The diffeomorphism group of a $K3$ surface and Nielsen realization

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THE DIFFEOMORPHISM GROUP OF A $K3$ SURFACE AND NIELSEN REALIZATION

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Abstract. We use moduli spaces of various geometric structures on a manifold M to probe the cohomology of the diffeomorphism group and mapping class group of M. The general principle is that existence of a moduli problem for which the Teichmüller space resembles a point implies that the homomorphism from the diffeomorphism group (or the mapping class group) to an appropriate discrete group resembles a retraction after applying the classifying space functor. Our main application of this idea is for K^4 a $K3$ surface; here the maps $B\mathrm{Diff}(K) \to B\mathrm{Aut}(H^2(K;\mathbb{Z}))$ and $B\pi_0\mathrm{Diff}(K) \to B\mathrm{Aut}(H^2(K;\mathbb{Z}))$ are injective on real cohomology in degrees $* \leq 9$. The work of Borel and Matsushima determines the real cohomology of $B\mathrm{Aut}(H^2(K;\mathbb{Z}))$ in these degrees. Using the above injections, the Borel classes provide cohomological obstructions to a generalized Nielsen realization problem which asks when subgroups of the mapping class group can be lifted to the diffeomorphism group. We conclude that the homomorphism $\mathrm{Diff}(M) \to \pi_0\mathrm{Diff}(M)$ does not admit a section if M contains a $K3$ surface as a connected summand.

1. Introduction

Let M be a smooth closed oriented manifold. We write $\mathrm{Diff}(M)$ for the group of orientation preserving C^∞ diffeomorphisms of M. It is a topological group with the C^∞ Fréchet topology. There is an associated *mapping class group* $\Gamma(M)$, which is the group of path components of $\mathrm{Diff}(M)$.

The main idea of this paper is to probe the algebraic topology of $B\mathrm{Diff}(M)$ and $B\Gamma(M)$ using various moduli spaces coming from differential and complex geometry. The key is to find a moduli problem for which the associated Teichmüller space is contractible. This occurs for instance in the following three examples:

- Flat pointed Riemannian manifolds of arbitrary dimension. This is essentially trivial.
- Finite volume hyperbolic manifolds of dimension greater than or equal to 3, for which the situation is trivial as a consequence of Mostow Rigidity.
- Surfaces, for which this approach was worked out completely by Earle and Eells [EE69].

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A novel example is provided by Einstein metrics on a $K3$ surface K^4. Relatively little is known about the diffeomorphism group and our method provides interesting new information. Here the Teichmüller space is *not* contractible. However, period theory shows us that the Teichmüller space is obtained from a contractible space in a straightforward way, and this turns out to be good enough for the purposes of detecting cohomology on $B\text{Diff}(K)$.

For a given manifold M it is an interesting (and generally difficult) problem to understand the cohomology of $B\text{Diff}(M)$ and $B\Gamma(M)$. One motivation is the following: the set of elements in the cohomology of $B\text{Diff}(M)$ is precisely the set of universal characteristic classes for smooth fibre bundles with fibre M. A bundle $E \to B$ is classified by a map $B \to B\text{Diff}(M)$, and the characteristic classes of π are obtained by pulling classes back from $B\text{Diff}(M)$ via the classifying map.

For a few types of manifolds M the cohomology of $B\text{Diff}(M)$ is well understood. There are Hatcher's theorems that $\text{Diff}(S^3) \simeq O(4)$ [Hat81] and $\text{Diff}(S^1 \times S^2) \simeq O(2) \times O(3) \times \Omega O(3)$ [Hat83] (for the full unoriented diffeomorphism group). For a surface F_g of genus g the integral cohomology of $B\text{Diff}(F_g)$ is computable in the Harer-Ivanov stable range $* \leq (g-1)/2$ by the Madsen-Weiss Theorem [MW02]. For M an aspherical manifold the rational homotopy groups of $B\text{Diff}(M)$ have been computed in Igusa's concordance stable range $* < (\dim M)/6 - 7$ by Farrell and Hsiang [FH78]. Finally there is the work of Weiss-Williams [WW01, WW88, WW89, WW06] which gives more general information in the concordance stable range. While this is certainly not an exhaustive list, it at least illustrates that for most manifolds M the cohomology of $B\text{Diff}(M)$ remains largely unknown outside the concordance stable range.

1.1. **Our strategy.** A naive way to produce cohomology classes on $B\text{Diff}(M)$ is to construct a map into a space X with computable cohomology and then define classes on $B\text{Diff}(M)$ by pulling back known classes on X. This leaves us with the problem of detecting whether or not the classes pulled back from X are actually nontrivial on $B\text{Diff}(M)$. A naive way to detect classes is to choose a space Y with known cohomology, construct a map of Y into $B\text{Diff}(M)$, and then try to compute the pullbacks on Y.

One can obtain interesting maps $B\text{Diff}(M) \to X$ by applying the classifying space functor to a homomorphism $\text{Diff}(M) \to G$ for some group G. A natural choice for G is the group $\text{Aut}(A_M)$ of automorphisms of some algebraic object A_M associated to M such as the fundamental group of M or the cohomology ring of M. A diffeomorphism of M induces an automorphism of A_M and so there is a homomorphism $\Psi : \text{Diff}(M) \to \text{Aut}(A_M)$. One then hopes to be able to compute the cohomology of the group $\text{Aut}(A_M)$.

To construct a map $Y \to B\text{Diff}(M)$ it suffices to look for a fibre bundle over Y with fibre M. One promising class of spaces which naturally come equipped with fibre bundles are moduli spaces $\mathcal{M}_?(M)$ of various types ? of geometric structures on

M. For example, one can take the moduli space $\mathcal{M}_{Cplx}(M)$ of complex structures on M. Here one should be careful to work with a *fine* moduli space, meaning that it carries a universal bundle. If one chooses a favorable type ? of structure then it might turn out that the cohomology of the moduli space $\mathcal{M}_?(M)$ is computable to some degree.

The key is to find a moduli problem for which the Teichmüller space resembles a point. In fact, contractibility of the Teichmüller space implies that the moduli space has the homotopy type of $B\mathrm{Im}\Psi$, and moreover the composition

$$\mathcal{M}_?(M) \to B\mathrm{Diff}(M) \to B\mathrm{Im}\Psi$$

is a homotopy equivalence. From here one can compute a piece of the cohomology of $B\mathrm{Diff}(M)$.

1.2. **K3 manifolds.** A highly nontrivial example of this method is provided by the moduli theory of Einstein metrics on a $K3$ surface. Recall that all $K3$ surfaces have the same diffeomorphism type; we shall call a 4-manifold K in this diffeomorphism type a $K3$ *manifold.* See section 5.1 for the basic facts about $K3$ manifolds.

Let Q_K denote the cup-form on $H^2(K,\mathbb{Z})$. Sending a diffeomorphism to the induced automorphism of cohomology induces a homomorphism Ψ from $\mathrm{Diff}(K)$ to the group $\mathrm{Aut}(Q_K)$ of automorphisms of $H^2(K;\mathbb{Z})$ which preserve the cup pairing. This factors through the mapping class group $\Gamma(K)$ since $\mathrm{Aut}(Q_K)$ is discrete. The image of Ψ is an index 2 subgroup $\mathrm{Aut}' \subset \mathrm{Aut}(Q_K)$ ([Don90], [Mat86]).

For this moduli problem the Teichmüller space is *not* contractible, but it is related closely enough to a contractible space that the reasoning above still provides information. Period theory identifies the homotopy type of the Teichmüller space and moduli space of Einstein metrics in terms of the arithmetic group Aut'. The Teichmüller space is homeomorphic to two copies of a Euclidean space of dimension 57 minus a countable family of transversely intersecting codimension 3 Euclidean submanifolds; the (fine) moduli space is the (homotopy) quotient of this by a natural proper action of Aut'.

Let $\mathcal{M}^f_E(K)$ denote the *fine* moduli space of unit volume Einstein metrics on K (this is defined in section 2 in terms of a homotopy quotient).

Theorem 1.1. *The composition*

$$\mathcal{M}^f_E(K) \to B\mathrm{Diff}(K) \to B\Gamma(K) \to B\mathrm{Aut}'$$

is injective on real cohomology in degrees $* \leq 9$*. Hence the homomorphisms*

$$H^*(B\mathrm{Aut}';\mathbb{R}) \to H^*(B\mathrm{Diff}(K);\mathbb{R}),$$
$$H^*(B\mathrm{Aut}';\mathbb{R}) \to H^*(B\Gamma(K);\mathbb{R})$$

are injective in degrees $* \leq 9$.

Remark 1.2. Note that for any simply connected closed oriented topological 4-manifold, Freedman and Quinn [Fre82], [Qui86] have shown that there is an isomorphism $\pi_0\mathrm{Homeo}(M) \cong \mathrm{Aut}(Q_M)$. However, in the smooth category Ruberman [Rub98], [Rub99] has constructed examples of smooth diffeomorphisms which are continuously isotopic but not smoothly isotopic.

The proof of Theorem 1.1 rests on a careful examination of the group cohomology of Aut' and certain other related arithmetic groups. Also, from this theorem we see that the group cohomology of Aut' can serve as an interesting source of characteristic classes for smooth fibre bundles with fibre a $K3$ manifold. Thus one is lead to try to compute the cohomology of groups such as this. The well-known techniques of Borel and Matsushima do the job by producing a nontrivial family of real cohomology classes, which we shall refer to as the *Borel-Matsushima classes.* There is an inclusion $\mathrm{Aut}(Q_K) \hookrightarrow O_{3,19}$ and this Lie group retracts onto the maximal compact subgroup $O_3 \times O_{19}$. Taking classifying spaces and projecting onto the first factor gives a map $B\mathrm{Aut}(Q_K) \to BO_3$. On real cohomology we then have:

Theorem 1.3. *The ring homomorphism*

$$H^*(BO_3;\mathbb{R}) \cong \mathbb{R}[p_1] \to H^*(B\mathrm{Aut}(Q_K);\mathbb{R})$$

is an isomorphism in degrees $* \leq 9$*, and in this range there is an isomorphism*

$$H^*(B\mathrm{Aut}(Q_K);\mathbb{R}) \cong H^*(B\mathrm{Aut}';\mathbb{R}).$$

Thus $\mathbb{R}[p_1]$ *injects into* $H^*(B\mathrm{Diff}(K);\mathbb{R})$ *and* $H^*(B\Gamma(K);\mathbb{R})$ *in degrees* $* \leq 9$.

See section 4 for a more general statement regarding arbitrary cup forms.

