Tubular Neighborhoods of Nodal Sets and Diophantine Approximation

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Abstract. We give upper and lower bounds on the volume of a tubular neighborhood of the nodal set of an eigenfunction of the Laplacian on a real analytic closed Riemannian manifold M. As an application we consider the question of approximating points on M by nodal sets, and explore analogy with approximation by rational numbers.

1. Introduction and Main Results

Let (M, g) be a real analytic closed Riemannian manifold. In the first part of this paper we give upper and lower bounds on the volume of tubular neighborhoods of nodal sets. Consider the eigenequation

$$\Delta\phi_\mu + \mu^2\phi_\mu = 0,$$

where Δ is the Laplace–Beltrami operator on M. We denote the nodal set $\{\phi_\mu = 0\}$ by $\mathcal{N}_\mu$. Consider the tubular neighborhood of the nodal set

$$T_{\mu,\delta} = \{x \in M \,:\, \mathrm{dist}(x, \mathcal{N}_\mu) < \delta\}, \tag{1.1}$$

where $\mu\delta < c$. Throughout this paper C, C_i denote positive constants which depend only on the Riemannian metric g. c, c_i denote positive constants which are small. We prove

Theorem 1.2. *Let (M, g) be a real analytic closed Riemannian manifold. Then there exist $C_1, C_2 > 0$ such that*

$$C_1\mu\delta \le \mathrm{Vol}(T_{\mu,\delta}) \le C_2\mu\delta,$$

whenever $\mu\delta < c_3$.

To put Theorem 1.2 in the right context we recall

Theorem 1.3 ([DF88, Theorem 1.2])**.** *Let (M, g) be a closed real analytic Riemannian manifold. Then, there exist $C_4, C_5 > 0$ such that $C_4\mu \le \mathrm{Vol}_{n-1}(\mathcal{N}_\mu) \le C_5\mu$, where Vol_{n-1} is the $(n-1)$-dimensional Hausdorff measure on M.*

One can regard this result as a generalization of the fact that the k-th eigenfunction of a Sturm-Liouville problem has $(k-1)$ zeros. From this perspective, we see that Theorem 1.2 describes a regularity property of the nodal set. For example, the upper bound implies that the nodal set does not have too many needles or very narrow branches, while the lower bound says that the nodal set doesn't curve too much.

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The proof of the lower bound in Theorem 1.2 is given in Section 2. It is based on the behavior of eigenfunctions in the wavelength scale and on the Brunn-Minkowski inequality.

The proof of the upper bound in Theorem 1.2 is given in Section 6. The idea of the proof was suggested to the authors by C. Fefferman. It is based on the upper bound in Theorem 1.3 and a study of the behavior of eigenfunctions in very small scales compared to the wavelength $1/\mu$. This study is the content of Sections 3, 4 and 5.

In Section 5 of [DF88] Donnelly and Fefferman study the behavior of eigenfunctions in scales *comparable* to the wavelength $1/\mu$. In this paper we generalize their study to *arbitrarily small* scales with respect to the wavelength. To that end, we follow closely the guidelines in Section 5 of [DF88]. Donnelly and Fefferman showed that this kind of problems can be treated on real analytic manifolds by considering polynomials in dimension one, and then applying an induction argument. We adopt this approach also here and adjust the proof in [DF88] to our case. The key proposition is Proposition 5.2. Most of its proof goes without change from the proof of Proposition 5.11 in [DF88]. We had to adjust the arguments from [DF88] in two main points. The first is the proof of Lemma 3.5 with explicit estimates of the Hilbert Transform. The second is in the proof of Proposition 4.3 where the change of variables argument is more subtle than the parallel argument in [DF88].

In Section 8 we consider the special case where $\dim(M) = 2$. We show that the lower bound is true for any *smooth* surface and the upper bound is true for any smooth surface which satisfies Yau's conjecture.

In the second part of the paper we make an attempt to look simultaneously on the ensemble of nodal sets which belong to different eigenvalues. Consider first a simple example: Eigenfunctions on $M = [0, \pi]$ with the standard metric and (say) with Dirichlet boundary conditions. Then

$$\mu_k = k,\ \phi_k(x) = \sin(kx),\ \mathcal{N}_k = \left\{ \frac{\pi j}{k} : 0 \le j \le k \right\}.$$

Accordingly, the set $\mathcal{N}_k$ is $\pi/(2k)$-dense in M. Interestingly, a similar result holds on any smooth Riemannian manifold (see e.g. [Brü78]):

Proposition 1.4. *There exists $C > 0$ (which depends only on M, g) such that*

$$B(x, C/\mu) \cap \mathcal{N}(\phi_\mu) \ne \emptyset$$

for any $x \in M$ and $\mu > 0$.

Here $B(x, r)$ denotes the ball of radius r centered at $x \in M$. Thus $\mathcal{N}_\mu$ is C/μ-dense in M.

To study the rate of approximation by $\mathcal{N}_\mu$ as $\mu \to \infty$ in more detail, consider again the case of $M = [0, \pi]$ where approximating by points in $\mathcal{N}_k$ is equivalent (after rescaling by π) to approximating by rationals with denominator k. It is well-known (see e.g. [Khi97]) that the distance from any $x \in [0, 1]$ to the m-th convergent of its continued fraction expansion p_m/q_m is $O(1/q_m^2)$. However, the *denominator* q_m of the m-th continued fraction grows exponentially in m for $x \notin \mathbb{Q}$ ([Khi97]).

Denote by $||x||$ the distance from $x \in \mathbb{R}$ to the nearest integer. The following proposition can be found in [Khi97] and is proved by an application of the Borel-Cantelli Lemma.

Proposition 1.5. *If $\sum_q \psi(q)$ converges, then for Lebesgue-almost all x, there exist only finitely many q such that $||qx|| < \psi(q)$.*

Taking $\psi(q) = C/q^{1+\varepsilon}$ in Proposition 1.5 we conclude that

Corollary 1.6. *Given $C, \varepsilon > 0$, for Lebesgue-almost all $x \in [0,1]$ the inequality*

$$|x - p/q| < C/q^{2+\varepsilon}$$

has finitely many integer solutions (p,q).

Equivalently, almost all $x \in M = [0,\pi]$ *cannot* be approximated by points in $\mathcal{N}_k$ to within $C/k^{2+\varepsilon}$ infinitely often. We prove an analogous statement for any real analytic manifold M.

