Local Asymmetry and the Inner Radius of Nodal Domains

Dan MANGOUBI

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Avril 2007

IHES/M/07/14

Local Asymmetry and the Inner Radius of Nodal Domains

Dan Mangoubi

Abstract

Let M be a closed Riemannian manifold of dimension n. Let φ_λ be an eigenfunction of the Laplace–Beltrami operator corresponding to an eigenvalue λ. We show that the volume of $\{\varphi_\lambda > 0\} \cap B$ is $\geq C|B|/\lambda^n$, where B is any ball centered at a point on the nodal set. We apply this result to prove that each nodal domain contains a ball of radius $\geq C/\lambda^n$. The results in this paper extend previous results of F. Nazarov, L. Polterovich, and M. Sodin, and of the author.

1 Introduction and Main Results

Let (M, g) be a closed Riemannian manifold of dimension n. Let $\Delta = -\mathrm{div} \circ \mathrm{grad}$ be the Laplace–Beltrami operator on M. We consider the eigenvalue equation

$$\Delta\varphi_\lambda = \lambda\varphi_\lambda \ . \tag{1.1}$$

A λ-*nodal domain* on M is any connected component of the set $\{\varphi_\lambda \neq 0\}$ (see Fig. 1). In this paper we study asymptotic local geometry of nodal domains. Let Ω_λ denote a λ-nodal domain on M. Let C_i, $i = 1, 2, \ldots$ denote constants which depend only on the Riemannian metric g. Our first result is

Theorem 1.2.

$$\frac{\mathrm{Vol}(\{\varphi_\lambda > 0\} \cap B)}{\mathrm{Vol}(B)} \geq \frac{C_1}{\lambda^{(n-1)/2}} \ ,$$

for all geodesic balls $B \subseteq M$ such that $\{\varphi_\lambda = 0\} \cap \frac{1}{2}B \neq \emptyset$. Here, $\frac{1}{2}B$ is a concentric ball of half the radius of B.

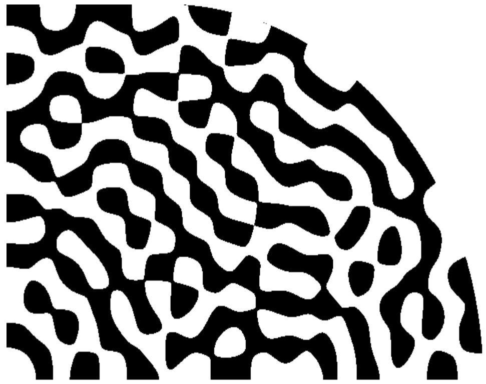

Figure 1: Nodal domains on a Quarter of a Stadium, Dirichlet boundary conditions. Courtesy of Sven Gnutzmann

One can think of Theorem 1.2 as measuring the local asymmetry of nodal domains. Namely, it measures the volumes ratio between the positivity and the negativity set of φ_λ in B. Our motivation to prove the local asymmetry estimate in Theorem 1.2 comes from two sources. The first one is the following local asymmetry estimate in dimension two:

Theorem 1.3 ([NPS05]). *Let Σ be a closed Riemannian surface. Then*

$$\frac{\mathrm{Vol}(\{\varphi_\lambda > 0\} \cap B)}{\mathrm{Vol}(B)} \geq \frac{C_2}{\log \lambda \sqrt{\log \log \lambda}} \ ,$$

for all geodesic balls $B \subseteq M$ such that $\{\varphi_\lambda = 0\} \cap \frac{1}{2}B \neq \emptyset$.

The proof of Theorem 1.3 is based on one-dimensional complex analysis. F. Nazarov, L. Polterovich and M. Sodin suggest in [NPS05] to explore local asymmetry in higher dimensions. The idea of the proof of Theorem 1.2 is based on a method of Carleman in [Car26]. Carleman finds a differential inequality which involves the rate of growth of a harmonic function in a two dimensional ball and its volume of positivity. In [NPS05], the authors indicate how to obtain a local asymmetry estimate for harmonic functions in dimensions $n \geq 3$ based on Carleman's method. In this paper we adapt Carleman's method to solutions of second order elliptic equations. As a result we can get a local asymmetry estimate also for eigenfuncions of the Laplace–Beltrami operator.