1.3. **Nielsen realization and obstructions.** Let M^n be a smooth oriented closed manifold. The *(generalized) Nielsen realization problem* asks when a subgroup of the mapping class group of M can be realized as (a.k.a lifted or rectified to) a subgroup of the diffeomorphism group of M. In the context of surfaces this problem has a long history—it originates with Nielsen [Nie43] who phrased it in in terms of isometries of hyperbolic surfaces. Kerckhoff [Ker83] showed that all *finite* subgroups of the mapping class group of a surface can be rectified, but Morita showed that the Miller-Morita-Mumford characteristic classes $\kappa_i \in H^*(B\mathrm{Diff}(F_g);\mathbb{Q}) \cong H^*(B\Gamma(F_g);\mathbb{Q})$ provide cohomological obstructions to rectifying infinite subgroups.

For M an arbitrary even dimensional closed oriented manifold the classes on $B\mathrm{Diff}(M)$ obtained by pulling the Borel-Matsushima classes back from $B\mathrm{Aut}(Q_M)$ have a more geometric origin. They are related through index theory to the integration along the fibres of the components of the Atiyah-Hirzebruch $\widetilde{\mathcal{L}}$-class of the vertical tangent bundle. Morita observed [Mor87, Theorem 8.1] that the Bott Vanishing Theorem [Bot70] implies that these classes vanish on $B\mathrm{Diff}(M)^\delta$ (the diffeomorphism group given the discrete topology). Hence the Borel-Matsushima classes provide obstructions to lifting infinite subgroups of mapping class groups in higher dimensions also.

Morita's argument, together with Theorems 1.1 and 1.3, implies:

Theorem 1.4. *If M contains a $K3$ manifold as a connected summand then the projection*

$$\mathrm{Diff}(M) \to \pi_0\mathrm{Diff}(M) = \Gamma(M)$$

does not admit a section as a group homomorphism.

Note that if one is instead interested in realizing (infinite subgroups of) the group of automorphisms of the cup form then a corresponding result is much more easily obtained.

Theorem 1.5. *Let M be a $4k$-dimensional smooth closed oriented manifold such that the natural map $\Psi : \mathrm{Diff}(M) \to \mathrm{Aut}(Q_M)$ is surjective and*

$$\lfloor \mathit{rank}(M)/2 \rfloor - 2, \mathit{rank}(M) - |\mathit{signature}(M)| \geq 4k + 4.$$

Then Ψ does not admit a section which is a group homomorphism.

Variants of this can easily be stated when M is $4k+2$ dimensional and/or when Ψ is not quite surjective.

Organization of the paper:

In section 2 we describe our point of view on Teichmüller spaces and moduli spaces. This leads to the very simple Meta-Theorem 2.1 which serves as a template for Theorem 1.1.

In section 3 we discuss the examples of flat metrics, finite volume hyperbolic metrics in dimensions greater than 2, and complex structures on surfaces of genus greater than 1.

Section 4 reviews the techniques of Borel and Matsushima which we use to understand the cohomology of the groups $\mathrm{Aut}(Q_M)$ and hence to prove Theorem 4.6 (of which Theorem 1.3 is a special case).

Section 5 recalls the elements of period theory for Einstein metrics on a $K3$ manifold. We then apply period theory together with a computation in group cohomology and the results of section 4 to prove Theorem 1.1.

Section 6 explores the geometry of the Borel-Matsushima classes and their role as obstructions in generalized Nielsen realization problems. Here we give the proof of Theorem 1.5.

Finally, in section 7 we collect some ideas about the extent to which the Borel-Matsushima classes are algebraically independent when pulled back to the cohomology of $B\mathrm{Diff}(M)$.

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2. Teichmüller space and moduli space

We adopt the following homotopical view of moduli theory. Fix an algebraic object A_M associated to M such that a diffeomorphism of M induces an automorphism of A_M. Then there is homomorphism

$$\Psi : \mathrm{Diff}(M) \to \mathrm{Aut}(A_M);$$

let $\mathrm{Diff}_0(M)$ denote the kernel of Ψ and let $\mathrm{Aut}' \subset \mathrm{Aut}(A_M)$ denote the image of Ψ. Note that if $\mathrm{Aut}(A_M)$ is discrete then Ψ automatically factors through the mapping class group,

$$\begin{array}{ccc} \mathrm{Diff}(M) & \xrightarrow{\Psi} & \mathrm{Aut}(A_M) \\ & \searrow \quad \nearrow & \\ & \Gamma(M) & \end{array}$$

Fix a class ? of geometric structure on M and let $S_?(M)$ denote the space of all such structures on M.

The *Teichmüller space* $\mathcal{T}_?(M)$ of ? structures on M is the quotient of $S_?(M)$ by the group $\mathrm{Diff}_0(M)$. The *moduli space* $\mathcal{M}_?(M)$ of ? structures on M is the quotient of $S_?(M)$ by $\mathrm{Diff}(M)$, which is the same as the quotient of the Teichmüller space by the action of Aut'. A *marking* is a fixed identification of A_M, so $\mathrm{Diff}_0(M)$ is the group of diffeomorphisms which preserve the marking. Thus the Teichmüller space is often thought of as the moduli space of marked structures.

A good moduli space should have a universal bundle over it. If it does then it is called a *fine* moduli space; if it does not then it is called a *coarse* moduli space.

The universal M bundle over $\mathcal{M}_?(M)$ ($\mathcal{T}_?(M)$) should be the $\mathrm{Diff}(M)$ ($\mathrm{Diff}_0(M)$ resp.) quotient of the product bundle

$$S_?(M) \times M \to S_?(M),$$

where $\mathrm{Diff}(M)$ acts diagonally on the total space. If the action is not free and proper then the ordinary quotient may no longer be a bundle, in which case the ordinary quotient yields a coarse Teichmüller/moduli space. The standard solution in topology is to replace the ordinary quotient by the homotopy quotient (a.k.a. Borel construction)

$$X/\!\!/G := X \times_G EG$$

where EG is a universal free G-space.

Definition 2.1. The *coarse/fine Teichmüller spaces* of ? structures on M are

$$\mathcal{T}^c_?(M) := S_?(M)/\mathrm{Diff}_0(M) \qquad \mathcal{T}^f_?(M) := S_?(M)/\!\!/\mathrm{Diff}_0(M)$$

Similarly, the *coarse/fine moduli spaces* are

$$\begin{aligned}\mathcal{M}^c_?(M) &:= S_?(M)/\mathrm{Diff}(M) & \mathcal{M}^f_?(M) &:= S_?(M)/\!\!/\mathrm{Diff}(M)\\ &\cong \mathcal{T}^c_?(M)/\mathrm{Aut}' & &\cong \mathcal{T}^f_?(M)/\!\!/\mathrm{Aut}'.\end{aligned}$$

Definition 2.2. The universal bundle $U_{\mathcal{T}}$ over the fine Teichmüller space is

$$\begin{array}{c}(S_?(M)\times M)/\!\!/\mathrm{Diff}_0(M) = U_{\mathcal{T}}\\ \downarrow\\ S_?(M)/\!\!/\mathrm{Diff}_0(M) \;\; = \mathcal{T}^f_?(M);\end{array}$$

the universal bundle $U_{\mathcal{M}}$ over the fine moduli space is obtained by replacing $\mathrm{Diff}_0(M)$ with $\mathrm{Diff}(M)$.

Remark 2.3. In using the homotopy quotient one loses sight of any geometry on the fine moduli space. This will not matter for our purposes since we are concerned only with the homotopy type. To work with the geometry of the fine moduli space one should instead use a stack quotient.

Example 2.4. Take M to be a surface of genus $g > 1$, take ? to be the class *Cplx* of complex structures, and take A_M to be $\pi_1(M)$. On a surface two diffeomorphisms are isotopic if and only they induce the same outer automorphism of $\pi_1 M$, so $\mathrm{Diff}_0(M)$ is the identity component of the diffeomorphism group. The action of $\mathrm{Diff}_0(M)$ is free, and in fact the fine Teichmüller space is homotopy equivalent to the coarse space, which is the familiar $6g-6$ dimensional space homeomorphic to a ball. The action of $\mathrm{Diff}(M)$ is not free. Our coarse moduli space is the familiar coarse moduli space of genus g curves, and our fine moduli space has the homotopy type of $B\Gamma(M)$. See section 3.3 for more detail.

In practice one usually sets up a moduli problem so that $\mathrm{Diff}_0(M)$ acts freely on $S_?(M)$. If the action is also proper and admits local sections then the projection $S_?(M) \to \mathcal{T}^c_?(M)$ is a principal $\mathrm{Diff}_0(M)$-bundle and hence the map $\mathcal{T}^f_?(M) \to \mathcal{T}^c_?(M)$ is a homotopy equivalence. In this case we will drop the distinction between the coarse and fine Teichmüller spaces from the notation.

For the four moduli problems that we shall consider in this paper the action of $\mathrm{Diff}_0(M)$ is indeed free and proper with local sections. The key point is the following theorem of Ebin and Palais (restated in the form we shall use).

Theorem 2.5 (The Ebin-Palais Slice Theorem)**.** *Let* $\mathrm{Riem}(M)$ *denote the space of all Riemannian metrics on* M *and let* $S \subset \mathrm{Riem}(M)$ *be a closed subset for which* $\mathrm{Diff}_0(M)$ *acts freely. Then the action of* $\mathrm{Diff}_0(M)$ *on* S *is proper and admits local sections and hence the projection*

$$S \to S/\mathrm{Diff}_0(M)$$

is a principal $\mathrm{Diff}_0(M)$*-bundle.*

(Theorem 7.1 of [Ebi70] asserts the existence of local sections. Properness of the action follows from pages 29-30 of [Ebi70], as explained in Lemma 8.14 of [FT84].)

2.1. **The universal property of universal bundles.** Our use of the terminology defined the above is justified by the following proposition. Two bundles $E_0, E_1 \to X$ are *concordant* if they are isomorphic to the restrictions to $X \times \{0\}$ and $X \times \{1\}$ of a bundle over $X \times I$.