To characterize the *rate* of approximation by nodal sets, we make the following definition:

Definition 1.7. Given $b > 0$ *(exponent)*, and $C > 0$ *(constant)*, let $M(b,C)$ be the set of all $x \in M$ such that there exists an infinite sequence of eigenvalues $\mu_k \to \infty$ for which

$$B\left(x, \frac{C}{\mu_k^b}\right) \cap \mathcal{N}(\phi_{\mu_k}) \neq \emptyset.$$

For example, Proposition 1.4 implies that $M(1,C) = M$ for some $C > 0$. Also, Corollary 1.6 implies that for $M = [0,\pi]$, we have $\mathrm{Vol}(M(2+\varepsilon, C)) = 0\ \forall C, \varepsilon > 0$. We prove

Theorem 1.8. *Let (M,g) be a closed real analytic Riemannian manifold of dimension n. Then for any $C > 0, \varepsilon > 0$,*

$$\mathrm{Vol}(M(n+1+\varepsilon, C)) = 0.$$

The proof consists of Theorem 1.2, the Borel–Cantelli Lemma and Weyl's asymptotics of eigenvalues.

1.1. A Reader's Guide.

In Section 2 we prove the lower bound in Theorem 1.2. It is independent of the other sections. Section 6 gives the line of proof of the upper bound in Theorem 1.2. On a first reading one may start with this section and then move to section 5. Section 5 is a study of eigenfunctions in small scales. It is based on the study of holomorphic functions in Section 3 and Section 4. In Section 7 we combine the upper bound in Theorem 1.2 with Weyl's Law and the Borel-Cantelli Lemma in order to establish Theorem 1.8. In Section 8 we discuss Theorem 1.2 for smooth surfaces. In Section 9 we discuss possible extensions of the approximation result.

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2. Proof of the Lower Bound in Theorem 1.2

Given a ball $B \subseteq M$, let $B^+ := \{\phi_\mu > 0\} \cap B$ and $B^- := \{\phi_\mu < 0\} \cap B$. The following proposition is proved in [DF88, pp. 164–165].

Proposition 2.1. *There exists a finite collection of balls $B_i = B(x_i, r)$ centered at x_i of radius $r = C_1/\mu$ which satisfy*

(i) $\phi_\mu(x_i) = 0$,
(ii) *their doubles $2B_i = B(x_i, 2r)$ are pairwise disjoint,*
(iii)
$$C_2 < \frac{\mathrm{Vol}(B_i^+)}{\mathrm{Vol}(B_i^-)} < C_3 \ ,$$
(iv) $\sum_i \mathrm{Vol}(B_i) > C_4 \mathrm{Vol}(M)$.

Proposition 2.2. *Let $B(x, r)$ be one of the balls described above. Then we have $\mathrm{Vol}(T_{\mu,\delta} \cap 2B) \geq C_5 r^{n-1}\delta$, whenever $\mu\delta < c_6$.*

Proof. Let $(B^+)_\delta$ be a δ-neighborhood of B^+, and similarly for $(B^-)_\delta$. Since $T_{\mu,\delta} \cap 2B \supseteq (B^+)_\delta \cap (B^-)_\delta$, it is clear that
$$\mathrm{Vol}(T_{\mu,\delta} \cap 2B) \geq \mathrm{Vol}(B^+)_\delta + \mathrm{Vol}(B^-)_\delta - \mathrm{Vol}(B(x, r + \delta)) \ .$$
Assume first that the metric g is flat on $2B$. By the Brunn-Minkowski Inequality [Fed69, §3.2.41] we know
$$\mathrm{Vol}(B^+)_\delta \geq \mathrm{Vol}(B^+) + n\omega_n^{1/n}\delta\,\mathrm{Vol}(B^+)^{1-1/n} \ ,$$
where ω_n is the volume of the n-dimensional unit ball. We have the same inequality for $(B^-)_\delta$. Set $\mathrm{Vol}(B^+) = \alpha\,\mathrm{Vol}(B)$, and $\mathrm{Vol}(B^-) = (1-\alpha)\,\mathrm{Vol}(B)$. We have

$$\begin{aligned} \mathrm{Vol}(T_{\mu,\delta} \cap 2B) \geq{} & \mathrm{Vol}(B) - \mathrm{Vol}(B(x, r+\delta)) + \\ & n\omega_n^{1/n}\delta\,\mathrm{Vol}(B)^{1-1/n}\left(\alpha^{1-1/n} + (1-\alpha)^{1-1/n}\right) \geq \\ & \omega_n(r^n - (r+\delta)^n) + n\omega_n r^{n-1}\delta\left(\alpha^{1-1/n} + (1-\alpha)^{1-1/n}\right) \ . \end{aligned} \tag{2.3}$$

At this point one observes that when α is bounded away from 0 and 1 we have $\alpha^{1-1/n} + (1-\alpha)^{1-1/n} > 1 + c_8$. So, if we take $\delta/r = C\mu\delta$ small enough then the last expression in (2.3) is positive and we obtain
$$\mathrm{Vol}(T_{\mu,\delta} \cap 2B) \geq c_9 n\omega_n r^{n-1}\delta \ .$$

Finally, since the metric g is comparable to a flat metric on a small ball, we have a similar inequality also for g. □

To conclude the proof of the lower bound in Theorem 1.2 we observe that due to Proposition 2.1 (iv) the number of balls in Proposition 2.1 is $> c\mu^n$. So, $\mathrm{Vol}(T_{\mu,\delta}) > c\delta/\mu^{n-1} \cdot \mu^n = c\mu\delta$.

3. Holomorphic Functions in Small Scales - dimension 1

The next three sections are a preparation for Section 6. The reader may prefer to begin in Section 6 and come back here when necessary. In this section we describe the behavior of holomorphic functions of one variable in small scales. The proofs in this section follow closely the proofs in section 5 of [DF88]. We will omit the proofs when they are identical to the proofs in [DF88].