Our second source of motivation comes from our work [Man05]. In that work we gave a lower bound for the inner radius of nodal domains based on a growth bound for eigenfunctions by H. Donnelly and C. Fefferman and the Local Courant's Nodal Domain Theorem:

Theorem 1.4 ([DF90, CM91]). *Let M be a closed Riemannian manifold of dimension n. Let Ω_λ be a λ-nodal domain. Then*

$$\frac{\mathrm{Vol}(\Omega_\lambda \cap B)}{\mathrm{Vol}(B)} \geq \frac{C_3}{\lambda^{3n^2}} ,$$

for all geodesic balls $B \subseteq M$ such that $\Omega_\lambda \cap \frac{1}{2}B \neq \emptyset$.

In the present paper Theorem 1.2 replaces Theorem 1.4. Namely, we now consider the union of all components of the positivity set of φ_λ in B, while in Theorem 1.4 only one deep (i.e. which intersects $\frac{1}{2}B$) component in B is considered. We believe that the lower bound $1/\lambda^{C_4 n}$ is true also for the volume of one deep component. This lets us improve our estimate on the inner radius significantly, and make the proof of our result more self-contained. We prove:

Theorem 1.5.

$$\frac{C_5}{\lambda^{\alpha(n)}} \leq \mathrm{inrad}(\Omega_\lambda) \leq \frac{C_6}{\sqrt{\lambda}},$$

where $\alpha(n) = \frac{1}{4}(n-1) + \frac{1}{2n}$.

The proof of the upper bound and of the two dimensional case is given in [Man05]. In this paper we assume $n \geq 3$.

Organization of the Paper. In Section 2 we explain the principle that in small scales compared with the wavelength $1/\sqrt{\lambda}$ an eigenfunction behaves like a harmonic function. In Section 3 we present some results and estimates for solutions of elliptic equations, which we frequently use. We prove these estimates in Section 7. In Section 4 we give an estimate of the volume of positivity for solutions of the Schrödinger equation with small potential in the unit ball. Our estimate will be given in terms of the growth of the solution, and its proof is based on Carleman's method. In Section 5 we combine our estimate from section 4 and a growth bound by Donnelly and Fefferman in order to prove Theorem 1.2. In section 6 we prove that the asymmetry estimate in Theorem 1.2 implies the estimate on the inner radius of a nodal domain in Theorem 1.5.

Acknowledgements. I would like to express my deep gratitude and appreciation to Leonid Polterovich under the direction of whom this work began, and also to Misha Sodin for his invaluable guidance and extensive discussions.

I thank Fëdor Nazarov and Misha Sodin for drawing my attention to the asymmetry result in [NPS05], and the suggestion to prove an estimate in higher dimensions.

I thank Dima Jakobson for his continuous encouragement and support. I thank Pengfei Guan, Iosif Polterovich and John Toth for discussions concerning elliptic theory, while visiting the CRM. I thank Johannes Sjöstrand for explaining to me some estimates on the Poisson Kernels in a previous version of this paper. I thank Jürgen Jost for discussing with me the Harnack Inequality, and finally I thank Sven Gnutzmann for giving me pictures of nodal domains he generated with his computer program. One of these pictures is shown in Fig. 1.

This work was done while visiting the CRM, Montréal, the University of Mcgill, Montréal and the IHÉS, Bures-sur-Yvette. The hospitality of these institutions is gratefully acknowledged.

2 Eigenfunctions on the Wavelength Scale

In this section we explain the following principle.

Principle: On a small scale comparable to the wavelength ($1/\sqrt{\lambda}$), eigenfunctions behave like harmonic functions.

The above principle was extensively used in the works of H. Donnelly, C. Fefferman and N. Nadirashvili. We start by fixing an atlas on M.