Proposition 2.6. *For any CW-space X there is a canonical bijection between (i) the set of homotopy classes of maps $[X, \mathcal{M}^f_?(M)]$ and (ii) the set of concordance classes of M-bundles over X equipped with fibrewise ?-structures. Similarly, the fine Teichmüller space classifies concordance classes of M-bundles over X with fibrewise ?-structures whose associated principal* $\mathrm{Aut}(A_M)$ *bundle is trivial.*

Proof. Let D be either $\mathrm{Diff}(M)$ or $\mathrm{Diff}_0(M)$. A map $f : X \to ED \times_D S_?(M)$ produces a principal D-bundle $P \to X$ together with a section σ_f of the associated bundle $P \times_D S_?(M)$. This amounts to an M-bundle over X equipped a fibrewise ?-structure. Conversely, such a bundle determines up to homotopy a map $X \to BD$ together with a lift to $ED \times_D S_?(M)$. It is not hard to check that this relation becomes an isomorphism at the level of homotopy classes and concordance classes. ∎

2.2. **The Meta-Theorem.** The universal bundle over the fine moduli space has an associated principal Aut'-bundle:

$$\begin{array}{c} (S_?(M) \times \mathrm{Aut}')/\!/\mathrm{Diff}(M) = (\mathcal{T}^f_?(M) \times \mathrm{Aut}')/\!/\mathrm{Aut}' \\ \downarrow \\ \mathcal{M}^f_?(M). \end{array}$$

This bundle is classified by a map

$$\mathcal{M}^f_?(M) \to B\mathrm{Aut}'. \tag{1}$$

A model for the classifying map (1) is the map,

$$\mathcal{M}^f_?(M) = \mathcal{T}^f_?(M)/\!/\mathrm{Aut}' \to */\!/\mathrm{Aut}' = B\mathrm{Aut}',$$

induced by collapsing the Teichmüller space to a point.

Lemma 2.7. *If $\mathcal{T}^f_?(M) \to *$ is a (weak) homotopy equivalence then the classifying map (1) is a (weak) homotopy equivalence. If the Teichmüller space is homology equivalent to a point then (1) is a homology equivalence.*

Proof. The first statement follows from the five-lemma applied to the homotopy long exact sequences of the fibrations

$$\mathcal{T}^f_?(M) \to \mathcal{T}^f_?(M)/\!/\mathrm{Aut}' \to B\mathrm{Aut}'$$

and

$$* \to */\!/\mathrm{Aut}' \to B\mathrm{Aut}'.$$

For the second statement, observe that the induced map of Serre spectral sequences is an isomorphism on the E_2 page. This is true because $H_*(\mathcal{T}^f_?(M)) \to H_*(*)$ is an isomorphism of Aut'-modules. □

Another model for the classifying map of the associated Aut' principal bundle is the composition

$$\mathcal{M}^f_?(M) \to B\mathrm{Diff}(M) \to B\Gamma(M) \to B\mathrm{Aut}'$$

given by first classifying the universal bundle. Hence,

Meta-Theorem 2.1. *If $\mathcal{T}^f_?(M) \to *$ is a (weak) homotopy equivalence then*

$$B\mathrm{Diff}(M) \to B\mathrm{Aut}' \text{ and } B\Gamma(M) \to B\mathrm{Aut}'$$

are (weakly equivalent to) retractions. If $\mathcal{T}^f_?(M) \to *$ is a homology equivalence then $H^*(B\mathrm{Aut}') \to H^*(B\mathrm{Diff}(M))$ and $H^*(B\mathrm{Aut}') \to H^*(B\Gamma(M))$ are split-injective ring homomorphisms.

3. Flat manifolds, hyperbolic manifolds, and Riemann surfaces

In this section we discuss three examples of moduli problems which result in contractible Teichmüller spaces. The examples of flat manifolds and hyperbolic manifolds are essentially trivial. The example of Riemann surfaces was worked out completely by Earle, Eells and Schatz [EE69], [ES70].

3.1. **Flat manifolds.** Let M be a closed manifold admitting a Riemannian metric with vanishing sectional curvature, i.e. a flat manifold. The moduli problem we consider is the moduli of (pointed) flat metrics on M.

Let $S_{Flat}(M)$ be the space of all unit volume flat Riemannian metrics on M. This is a closed subspace $\mathrm{Riem}(M)$. We take the algebraic object A_M to be $\pi_1 M$ (which is a Bieberbach group).

Fix a basepoint $*$ on M and let $\mathrm{Diff}(M, *)$ denote the group of diffeomorphisms which preserve the basepoint. Such a diffeomorphism induces an automorphism of the fundamental group. The assignment $\phi \mapsto \phi^* \in \mathrm{Aut}(\pi_1 M)$ determines a homomorphism $\Psi : \mathrm{Diff}(M, *) \to \mathrm{Aut}(\pi_1 M)$ which factors through $\pi_0\mathrm{Diff}(M, *)$ since the target is discrete.

The subgroup $\mathrm{Diff}_0(M, *) = \ker \Psi$ consists of those diffeomorphisms which induce the identity on $\pi_1 M$. The group of based affine diffeomorphisms of M is isomorphic to $\mathrm{Aut}(\pi_1 M)$, so Ψ is surjective.

Lemma 3.1. $\mathrm{Diff}_0(M,*)$ *acts freely on* $S_{Flat}(M)$.

Proof. If an element $\varphi \in \mathrm{Diff}_0(M,*)$ preserves a flat metric g then it lifts to an affine transformation of the universal cover. Since φ is the identity on π_1, the lift of φ is the identity on a lattice and so φ is the identity. □

Consequently the Ebin-Palais Theorem 2.5 applies and the coarse and fine Teichmüller spaces are equivalent.

Lemma 3.2. *The Teichmüller space* $\mathcal{T}_{Flat}(M)$ *is contractible.*

Proof. (See [Bes87, p. 345].) By the Bieberbach theorems M is covered by a torus T; let F denote the group of deck transformations.

First we determine the Teichmüller space of T. Choose a basepoint $* \in T$ lifting the basepoint of M. A flat metric on T determines a group structure and the metric is left-invariant with respect to this group structure. Any two compact abelian Lie groups of the same dimension are isomorphic and there is precisely one isomorphism within each isotopy class of basepoint-preserving diffeomorphisms between them. Hence after fixing a group structure on T $\mathcal{T}_{Flat}(M)$ identifies with the space of left-invariant metrics on T. This is homeomorphic to the space of symmetric $n \times n$ matrices of determinant 1, and by the Cholesky matrix decomposition $A = L \cdot e^D \cdot L^T$ one sees that this is convex open subset of a real vector space of dimension $n(n+1)/2 - 1$.

Now, flat metrics on M are in bijection with F-invariant flat metrics on T, and the subgroup of $\mathrm{Diff}_0(T,*)$ consisting of those diffeomorphisms which fix the F-invariant metrics set-wise is isomorphic to $\mathrm{Diff}_0(M,*)$. We now conclude that $\mathcal{T}_{Flat}(M)$ is homeomorphic to the the image of $S_{Flat}(T)^F$ inside $S_{Flat}(T)/\mathrm{Diff}_0(T,*) = \mathcal{T}_{Flat}(T)$. Thus the Teichmüller space for M is a convex open subset of a vector space and is therefore contractible. □

Meta-Theorem 2.1 applies here, but it provides no new information in light of the fact that $\mathrm{Aut}(\pi_1 M)$ embeds in $\mathrm{Diff}(M,*)$ as the group of based isometries of M with respect to any fixed flat metric.

3.2. **Hyperbolic metrics in dimensions 3 and larger.** Let (M^n) $(n \geq 3)$ be a manifold which admits a complete finite volume Riemannian metric of constant sectional curvature -1. Let $S_{Hyp}(M)$ denote the space of all such hyperbolic metrics on M. The object A_M is again the fundamental group of M, so the subgroup $\mathrm{Diff}_0(M) \subset \mathrm{Diff}(M)$ consists of those diffeomorphisms which induce the identity outer automorphism of $\pi_1 M$. Note that hyperbolic groups have no nontrivial inner automorphisms, so $\mathrm{Out}(\pi_1 M) \cong \mathrm{Aut}(\pi_1 M)$.

By the Mostow Rigidity Theorem, $\mathrm{Diff}(M) \to \mathrm{Out}(\pi_1 M)$ is surjective and the action of $\mathrm{Diff}_0(M)$ on $S_{Hyp}(M)$ is free and transitive. By the Ebin-Palais Theorem 2.5 a choice of a point in $S_{Hyp}(M)$ determines a homeomorphism $\mathrm{Diff}_0(M) \cong$

$S_{Hyp}(M)$. Thus the coarse Teichmüller space is a point, and the fine Teichmüller space is contractible.

Again Meta-Theorem 2.1 says nothing interesting here because $\mathrm{Out}(\pi_1 M)$ embeds into $\mathrm{Diff}(M)$ as the isometry group of any fixed metric.

3.3. **Surfaces.** In the case of complex structures on surfaces our viewpoint on moduli theory was completely worked out by Earle, Eells, and Schatz [EE69], [ES70].

Let F_g be a surface of genus $g > 1$. For the algebraic object A_F we take the fundamental group $\pi_1 F$. A diffeomorphism induces an automorphism of the fundamental group which is well-defined up to inner automorphisms, so there is a homomorphism

$$\Psi : \mathrm{Diff}(F) \to \mathrm{Out}(\pi_1 F).$$

Since we are tacitly only considering orientation preserving diffeomorphisms, the image of ϕ is contained in the subgroup $\mathrm{Out}_+(\pi_1 F)$ of outer automorphisms which act by identity on $H^2(\pi_1 F)$. A classical theorem of Dehn-Nielsen-Baer ([Nie27], [Bae28]) asserts that $\mathrm{Out}_+(\pi_1 F)$ is isomorphic to the mapping class group $\Gamma(F) = \pi_0 \mathrm{Diff}(F)$. Hence the group $\mathrm{Diff}_0(F)$, defined as the kernel of Ψ, consists of diffeomorphisms which are isotopic to the identity.

Let $S_{Cplx}(F)$ denote the set of complex structures on F. Earle and Eells identified the associated (coarse) Teichmüller space

$$\mathcal{T}^c_{Cplx}(F) := S_{Cplx}(F)/\mathrm{Diff}_0(F)$$

with the classical Teichmüller space of surface theory, which is a ball of dimension $6g - 6$ (*a priori* the definitions are different). The action of $\mathrm{Diff}_0(F)$ on $S_{Cplx}(F)$ is free. Earle and Eells proved, *a fortiori*, that the quotient map $S_{Cplx}(F) \to \mathcal{T}^c_{Cplx}(F)$ is a principal $\mathrm{Diff}_0(F)$-bundle, so the coarse and fine Teichmüller spaces are homotopy equivalent. (Alternatively one could replace complex structures with hyperbolic metrics and appeal again to the Ebin-Palais Theorem.)

In this setting Meta-Theorem 2.1 says that the map

$$B\mathrm{Diff}(F) \to B\Gamma(F) \tag{2}$$

is a retraction. Of course this is only half of the theorem of Earle and Eells [EE69] that $\mathrm{Diff}_0(F)$ is contractible, so (2) is actually a homotopy equivalence. The other half of their theorem comes from the observation that $S_{Cplx}(F)$ is contractible; the homotopy long exact sequence for the $\mathrm{Diff}_0(F)$-principal bundle $S_{Cplx}(F) \to \mathcal{T}^c_{Cplx}(F)$ then gives their entire theorem at once.

4. The real cohomology of arithmetic groups

In this section we review techniques used to study the real cohomology of arithmetic groups such as automorphism groups of unimodular lattices. The technique is due to Matsushima [Mat62] and Borel [Bor74] [Bor81].