We denote by $B_r \subset \mathbb{C}$ the disk $|z| \le r$. Suppose F is holomorphic on B_3 and satisfies the following growth assumption:

$$\sup_{B_2} |F| \le |F(0)| e^{C_1\mu} \ . \tag{3.1}$$

Let $I \subset \mathbb{R}$ denote the interval $[-1,1]$. Let $\delta > 0$ be small enough. We decompose I into disjoint subintervals of sizes $C_2\delta < |I_\nu| < C_3\delta$. We call I_ν a *tame* interval if

$$\frac{|F(x)|}{|F(y)|} < C_4 \ \forall x, y \in I_\nu .$$

Otherwise, I_ν is called a *wild* interval. Given $x \in I$, we denote by I_x the subinterval to which x belongs (I_x is defined outside a set of measure 0). The main proposition of this section is

Proposition 3.2. *Assume F satisfies (3.1). Then*

(a) *There exist at most $C_5\mu$ wild intervals.*
(b) *There exists a set $E \subseteq I$ of measure $|E| < C_6\mu\delta$ such that for all $x \in I \setminus E$ we have*

$$\frac{1}{C_4} \le \frac{|F(x)|}{\mathrm{Av}_{I_x}|F|} \le C_4 \ ,$$

where I_x is the subinterval of I which contains x. E may depend on F.

The point of part (a) is that the number of wild intervals stays bounded as $\delta \to 0$. Part (b) is an immediate corollary of part (a).

Proposition 3.2 generalizes Proposition 5.1 from [DF88]. The main new point here is the introduction of the parameter δ of the subdivision, while in [DF88] the size of the subdivision is taken to be comparable to $1/\mu$. A minor technical difference is that here we also allow subdivisions with non-fixed size of the subintervals. This will serve us in the change of variable argument in the proof of Proposition 4.3.

The first step we make is a reduction to polynomials. It is shown in Section 5 of [DF88]

Lemma 3.3 ([DF88, Lemma 5.2])**.** *F has at most $C_7\mu$ zeroes in B_2.*

Let $P(z) := \prod_{|\alpha|\le 2, F(\alpha)=0}(z-\alpha)$. P is a polynomial of degree $d \le C_7\mu$. Let $f(z) = \log|P(z)|$. The next lemma shows that we can assume $F(z) = P(z)$.

Lemma 3.4 ([DF88, Lemma 5.3])**.** *For all $x, y \in I_\nu$ we have*

$$|\log|F(x)| - \log|F(y)|| \le |f(x) - f(y)| + C_8\mu\delta \ .$$

We now turn to bound from above $|f(x) - f(y)|$.

Lemma 3.5. *There exists a set $\tilde{E} \subseteq \mathbb{R}$ of measure $|\tilde{E}| < C_9\mu\delta$ such that $|f'(x)| \le c_{10}/\delta$ for all $x \in \mathbb{R} \setminus \tilde{E}$.*

Proof. This follows from the properties of the Hilbert Transform. We imitate the proof of Lemma 5.4 in [DF88] with a little more details.

We recall the definition and some basic properties of the Hilbert Transform. Let $u \in L^2(\mathbb{R})$. Let sgn be the sign function on $\mathbb{R}$. Let $\mathcal{F}$ be the Fourier Transform on $L^2(\mathbb{R})$. Define the Hilbert Transform $\mathcal{H}u$ by

$$\mathcal{F}(\mathcal{H}u) = \frac{i}{2}\mathrm{sgn} \cdot \mathcal{F}(u) \ .$$

From this definition it is clear that $\mathcal{H}$ is a bounded operator on $L^2(\mathbb{R})$. Observe that

$$f'(x) = \sum_\alpha \Re\left(\frac{1}{x-\alpha}\right) \ .$$

We may assume $\forall\alpha,\ \Im\alpha \leq 0$. Consider first the case where $\forall\alpha,\ \alpha \notin \mathbb{R}$. Let $q_\alpha(x) = -\Im(1/(x-\alpha))$, and $q = \sum_\alpha q_\alpha$. Then, $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and by Theorem 3 in III.2.3 of [Ste70] $\mathcal{H}q = f'$. By observing that $\mathrm{sgn}' = 2\delta_0$ and by basic properties of the Fourier Transform one sees that if $u \in L^2(\mathbb{R})$ has a compact support and $x \notin \mathrm{Supp}\, u$, then

$$(\mathcal{H}u)(x) = \int_{\mathbb{R}} \frac{u(y)}{x-y}\, dy$$

(See also exc. 1.9 in [GS94] and Theorem 5 in III.3.3 of [Ste70]). Thus, by Theorem 3 in I.5 of [Ste93] the Hilbert Transform is of weak type (1,1). We get

$$|\{|f'| > c_{10}/\delta\}| \leq C_{11}\delta\|q\|_1 \leq C_{12}\mu\delta \ . \tag{3.6}$$

Finally, we move to the case where $\exists\alpha \in \mathbb{R}$. Define $g_\varepsilon(x) := f'(x - i\varepsilon)$. A small calculation shows that $g_\varepsilon \to f'$ in measure as $\varepsilon \to 0$. Since we can apply the considerations above to g_ε we conclude that (3.6) is true also in this case. □

We call I_ν a *bad interval of type I* if $I_\nu \subseteq \tilde{E}$. We denote the set of all bad intervals of type I by B_1. From Lemma 3.5 we know

Lemma 3.7. *$\#B_1 \leq C_{13}\mu$.*

We call I_ν a *bad interval of type II* if I_ν or one of its adjacent subintervals contains $\Re\alpha$ for some α, or I_ν is one of the two extreme subintervals. We denote by I_ν^* the union of the subinterval I_ν with its two adjacent subintervals. We denote the set of all bad subintervals of type II by B_2. It is clear that

Lemma 3.8. *$\#B_2 \leq C_{14}\mu$.*

We call I_ν a *bad interval of type III* if $\int_{I_\nu} |f''(x)|\, dx > c_{10}/\delta$. We denote the set of all bad intervals of type III by B_3.