Lemma 2.1. *We can find on M a finite atlas such that in each chart the coefficients of g, g^{-1} are given by bounded functions in the C^1-norm, and g^{-1} is uniformly elliptic in each coordinate chart.*

□

In each chart we have

$$\|g^{ij}\|_{C^1} \leq K_1, \quad g = \det g_{ij} \leq K_2, \tag{2.2}$$

and an ellipticity bound

$$g^{ij}(x)\xi_i\xi_j \geq \kappa|\xi|^2. \tag{2.3}$$

The eigenequation (1.1) expressed in local coordinates is

$$-\frac{1}{\sqrt{g}}\partial_i(g^{ij}\sqrt{g}\partial_j\varphi_\lambda) = \lambda\varphi_\lambda \ . \tag{2.4}$$

We consider equation (2.4) in balls $B_r = B(0,r)$, where $r < \sqrt{\varepsilon_0/\lambda}$ and ε_0 is a small positive number to be chosen later. When we rescale it to an equation in the unit ball B_1, we get

$$-\partial_i(g_r^{ij}\sqrt{g_r}\partial_j\varphi_{\lambda,r}) = \varepsilon_0\sqrt{g_r}\varphi_{\lambda,r} \text{ on } B_1. \tag{2.5}$$

Here, a subindex r denotes a scaled function, $f_r(x) = f(rx)$. Since $r < 1$, the bounds (2.2) and (2.3) remain true also for the rescaled metric coefficients.

Throughout this paper we let

$$\varphi = \varphi_{\lambda,r}, \quad a^{ij} = g_r^{ij}\sqrt{g_r}, \quad q = \sqrt{g_r} \ .$$

We set

$$Lu = -\partial_i(a^{ij}\partial_j u) - \varepsilon_0 q u. \tag{2.6}$$

Equation (2.4) takes now the form

$$L\varphi = 0 \text{ in } B_1, \tag{2.7}$$

with the bounds

$$\|a^{ij}\|_{C^1(\overline{B_1})} \le K_3, 0 \le q \le K_4, \tag{2.8}$$

and an ellipticity bound

$$a^{ij}\xi_i\xi_j \ge K_5|\xi|^2 \ . \tag{2.9}$$

3 Estimates for Solutions of Elliptic Equations

In this section we present some properties of solutions, subsolutions and supersolutions of second order elliptic equations which will be useful in the next sections. The proofs are postponed to Section 7. L is the operator given in (2.6) in the unit ball B_1.

The following theorem is a local maximum principle.

Theorem 3.1 ([GT83, Theorem 9.20]). *Suppose $Lu \leq 0$ on B_1. Then*

$$\sup_{B(y,r_1)} u \leq C_1(r_1/r_2, p) \left(\frac{1}{\mathrm{Vol}(B(y,r_2))} \int_{B(y,r_2)} (u^+(x))^p \, dx \right)^{1/p},$$

for all $p > 0$, whenever $0 < r_1 < r_2$ and $B(y,r_2) \subseteq B_1$.

We will also need the weak Harnack Inequality

Theorem 3.2 ([GT83, Theorem 9.22]). *Suppose $Lu \geq \delta$ in B_1, and $u \geq 0$ in $B(y,r_2) \subseteq B(0,1)$. Then $\exists p > 0$ such that*

$$\left(\frac{1}{\mathrm{Vol}(B(y,r_1))} \int_{B(y,r_1)} u^p \right)^{1/p} \leq C_2(r_1,r_2) \inf_{B(y,r_1)} u + C_3(r_1,r_2)\delta,$$

where $r_1 < r_2$.

We let

$$L_0 u = -\partial_i (a^{ij} \partial_j u) \ .$$

Then $L = L_0 - \varepsilon_0 q$. A maximum principle for L_0 is

Theorem 3.3 ([GT83, Theorem 3.7]). *Let u satisfy $L_0 u \leq \delta$ on a ball $B \subseteq B_1$. Then*

$$\sup_{\partial B} u \geq \sup_{B} u - C_4 \delta \ ,$$

where C_4 depends only on the C^1-bounds and the ellipticity bounds of the coefficients a^{ij}.