4.1. **The Borel-Matsushima homomorphism.** First we review the general construction, due to Borel and Matsushima, of a homomorphism from the cohomology of a compact symmetric space to the cohomology of a related arithmetic group.

Suppose G is a connected semisimple linear Lie group and $A \subset G$ an arithmetic subgroup for which we would like to understand the cohomology with real coefficients. We have in mind $G = \mathbb{G}(\mathbb{R})$ for an algebraic group $\mathbb{G}$ and $A = \mathbb{G}(\mathbb{Z})$ is the integer points in $\mathbb{G}$.

The group G admits a maximal compact subgroup K; let $X = G/K$ be the associated symmetric space of non-compact type. The discrete group A acts on X from the left with finite isotropy subgroups and X is contractible, so $H^*(A\backslash X;\mathbb{R}) \cong H^*(BA;\mathbb{R})$. Let G_u be a maximal compact subgroup of the complexification $G_\mathbb{C}$ which contains K. The quotient $X_u = G_u/K$ is a compact symmetric space known as the *compact dual* of X. Matsushima [Mat62] defined a ring homomorphism

$$j^* : H^*(X_u;\mathbb{R}) \to H^*(A\backslash X;\mathbb{R}) \cong H^*(BA;\mathbb{R}) \tag{3}$$

and studied the extent to which this map is injective and/or surjective when A is a cocompact subgroup. Borel [Bor74] later extended these results to the case of general arithmetic subgroups. We refer to (3) as the *Borel-Matsushima homomorphism.*

The construction of the homomorphism (3) proceeds as follows. The cohomology of $A\backslash X$ can be computed using de Rham cohomology. If A is torsion free then the de Rham complex $\Omega^*(A\backslash X)$ is easily seen to be isomorphic as a dga to the ring $\Omega(X)^A$ of A-invariant forms on X, and when A is not torsion free it is true (by a standard argument) that $\Omega^*(X)^A$ still computes the cohomology of $A\backslash X$.

An easy way to produce A-invariant forms on X is take G-invariant forms on X. The inclusion

$$\Omega^*(X)^G \hookrightarrow \Omega^*(X)^A \tag{4}$$

induces a map on cohomology. A G-invariant form on X is entirely determined by its value on the tangent space at a single point since G acts transitively, and hence the complex $\Omega^*(X)^G$ is entirely a Lie algebra theoretic object. Let $\mathfrak{g}$, $\mathfrak{g}_u$, $\mathfrak{k}$ denote the Lie algebras of G, G_u, K respectively. Then there are Cartan decompositions

$$\mathfrak{g} \cong \mathfrak{k} \oplus \mathfrak{p}$$
$$\mathfrak{g}_u \cong \mathfrak{k} \oplus i\mathfrak{p},$$

and hence there are canonical isomorphisms

$$\Omega^*(X)^G \cong \left(\bigwedge \mathfrak{p}^*\right)^K \cong \Omega^*(X_u)^{G_u}. \tag{5}$$

Since X_u is a compact manifold and $\Omega^*(X_u)^{G_u}$ consists of harmonic forms, Hodge theory implies that

$$\Omega^*(X_u)^{G_u} \cong H^*(X_u;\mathbb{R}). \tag{6}$$

Combining (4), (5), and (6), one obtains the homomorphism (3).

Borel proved that this homomorphism is injective and surjective in ranges of degrees depending only on the root system of G. In particular, for the B_n and D_n root systems we have:

Theorem 4.1 ([Bor81, Theorem 4.4])**.** *For A an arithmetic subgroup of a group G with root system of type D_n (B_n), the homomorphism (3) is bijective in degrees $* < n-1$ (resp. $* < n$) and injective for $* = n-1$ (resp. $* = n$).*

Remark 4.2. In particular, $SO^+_{p,q}$ has root system of type $D_{(p+q)/2}$ if $p+q$ is even, and $B_{\lfloor (p+q)/2 \rfloor}$ if $p+q$ is odd. Therefore the bijective range for $SO^+_{p,q}$ is $* \leq \lfloor (p+q)/2 \rfloor - 2$.

4.2. **A reinterpretation of Borel-Matsushima.** The Borel-Matsushima homomorphism (3) is closely related to another map which can be described more directly. Precomposition of (3) with the classifying map $c_u : X_u \to BK$ for the principal K-bundle $G_u \to G_u/K = X_u$ gives a homomorphism

$$H^*(BK;\mathbb{R}) \to H^*(BA;\mathbb{R}).$$

On the other hand, one has

$$A \hookrightarrow G \simeq K$$

which also induces a map from the cohomology of BK to the cohomology of BA.

Lemma 4.3. *These two homomorphisms coincide.*

We'll need the following result for the proof of this lemma. The principal K-bundle $G_u \to G_u/K = X_u$ is classified by a map $c_u : X_u \to BK$. Suppose A is torsion free, so $A\backslash G \to A\backslash G/K = A\backslash X$ is a principal K bundle classified by a map $c : A\backslash X \to BK$. In this situation we have:

Lemma 4.4 ([Bor77], Proposition 7.2)**.** *Then the diagram*

$$\begin{array}{ccc} H^*(BK;\mathbb{R}) & \xrightarrow{c_u^*} & H^*(X_u;\mathbb{R}) \\ {\scriptstyle c^*}\downarrow & \swarrow{\scriptstyle j^*} & \\ H^*(A\backslash X;\mathbb{R}) & & \end{array}$$

commutes.

Proof of Lemma 4.3. By a well-known result of Selberg, the arithmetic group A admits a finite index subgroup $\widetilde{A}$ which is torsion free. Since

$$H^*(BA;\mathbb{R}) \cong H^*(B\widetilde{A};\mathbb{R})^{A/\widetilde{A}} \subset H^*(B\widetilde{A};\mathbb{R}),$$

and the Borel-Matsushima homomorphism (3) is natural with respect to inclusions, it suffices to verify the claim for torsion free arithmetic groups. So we now assume that A is torsion free.

In this case the quotient $A\backslash G \to A\backslash G/K$ is a principal K-bundle. The classifying map $A\backslash G/K \to BK$ of this bundle fits into the commutative diagram

$$\begin{array}{ccccccc} BA & \xleftarrow{\simeq} & (A\backslash G)\times_G EG & \xleftarrow{\simeq} & (A\backslash G)\times_K EG & \xrightarrow{\simeq_{\mathbb{R}}} & A\backslash G/K \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & BG & \xleftarrow{\simeq} & BK & & \end{array}$$

in which the left diagonal arrow is induced by the inclusion $A \hookrightarrow G$, and the rightmost horizontal arrow is a real cohomology isomorphism. Hence $BA \hookrightarrow BG \simeq BK$ agrees with with $A\backslash G/K \to BK$ on real cohomology. The statement now follows from Lemma 4.4. □

4.3. **The automorphism groups of intersection forms.** Let M be a $4k$ dimensional oriented manifold. Consider the unimodular form

$$Q_M : H^{2k}(M;\mathbb{Z})/\text{torsion} \otimes H^{2k}(M;\mathbb{Z})/\text{torsion} \to H^{4k}(M;\mathbb{Z}) \cong \mathbb{Z}$$

given by the cup product pairing. Let $p = b^+_{2k}$, $q = b^-_{2k}$ be the dimensions of the maximal positive and negative definite subspaces of $H^{2k}(M;\mathbb{R})$, and assume that these numbers are both strictly positive. We now study the cohomology $H^*(B\mathrm{Aut}(Q_M);\mathbb{R})$ using the Borel-Matsushima homomorphism.

The group $\mathrm{Aut}(Q_M)$ can be regarded as the integer points of a linear algebraic group with real points $O_{p,q} = O_{p,q}(\mathbb{R})$ (defined over $\mathbb{Q}$). This group has four components indexed by the spinor norm and the determinant (each of which can take the values ± 1). Let $\mathrm{Aut}' \subset \mathrm{Aut}(Q_M)$ be the subgroup where (determinant) $\cdot$ (spinor norm) $= +1$, and $\mathrm{Aut}'' \subset \mathrm{Aut}'$ the subgroup where the spinor norm and determinant are both $+1$. It is slightly more convenient to work with Aut'' because it sits inside the connected component $SO^+_{p,q}$ of $O_{p,q}$.

Proposition 4.5. *The extensions*

$$\mathrm{Aut}' \hookrightarrow \mathrm{Aut}(Q_M) \xrightarrow{\det\cdot\mathrm{spin}} \mathbb{Z}/2$$

$$\mathrm{Aut}'' \hookrightarrow \mathrm{Aut}' \xrightarrow{\det} \mathbb{Z}/2$$

are both split.

Proof. By hypothesis Q_M is indefinite so the Hasse-Minkowski classification implies that Q_M contains either a hyperbolic plane H or the form $(1)\oplus(-1)$ as a summand. On the hyperbolic plane there are reflections $R_\pm$ through the vectors $(1,\pm 1)$. These have determinant -1 and spinor norm ± 1. Hence the homomorphisms $f_\pm : -1 \mapsto R_\pm \oplus \mathrm{id}_{H^\perp}$ are sections of $\mathrm{Aut}(Q_M) \twoheadrightarrow \mathbb{Z}/2$ and $\mathrm{Aut}' \twoheadrightarrow \mathbb{Z}/2$ respectively. One can easily construct similar splittings for the case of $(1)\oplus(-1)$. □

Now consider the connected linear algebraic group $G = SO^+_{p,q}$ with maximal compact subgroup $K = SO_p \times SO_q$ and arithmetic subgroup Aut''. The complexification of $SO^+_{p,q}$ is $G_{\mathbb{C}} = SO_{p+q}(\mathbb{C})$ which contains $G_u = SO_{p+q}$ as a maximal compact subgroup. Hence

$$X_u = SO_{p+q}/SO_p \times SO_q.$$

By Remark 4.2 the bijective range for the Borel-Matsushima homomorphism (3) is $\lfloor (p+q)/2 \rfloor - 2$.

The natural map $X_u \to BK = BSO_p \times BSO_q$ is $2q+1$-connected after projection onto the first factor and $2p+1$-connected after projection onto the second.

Theorem 4.6. *The composition*

$$H^*(BSO_p;\mathbb{R}) \to H^*(BK;\mathbb{R}) \to H^*(X_u;\mathbb{R}) \to H^*(B\mathrm{Aut}'';\mathbb{R})$$

is an isomorphism in degrees $* \leq \min(2q, \lfloor (p+q)/2 \rfloor - 2)$.