Lemma 3.9. *$\#B_3 \leq C_{15}\mu$.*

Proof. We observe that

$$f''(x) = -\sum_\alpha \Re\left(\frac{1}{(x-\alpha)^2}\right) \ .$$

Hence,

$$\sum_{\nu, I_\nu \notin B_2} \int_{I_\nu} |f''(x)|\, dx \leq \sum_{\nu, I_\nu \notin B_2} \sum_{\alpha} \int_{I_\nu} \frac{1}{|x-\alpha|^2}\, dx \leq$$

$$\sum_{\alpha} \sum_{\nu, I_\nu^* \not\ni \Re\alpha} \int_{I_\nu} \frac{1}{|x-\alpha|^2}\, dx \leq \sum_{\alpha} \int_{|x-\alpha|>C_2\delta} \frac{1}{|x-\alpha|^2}\, dx \leq C_{16}\mu/\delta.$$

On the other hand

$$\sum_{\nu, I_\nu \notin B_2} \int_{I_\nu} |f''(x)|\, dx \geq \sum_{\nu, I_\nu \in B_3 \setminus B_2} \int_{I_\nu} |f''(x)|\, dx \geq c_{10} \#(B_3 \setminus B_2)/\delta\ .$$

Together, we get that $\#(B_3 \setminus B_2) \leq C_{17}\mu$. Since $\#(B_2) \leq C_{14}\mu$, we also obtain $\#(B_3) \leq C_{15}\mu$. □

I_ν is called *good* if it is not in $B_1 \cup B_2 \cup B_3$.

Lemma 3.10. *If I_ν is a good interval we have*

$$\sup_{x,y \in I_\nu} |f(x) - f(y)| \leq c_{16}\ .$$

Proof. Since $I_\nu \notin B_1$, there exists $x_\nu \in I_\nu$ such that $|f'(x_\nu)| < c_{10}/\delta$. Since $I_\nu \notin B_3$

$$\sup_{x,y \in I_\nu} |f'(x) - f'(y)| \leq c_{10}/\delta\ .$$

Together, we get that $\sup_{I_\nu} |f'(x)| \leq 2c_{10}/\delta$. This gives that $\sup_{x,y\in I_\nu} |f(x) - f(y)| \leq c_{16}$. □

This completes the proof of Proposition 3.2.

4. Holomorphic Functions in Small Scales - dimension $n > 1$

In this section we continue the study of holomorphic functions. We prove an analog of Proposition 3.2 in dimension $n > 1$. We adjust the proof of Proposition 5.11 in [DF88].

Let F be a holomorphic function on the polydisk $B_3 \times \ldots \times B_3 \subseteq \mathbb{C}^n$. Let $Q = I^n$. Assume F satisfies

$$\sup_{B_2^n} |F| \leq |F(0)| \mathrm{e}^{C_1\mu}\ . \tag{4.1}$$

Subdivide Q into subcubes Q_ν of sides $C_2\delta < l_\nu < C_3\delta$.

The following proposition follows from Proposition 3.2 by induction in the same way as Proposition 5.9 follows from Proposition 5.1 in section 5 of [DF88].

Proposition 4.2. *Let F satisfy (4.1) and $F \geq 0$ on Q. Assume that $F \equiv 1$ on each of the hyperplanes $z_i = 0$. There exists a set $E \subseteq Q$ of measure $|E| \leq C_4\mu\delta$ such that*

$$C_5 \leq \frac{F(x)}{\mathrm{Av}_{Q_x} F} \leq C_6 \quad \forall x \in Q \setminus E,$$

where Q_x is the subcube containing x (Q_x is defined outside a set of measure 0).

We now remove the technical assumption in proposition 4.2. The main proposition of this section is

Proposition 4.3. *Let F satisfy (4.1) and $F \geq 0$ on Q. There exists a cube $R \subseteq Q$ independent of F with the following property: Suppose $\mu\delta < c_7$. We subdivide R into subcubes R_α of sides $C_8\delta < l_\alpha < C_9\delta$. Then, there exists a subset $E \subseteq R$ of measure $|E| \leq C_{10}\mu\delta$ such that*

$$C_5 \leq \frac{F(x)}{\mathrm{Av}_{R_x} F} \leq C_6 \quad \forall x \in Q \setminus E,$$

where R_x is the subcube containing x.

We set R in the same way as in [DF88]:

Lemma 4.4 ([DF88, Lemma 5.10])**.** *There exists a map $W : \mathbb{R}^n \to \mathbb{R}^n$ with the following properties:*

(1) *W is a polynomial map.*
(2) *$W(Q) \subseteq Q$.*
(3) *W maps the hyperplanes $x_i = 0$ to 0.*
(4) *W is a local diffeomorphism outside the hyperplanes $x_i = 0$.*

Let $U \subseteq Q$ be an open set which is mapped diffeomorphically onto $W(U)$ and has a positive distance from any hyperplane $x_i = 0$. Let $R \subseteq W(U) \subseteq W(Q) \subseteq Q$ be a cube. We show that R satisfies the property stated in Proposition 4.3.

We begin with the following: Given a subdivision $\mathcal{D}$ of Q and a subset $A \subseteq Q$, we say that A is *adapted* to $\mathcal{D}$, if A is contained in one of the subcubes of $\mathcal{D}$. The next lemma is valid when $C_{11} \ll C_2$.

Lemma 4.5. *There exist a finite number of subdivisions $\mathcal{D}_i$ of Q into cubes Q_ν of sides $C_2\delta < l_\nu < C_3\delta$ such that every set of diameter $< C_{11}\delta$ is adapted to at least one of the $\mathcal{D}_i$'s.*

We can now finish the proof of Proposition 4.3. We may assume that $F(0) = 1$. The function $\tilde{F} = F \circ W$ satisfies the conditions of Proposition 4.2. So, given any of the subdivisions $\mathcal{D}_i$ of Lemma 4.5 we can find an exceptional set $\tilde{E}_i \subseteq Q$ corresponding to $\tilde{F}$. Let $\tilde{E} = \cup_i \tilde{E}_i$.

Call Q_ν a *bad* subcube if $|\tilde{E} \cap Q_\nu|/|Q_\nu| > \gamma_0$. Let B be the union of all bad subcubes Q_ν. Finally, set $E = W((\tilde{E} \cup B) \cap U)$.