We recall that we denote by φ a solution of the Schrödinger equation (2.7). As a corollary of Theorem 3.3 we obtain

Corollary 3.4. *We have*

$$\sup_{\partial B} \varphi^+ \geq 0.9 \sup_{B} \varphi \ ,$$

for all balls $B \subseteq B_1$, and for all ε_0 small enough.

We have also a Mean Value Property

Theorem 3.5. *Suppose $\varphi(0) = 0$. Then*

$$\sup_{B_{r_1}} \varphi^- \leq C_5(r_1,r_2) \sup_{B_{r_2}} \varphi^+ \ ,$$

where $r_1 < r_2 \leq 1$.

4 Positivity Volume for Solutions of Schrödinger's Equation

We recall that φ is a solution of the Schrödinger equation (2.7) in the unit ball B_1, under the conditions (2.8)–(2.9). We estimate the positivity volume of φ in terms of its growth.

Let $0 < r < 1$. Denote by $\beta_r^+(\varphi)$ the growth exponent of φ:

$$\beta_r^+(\varphi) := \log \left| \frac{\sup_{|x|\leq 1} \varphi(x)}{\sup_{|x|\leq r} \varphi(x)} \right| \ .$$

Set $\langle \beta_r^+ \rangle = 1 + \beta_r^+$. We prove

Theorem 4.1. *Suppose $\varphi(0) = 0$ and ε_0 is small enough. Then*

$$\mathrm{Vol}(\{\varphi > 0\}) \geq \frac{C_1(r)}{\langle \beta_r^+ \rangle^{n-1}}.$$

We start by considering the case $\varphi(0) \neq 0$.

Proposition 4.2. *Let $|x_0| < 1$. Suppose $\varphi(x_0) > 0$ and $\varphi(x) \leq \gamma\varphi(x_0)$ for all $x \in B = B(x_0, r) \subseteq B(0, 1)$. Then*

$$\frac{\mathrm{Vol}(\{\varphi > 0\} \cap B)}{\mathrm{Vol}(B)} \geq \frac{C_2}{\gamma} \ .$$

Proof of Proposition 4.2. We apply to φ Theorem 3.1.

$$\begin{aligned} \varphi(x_0) &\leq \sup_{B(x_0, r/2)} \varphi \leq \frac{C_3}{\mathrm{Vol}(B)} \int_B \varphi^+(x)\, dx = \frac{C_3}{\mathrm{Vol}(B)} \int_{B\cap\{\varphi>0\}} \varphi(x)\, dx \leq \\ &\leq \frac{C_3\gamma}{\mathrm{Vol}(B)} \int_{B\cap\{\varphi>0\}} \varphi(x_0)\, dx = C_3\gamma \frac{\mathrm{Vol}(\{\varphi > 0\} \cap B)}{\mathrm{Vol}(B)} \varphi(x_0) \ . \end{aligned} \tag{4.3}$$

Dividing by $\varphi(x_0)$ gives us the result. □

We now treat the case $\varphi(0) = 0$.

Proof of Theorem 4.1. Let $m = \lfloor \langle \beta_r^+ \rangle \rfloor$. Decompose the annulus $r < |x| < 1$ into m annuli $r_k < |x| < r_{k+1}$, where $r_k = r + (1-r)k/m$ for $k = 0, \ldots m$. Define

$$\beta_k = \log \frac{\sup_{|x| \le r_{k+1}} \varphi(x)}{\sup_{|x| \le r_k} \varphi(x)}, \quad (0 \le k \le m-1).$$

Let $S = \{k : \beta_k \le 2\beta_r^+/m\}$. Observe that $\sum_k \beta_k = \beta_r^+$. Therefore, $|S| \ge m/2$. Let S' be a maximal subset of S such that for all $k_1, k_2 \in S'$ we have $|r_{k_1} - r_{k_2}| \ge 2(1-r)/m$. Notice that $|S'| \ge m/4$.