Finally, note that the action of $\mathrm{Aut}'/\mathrm{Aut}'' \cong \mathbb{Z}/2$ on the Borel-Matsushima classes on $B\mathrm{Aut}''$ is trivial. Similarly, the action of $\mathrm{Aut}(Q_M)/\mathrm{Aut}' \cong \mathbb{Z}/2$ on the Borel-Matsushima classes of $B\mathrm{Aut}'$ is trivial. Thus the inclusions

$$\mathrm{Aut}'' \hookrightarrow \mathrm{Aut}' \hookrightarrow \mathrm{Aut}(Q_M)$$

induce isomorphisms on the subalgebras of the cohomology spanned by the Borel-Matsushima classes.

Theorem 1.3 now follows as a special case of Theorem 4.6.

5. Einstein metrics on a $K3$ manifold

In this section we recall some necessary background material on Einstein metrics on a $K3$ manifold, their moduli, and the associated period map. This material comes from [Bes87]. The proof of Theorem 1.1 uses the Global Torelli Theorem (described below) to relate the homotopy type of the moduli space of Einstein metric to an arithmetic group.

5.1. **Basic facts about $K3$ manifolds.** Recall that a $K3$ surface is a simply connected compact complex surface such that the canonical bundle (i.e. the top exterior power of the holomorphic cotangent bundle) is trivial. When considered with their complex structure there are many non-isomorphic $K3$ surfaces, but as smooth 4-manifolds they are all diffeomorphic [Kod64]. We shall call a smooth manifold of this diffeomorphism type a $K3$ *manifold* when it does not come with a chosen complex structure.

Let K be a $K3$ manifold. The middle integral cohomology of K is free abelian of rank 22. The cup product gives a non-degenerate symmetric bilinear pairing on the middle cohomology

$$Q_K : H^2(M;\mathbb{Z}) \otimes H^2(M;\mathbb{Z}) \to \mathbb{Z}.$$

The form Q_K is isomorphic to $H \oplus H \oplus H \oplus -E_8 \oplus -E_8$, where H is the hyperbolic plane (i.e. the unique rank 2 even indefinite form), and E_8 is the unique even positive definite rank 8 form (it is given by the Cartan matrix for the E_8 Dynkin diagram). The form Q_K has signature $(3, 19)$.

5.2. **Teichmüller space and Moduli space for Einstein metrics.** Let K^4 be a $K3$ manifold. An Einstein metric g on K is a Riemannian metric satisfying the Einstein condition

$$\mathrm{Ric}(g) = \frac{\lambda}{4} \cdot g$$

where λ is the scalar curvature constant of g. According to [Hit74], every Einstein metric on a $K3$ manifold has vanishing scalar curvature constant λ, so Einstein metrics are precisely the same as Ricci flat metrics in this setting.

We take ? to be the class E of unit volume Einstein metrics. The set $S_E(K)$ of all unit volume Einstein metrics on K is topologized with the C^∞ topology as a subspace of the sections of $T^*K \otimes T^*K$.

For A_K we take the lattice represented by the middle cohomology equipped with the cup product pairing: $(H^2(K; \mathbb{Z}), Q_K)$. We write $\mathrm{Aut}(Q_K)$ for the group of automorphisms of this free abelian group which preserve the cup pairing. Thus $\mathrm{Diff}_0(K)$ is the group of diffeomorphisms acting trivially on cohomology. It is known ([Mat86] or [Bes87, p. 367]) that the image of $\Psi : \mathrm{Diff}(K) \to \mathrm{Aut}(Q_K)$ is an index 2 subgroup $\mathrm{Aut}' \subset \mathrm{Aut}(Q_K)$ consisting of those automorphisms for which

$$(\text{determinant}) \cdot (\text{spinor norm}) = +1,$$

(cf. Remark 1.2).

As in section 2 there is a coarse/fine Teichmüller space and a coarse/fine moduli space of Einstein metrics on K.

5.3. **The period map for smooth Einstein metrics.** We now review the period map which identifies the coarse Teichmüller space with an open dense subset of a homogeneous space.

We continue to let K be a $K3$ manifold. An Einstein metric g determines a subspace $\mathscr{H}^+(g) \subset H^2(K; \mathbb{R})$ of harmonic self-dual real 2-forms on K. Since the cup product on $H^2(K; \mathbb{R})$ has signature $(3, 19)$ the Hodge Theorem implies that this space has dimension 3 and is positive definite.

There is a bijection between unit-norm self-dual harmonic 2-forms ω and complex structures J compatible with g. In one direction the bijection is given by

$$J \mapsto \omega_J = g(-, J-).$$

Every Einstein metric on K is hyperkähler with respect to some triple of complex structures (I, J, K) (see [Bes87, Theorem 6.40]). Therefore $\mathscr{H}^+(g)$ has a well-defined orientation determined by bases of the form $(\omega_I, \omega_J, \omega_{IJ})$.

The assignment

$$g \mapsto \mathscr{H}^+(g)$$

defines a continuous map

$$p : S_E(K) \to Gr_3^+(\mathbb{R}^{3,19})$$

from the space of Einstein metrics to the Grassmanian $Gr_3^+(\mathbb{R}^{3,19})$ of positive oriented 3-planes in $H^2(K;\mathbb{R}) \cong \mathbb{R}^{3,19}$. Written as a homogeneous space,

$$Gr_3^+(\mathbb{R}^{3,19}) \cong O(3,19)/SO(3) \times O(19)$$

and one sees that this space has two connected components, each of which is contractible.

The mapping p is $\mathrm{Diff}(K)$-equivariant, where the action on the source is by $g \mapsto \phi^* g$ and the action on the target is induced by the natural action of $\mathrm{Diff}(K)$ on the cohomology of M. The map p is constant on the orbits of $\mathrm{Diff}_0(K)$ and so descends to a map

$$P_E : \mathcal{T}_E^c(K) \to Gr_3^+(\mathbb{R}^{3,19}). \tag{7}$$

This map is called the *period map for Einstein structures*, and the image of a given metric is called its *period*.

The set of *roots* is

$$\Delta = \{\delta \in H^2(K;\mathbb{Z}) \mid \delta^2 = -2\}.$$

It is not difficult to see that the image of the Einstein period map is contained within the set

$$W = \{\tau \in Gr_3^+(\mathbb{R}^{3,19}) \mid \tau^\perp \cap \Delta = \emptyset\}.$$

The argument is as follows. Let g be an Einstein metric; a 2-plane $\eta \subset \mathscr{H}^+(g)$ determines a Kähler structure (g, J, ω_J). If a root δ is orthogonal to η then δ is contained in $H^{1,1}(J)$. The Lefschetz Theorem on $(1,1)$ cohomology implies that any integral class in $H^{1,1}(J)$ is the first Chern class of some divisor C, and by the Riemann-Roch formula, since $\delta^2 = -2$, either C of $-C$ is effective. Finally, ω_J is a unit-norm Kähler class so $\omega_J \cdot \delta = \mathrm{Area}(C) \neq 0$. Hence δ is not orthogonal to $\mathscr{H}^+(g)$.

Theorem 5.1 (The Global Torelli Theorem for Einstein metrics [Bes87, p. 366])**.** *The Einstein period map*

$$P_E : \mathcal{T}_E^c(K) \to Gr_3^+(\mathbb{R}^{3,19})$$
$$g \mapsto \mathscr{H}^+(g)$$

is a homeomorphism onto the open dense subspace W consisting of 3-planes not orthogonal to any root.

Thus the coarse Teichmüller space $\mathcal{T}_E^c(K)$ is a homogeneous space with certain codimension 3 holes. We now show that the fine Teichmüller space has the same homotopy type.

Lemma 5.2. *The action of* $\mathrm{Diff}_0(K)$ *on* $S_E(K)$ *is free.*

Proof. Suppose $\varphi \in \mathrm{Diff}_0(K)$ fixes a metric $g \in S_E(K)$, which is to say that φ is an isometry of g. The metric g is hyperkählerian, so let $S^2_g \cong S(\mathcal{H}^+(g))$ be the 2-sphere of complex structures. The isometry φ induces an orthogonal transformation of S^2_g. Such a transformation must have a fixed point $J \in S^2_g$ (use the Lefschetz Fixed Point Formula). Therefore φ is a holomorphic automorphism of the complex $K3$ surface (K, J). Since φ is the identity on homology, it follows from the Burns-Rapoport Uniqueness theorem [BR75, Proposition 1] that φ is the identity. □

Since $S_E(K)$ is a closed subset of $\mathrm{Riem}(K)$, the Ebin-Palais Theorem 2.5 applies:

Lemma 5.3. *The quotient map* $S_E(K) \to S_E(K)/\mathrm{Diff}_0(K) = \mathcal{T}^c_E(K)$ *is a principal* $\mathrm{Diff}_0(K)$*-bundle, and hence the homotopy-to-geometric quotient map* $\mathcal{T}^f_E(K) \to \mathcal{T}^c_E(K)$ *is a homotopy equivalence.*

In light of this we will drop the distinction between coarse and fine Teichmüller spaces from the notation.

5.4. **Proof of Theorem 1.1.** For the moduli problem of Einstein metrics on a $K3$ surface the Teichmüller space is not contractible and so Meta-Theorem 2.1 does not quite apply immediately. However, the argument is easily modified to show that the composition

$$\mathcal{M}^f_E(K) \to B\mathrm{Diff}(K) \to B\mathrm{Aut}' \tag{8}$$

is induced by collapsing the Teichmüller space to a point. The Teichmüller space has precisely two connected components since it is the complement of a codimension 3 subset in $Gr^+_3(\mathbb{R}^{3,19})$, and this Grassmanian clearly has two components. Let $\mathcal{T}_0$ denote one of these components, and let $\mathrm{Aut}'' \subset \mathrm{Aut}'$ denote the index 2 subgroup which acts trivially on π_0 (see Proposition 4.5 above), so $\mathcal{M}^f_E(K) = \mathcal{T}_0 /\!\!/ \mathrm{Aut}''$. The composition (8) thus factors up to homotopy as

$$\mathcal{M}^f_E(K) \to B\mathrm{Aut}'' \to B\mathrm{Aut}',$$

where the first arrow is induced by collapsing $\mathcal{T}_0$ to a point, and the second arrow is induced by the inclusion.

Theorem 1.1 will follow once we prove:

Theorem 5.4. *The map* $\mathcal{M}^f_E = \mathcal{T}_0 /\!\!/ \mathrm{Aut}'' \to B\mathrm{Aut}''$ *induced by collapsing* $\mathcal{T}_0$ *to a point is injective on real cohomology in degrees* $* \leq 9$.

To establish this theorem we will need to study the (co)homology of $\mathcal{T}_0$ as an Aut''-module. Recall that the set of roots Δ is the set of all vectors in the $K3$ lattice of length -2. Let Δ_n denote the set of unordered n-tuples of distinct elements of Δ.