Lemma 4.6. *$|E| \leq C_{12}\mu\delta$.*

Proof of Lemma. We estimate $|B|$:

$$\begin{aligned} C_{13}\mu\delta \geq |\tilde{E}| \geq 3^{-n} \sum_{\text{bad } Q_\nu\text{'s}} |\tilde{E} \cap Q_\nu| \geq \\ 3^{-n} \cdot \gamma_0 \#(\text{bad } Q_\nu\text{'s})|Q_\nu| \geq C_{14}\delta^n \#(\text{bad } Q_\nu\text{'s}). \end{aligned}$$

Hence, the number of bad Q_ν's is $\leq C_{15}\mu/\delta^{n-1}$ and their total volume is $\leq C_{16}\mu\delta$. So $|\tilde{E} \cup B| \leq C_{17}\mu\delta$. Since the Jacobian of the map W is bounded on U we conclude that $|E| \leq C_{18}\mu\delta$. □

Let R_α be a subcube. Look at $\tilde{R}_\alpha = W^{-1}(R_\alpha)$. Since W^{-1} has a bounded Jacobian, and we may assume $C_9 \ll C_2$, $\tilde{R}_\alpha$ is a set of diameter $< C_{11}\delta$. Let $\mathcal{D}_i$ be one of the subdivisions of Q from Lemma 4.5 which is adapted to $\tilde{R}_\alpha$. $\tilde{R}_\alpha \subseteq Q_\nu$ for some ν.

It follows from Proposition 4.2 that $\tilde{F}(y_1)/\tilde{F}(y_2) \le C_{19}\ \forall y_1, y_2 \in Q_\nu \setminus \tilde{E}$. Hence, if we let $x_0 \in R_\alpha \setminus E$ and $y_0 = W^{-1}(x_0)$, then $y_0 \in \tilde{R}_\alpha \setminus (\tilde{E} \cup B)$ and we obtain

$$\text{(4.7)}\quad \mathrm{Av}_{R_\alpha} F = \frac{1}{|R_\alpha|}\int_{R_\alpha} F(x)\,dx \ge \frac{1}{|R_\alpha|}\int_{R_\alpha\setminus E} F(x)\,dx = \frac{1}{|R_\alpha|}\int_{\tilde{R}_\alpha\setminus\tilde{E}} \tilde{F}(y)|J_W|\,dy \ge \frac{C_{20}}{|R_\alpha|}\int_{\tilde{R}_\alpha\setminus\tilde{E}} \tilde{F}(y_0)|J_W|\,dy = C_{20}\frac{|R_\alpha\setminus E|}{|R_\alpha|}F(x_0) \ge C_{21}F(x_0).$$

The last inequality is true, since $\tilde{R}_\alpha \subseteq Q_\nu$, Q_ν is not bad and $|R_\alpha|/|Q_\nu| > C_{22}$.

On the other hand,

$$\text{(4.8)}\quad \mathrm{Av}_{R_\alpha} F = \frac{1}{|R_\alpha|}\int_{R_\alpha} F(x)\,dx = \frac{1}{|R_\alpha|}\int_{\tilde{R}_\alpha} \tilde{F}(y)|J_W|\,dy \le \frac{1}{|R_\alpha|}\int_{Q_\nu} \tilde{F}(y)|J_W|\,dy \le \frac{C_{23}|Q_\nu|}{|R_\alpha|}\frac{1}{|Q_\nu|}\int_{Q_\nu} \tilde{F}(y)\,dy \le C_{24}\mathrm{Av}_{Q_\nu}\tilde{F} \le C_{25}\tilde{F}(y_0) = C_{25}F(x_0).$$

Inequalities (4.7) and (4.8) complete the proof of Proposition 4.3. □

5. Eigenfunctions in Small Scales on Real Analytic Manifolds

Let ϕ_μ be an eigenfunction. Let V be a small open set in which the metric g can be developed in power series. We identify V with a ball $B(0,\rho_0) \subseteq \mathbb{R}^n$.

It is proved in Section 7 of [DF88] that Proposition 4.3 and the growth property of eigenfunctions imply

Proposition 5.1. *There exists a cube $R \subseteq V$ with the following property: Suppose $\mu\delta < c_1$. We subdivide R into cubes R_α of sides $C_2\delta < l_\alpha < C_3\delta$. There exists a subset $E \subseteq R$ of measure $|E| \le C_4\mu\delta$ such that*

$$C_5 \le \frac{\phi_\mu(x)^2}{\mathrm{Av}_{R_x}\phi_\mu^2} \le C_6 \quad \forall x \in R\setminus E.$$

Here, R_x denotes the subcube which contains x (defined outside a set of measure 0).

We need a slightly different version of this proposition. We say that R_α touches R_β if they have at least one vertex in common. Each cube R_α touches at most 3^n cubes. Let us denote by R^*_α the cube R_α together with the $3^n - 1$ cubes which touch R_α. There exist 3^n subdivisions $\mathcal{D}_i$ of R such that each subcube of $\mathcal{D}_i$ is equal to R^*_α for some α. Let E be the union of all the sets E_i corresponding to the subdivision $\mathcal{D}_i$ according to Proposition 5.1. $|E| \le C_7\mu\delta$. These considerations prove the following version of Proposition 5.1.

Proposition 5.2. *There exists a cube $R \subseteq V$ with the following property: Suppose $\mu\delta < c_1$. We subdivide R into cubes R_α of sides $C_8\delta < l_\alpha < C_9\delta$. There exists a subset $E \subseteq R$ of measure $|E| \le C_{10}\mu\delta$ such that*

$$C_5 \le \frac{\phi_\mu(x)^2}{\mathrm{Av}_{R^*_x}\phi_\mu^2} \le C_6 \quad \forall x \in R\setminus E.$$

For later reference we divide the subcubes R_α into *good* and *bad*. We show that in the vicinity of good subcubes we have a bounded L^2-growth, and that the proportion of bad cubes is $\leq c\mu\delta$.

Definition 5.3. We say that R_α is *good* if $|E \cap R_\alpha|/|R_\alpha| < 0.9$. Otherwise, R_α is called *bad*.

