Operator-Valued Involutive Distributions of Evolutionary Vector Fields and their Affine Geometry

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Abstract. Involutive distributions of evolutionary vector fields that belong to images of matrix operators in total derivatives are considered and some classifications of the operators are obtained. The weak compatibility of these operators is an analog of the Poisson pencils for the Hamiltonian structures, while the commutation closure of sums of the images for N-tuples of the operators, which is the strong compatibility, suggests a generalization of the affine geometry such that a flat connection is determined by bi-differential operators. We assign a class of matrix operators whose images are closed w.r.t. the commutation and the Koszul brackets induced in their pre-images to integrable KdV-type hierarchies of symmetry flows on hyperbolic Euler–Lagrange Liouville-type systems (e.g., the 2D Toda lattices associated with semi-simple Lie algebras).

Introduction. Relations between completely integrable Hamiltonian systems of PDE and Lie algebras are well acknowledged in mathematical physics, see [6, 7, 12, 28, 30] and references therein. The Hamiltonian structures for evolution equations are inherited from the algebras, while the bi-Hamiltonianity w.r.t. a Poisson pencil $A_{1,2}$ and triviality of the first Poisson cohomology [10] w.r.t. A_1 mean the complete itegrability [25]. Reciprocally, the higher Hamiltonian operators for KdV-type equations [7] determine the W-algebra structures on the Fourier decompositions of the evolving fields (e.g., the Virasoro algebra for KdV itself, W_3 for the Boussinesq equation, $w_{1+\infty}$ for KP, etc.).

In this paper we consider a new algebraic construction in geometry of integrable systems: the class of matrix operators in total derivatives whose images in the spaces of evolutionary vector fields are closed with respect to the commutation, that is, determine involutive distributions in the infinite-dimensional Lie algebras of evolutionary fields on jet spaces. These differential operators, which we call *Frobenius*, induce the Koszul brackets in their pre-images; the bracket is trivial in the ODE case and has been known explicitly only for the Hamiltonian operators.

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The Frobenius operators extend the notion of the Poisson structures for Hamiltonian evolutionary PDE. There extensions were suggested in [34] and were studied in [31] in a scalar, scaling-invariant case using Gelfand's approach [9]. Multi-component analogs of known Frobenius operators related to KdV-type and Toda systems were obtained in [16]. Independently, examples of the operators in total derivatives that factor symmetries of the Liouville-type systems and whose images are closed w.r.t. the commutation were constructed in [4, 33].

We say that the Frobenius operators are weakly compatible if their arbitrary linear combinations retain the same property. The operators A_1, …, A_N are strongly compatible if the commutators of any evolutionary fields $\partial_{A_k(\cdot)}$ in their images belong to the sum of these images:

$$\begin{aligned}[A_i(p), A_j(q)] &= \partial_{A_i(p)}\bigl(A_j(q)\bigr) - \partial_{A_i(p)}\bigl(A_j(q)\bigr)\\ &= A_j\bigl(\partial_{A_i(p)}(q)\bigr) - A_i\bigl(\partial_{A_j(q)}(p)\bigr) + \sum\nolimits_{k=1}^{N} A_k\bigl(\Gamma^k_{ij}(p,q)\bigr).\end{aligned} \tag{I.1}$$

The analogs Γ^k_{ij} of the Christoffel symbols in pre-images of $A_1, \ldots, A_N$ are bi-differential operators of $\mathbb{Z}$-graded arguments. The standard symmetric affine connections appear as the zero order terms in the geometry described by Frobenius operators on supermanifolds and the completely integrable hierarchies are the geodesics in it. The arising new connection is flat owing to the Jacobi identity for the vector fields.

The Frobenius recursion operators are the automorphisms of evolutionary vector fields specified by differential operators whose images are involutive distributions. The associative composition of strongly compatible Frobenius recursions and the commutator expansions (I.1) for fields in their images suggest higher-order finite-dimensional generalizations with bi-differential structureal constants for the Lie groups and algebras. In this paper we give examples of the Frobenius recursions that are factored using non-Hamiltonian Frobenius operators.