Lemma 5.5. *Given an element $x \in \Delta_n$, the stabilizer* $\mathrm{Stab}(x) \subset \mathrm{Aut}''$ *is an extension of a finite group by an arithmetic subgroup of $SO^+_{3-n_1,19-n_2}$, where (n_1, n_2) is the signature of the sublattice generated by the roots in x and $n_1 + n_2 = n$.*

Proof. The element x consists of n distinct roots $\{\delta_1, \dots, \delta_n\}$. The group $\mathrm{Stab}(x)$ permutes the δ_i so there is a homomorphism $\mathrm{Stab}(x) \to \Sigma_n$; let G_x be the image of this homomorphism. Thus there is a group extension

$$A_x := \bigcap_{i=1}^{n} \mathrm{Stab}(\delta_i) \hookrightarrow \mathrm{Stab}(x) \twoheadrightarrow G_x. \tag{9}$$

It remains to show that the kernel A_x of (9) is an arithmetic subgroup of $SO^+_{3-n_1,19-n_2}$. Let P_x denote the sublattice of L_{K3} generated by the δ_i. We may express A_x as the subgroup of Aut'' consisting of those automorphisms which restrict to the identity on P_x. Let $\mathrm{Aut}''(P_x^\perp)$ denote the group of all automorphisms of $P_x^\perp$ having spinor norm and determinant both equal to 1. An element of A_x determines an element of $\mathrm{Aut}''(P_x^\perp)$, and since $(P_x \oplus P_x^\perp) \otimes \mathbb{Q} \cong L_{K3} \otimes \mathbb{Q}$ there is in fact an inclusion $A_x \hookrightarrow \mathrm{Aut}''(P_x^\perp)$.

The group $\mathrm{Aut}''(P_x^\perp)$ is an arithmetic subgroup of $SO^+_{3-n_1,19-n_2}$, where (n_1, n_2) is the signature of P_x. Therefore we need only verify that A_x is of finite index in $\mathrm{Aut}''(P_x^\perp)$. The lattice $P_x \oplus P_x^\perp$ is of finite index in L_{K3}, so for some integer k there are finite index inclusions

$$P_x \oplus P_x^\perp \subset L_{K3} \subset \frac{1}{k}(P_x \oplus P_x^\perp).$$

Let B_x denote the group of automorphisms of $\frac{1}{k}(P_x \oplus P_x^\perp)$ which restrict to the identity on P_x and have spinor norm and determinant 1, and let $B_x(L_{K3})$ denote the subgroup of B_x which sends the lattice L_{K3} onto itself. Since L_{K3} is of finite index in $\frac{1}{k}(P_x \oplus P_x^\perp)$ it follows that $B_x(L_{K3})$ is of finite index in B_x.

Observe that $B_x \cong \mathrm{Aut}''(P_x^\perp)$; this is because the automorphism group of $\frac{1}{k}P_x^\perp$ is precisely the automorphism of $P_x^\perp$. Furthermore, $B_x(L_{K3})$ is isomorphic to A_x; this is because the homomorphism $B_x(L_{K3}) \to A_x$ given by restriction to L_{K3} is surjective (it admits a section) and there is a commutative diagram

$$\begin{array}{ccc} B_x(L_{K3}) & \hookrightarrow & B_x \\ \downarrow\!\!\downarrow & \nearrow & \\ A_x & & \end{array}$$

which shows that the left vertical arrow must also be injective. Thus A_x is a finite index subgroup of the arithmetic group $\mathrm{Aut}''(P_x^\perp) \subset SO^+_{3-n_1,19-n_2}$, and hence A_x is an arithmetic subgroup itself. □

Given an orbit $\sigma \in \Delta_n/\mathrm{Aut}''$, we write $\mathrm{Stab}(\sigma) \subset \mathrm{Aut}''$ for the stabilizer of any point in the orbit. Note that while Δ_1/Aut'' is finite, Δ_n/Aut'' is in general countably infinite.

Lemma 5.6. *The integral homology of $\mathcal{T}_0$ is concentrated in even degrees. Furthermore, for $n \leq 14$,*

$$H_{2n}(\mathcal{T}_0;\mathbb{Z}) \cong \mathbb{Z}[\Delta_n] \cong \bigoplus_{\sigma\in\Delta_n/\mathrm{Aut}''} \mathbb{Z}[\mathrm{Aut}''] \otimes_{\mathrm{Stab}(\sigma)} \mathbb{Z}$$

The action of Aut'' *on* $H_{2n}(\mathcal{T}_0;\mathbb{Z})$ *is determined by the action of* Aut'' *on* Δ_n.

Proof. A root δ determines a codimension 3 totally geodesic submanifold $A_\delta \subset Gr_3^+(\mathbb{R}^{3,19})$. Note that $Gr_3^+(\mathbb{R}^{3,19})$ is diffeomorphic to a Euclidean space $\mathbb{R}^{57}$. The submanifolds $\{A_\delta\}_{\delta\in\Delta}$ intersect each other pairwise transversally and any finite intersection $A_{\delta_1} \cup \cdots \cup A_{\delta_k}$ is diffeomorphic to $\mathbb{R}^{57-3k}$.

Goresky and MacPherson [GM88, Theorem B, p. 239] compute the homology of the complement of an arrangement of finitely many affine subspaces A_i in $\mathbb{R}^N$ using stratified Morse theory. The Morse function they use is $f(x) = \mathrm{dist}(x,p)^2$ for a generic point p. If the codimension of each A_i is 2 and they all intersect pairwise transversally then the result of their computation is that the odd homology of the complement vanishes and the homology in degree $2n$ (for $n \leq N/4$) is free abelian with generators corresponding to the unordered n-tuples of distinct subspaces.

Pick a generic point p in $Gr_3^+(\mathbb{R}^{3,19})$ and let $B_r(p)$ denote the ball of radius r centered at p. Note that only finitely many of the submanifolds A_δ intersect the ball since its closure is compact. The Goresky-MacPherson computation carries over essentially verbatim if the ambient $\mathbb{R}^N$ is replaced by $B_r(p) \subset Gr_3^+(\mathbb{R}^{3,19})$ and the affine subspaces are replaced by the totally geodesic submanifolds $A_\delta \cap B_r(p)$. Taking the colimit as the radius goes to infinity produces the desired result. □

Applying the Universal Coefficient Theorem gives,

Lemma 5.7. *The cohomology $H^*(\mathcal{T}_0;\mathbb{R}) = 0$ vanishes in odd degrees, and for $n \leq 14$,*

$$H^{2n}(\mathcal{T}_0;\mathbb{R}) \cong \prod_{\sigma\in\Delta_n/\mathrm{Aut}''} \mathrm{CoInd}^{\mathrm{Aut}''}_{\mathrm{Stab}(\sigma)}(\mathbb{R}),$$

where $\mathbb{R}$ *is the trivial* $\mathrm{Stab}(\sigma)$*-representation.*

Here, for $H \subset G$, $\mathrm{CoInd}^G_H(M)$ is the G-module coinduced up from an H-module M.

Proof of Theorem 5.4. Consider the real cohomology Serre spectral sequence for the fibration

$$\mathcal{T}_0 \hookrightarrow \mathcal{T}_0/\!\!/\mathrm{Aut}'' \to B\mathrm{Aut}''.$$

The E_2 page has $E_2^{p,2q+1} = 0$, and for $q \leq 14$

$$\begin{aligned} E_2^{p,2q} &= H^p(B\mathrm{Aut}''; H^{2q}(\mathcal{T}_0; \mathbb{R})) \\ &\cong \prod_{\sigma \in \Delta_q/\mathrm{Aut}''} H^p(B\mathrm{Stab}(\sigma); \mathbb{R}), \end{aligned}$$

where second line follows from Lemma 5.7 together with Shapiro's Lemma. By Lemma 5.5 the group $\mathrm{Stab}(\sigma)$ is an extension of a finite group by an arithmetic subgroup of $SO^+_{3-q_1,19-q_2}$ for some partition $q_1 + q_2 = q$. Since the real cohomology of an arithmetic subgroup of $SO^+_{3-q_1,19-q_2}$ vanishes for in odd degrees satisfying $* \leq \lfloor (22-q)/2 \rfloor - 2$ (by Theorem 4.6), it follows from taking invariants that the real cohomology of $B\mathrm{Stab}(\sigma)$ also vanishes in odd degrees in this range.

In the region of total degree $* \leq 9$ on the E^2 page all nonzero terms occur in even bidegree, so in this region there can be no nontrivial differentials. Hence the spectral sequence degenerates in this region and it now follows that $H^*(B\mathrm{Aut}''; \mathbb{R}) \to H^*(\mathcal{M}^f_E; \mathbb{R})$ is injective for degrees $* \leq 9$. □

Remark 5.8. The above spectral sequence actually shows that there are many more classes than just those coming from $B\mathrm{Aut}''$. It would be interesting to investigate whether or not any of these classes can be pulled back from $B\mathrm{Diff}(K)$.

6. Characteristic classes and the Nielsen realization problem

In this section we compare the characteristic classes of M-bundles which come from the automorphism group of Q_M to those classes which come from integrating characteristic classes of the vertical tangent bundle over the fibres. A consequence of the Atiyah-Singer Index Theorem is that these two recipes are equivalent over the rationals. Thus the classes detected for a $K3$ manifold by Theorems 1.1 and 1.3 can be given a geometrical interpretation in terms of the vertical tangent bundle. From here we show, following Morita [Mor87], that these classes provide obstructions to solving the generalized Nielsen realization problem on a $K3$ manifold.

For M a smooth manifold, the methods for constructing classes in $H^*B\mathrm{Diff}(M)$ (or even on $H^*B\mathrm{Homeo}(M)$) generally fall into two categories:

- (i) One can construct a map from $B\mathrm{Diff}(M)$ to some better-understood space and pull classes back from there.
- (ii) One can describe a geometric recipe for producing classes.

Thus far in this paper we have focused on method (i). The standard example of method (ii) is the usual construction of the Miller-Morita-Mumford classes. There is the following recipe. Given an orientable fibre bundle

$$M \to E \xrightarrow{\pi} B$$

with fibre M a closed manifold there is the vertical tangent bundle $T^\nu E \to E$ (if the structure group is $\mathrm{Homeo}(M)$ then this is a microbundle rather than a vector bundle). Apply a characteristic class P (e.g. a polynomial in the Pontrjagin classes) to $T^\nu E$ and then integrate over the fibres to obtain a class $\pi_* P(T^\nu E) \in H^*(B)$. This method has been explored for instance in [KJ02].

For oriented smooth (or C^1) surface bundles the vertical tangent bundle is rank 2, so the only interesting vector bundle characteristic classes one can apply are the powers of the Euler class. When the fibre is a 4-manifold the vertical tangent bundle is rank 4 and so one has a little bit more choice: there is the first Pontrjagin class p_1 and the Euler class e (with $e^2 = p_2$), as well as all polynomials formed from these.