Lemma 5.4. *Good subcubes R_α satisfy*

$$\int_{R_\alpha} \phi_\mu^2 \, dx \geq C_{11} \int_{R_\alpha^*} \phi_\mu^2 \, dx \tag{5.5}$$

Proof. By Proposition 5.2

$$\int_{R_\alpha} \phi_\mu^2(x) \, dx \geq \int_{R_\alpha \setminus E} \phi_\mu^2(x) \, dx \geq C_5 \int_{R_\alpha \setminus E} \mathrm{Av}_{R_\alpha^*} \phi_\mu^2 \, dx =$$
$$C_5 \frac{|R_\alpha \setminus E|}{|R_\alpha^*|} \int_{R_\alpha^*} \phi_\mu^2(x) \, dx \geq 0.1 C_5 \frac{|R_\alpha|}{|R_\alpha^*|} \int_{R_\alpha^*} \phi_\mu^2(x) \, dx \geq C_{11} \int_{R_\alpha^*} \phi_\mu^2(x) \, dx \; .$$

□

Lemma 5.6. *The number of bad subcubes is $\leq C_{12}\mu/\delta^{n-1}$.*

Proof.

$$C_{10}\mu\delta \geq |E| \geq 3^{-n} \sum_{\text{bad } R_\alpha\text{'s}} |E \cap R_\alpha| \geq$$
$$0.9\#(\text{bad } R_\alpha\text{'s})|R_\alpha| \geq 0.9 C_8^n \delta^n \#(\text{bad } R_\alpha\text{'s}).$$

□

6. Proof of the Upper Bound in Theorem 1.2

In this section we estimate from above the volume of a tubular neighborhood of the nodal set. The proof is based on the study of eigenfunctions in small scale in Section 5.

Let $\mathcal{V} = \{V_k\}$ be a covering of M by small open sets. Let $R_k \subseteq V_k$ be a cube preferred by Proposition 5.2. The next lemma shows that it is enough to estimate the volume of $T_{\mu,\delta}$ in preferred cubes.

Lemma 6.1. *There exists a covering $\mathcal{V} = \{V_k\}$ on M with the following properties*

(a) *$\mathcal{V}$ is a finite covering.*
(b) *the metric g can be developed in power series in each chart V_k.*
(c) *$M = \cup_k R_k$ for some choice of cubes $R_k \subseteq V_k$ preferred by Proposition 5.2.*

We defer the proof of this Lemma to Section 6.1.

Now, let $R \subseteq V$ be a preferred cube. We denote by R_α its subcubes. The side of R_α is $C_1\delta < l_\alpha < C_2\delta$.

Definition 6.2. We call R_α a *nodal* cube if $\mathcal{N}_\mu \cap R_\alpha \neq \emptyset$.

Let us denote the set of nodal subcubes R_α by Nod. R_α^* denotes the union of R_α with its $3^n - 1$ neighbors.

Lemma 6.3. $T_{\mu,c_3\delta} \subseteq \cup_{R_\alpha \in \mathsf{Nod}} R_\alpha^*$.

It remains to estimate the number of nodal cubes.

Lemma 6.4. *The number of good nodal cubes is* $\le C_4 \operatorname{Vol}_{n-1}(\mathcal{N}_\mu)/\delta^{n-1}$.

Proof. Let R_α be a good nodal cube. It is proved in pp. 181–182 of [DF88] that the growth assumption (5.5) implies

$$\operatorname{Vol}_{n-1}(\mathcal{N}_\mu \cap R^*_\alpha) \ge C_5\delta^{n-1} \ . \tag{6.5}$$

Another proof of this fact follows from Theorem 4.1 in [Man07].

Summing up (6.5) over all good nodal cubes we arrive at

$$\begin{aligned} 3^n \operatorname{Vol}_{n-1}(\mathcal{N}_\mu) &\ge \sum_\alpha \operatorname{Vol}_{n-1}(\mathcal{N}_\mu \cap R^*_\alpha) \ge \\ &\sum_{\text{good nodal } R_\alpha\text{'s}} \operatorname{Vol}_{n-1}(\mathcal{N}_\mu \cap R^*_\alpha) \ge C_5\#(\text{good nodal } R_\alpha\text{'s})\delta^{n-1} \ . \end{aligned} \tag{6.6}$$

□

Proof of Theorem 1.2. By Lemma 5.6 we know that the number of bad nodal cubes is $\le C_6\mu/\delta^{n-1}$. By Lemma 6.4 and Theorem 1.3 the number of good nodal cubes is $\le C_7\mu/\delta^{n-1}$. Together, we get that the number of nodal cubes is $\le C_8\mu/\delta^{n-1}$. By Lemma 6.3

$$\operatorname{Vol}(T_{\mu,\delta}) \le C_9\#(\mathsf{Nod})\delta^n \le C_{10}\mu\delta \ .$$

□

6.1. **Proof of Lemma 6.1.** The following lemma is clear by compactness of M.

Lemma 6.7. *There exists* $\rho_0 > 0$ *such that for all* p*, the metric* g *can be developed in power series in* $B(p, \rho_0)$.

Let $\rho_1 = \rho_0/C$ with C large enough.

Lemma 6.8. *Every ball* $B(p, \rho_1)$ *contains a preferred cube* R *which contains* p.

Proof. We identify $B(p, \rho)$ with the Euclidean ball $B(0, \rho)$ by working in geodesic coordinates. Suppose that the point $x_0 \in R \subseteq B(0, \rho_0)$. Let $x_1 \in B(0, \rho_0)$ with $|x_1| = |x_0| =: r$. We can construct a preferred cube $R' \subseteq B(0, \rho_0)$ which contains x_1 by using an orthogonal transformation in $B(0, \rho_0)$.

Now, given p, let q be any point on M such that $\operatorname{dist}(p, q) = r$. The geodesic ball $B(q, \rho_0)$ contains a preferred cube R_1 which contains p. Take a cube R in $R_1 \cap B(p, \rho_1)$ which contains p. □

Proof of Lemma 6.1. By lemma 6.8 we can cover M by preferred cubes. Then by compactness of M we can extract a finite covering by preferred cubes. □

7. Approximation by Nodal Sets

Proof of Theorem 1.8. The proof proceeds similarly to the proof of Corollary 1.6. Fix $C, \varepsilon > 0$. Let $T_{k,\delta}$ be the tubular neighborhood of $\mathcal{N}(\phi_k)$ of radius $\delta_k = C/\mu_k^{n+1+\varepsilon}$. By Theorem 1.2 $\operatorname{Vol}(T_{k,\delta_k}) \le C/\mu_k^{n+\varepsilon}$. We conclude that

$$\sum_k \operatorname{Vol}(T_{k,\delta_k}) \le C \sum_k \mu_k^{-n-\varepsilon}. \tag{7.1}$$

By Weyl's Law [Wey12, Hör68] we know that

$$\mu_k \asymp Ck^{1/n}.$$

Hence

$$\sum_k \mathrm{Vol}\,(T_{k,\delta_k}) \le C \sum_k k^{-1-\varepsilon/n}$$

is finite. So, by the Borel-Cantelli Lemma (see e.g. [Fel68]) we obtain

$$\mathrm{Vol}(\cap_{j=1}^{\infty} \cup_{k=j}^{\infty} T_{k,\delta_k}) = 0\ .$$

□

8. Dimension two

Theorem 8.1. *Let (Σ, g) be a smooth (i.e. C^∞) closed Riemannian surface. Then there exist $C_1, C_2 > 0$ such that*

$$C_1 \mu\delta \le \mathrm{Vol}(T_{\mu,\delta}) \le C_2\, \mathrm{length}(\mathcal{N}_\mu)\delta\ .$$

In particular, Theorem 1.2 is true for surfaces which satisfy Yau's conjecture.