Fix $k \in S'$. By Corollary 3.4, we can find x_0 such that $|x_0| = r_k$ and

$$\varphi(x_0) \ge 0.9 \sup_{|x| \le r_k} \varphi(x) \ .$$

Consider the ball $B = B(x_0, (1-r)/m)$. For all $x \in B$ we have $\varphi(x) \le 0.9 \mathrm{e}^{2\beta_r^+/m} \varphi(x_0)$. Hence, from Proposition 4.2 we know that

$$\frac{\mathrm{Vol}(\{\varphi > 0\} \cap B)}{\mathrm{Vol}(B)} \ge C_4 \mathrm{e}^{-2\beta_r^+/m} \ge C_4 \mathrm{e}^{-2} \ge C_5 \ .$$

If we run over all $k \in S'$, we obtain the following estimate

$$\begin{aligned} \mathrm{Vol}(\{\varphi > 0\}) &\ge m \mathrm{Vol}(\{\varphi > 0\} \cap B)/4 \ge C_5 \mathrm{Vol}(B) m/4 \ge \\ &\ge C_6 (1-r)^n / m^{n-1} \ge \frac{C_6 (1-r)^n}{\langle \beta_r^+ \rangle^{n-1}} \ . \end{aligned}$$

□

Remark. In the above proof if we avoid the use of the Maximum Principle, we get a lower bound of $C(r)/(\beta_r^+)^n$.

Different Variants of the Growth Exponent. We now replace β_r^+ in Theorem 4.1 by a more conventional growth constant:

$$\beta_r(\varphi) := \log \frac{\sup_{|x| \le 1} |\varphi(x)|}{\sup_{|x| \le r} |\varphi(x)|} \ . \tag{4.4}$$

We let $\langle \beta_r \rangle = 1 + \beta_r$.

Proposition 4.5. *Suppose $\varphi(0) = 0$. Let $0 < r_1 < r_2 < 1$. Then*

$$\beta_{r_1}^+(\varphi) \le C_7(r_1, r_2) \beta_{r_2}(\varphi) \ .$$

Proof. The proposition amounts to proving

$$\sup_{B_{r_1}} |\varphi| \leq C_8(r_1, r_2) \sup_{B_{r_2}} \varphi \ . \tag{4.6}$$

We may assume $\sup_{B_{r_1}} |\varphi| = \sup_{B_{r_1}} \varphi^-$. But then, inequality (4.6) is just Theorem 3.5. □

An immediate consequence of Proposition 4.5 and Theorem 4.1 is

Theorem 4.7. *Suppose $\varphi(0) = 0$. Then*

$$\mathrm{Vol}(\{\varphi > 0\}) \geq \frac{C_9(r)}{\langle \beta_r \rangle^{n-1}},$$

for $0 < r < 1$ and ε_0 small enough.

5 Local Asymmetry of Nodal Domains

We take the positivity volume estimate in Section 4, and a growth estimate by Donnelly and Fefferman in order to prove Theorem 1.2.

Proof of Theorem 1.2. First, we consider balls $B \subseteq M$ in scales small compared with the wavelength $1/\sqrt{\lambda}$, i.e. balls whose radius $r \leq \sqrt{\varepsilon_0/\lambda}$. We can assume that B is the Euclidean ball $B(0, r)$. Let x_0 be such that $\varphi_\lambda(x_0) = 0$ and $|x_0| < r/2$. We consider the eigenfunction φ_λ on the ball $\tilde{B} = B(x_0, r/2)$. We apply Theorem 4.7 with the function $\varphi(x) = \varphi_\lambda(rx/2)$ which is defined on the unit ball B_1. We learn that

$$\begin{aligned} \frac{\mathrm{Vol}(\{\varphi_\lambda > 0\} \cap B)}{\mathrm{Vol}(B)} \geq \frac{\mathrm{Vol}(\{\varphi_\lambda > 0\} \cap \tilde{B})}{2^n \mathrm{Vol}(\tilde{B})} &= \frac{\mathrm{Vol}(\{\varphi > 0\} \cap B_1)}{2^n \mathrm{Vol}(B_1)} \\ &\geq \frac{C_1}{\langle \beta_{1/2}(\varphi) \rangle^{n-1}} \ . \end{aligned} \tag{5.1}$$