It will be more convenient to work with a different generating set for the Pontrjagin ring, namely the components $\widetilde{\mathcal{L}}_i$ of the $\widetilde{\mathcal{L}}$-class (the Atiyah-Singer modification of Hirzebruch's L-class.) The $\widetilde{\mathcal{L}}$-class is the rational formal power series in the Pontrjagin classes given by the formal power series expansion of

$$\prod_i \frac{x_i}{tanh(x_i/2)},$$

where as usual the x_i are the Chern roots and the j^{th} Pontrjagin class p_j is interpreted as the j^{th} elementary symmetric function in the x_i^2. Thus $\widetilde{\mathcal{L}}_i$ is a homogeneous polynomial of degree 4i with rational coefficients in $p_1, \dots, p_i$.

Definition 6.1. For M a $4k$-dimensional manifold, define characteristic classes

$$\ell_i := \pi_* \widetilde{\mathcal{L}}_{i+k}(T^\nu E) \in H^{4i}(B\mathrm{Diff}(M), \mathbb{Q}),$$

where π_* is the integration along the fibres map for the tautological M-bundle $E \to B\mathrm{Diff}(M)$.

We regard these classes as analogues of the Miller-Morita-Mumford characteristic classes κ_i of surface bundles. Note that they could just as easily be defined on $B\mathrm{Homeo}(M)$ by using the Pontrjagin classes of the vertical tangent microbundle.

Let $\widetilde{ch}$ denote the pullback of the Chern character $ch \in H^*(BO_p; \mathbb{R})$ by the composition

$$\begin{array}{ccccc} B\mathrm{Diff}(M) & \longrightarrow & B\mathrm{Aut}(Q_M) & \longrightarrow & BO_{p,q} \simeq BO_p \times BO_q \\ & & & & \downarrow \mathrm{proj} \\ & & & & BO_p \end{array} \tag{10}$$

(Here p, q are b_{2k}^+, b_{2k}^- respectively.)

Proposition 6.2. *The relation $\ell = 2\widetilde{ch}$ holds in $H^*(B\mathrm{Diff}(M); \mathbb{R})$.*

Proof. This is a consequence of the Atiyah-Singer families index theorem. Consider a fibre bundle $M \to E \to B$, and let η denote the associated vector bundle formed by replacing M with $H^{2k}(M, \mathbb{R})$. A choice of a fibrewise Riemannian metric on E

induces a Hodge star operator $* : H^{2k}(M;\mathbb{R}) \to H^{2k}(M;\mathbb{R})$ which satisfies $*^2 = 1$. Hence this bundle splits as a sum of positive and negative eigenspaces $\eta = \eta_+ \oplus \eta_-$. The Atiyah-Singer index theorem for families applied to the signature operator gives the equation

$$ch(\eta_+ - \eta_-) = \pi_* \widetilde{\mathcal{L}}(T^\nu E).$$

The real vector bundle η has structure group $\mathrm{Aut}(Q_M)$, where Q_M is the symmetric bilinear cup product pairing on $H^{2k}(M;\mathbb{Z})$. This automorphism group is discrete and hence η is flat, so by the Chern-Weil curvature construction all Pontrjagin classes of η vanish. Therefore

$$\begin{aligned} 0 &= ch(\eta) \\ &= ch(\eta_+) + ch(\eta_-) \end{aligned}$$

and so

$$\ell(E) = \pi_* \widetilde{\mathcal{L}}(T^\nu E) = ch(\eta_+ - \eta_-) = 2ch(\eta_+). \tag{11}$$

Finally, observe that the characteristic classes of the bundle η_+ coincide with the classes pulled back from BO_p along the composition of (10). □

The following is an argument essentially due to Morita [Mor87]. Let $\mathrm{Diff}(M)^\delta$ denote the diffeomorphism group endowed with the discrete topology and consider the natural map $\epsilon : B\mathrm{Diff}(M)^\delta \to B\mathrm{Diff}(M)$.

Theorem 6.3. *For dim $M = 4k$ and $i > k$ the relation*

$$\epsilon^* \ell_i = 0$$

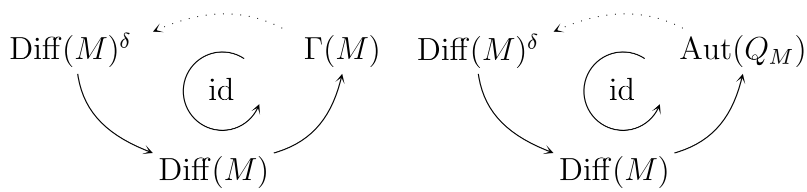
holds in $H^*(B\mathrm{Diff}(M)^\delta;\mathbb{R})$.

Proof. The space $B\mathrm{Diff}(M)^\delta$ is the classifying space for smooth M bundles which are *flat*, which is to say bundles equipped with a foliation transverse to the fibres and of codimension equal to the dimension of M (the leaves of the foliation are parallel to the base). Let $M \to E \to B$ be a fibre bundle with structure group $\mathrm{Diff}(M)^\delta$ and let $\mathscr{F}$ denote the corresponding foliation. Then the normal bundle to $\mathscr{F}$ can be canonically identified with the vertical tangent bundle. Now Bott's Vanishing Theorem [Bot70] states that the rational Pontrjagin ring of $T^\nu E$ vanishes in degrees greater than $8k$. In particular, $\widetilde{\mathcal{L}}_{i+k}(T^\nu E) = 0$ for $4(i+k) > 8k$, and therefore $\ell_i(E) = 0$ for $i > k$. Finally, since this holds for any flat M-bundle where the base and total space are manifolds, it holds in the universal case on $B\mathrm{Diff}(M)^\delta$. □

Let ch_{2i} denote the $2i^{th}$ component of the Chern character, regarded as an element of $H^{4i}(B\mathrm{Aut}(Q_M);\mathbb{R})$ via the homomorphism of Lemma 4.3 (or as an element of $H^{4i}(B\Gamma(M);\mathbb{R})$ by pullback).

Corollary 6.4. *For M a $4k$-dimensional manifold, the classes $ch_i \in H^{4i}(B\Gamma(M);\mathbb{R})$ (resp. $H^{4i}(B\mathrm{Aut}(Q_M);\mathbb{R})$) for $i > k$ are obstructions to the existence of a section of the group homomorphism $\mathrm{Diff}(M) \to \Gamma(M)$ (resp. $\mathrm{Diff}(M) \to \mathrm{Aut}(Q_M)$).*

Proof. Existence of such a section means that the identity on $\Gamma(M)$ (resp. $\mathrm{Aut}(Q_M)$) factors through $\mathrm{Diff}(M)$. Since $\Gamma(M)$ (resp. $\mathrm{Aut}(Q_M)$) is discrete the identity actually factors through $\mathrm{Diff}(M)^\delta$, as below.



By Proposition 6.2 the ℓ classes on $B\mathrm{Diff}(M)$ are pulled back from components of the Chern character on $B\Gamma(M)$ or $B\mathrm{Aut}(Q_M)$, and by Theorem 6.3 they are zero when pulled back to $B\mathrm{Diff}(M)^\delta$. Hence if they are nonzero on $B\Gamma(M)$ (resp. $B\mathrm{Aut}(Q_M)$) then no section can exist. $\square$

In particular, Theorem 1.5 now follows, as does the special case of Theorem 1.4 when M is a $K3$ manifold; the slightly more general connected sum case is discussed below in section 7.2.

Remark 6.5. Corollary 6.4 still holds if $\mathrm{Diff}(M)$ is replaced by the C^1 diffeomorphism group. Hilsum [Hil89] provides a version of the Index Theorem which is valid even in the Lipschitz setting, and the proof of Bott's Vanishing Theorem works verbatim in the C^1 setting (although it is unknown if Bott's theorem holds in the Lipschitz category). However, As Morita points out, the above method provides no information about lifting mapping class groups to homeomorphisms in light of the fact [McD80] that $B\mathrm{Homeo}(M)^\delta \to B\mathrm{Homeo}(M)$ is a homology isomorphism.

7. Algebraic independence of the ℓ_i classes

In addition to the question of when the ℓ_i classes are nontrivial, one can ask when they are algebraically independent. If M is a $K3$ manifold then we have seen (Theorems 1.1, 1.3, and Proposition 6.2) that ℓ_1, ℓ_2 are nonzero in $H^*(B\mathrm{Diff}(M);\mathbb{R})$. However, they are *not* algebraically independent; they satisfy

$$\ell_1^2 = \ell_2.$$

In this section we give two situations in which the ℓ_i classes become algebraically independent. The first is when M is a product of two surfaces of large genus, in which case algebraic independence follows easily from the algebraic independence of the Miller-Morita-Mumford classes on large-genus surfaces. The second situation is an iterated self-connected sum $M\#\cdots\#M$ of a $4k$-dimensional manifold

M; nontriviality of the ℓ_i classes on M in a range of degrees implies algebraic independence on the connected sum in a range of degrees.

7.1. **A product of two surfaces.** Consider $M = F_{g_1} \times F_{g_2}$ a product of two surfaces.

Proposition 7.1. *The ring homomorphism* $\mathbb{R}[\ell_1, \ell_2, \ldots] \to H^*(B\mathrm{Diff}(M); \mathbb{R})$ *is injective in degrees* $* \leq (\min(g_1, g_2) - 1)/2$.