We recall from [DF90] that for any smooth surface $\mathrm{length}(\mathcal{N}_\mu) \le C_3\mu^{3/2}$. Hence, if we modify the proof of Theorem 1.8 according to Theorem 8.1 we obtain

Proposition 8.2. *Let (Σ, g) be a closed compact surface with a* smooth *metric g. Then we have* $\mathrm{Vol}(M(7/2+\varepsilon, C)) = 0$ *for all $C, \varepsilon > 0$.*

8.1. **Lower Bound in Theorem 8.1.** This is basically Brüning's argument. We can cover a fixed portion of Σ with pairwise disjoint balls $B_i = B(x_i, r)$ of radius $r = c/\mu$ and such that $\phi_\mu(x_i) = 0$. The set $\mathcal{N}_\mu \cap B(x_i, r)$ is of length $\ge r$. Moreover, in local coordinates it has a projection of length $\ge cr$ on one of the axes. This implies that $T_{\mu,\delta} \cap B(x_i, r)$ has area $\ge cr\delta$. Summing up over all the balls B_i we obtain

$$\mathrm{Vol}(T_{\mu,\delta}) \ge c_1\mu^2 \cdot c_2\delta/\mu = c_3\mu\delta\ .$$

8.2. **Upper Bound in Theorem 8.1- First Proof.** Let an eigenfunction ϕ_μ have nodal domains $\Omega_1, \ldots, \Omega_{N(\mu)}$. Given $\partial\Omega_j \subset \mathcal{N}_\mu$, let $L_j(t)$ denote the *interior parallel* of $\partial\Omega_k$ at the distance t inside Ω_j. It is clear that

$$\mathrm{area}(A_\mu) = \sum_{j=1}^{N} \int_{t=0}^{\delta} \mathrm{length}(L_j(t))\, dt. \tag{8.3}$$

The following inequality can be found in [Sav01, Proposition A.1.iv]:

$$\mathrm{length}(L_j(t)) \le \mathrm{length}(\partial\Omega_j) + R(\Omega_j) \max\left\{ \int_{\Omega_j} K^+ - 2\pi\chi(\Omega_j), 0 \right\}. \tag{8.4}$$

Here K^+ denotes the positive part of the Gauss curvature, $\chi(\Omega_j)$ is proportional to the number $m_j = m_j(\mu)$ of connected components of $\partial\Omega_j$, and $R(\Omega_j)$ denotes the inner radius of Ω_j. We substitute (8.4) into (8.3) and sum over $1 \le j \le N$. By Proposition 1.4 we know that $R(\Omega_j) \le C/\mu$. We get the estimate

$$\frac{\mathrm{area}(A_\mu)}{\delta} \le 2 \cdot \mathrm{length}(\mathcal{N}_\mu) + \frac{C\int_M K^+}{\mu} + \frac{4\pi C}{\mu} \sum_{j=1}^{N(\mu)} m_j(\mu) \tag{8.5}$$

As $\mu_j = \mu \to \infty$, the second term goes to zero. It remains to estimate the third term. One can construct a connected graph on M whose edges will include all arcs of $\mathcal{N}_\mu$, and show using Euler's formula that

$$\sum_{j=1}^{N} m_j \leq 2(N+g-1),$$

where g denotes the genus of the surface M. Also, by Courant's nodal domain theorem

$$N = N(\mu_k) \leq k+1.$$

We recall that by [Wey12, Hör68] in dimension two $\mu_k \asymp C\sqrt{k}$, hence $N(\mu_k) \leq C\mu_k^2$. It follows that the third term in the right-hand side of (8.5) is less than $C\mu$. Substituting everything back into (8.5) and recalling that $\text{length}(\mathcal{N}_\mu) \geq C\mu$ (see [Brü78]) we get the desired estimate. □

8.3. **Upper Bound in Theorem 8.1- Second Proof.** This proof was communicated by M. Sodin and I. Polterovich.

It suffices to give a proof for the neighborhood of $\mathcal{N}_\mu$ of size $\delta/3$. We cover M with cubes of side $C\delta$ (*large cubes*), as well as by cubes of side $C\delta/3$ (*small cubes*). One can easily arrange that each cube intersects a bounded number of other cubes. For every small cube, there exists a unique concentric large cube whose side is three times larger. To estimate the area of $T_{\mu,\delta}$, it suffices to estimate the volume of the union B_j of all small cubes which intersect the nodal set $\mathcal{N}_\mu$. Indeed, if $x \in T_{\mu,\delta}$, then $\mathcal{N}_\mu$ intersects either the small cube containing x, or one of the 8 neighboring small cubes, so the volume of $T_{\mu,\delta}$ is at most $9 \cdot \text{vol}(B_j)$.

We distinguish several cases

- i) $\mathcal{N}_\mu$ intersects a small cube Q, but any connected component of $\mathcal{N}_\mu \cap Q$ doesn't intersect the boundary of the big concentric cube Q'.
- ii) $\mathcal{N}_\mu$ intersects a small cube Q, and there exists a connected component of $\mathcal{N}_\mu \cap Q$ that intersects the boundary of the big concentric cube Q'.

In case (i) there is at least one nodal domain contained in Q', so by the Faber-Krahn Inequality (see [EK96, Ch. 7, Th. 1]) we get that the area of this nodal domain is $> C/\mu^2$. By the Isoperimetric Inequality, the length of $\mathcal{N}_\mu \cap Q'$ is at least $C/\mu \geq C\delta$.

In case (ii), the length of $\mathcal{N}_j \cap Q'$ is at least $\delta/3$.