The paper is organized as follows. In sec. 1 we summarize important properties of the Hamiltonian operators, which are substantially used in the sequel. Then, in sec. 2 we define the Frobenius operators, describe the properties of the Koszul brackets in their pre-images, and interpret these structures as the Lie algebroids over the infinite jet spaces. In sec. 3 we propose the definitions of weakly and strongly compatible Frobenius operators; the former is an analog of the Poisson pencils and the latter is based on the collective commutation closure of the images. In sec. 4 we recall the construction of a natural bracket of conservation laws for PDE that yields another, non-Koszul bracket in pre-images of the Frobenius operators; the dispersionless 3-component Boussinesq system and its recently discovered properties provide an illustration. Further, we derive a one-parametric family of Frobenius operators from the symmetry structure of the extended Liouville equation (see [17]), which is obtained using the Gardner deformation of the KdV equation.

Our main result is contained in sec. 5 where we assign the infinite class of matrix Frobenius operators and the corresponding Koszul brackets in their pre-images to the Liouville-type Euler–Lagrange systems; to this end, we describe the generators of higher symmetry algebras for these Darboux-integrable nonlinear equations (in particular, for the 2D Toda lattice associated with the root systems of semi-simple Lie algebras [7, 24, 32]). We demonstrate that the Frobenius operators determine the factorizations of higher Hamiltonian structures in the hierarchies whose Magri's schemes are correlated.

In auxiliary appendix A we recall the parallel between ordinary and partial differential Hamiltonian equations, fix the notation used throughout the text, and formulate the proofs of main lemmas. In appendix B we describe a method for reconstructing the Koszul brackets in the inverse images of the Frobenius operator.

1. Basic properties of the Hamiltonian operators for PDE

Let us first recall necessary definitions and introduce some notation [6, 8, 14, 23]. To facilitate the exposition, in appendix A we summarize the notation and its analogy with the structures for Hamiltonian ordinary differential equations. The proofs of Lemma 2, Proposition 5, and Theorem 7 are recalled there, too; a reformulation of Lemma 4 is proved on p 19.

Let $\pi\colon E^{n+m} \to M^n$ be a vector bundle over an n-dimensional manifold M, $J^\infty(\pi)$ be the infinite jets over this bundle, $\mathfrak{g}(\pi) \subset \mathrm{D}^v(J^\infty(\pi))$ be the Lie algebra of evolutionary derivations ∂_φ assigned to the sections $\varphi \in \varkappa(\pi)$ that belong to the $C^\infty(J^\infty(\pi))$-module $\varkappa(\pi) = \Gamma(\pi_\infty^*(\pi))$, and let $\hat{\varkappa}(\pi) = \mathrm{Hom}_{C^\infty(M)}(\varkappa, \bar{\Lambda}^n(\pi))$ be the dual of $\varkappa$ taking values in the horizontal forms of senior degree. By definition, put $f[u] = f(u, u_x, \ldots, u_\sigma)$, here u denotes the components of a vector ${}^t(u^1, \ldots, u^m)$ and $|\sigma| < \infty$.

Consider the space $\bar{H}^n(\pi) = \bar{\Lambda}^n(\pi)/\{\mathrm{im}\,\mathrm{d}_h\}$ of functionals $\mathcal{H} = \int H[u]\,\mathrm{d}x$ such that the elements of the space are equivalent if they differ by the exact terms $\mathrm{d}_h\eta$, $\mathrm{d}_h = \sum_{i=1}^n \mathrm{d}x^i \otimes D_{x^i}$ being the horizontal differential. Let $\mathbf{E}$ denote the variational derivative with respect to u; we recall that the variational derivative $\mathbf{E}$ is determined by the restriction of the Cartan differential $\mathrm{d}_\mathcal{C} = \mathrm{d} - \mathrm{d}_h$ onto the senior cohomology $\bar{H}^n(\pi)$.

