C. R. Acad. Sci. Paris, Ser. I, **335** (2002), pp. 993–996

[21] S. Kumar, N. Lauritzen, J.F.Thomsen, “Frobenius splitting of cotangent bundles of flag varieties”, Invent. Math. **136** (1999), pp. 603–621

[22] N. Lauritzen, U. Raben–Pedersen, J.F. Thomsen, "Global F-regularity of Schubert varieties with applications to $\mathcal{D}$-modules", Journal of the Amer. Math .Soc., **19** (2005), no.2, pp. 345–355

[23] V. Mehta, A. Ramanathan, "Frobenius splitting and cohomology vanishing for Schubert varieties", Ann. of Math. **122** (1985), pp. 27 – 40

[24] V. Mehta, W. Van der Kallen, "A simultaneous Frobenius splitting for closures of conjugacy classes of nilpotent matrices", Compositio Math. **84** (1992) no. 2, pp. 211–221

[25] A. Samokhin, "On the D-affinity of quadrics in positive characteristic", arXiv:math.AG/0612417, to appear in C.R.Acad. Sci. Paris

[26] A. Samokhin, "Tilting bundles on some Fano varieties via the Frobenius morphism", in preparation

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