Next, we recall the growth estimate for eigenfunctions by Donnelly and Fefferman:

Theorem 5.2 ([DF88]). *$\beta_{1/2}(\varphi_\lambda; \tilde{B}) \leq C_2\sqrt{\lambda}$, where $\beta_{1/2}(\varphi_\lambda; \tilde{B})$ is by definition $\beta_{1/2}(\varphi)$.*

Together with (5.1) we get

$$\frac{\mathrm{Vol}(\{\varphi_\lambda > 0\} \cap B)}{\mathrm{Vol}(B)} \geq \frac{C_3}{\lambda^{(n-1)/2}} \ . \tag{5.3}$$

We now consider large balls B. Let $r > \sqrt{\varepsilon_0/\lambda}$. We know that the inner radius of nodal domains is $< C_4/\sqrt{\lambda}$ (see e.g. [Man05]). From this fact it follows

Lemma 5.4. *We can find a maximal set of disjoint balls $B_i = B_i(x_i, r_0)$ contained in B, such that $r_0 < \sqrt{\varepsilon_0/\lambda}$, $\varphi_\lambda(x_i) = 0$, and $\mathrm{Vol}(\cup_i B_i)/\mathrm{Vol}(B) \geq C_5$.*

□

The balls B_i are small. Hence, by (5.3)

$$\mathrm{Vol}(\{\varphi_\lambda > 0\} \cap B_i) \geq C_6 \mathrm{Vol}(B_i)/\lambda^{(n-1)/2} \ .$$

Summing over all balls B_i gives us

$$\mathrm{Vol}(\{\varphi_\lambda > 0\} \cap B) \geq C_7 \mathrm{Vol}(\cup_i B_i)/\lambda^{(n-1)/2} \geq C_8 \mathrm{Vol}(B)/\lambda^{(n-1)/2} \ ,$$

as desired. □

6 Local Asymmetry implies Inner Radius Estimate

In this section we prove that a local asymmetry of a domain $\Omega \subseteq M$ implies a lower bound on its first eigenvalue. Then, we apply this result to a nodal domain in order to establish Theorem 1.5.

Definition 6.1. *Let $\Omega \subseteq M$ be a domain. We say that Ω satisfies* (ASym-α) *if*

$$\frac{\mathrm{Vol}(B \setminus \Omega)}{\mathrm{Vol}(B)} \geq \alpha.$$

for all balls $B \subseteq M$ such that $(\frac{1}{2}B \setminus \Omega) \neq \emptyset$.

We prove

Theorem 6.2. *Let M be of dimension $n \geq 3$. If $\Omega \subseteq M$ satisfies* (ASym-α), *then*

$$\lambda_1(\Omega) \geq C_1 \frac{\alpha^{1-2/n}}{\mathrm{inrad}(\Omega)^2} \ .$$

Remark. In dimension two, one can prove that if each connected component of the complement has area $\geq \alpha$, then $\lambda_1(\Omega) \geq C_2/\sqrt{\alpha}$. The full argument is given in [Man05].

Proof. We may assume that $\alpha > 0$. Let ψ be the first Dirichlet eigenfunction on Ω. Notice that ψ is not constant.

Let us fix a finite atlas $\{U_i, \kappa_i\}$ on M as in Section 2. Here $\kappa_i : U_i \to \mathbb{R}^n$, are the coordinate maps. The metric on each chart U_i is comparable to the Euclidean metric on the unit ball.

We divide $\kappa_i(U_i)$ into small non-overlapping small cubes Q_{ij} of size h to be chosen later. Define the local Rayleigh Quotient by

$$R_{ij}(\psi) = \frac{\int_{\kappa_i^{-1}(Q_{ij})} |\nabla\psi|^2 \, d(\mathrm{vol})}{\int_{\kappa_i^{-1}(Q_{ij})} |\psi|^2 \, d(\mathrm{vol})} \ . \tag{6.3}$$

Claim 6.4.

$$R_{ij}(\psi) \leq K\lambda_1(\Omega) \ , \tag{6.5}$$

for some i, j, where K is the number of charts in the atlas.