Lemma 1 ([23]). Let $\mathcal{E} = \{F \equiv u_t - f = 0\}$ be an evolutionary system. For any conservation law $\eta = \rho\,\mathrm{d}x + \cdots$ such that $\mathrm{d}_h(\eta) = \nabla(F)$, its generating section $\psi = \nabla^*(1)$ is the 'gradient' $\psi = \mathbf{E}_u(\rho)$ of the conserved density ρ.

Lemma 2 ([23]). The relation

$$\mathbf{E}(\partial_\varphi(\rho)) = \partial_\varphi(\mathbf{E}(\rho)) + \ell_\varphi^*(\mathbf{E}(\rho))$$

holds for any $\varphi \in \varkappa(\pi)$ and $\rho \in \bar{\Lambda}^n(\pi)$.

The standard concept of Hamiltonian dynamics for PDE (see [6, 14, 23]) is based on the notion of the Poisson structure $\{\mathcal{H}_1, \mathcal{H}_2\}_A = \langle \mathbf{E}(\mathcal{H}_1), A(\mathbf{E}(\mathcal{H}_2))\rangle$ of two Hamiltonians $\mathcal{H}_{1,2}$, which is induced by a skew-adjoint Hamiltonian operator A such that the bracket $\{\,,\,\}_A$ satisfies the Jacobi identity. The Lie algebra of Hamiltonian evolutionary vector fields $\partial_{A(\mathbf{E}(\mathcal{H}))} \in \mathfrak{g}(\pi)$ is correlated by A with the Lie algebra of Hamiltonians $\mathcal{H}$.

Proposition 3 ([6, §27])**.** The Euler derivative $\mathbf{E}$ and the Hamiltonian operators A determine the Lie algebra morphisms

$$\bigl(\bar{H}^n(\pi), \{\,,\,\}_A\bigr) \xrightarrow{\mathbf{E}} \bigl(\Omega^1(\pi), [\,,\,]_A\bigr) \xrightarrow{\partial_{A(\cdot)}} (\mathfrak{g}(\pi), [\,,\,])$$

such that $A\big([\psi_1,\psi_2]_A\big) = [A\psi_1, A\psi_2]$ for ψ_1, $\psi_2 \in \Omega^1(\pi)$ and[1] $\big[\mathbf{E}(\mathcal{H}_1),\mathbf{E}(\mathcal{H}_2)\big]_A = \mathbf{E}\big(\{\mathcal{H}_1,\mathcal{H}_2\}_A\big)$ for any $\mathcal{H}_1$, $\mathcal{H}_2 \in \bar{H}^n(\pi)$. The correlation between the Koszul–Dorfman–Daletsky bracket $[\,,\,]_A$, the Poisson bracket $\{\,,\,\}_A$, and the commutator $[\,,\,]$ of the evolutionary fields $\partial_{A(\cdot)}$ is such that

$$[\psi_1,\psi_2]_A = \partial_{A(\psi_1)}(\psi_2) - \partial_{A(\psi_2)}(\psi_1) + \{\{\psi_1,\psi_2\}\}_A = \mathbf{E}\big(\langle\psi_1, A(\psi_2)\rangle\big), \tag{1}$$

where $A\big(\{\{\psi_1,\psi_2\}\}_A\big) = \big(\partial_{A\psi_1}(A)\big)(\psi_2) - \big(\partial_{A\psi_2}(A)\big)(\psi_1)$.

For A Hamiltonian, both $\ker A$ and $\operatorname{im} A$ are ideals in $\Omega^1(\pi)$ and $\mathfrak{g}(\pi)$ w.r.t. $[\,,\,]_A$ and $[\,,\,]$, respectively.