Hence, we conclude that the number of Q' for which Q satisfies case (i) or case (ii) is $\ll \text{length}(\mathcal{N}_\mu)/\delta$. Accordingly, the sum of the areas of those cubes is

$$\ll \text{length}(\mathcal{N}_\mu)/\delta \cdot \delta^2 \leq C\,\text{length}(\mathcal{N}_\mu)\delta. \tag{8.6}$$

□

9. Discussion

For a given M it seems interesting to find

$$E(M) := \sup\{b : \text{vol}(M(b,C)) > 0 \ \textit{for some}\ C > 0\}.$$

Theorem 1.8 implies that on real-analytic n-dimensional manifolds, $E(M) \leq n+1$. In dimension one, it follows from the theory of continued fractions that $E(M) = 2$ for $M = [0,\pi]$. In fact, $M(2,\pi) = M$ while $\text{Vol}(M(2+\varepsilon, C)) = 0\ \forall\varepsilon > 0$.

The same result likely holds for *separable systems*.[1] In such systems one can separate variables and choose a basis of eigenfunctions that (in appropriate coordinates) have the form $\phi(x_1,\ldots,x_n)=\prod\psi_j(x_j)$, where ψ_j are solutions of 2nd order differential equations. Accordingly, $\mathcal{N}(\phi)$ forms a "grid" of hypersurfaces determined by zeros of ψ_j-s, and approximation by $\mathcal{N}(\phi)$ reduces to a series of one-dimensional problems.

As a model example we consider an n-dimensional cube

$$M(n)=\prod_{j=1}^{n}[0,\pi/\alpha_j],$$

with Dirichlet boundary conditions, where for simplicity we assume $\{\alpha_j^2\}_{j=1}^n$ are linearly independent over $\mathbb{Q}$. Then the eigenvalues have the form $\sum_{j=1}^n \alpha_j^2 m_j^2$ (where $m_j\in\mathbb{N}$) and are simple, while the corresponding eigenfunctions have the form

$$\phi(m_1,\ldots,m_n;x_1,\ldots,x_n)=\prod_{j:\,m_j\neq 0}\sin(m_j\alpha_j x_j).$$

Proposition 9.1. *$E(M(n))=2$ for all n.*

Proof of Proposition 9.1.

We first make a change of variables $y_j=\pi\alpha_j x_j$. This change of variables will only affect constants in the rate of approximation by nodal sets; it won't affect the exponent. In the rescaled coordinates, nodal sets have the form

$$\mathcal{N}(\phi(m_1,\ldots,m_n))=\cup_{j:m_j\neq 0}\mathcal{A}_j, \tag{9.2}$$

where $\mathcal{A}_j:=\{(y_1,\ldots,y_n): y_j=k_j/m_j,\ \ 0\le k_j\le m_j\}$. We first show that

Claim 9.3. *$E(M(n))\ge 2$.*

Proof. Let $(y_1,\ldots,y_n)\in M$ be an arbitrary point on M; we have $0\le y_j\le 1$. We can assume without loss of generality that $y_j\notin\mathbb{Q},\forall 1\le j\le n$, since the set of such points has the full measure. Consider next the continued fraction expansion of its first (say) coordinate,

$$y_1=[0;a_1,a_2,\ldots],$$

where we use the notation of [Khi97]. Let $p_k/q_k,\ k=1,2,\ldots$ be the corresponding continued fractions. Then the points $(p_k/q_k,y_2,\ldots,y_n)\in\mathcal{N}(\phi(q_k,0,\ldots,0))$, and the Claim follows from the well-known inequality [Khi97]

$$|y_1-p_k/q_k|<1/q_k^2.$$

□

We next show that

Claim 9.4. *$E(M(n))\le 2$.*

Proof. It suffices to show that $\mathrm{Vol}(M(2+\varepsilon,C))=0$ for all $C,\varepsilon>0$. Let $\mathbf{y}=(y_1,\ldots,y_n)\in M(2+\varepsilon,C)$. As before, we may assume that $y_j\notin\mathbb{Q}$. We know that there exists a sequence of eigenvalues $\mu_k\to\infty$ such that $d(\mathbf{y},\mathcal{N}(\phi_{\mu_k}))<C/\mu_k^{2+\varepsilon}$. Since all distances on $[0,1]^n$ are equivalent, we may define $d(\mathbf{x},\mathbf{y})=\max_{1\le j\le n}|x_j-y_j|$.

[1]Examples include surfaces of revolution, Liouville tori and *quantum completely integrable* systems [TZ02].

In view of (9.2), it follows that for some $1 \le j \le n$ (say, for $j = 1$), there exists a sequence of integers $q_k, k = 1, 2, \ldots$, such that $q_k \to \infty$ and $|y_1 - p_k/q_k| < C/q_k^{2+\varepsilon}$ for some $0 \le p_k \le q_k$. The Claim now follows from Corollary 1.6. This also finishes the proof of Proposition 9.1. □

For manifolds with ergodic geodesic flows (e.g. in negative curvature), eigenfunction behavior has been studied using *random wave model* [Ber77]. In addition, *percolation model* [BS02] has been used to study the statistics of nodal domains in chaotic systems. [2]

In the opinion of the authors, it would be difficult to use these models directly to predict the "best possible" rate of approximation by nodal sets. The reason is that these models describe a *single* eigenfunction on a scale of C/μ (several wavelengths). However (as shown by the example of $M = [0, \pi]$) for a given $x \in M$ the values of μ giving the best approximation of x by $\mathcal{N}(\phi_\mu)$ can grow exponentially. It thus seems difficult to take into account simultaneous behavior of all eigenfunctions in such a large energy range. However, one can probably expect that $E(M) > 2$ for such manifolds (in contrast to the integrable case), due to irregularity of nodal lines for such systems.

It also seems interesting to study "level sets" $M(b)$ for the approximation exponent b, e.g. defined by

$$M(b) := \cup_C M(b, C) \setminus (\cup_{a<b} \cup_C M(a, C)) .$$

Remark 9.5. It should follow from the results of [JL99] that the conclusion of Theorem 1.8 should also hold for *level sets* of eigenfunctions (since the level set of an eigenfunction is a nodal set of a linear combination of that eigenfunction with a constant eigenfunction). It seems interesting to determine which level sets are C/μ-dense (like nodal sets).

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[2] We refer the reader to [FGS04] and references therein for a nice discussion about applicability of those models for studying various questions about eigenfunctions of chaotic systems.

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