Proof of Claim. Assume the contrary

$$\int_{\kappa_i^{-1}(Q_{ij})} |\nabla\psi|^2 \ d(\mathrm{vol}) > K\lambda_1(\Omega) \int_{\kappa_i^{-1}(Q_{ij})} |\psi|^2 \ d(\mathrm{vol}) \ . \tag{6.6}$$

We sum up inequalities (6.6) over all cubes Q_{ij}.

$$\begin{aligned} &\int_\Omega |\nabla\psi|^2 \, d(\mathrm{vol}) \geq \frac{1}{K} \sum_{i,j} \int_{\kappa_i^{-1}(Q_{ij})} |\nabla\psi|^2 \, d(\mathrm{vol}) \\ &> \lambda_1(\Omega) \sum_{i,j} \int_{\kappa_i^{-1}(Q_{ij})} |\psi|^2 \, d(\mathrm{vol}) \geq \lambda_1(\Omega) \int_\Omega |\psi|^2 \, d(\mathrm{vol}) \ . \end{aligned} \tag{6.7}$$

Hence, we obtain the following contradiction

$$\lambda_1(\Omega) = \frac{\int_\Omega |\nabla\psi|^2 \, d(\mathrm{vol})}{\int_\Omega |\psi|^2 \, d(\mathrm{vol})} > \lambda_1(\Omega) \ .$$

□

We now make a particular choice of h. Set $\Omega_i = \Omega \cap U_i$, and let r_i be the Euclidean inner radius of $\kappa_i(\Omega_i)$. Let $h = 8 \max_i r_i$. We note that

$$h < C_3 \mathrm{inrad}(\Omega), \tag{6.8}$$

where C_3 depends only on g and the atlas chosen.

Take $Q = Q_{ij}$ from Claim 6.4. Let $\frac{1}{2}Q$ be a concentric cube with parallel edges of size $h/2$. Since $r_i < h/4$

$$\frac{1}{2}Q \setminus \kappa_i(\Omega_i) \neq \emptyset \ . \tag{6.9}$$

So, the asymmetry assumption on Ω tells us that

$$\frac{\mathrm{Vol}(Q \setminus \kappa_i(\Omega_i))}{\mathrm{Vol}(Q)} \geq C_4 \alpha \ . \tag{6.10}$$

Observe that the function $\psi \circ \kappa_i^{-1}$ vanishes on the set $Q \setminus \kappa_i(\Omega_i)$. We now apply to $\psi \circ \kappa_i^{-1}$ the following Poincaré type inequality due to Maz'ya.

Theorem 6.11 ([Maz85, §10.1.2])**.** *Let $Q \subset \mathbb{R}^n$ be a closed cube whose edge is of length a. Then,*

$$\int_Q |u|^2 \ dx \leq \frac{C_5 a^n}{\mathrm{cap}_2(F, 2Q)} \int_Q |\nabla u|^2 \ dx$$

for all $u \in C^\infty(Q)$ and where $F = \{u = 0\}$.

We also recall

Theorem 6.12 ([Maz85, §2.2.3])**.** $\mathrm{cap}_2(F, 2Q) \geq C_6 \mathrm{Vol}(F)^{(n-2)/n}$ *for $n \geq 3$.*

From inequality (6.10), Theorem 6.11, Theorem 6.12 and the fact that the metric g is comparable to the Euclidean metric on each chart, we immediately obtain

$$\int_{\kappa_i(Q)} |\psi|^2 \ d(\mathrm{vol}) \leq C_7(\alpha) h^2 \int_{\kappa_i(Q)} |\nabla \psi|^2 \ d(\mathrm{vol}), \tag{6.13}$$

where $C_7(\alpha) = C_8/\alpha^{1-2/n}$. Combining inequalities (6.5) and (6.13) we arrive at $\lambda_1(\Omega) \geq C_9/(C_7(\alpha)h^2)$. To conclude, we recall inequality (6.8). □

Application to the Inner Radius of Nodal Domains:

Proof of Theorem 1.5. We notice that $\lambda_1(\Omega_\lambda) = \lambda$. This is true since φ_λ is a Dirichlet eigenfunction for Ω_λ with constant sign. Now, Theorem 1.5 is a consequence of Theorem 1.2 and Theorem 6.2. □

7 Proofs of Elliptic Estimates

In this section we give the proofs of the elliptic estimates from Section 3.