Proof. Let $\pi_i : E_i \to B\mathrm{Diff}(F_{g_i})$ be the universal F_{g_i}-bundle and consider the ℓ_j classes of the product bundle

$$E = E_1 \times E_2 \xrightarrow{\pi_1 \times \pi_2} B\mathrm{Diff}(F_{g_1}) \times B\mathrm{Diff}(F_{g_2}),$$

which has fibre $F_{g_1} \times F_{g_2}$. The vertical tangent bundle can be written as an external product $T^\nu E \cong T^\nu E_1 \times T^\nu E_2$. Hence

$$\begin{aligned} \ell(E) = (\pi_1 \times \pi_2)_* \widetilde{\mathcal{L}}(T^\nu E) &= (\pi_1 \times \pi_2)_* (\widetilde{\mathcal{L}}(T^\nu E_1) \times \widetilde{\mathcal{L}}(T^\nu E_2)) \\ &= (\pi_1)_* \widetilde{\mathcal{L}}(T^\nu E_1) \times (\pi_2)_* \widetilde{\mathcal{L}}(T^\nu E_2). \end{aligned}$$

Since $T^\nu E_i$ is a rank 2 vector bundle,

$$\widetilde{\mathcal{L}}_j(T^\nu E_i) = (\text{constant}) \cdot (e(T^\nu E_i))^{2j}$$

and so

$$(\pi_i)_* \widetilde{\mathcal{L}}_j(T^\nu E_i) = (\text{constant}) \cdot \kappa_{2i-1}.$$

It is well known that the κ_i classes are algebraically independent in $H^*(B\mathrm{Diff}(F_g); \mathbb{R})$ up to degree $(g-1)/2$. Hence the classes $\ell_i(E)$ are nontrivial and algebraically independent up to the desired degree. $\square$

7.2. **Iterated connected sums.** Let $M_1, \ldots M_n$ be $4k$-manifolds each having a $(4k-1)$-sphere as boundary, and let $\mathrm{Diff}(M_i, \partial M_i)$ denote the group of diffeomorphisms which fix a collar neighborhood of the boundary pointwise. Now let $\overline{M}$ be the result of gluing the M_i onto the boundary components of a $4k$-sphere with the interiors of n discs deleted. This is essentially a connected sum, so we abuse notation by writing $\overline{M} = M_1 \# \cdots \# M_n$. Extending diffeomorphisms by the identity on the sphere induces a map

$$\mu : B\mathrm{Diff}(M_1, \partial M_1) \times \cdots \times B\mathrm{Diff}(M_n, \partial M_n) \to B\mathrm{Diff}(\overline{M}).$$

Lemma 7.2. $\mu^* \ell_i(\overline{M}) = \sum_{j=1}^n 1 \times \cdots \times \ell_i(M_j) \times \cdots \times 1$, *where in the* j^{th} *term of the summation* ℓ_i *occurs in the* j^{th} *position.*

Proof. This follows immediately from the commutative square

$$\begin{array}{ccc} B\mathrm{Diff}(M_1, \partial M_1) \times \cdots \times B\mathrm{Diff}(M_n, \partial M_n) & \xrightarrow{\mu} & B\mathrm{Diff}(\overline{M}) \\ \downarrow & & \downarrow \\ B\mathrm{Aut}(Q_{M_1}) \times \cdots \times B\mathrm{Aut}(Q_{M_n}) & \longrightarrow & B\mathrm{Aut}(Q_{M_1} \oplus \cdots \oplus Q_{M_n}) \end{array}$$

together with Proposition 6.2. □

For a given manifold M with boundary a sphere, if one can show nontriviality of the ℓ classes on $B\mathrm{Diff}(M, \partial M)$ (or $B\Gamma(M, \partial M)$) then taking connected sums turns nontriviality into algebraic independence.

Theorem 7.3. *Suppose $\partial M \cong S^{4k-1}$ and suppose that the classes $\ell_1, \ldots, \ell_n$ are all nonzero in $H^*(B\mathrm{Diff}(M, \partial M))$. Then the monomials $\{\ell_1^{m_1} \cdots \ell_n^{m_n} | \sum m_i \leq N\}$ are all linearly independent in $H^*(B\mathrm{Diff}(\#^N M))$. In particular, on $\#^N M$ the classes $\ell_1, \ldots, \ell_n$ satisfy no polynomial relations of degree $\leq 4kN$. This holds also for mapping class groups.*

Proof. Define the *length* of an element

$$a_1 \times \cdots \times a_N \in H^*(B\mathrm{Diff}(M, \partial M) \times \cdots \times B\mathrm{Diff}(M, \partial M))$$

to be the number of components a_i which are not equal to 1. A maximal length term in $\mu^*(\ell_1^{m_1} \cdots \ell_n^{m_n})$ will have m_i components which are of degree $4ki$, for each i between 1 and n. Hence if $\mu^*(\ell_1^{m_1} \cdots \ell_n^{m_n})$ is proportional to $\mu^*(\ell_1^{m'_1} \cdots \ell_n^{m'_n})$ then it must be the case that $(m_1, \ldots, m_n) = (m'_1, \ldots, m'_n)$. □

Thus far we have only shown the ℓ_i classes to be nontrivial for products of surfaces and for the $K3$ manifold; in neither of these cases is there a boundary sphere to which one can apply Theorem 7.3. We must therefore add a boundary sphere. The case $N = 1$ above corresponds to gluing a disc in to the boundary sphere. One can pull the ℓ_i class back along the map $B\mathrm{Diff}(M, \partial M) \to B\mathrm{Diff}(\overline{M})$, but it seems difficult to say whether or not these pullbacks will be nonzero even if they are nonzero on $B\mathrm{Diff}(\overline{M})$. There is a homotopy fibre sequence

$$V(T\overline{M}) \to B\mathrm{Diff}(M, \partial M) \to B\mathrm{Diff}(\overline{M}),$$

where $V(T\overline{M})$ is the frame bundle of the tangent bundle, and one might hope to understand the Serre spectral sequence well enough to conclude that the ℓ_i classes are indeed nontrivial on $B\mathrm{Diff}(M, \partial M)$ if M is a $K3$ manifold. This seems somewhat difficult.

However, the kernel of $\Gamma(M, \partial M) \to \Gamma(\overline{M})$ is either trivial or $\mathbb{Z}/2$ (it is generated by the Dehn twist around the boundary sphere). Thus

$$H^*(B\Gamma(M, \partial M); \mathbb{Z}[1/2]) \cong H^*(B\Gamma(\overline{M}); \mathbb{Z}[1/2]).$$

The general connected sum case of Theorem 1.4 follows from this isomorphism and Lemma 7.2.

References

[Bae28] R. Baer, *Isotopien auf Kurven von orientierbaren, geschlossenen Flächen*, J. Reine Angew. Math. **159** (1928), 101–116.

[Bes87] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987.

[Bor74] A. Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 235–272 (1975).

[Bor77] ———, *Cohomologie de* SL_n *et valeurs de fonctions zeta aux points entiers*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **4** (1977), no. 4, 613–636.

[Bor81] Armand Borel, *Stable real cohomology of arithmetic groups. II*, Manifolds and Lie groups (Notre Dame, Ind., 1980), Progr. Math., vol. 14, Birkhäuser Boston, Mass., 1981, pp. 21–55.

[Bot70] R. Bott, *On a topological obstruction to integrability*, Global Analysis (Proc. Sympos. Pure Math., Vol. XVI, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 127–131.

[BR75] D. Burns, Jr. and M. Rapoport, *On the Torelli problem for kählerian* K*-3 surfaces*, Ann. Sci. École Norm. Sup. (4) **8** (1975), no. 2, 235–273.

[Don90] S. K. Donaldson, *Polynomial invariants for smooth four-manifolds*, Topology **29** (1990), no. 3, 257–315.

[Ebi70] D. G. Ebin, *The manifold of Riemannian metrics*, Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 11–40.

[EE69] C. J. Earle and J. Eells, *A fibre bundle description of Teichmüller theory*, J. Differential Geometry **3** (1969), 19–43.

[ES70] C. J. Earle and A. Schatz, *Teichmüller theory for surfaces with boundary*, J. Differential Geometry **4** (1970), 169–185.

[FH78] F. T. Farrell and W. C. Hsiang, *On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978, pp. 325–337.

[Fre82] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), no. 3, 357–453.

[FT84] A. E. Fischer and A. J. Tromba, *On a purely "Riemannian" proof of the structure and dimension of the unramified moduli space of a compact Riemann surface*, Math. Ann. **267** (1984), no. 3, 311–345.

[GM88] M. Goresky and R. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 14, Springer-Verlag, Berlin, 1988.

[Hat81] A. Hatcher, *On the diffeomorphism group of* $S^1 \times S^2$, Proc. Amer. Math. Soc. **83** (1981), no. 2, 427–430.

[Hat83] ———, *A proof of a Smale conjecture,* $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$, Ann. of Math. (2) **117** (1983), no. 3, 553–607.

[Hil89] M. Hilsum, *Fonctorialité en* K*-théorie bivariante pour les variétés lipschitziennes*, K-Theory **3** (1989), no. 5, 401–440.

[Hit74] N. Hitchin, *Compact four-dimensional Einstein manifolds*, J. Differential Geometry **9** (1974), 435–441.

[Ker83] S. P. Kerckhoff, *The Nielsen realization problem*, Ann. of Math. (2) **117** (1983), no. 2, 235–265.

[KJ02] J. Kedra and T. Januszkiewicz, *Characteristic classes of smooth fibrations*, ArXiv: math.AG/0209288, 2002.

[Kod64] K. Kodaira, *On the structure of compact complex analytic surfaces. I*, Amer. J. Math. **86** (1964), 751–798.

[Mat62] Y. Matsushima, *On Betti numbers of compact, locally sysmmetric Riemannian manifolds*, Osaka Math. J. **14** (1962), 1–20.

[Mat86] T. Matumoto, *On diffeomorphisms of a* $K3$ *surface*, Algebraic and topological theories (Kinosaki, 1984), Kinokuniya, Tokyo, 1986, pp. 616–621.

[McD80] D. McDuff, *The homology of some groups of diffeomorphisms*, Comment. Math. Helv. **55** (1980), no. 1, 97–129.

[Mor87] S. Morita, *Characteristic classes of surface bundles*, Invent. Math. **90** (1987), no. 3, 551–577.

[MW02] I. Madsen and M. Weiss, *The stable moduli space of Riemann surfaces: Mumford's conjecture*, ArXiv preprint math.AT/0212321, 2002.

[Nie27] J. Nielsen, *Untersuchungen zur Theorie der geschlossenen zweiseitigen Flächen I*, Acta Math. **50** (1927), 189–358.

[Nie43] ———, *Abbildungklassen endliche ordung*, Acta Math. **75** (1943), 23–115.

[Qui86] F. Quinn, *Isotopy of* 4*-manifolds*, J. Differential Geom. **24** (1986), no. 3, 343–372.

[Rub98] D. Ruberman, *An obstruction to smooth isotopy in dimension* 4, Math. Res. Lett. **5** (1998), no. 6, 743–758.

[Rub99] ———, *A polynomial invariant of diffeomorphisms of 4-manifolds*, Proceedings of the Kirbyfest (Berkeley, CA, 1998), Geom. Topol. Monogr., vol. 2, Geom. Topol. Publ., Coventry, 1999, pp. 473–488 (electronic).

[WW88] M. Weiss and B. Williams, *Automorphisms of manifolds and algebraic K-theory. I*, K-Theory **1** (1988), no. 6, 575–626.

[WW89] ———, *Automorphisms of manifolds and algebraic K-theory. II*, J. Pure Appl. Algebra **62** (1989), no. 1, 47–107.

[WW01] ———, *Automorphisms of manifolds*, Surveys on surgery theory, Vol. 2, Ann. of Math. Stud., vol. 149, Princeton Univ. Press, Princeton, NJ, 2001, pp. 165–220.

[WW06] ———, *Automorphisms of manifolds and algebraic K-theory. III*, Available at http://www.maths.abdn.ac.uk/ mweiss/preprints/ww003.pdf, 2006.

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