Lemma 4 ([23, §7.8]). Consider an operator $J \in \mathcal{C}\mathrm{Diff}_{(l)}(\Omega^1(\pi), P)$ which is skew-symmetric w.r.t. its l arguments that belong to $\Omega^1(\pi)$ and which takes values in a $C^\infty(J^\infty(\pi))$-module P. If for all $\mathcal{H}_1$, $\ldots$, $\mathcal{H}_l \in \bar{H}^n(\pi)$ one has $J(\mathbf{E}(\mathcal{H}_1),\ldots,\mathbf{E}(\mathcal{H}_l)) = 0$, then $A \equiv 0$.

Thus the following reformulation of the Jacobi identity $J(\psi_1,\psi_2,\psi_3) = 0$ can be verified for elements $\psi_i \in \operatorname{im}\mathbf{E}$ only.

Proposition 5 (A criterion of the Hamiltonianity, [23])**.** By definition, put $\ell_{A,\psi}(\varphi) := \big(\partial_\varphi(A)\big)(\psi)$. A skew-adjoint operator $A \in \mathcal{C}\mathrm{Diff}(\Omega^1(\pi), \varkappa(\pi))$ in total derivatives is Hamiltonian if and only if the relation

$$\ell_{A,\psi_1}(A(\psi_2)) - \ell_{A,\psi_2}(A(\psi_1)) = A\big(\ell^*_{A,\psi_2}(\psi_1)\big) \tag{2}$$

holds for all ψ_1, $\psi_2 \in \hat{\varkappa}(\pi)$. Note that the r.h.s. of the above formula is skew-symmetric w.r.t. ψ_1, ψ_2 and equals $-A\big(\{\{\psi_1,\psi_2\}\}_A\big)$, see (1).

Proposition 5 provides the formula $\{\{\psi_1,\psi_2\}\}_A = \ell^*_{A,\psi_1}(\psi_2)$ valid for any Hamiltonian operator A. Another way, see (17b), of calculating the Koszul bracket $[\,,\,]_A$ is based on the relation $\{\mathcal{H}_1,\mathcal{H}_2\}_A = -\partial_{A(\mathbf{E}(\mathcal{H}_1))}(\mathcal{H}_2)$ for the Poisson bracket and further use of Lemma 2. This allows to define a new bracket $[\![\,,\,]\!]_A$ on $\Omega^1(\pi)$, which coincides with the Koszul bracket in the Hamiltonian case, see section 4.

Lemma 6 ([23]). The Hamiltonian operators A take the generating sections (the 'gradients' $\mathbf{E}(\rho)$ of conserved densities ρ) $\psi \in \Omega^1(\mathcal{E})$ of conservation laws $\eta = \rho\,\mathrm{d}x + \cdots$ for $\mathcal{E} = \{u_t = A(\mathbf{E}(\mathcal{H}))\}$ to symmetries $\partial_\varphi \in \mathfrak{g}(\mathcal{E})$ of these systems, that is, $\partial_{A(\cdot)}\colon \Omega^1(\mathcal{E}) \to \mathfrak{g}(\mathcal{E})$.

For any $\mathbb{R}$-algebra[2] $\mathfrak{g}$, let $\Delta \in \mathrm{Hom}_{\mathbb{R}}(\bigwedge^k \mathfrak{g},\mathfrak{g})$ and $\nabla \in \mathrm{Hom}_{\mathbb{R}}(\bigwedge^l \mathfrak{g},\mathfrak{g})$. Denote by $\Delta[\nabla] \in \mathrm{Hom}_{\mathbb{R}}(\bigwedge^{k+l-1} \mathfrak{g},\mathfrak{g})$ the *action* $\Delta[\cdot]\colon \mathrm{Hom}_{\mathbb{R}}(\bigwedge^N \mathfrak{g},\mathfrak{g}) \to \mathrm{Hom}_{\mathbb{R}}(\bigwedge^{N+k-1} \mathfrak{g},\mathfrak{g})$ of Δ on ∇, which is given by the formula