We begin by the proof of the maximum principle.

Proof of Corollary 3.4. If $\sup_B \varphi \le 0$ the theorem is trivial. Otherwise, define $w = \varphi / \sup_B \varphi$. Then $L_0 w = \varepsilon_0 q w \le \varepsilon_0 q \sup_B w \le \varepsilon_0 q \le \varepsilon_0 K_4$.

Hence, by Theorem 3.3 we know

$$\sup_{\partial B} w \ge \sup_B w - C_1 K_4 \varepsilon_0 \ge 1 - C_2 \varepsilon_0 .$$

Hence, for all ε_0 small enough we have $\sup_{\partial B} w \ge 0.9$, from which we conclude $\sup_{\partial B} \varphi \ge 0.9 \sup_B \varphi$. □

We give now the proof of the Mean Value Property:

Proof of Theorem 3.5. Let $M = \sup_{B_{r_2}} \varphi^-$. Observe that $L(\varphi + M) = L\varphi + LM = -\varepsilon_0 q M$. Hence,

$$-\varepsilon_0 K_4 M \le L(\varphi + M) \le 0 \ .$$

By Theorem 3.2 we have for some $p > 0$,

$$\left(\frac{1}{\mathrm{vol}(B_{(r_1+r_2)/2})} \int_{B_{(r_1+r_2)/2}} (\varphi + M)^p \right)^{1/p} \le \\ \le C_3(r_1, r_2) (\inf_{B_{(r_1+r_2)/2}} \varphi + M) \le C_3(r_1, r_2) M \ . \quad (7.1)$$

By Theorem 3.1 we know that

$$\sup_{B_{r_1}} (\varphi + M) \le C_4(r_2/r_1, p) \left(\frac{1}{\mathrm{vol}(B_{(r_1+r_2)/2})} \int_{B_{(r_1+r_2)/2}} (\varphi + M)^p \right)^{1/p} \ . \quad (7.2)$$

Combining (7.1) and (7.2) we obtain

$$\sup_{B_{r_1}} (\varphi + M) \le C_5(r_1, r_2) M \ . \quad (7.3)$$

Recalling the definition of M we get,

$$\sup_{B_{r_1}} \varphi^+ \le C_5(r_1, r_2) \sup_{B_{r_2}} \varphi^- \ .$$

□

References

[Car26] T. Carleman, *Extension d'un théorème de Liouville*, Acta Math. **48** (1926), 363–366.

[CM91] S. Chanillo and B. Muckenhoupt, *Nodal geometry on Riemannian manifolds*, J. Differential Geom. **34** (1991), no. 1, 85–91.

[DF88] H. Donnelly and C. Fefferman, *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. Math. **93** (1988), no. 1, 161–183.

[DF90] ———, *Growth and geometry of eigenfunctions of the Laplacian*, Analysis and partial differential equations, Lecture Notes in Pure and Appl. Math., vol. 122, Dekker, New York, 1990, pp. 635–655.

[GT83] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, second ed., vol. 224, Springer-Verlag, Berlin, 1983.

[Man05] D. Mangoubi, *On the inner radius of nodal domains*, arXiv: math.SP/0511329, to appear in Canad. Math. Bull., 2005.

[Maz85] V. Maz'ya, *Sobolev spaces*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1985.

[NPS05] F. Nazarov, L. Polterovich, and M. Sodin, *Sign and area in nodal geometry of Laplace eigenfunctions*, Amer. J. Math. **127** (2005), no. 4, 879–910.

Dan Mangoubi,
IHÉS, Le Bois-Marie,
35, Route de Chartres,
F-91440 Bures-sur-Yvette,
France
`mangoubi@ihes.fr`