$$\Delta[\nabla](a_1,\ldots,a_{k+l-1}) = \sum_{\sigma\in S^l_{k+l-1}} (-1)^\sigma \Delta(\nabla(a_{\sigma(1)},\ldots,a_{\sigma(l)}), a_{\sigma(l+1)},\ldots,a_{\sigma(k+l-1)}),$$

where $a_i \in \mathfrak{g}$ and $S^k_m \subset S_m$ denotes the *unshuffles*, which are the permutations such that $\sigma(1) < \sigma(2) < \ldots < \sigma(k)$ and $\sigma(k+1) < \ldots < \sigma(m)$ for all $\sigma \in S^k_m$; note that

[1] Hence the image of the Euler operator $\mathbf{E}$ is closed with respect to the commutation.

[2] The real axis $\mathbb{R}$ is the field we operate with; the definition of the Schouten bracket is valid over any field $\Bbbk$ such that $\operatorname{char}\Bbbk \neq 2$.

$\sigma(i)$ is the index of the object placed onto the ith position under the permutation, unlike in [13]. The *Schouten bracket* $[\![\Delta,\nabla]\!]\in\mathrm{Hom}_{\mathbb{R}}(\bigwedge^{k+l-1}\mathfrak{g},\mathfrak{g})$ of Δ and ∇ is [6, 18]

$$[\![\Delta,\nabla]\!]=\Delta[\nabla]-(-1)^{(k-1)(l-1)}\nabla[\Delta].$$

Thus the commutator $[\![X,Y]\!]=[X,Y]$ of two evolutionary vector fields $X,Y\in\mathfrak{g}(\pi)$ is skew-symmetric and the bracket $[\![A_1,A_2]\!]$ of two bi-vectors is symmetric w.r.t. the arguments.

The Hamiltonian operators A are naturally regarded as the variational Poisson bi-vectors [6, 14] with the vanishing Schouten brackets $[\![A,A]\!]=0$ such that the Poisson bracket $\{\mathcal{H}_1,\mathcal{H}_2\}_A=[\![[\![A,\mathcal{H}_2]\!],\mathcal{H}_1]\!]$ is a derived bracket in the sense of [21]. The Schouten bracket itself satisfies the super-Jacobi identity

$$[\![[\![A_1,A_2]\!],A_3]\!]+[\![[\![A_2,A_3]\!],A_1]\!]+[\![[\![A_3,A_1]\!],A_2]\!]=0. \tag{3}$$

Hence the original Jacobi identity $[\![A,A]\!](\psi_1,\psi_2,\psi_3)=0$ for the arguments of A implies that $\partial_A=[\![A,\cdot\,]\!]$ is a differential, giving rise to the Poisson cohomology $H^k_A(\pi)$. Obviously, the Casimirs $\mathbf{c}\in\bar{H}^n(\pi)$ such that $[\![A,\mathbf{c}]\!]=0$ for a Hamiltonian operator A constitute $H^0_A(\pi)$.

Two Hamiltonian operators are compatible if their linear combinations $\lambda_1A_1+\lambda_2A_2$ are Hamiltonian; the compatibility condition for the Poisson bi-vectors is $[\![A_1,A_2]\!]=0$.

Theorem 7 (The Magri scheme [6, 25])**.** *Suppose $[\![A_1,A_2]\!]=0$, $\mathcal{H}_0\in H^0_{A_1}(\pi)$ is a Casimir of A_1, and $H^1_{A_1}(\pi)=0$. Then for any $k>0$ there is the Hamiltonian $\mathcal{H}_k\in\bar{H}^n(\pi)$ such that*

$$[\![A_2,\mathcal{H}_{k-1}]\!]=[\![A_1,\mathcal{H}_k]\!]. \tag{4}$$

Put $\varphi_k:=A_1\bigl(\mathbf{E}(\mathcal{H}_k)\bigr)$. The Hamiltonians $\mathcal{H}_i$, $i\geq 0$, Poisson commute w.r.t. either A_1 or A_2, the densities of $\mathcal{H}_i$ are conserved on any equation $u_{t_k}=\varphi_k$, and the evolutionary derivations ∂_{φ_k} pairwise commute.

We emphasize that Theorem 7 holds for the Hamiltonians which belong to the linear subspace of $\bar{H}^n(\pi)$ spanned by the descendants of the Casimirs $\mathcal{H}_0\in H^0_A(\pi)$.

The correlated Magri schemes for coupled hierarchies are further considered in section 5, see diagram (30).

2. Frobenius operators

By Proposition 3, the Poisson bracket $\{\,,\,\}_A$ in the Lie algebra of Hamiltonians is correlated with the commutator $[\,,\,]$ in the Lie algebra of the vector fields $\partial_{A(\cdot)}$; the skew-adjoint operator A in the Poisson bracket $\langle\mathbf{E}(\mathcal{H}_1),A(\mathbf{E}(\mathcal{H}_2))\rangle$ coincides with the operator A that defines a subalgebra of evolutionary fields $\partial_{A(\mathbf{E}(\mathcal{H}))}$. Thus we have

$$\bigl[\partial_{A(\mathbf{E}(\mathcal{H}_1))},\partial_{A(\mathbf{E}(\mathcal{H}_2))}\bigr]=\partial_{A(\mathbf{E}(\{\mathcal{H}_1,\mathcal{H}_2\}_A))}.$$

In this section we reverse the status of the two Lie algebra structures $[\,,\,]\big|_{\operatorname{im}A}$ and $\{\,,\,\}_A$, giving the priority to the commutators of vector fields and thus considering involutive distributions of evolutionary vector fields $\partial_\varphi\in\mathfrak{g}(\pi)$ whose generators $\varphi=A(\cdot)\in\varkappa(\pi)$ belong to images of matrix operators A in total derivatives.

Definition 1. A matrix operator A in total derivatives is *Frobenius* if its image in the module $\Gamma(\pi_\infty^*(\pi))$ of generators of evolutionary vector fields is closed w.r.t. the commutation: $[\operatorname{im} A, \operatorname{im} A] \subseteq \operatorname{im} A$. The commutator $[\,,\,]\big|_{\operatorname{im} A}$ induces the skew-symmetric Koszul bracket $[\,,\,]_A$ in the inverse image of A:

$$\big[A(\psi_1), A(\psi_2)\big] = A\big([\psi_1, \psi_2]_A\big). \tag{5a}$$

The Koszul bracket

$$[\psi_1, \psi_2]_A = \partial\!\!\!\raisebox{0pt}{}_{A(\psi_1)}(\psi_2) - \partial_{A(\psi_2)}(\psi_1) + \{\{\psi_1, \psi_2\}\}_A \tag{5b}$$

contains the two standard summands and the bracket $\{\{\,,\,\}\}_A$ that generally does not satisfy the Jacobi identity. The bracket $[\,,\,]_A$ is $\mathbb{R}$-bilinear, skew-symmetric, and transfers the Jacobi identity from the Lie algebra $\mathfrak{g}(\pi)$ of evolutionary vector fields onto $\Omega^1(\pi)$. Thus $[\,,\,]_A$ and hence $\{\{\,,\,\}\}_A$ are defined up to the kernel[3] $\ker A$, which is an ideal in $\Omega^1(\pi)$. The image $\operatorname{im} A$ of a Frobenius operator may not be an ideal in the space of evolutionary derivations although it is always a Lie subalgebra.

Thus the Frobenius operators A specify involutive distributions of vertical symmetries $\partial_{A(\cdot)} \in \mathrm{D}^v(J^\infty(\pi))$ of the horizontal Cartan distribution $\mathcal{C} = \langle \widehat{\partial/\partial x^i} \rangle \subset \mathrm{D}(J^\infty(\pi))$, which is Frobenius itself.

Remark 1 (The Lie algebroids)*.* A comment on the above definition is in order. Using one of equivalent definitions (see [21] and references therein), we recall that the *Lie algebroid* over a manifold M is a vector bundle $\Omega \to M$ whose sections $\Gamma\Omega$ are equipped with a Lie algebra structure $[\,,\,]_A$ and the morphism $A\colon \Omega \to TM$, called the *anchor*, such that the Leibnitz rule

$$[\psi_1, f \cdot \psi_2]_A = f \cdot [\psi_1, \psi_2]_A + \big(A(\psi_1) f\big) \cdot \psi_2 \tag{6}$$

holds for any $\psi_1, \psi_2 \in \Gamma\Omega$ and $f \in C^\infty(M)$. The anchor A maps the bracket $[\,,\,]_A$ between the sections of Ω to the natural Lie bracket $[\,,\,]$ on the tangent bundle. The Lie algebroids are conveniently described by the homological vector fields [1, 21], and this construction is further applied [2] to analysis of the QP-manifolds.

The Frobenius operators $A\colon \Omega^1(\pi) \to \varkappa(\pi)$ define the morphisms $\partial_{A(\cdot)}\colon \big(\Omega, [\,,\,]_A\big) \to \big(\mathfrak{g}, [\,,\,]\big)$ of the Lie algebras *but* the Leibnitz rule (6) is lost for either $f \in C^\infty(M^n)$ or $f \in C^\infty(J^\infty(\pi))$. However, this is a typical situation even in the Hamiltonian case when the Poisson bracket $\{f \cdot \mathcal{H}_1, \mathcal{H}_2\}_A$ is no longer a derivation w.r.t. f (the first structure for KdV is a counterexample). We further note that the evolutionary fields are π-vertical and thus are projected to zero under $\pi_{\infty,*}\colon \mathrm{D}(J^\infty(\pi)) \to TM$.

Hence we conclude that the Frobenius operators do specify the Lie algebroid structures over $J^\infty(\pi)$ such that $\pi\colon E^{n+m} \to M$ (not just the algebroids over M) in view that the loss of the derivation (6) is unavoidable.

Example 2.1. The most natural example of the Lie algebroids is given by an automorphism of the tangent bundle to a manifold. Let $\mathcal{E} \subset J^\infty(\pi)$ be a differential equation; consider the Lie algebra $\operatorname{sym} \mathcal{E}$ of its symmetries and set $\Omega := \operatorname{sym} \mathcal{E}$.

[3]In what follows, equation (7) specifies the case when the Jacobiator $J(\psi_1, \psi_2, \psi_3)$ for $[\,,\,]_A$ vanishes identically, not being a nontrivial element of $\ker A$ as in [29].

Then the *recursions* $R \in \operatorname{Rec}\mathcal{E}$ whose images are closed under the commutation are Frobenius. The Lie algebra structure $[\,,\,]_R$ transferred to the pre-image of R by

$$[R\varphi_1, R\varphi_2] = R\bigl([\varphi_1,\varphi_2]_R\bigr) \tag{5a$'$}$$

is generally different from the original commutators $[\,,\,]$ of evolutionary vector fields in the pre-image.[4] Any recursion $R = A_2 \circ A_1^{-1}$ factored by two Hamiltonian operators is frobenius but the bracket $[\,,\,]_R$ in its pre-image is generally nonlocal; the bracket $\{\{\,,\,\}\}_R$ is then reconstructed from $\{\{\,,\,\}\}_{A_2}$ using the chain rule (see Proposition 8 below). Another example of the Frobenius recursion that is not factored by the Hamiltonian operators is given in section 4, see (20).

The Frobenius recursions with the vanishing Nijenhuis torsion generate commutative Lie algebras of (higher) flows using a seed symmetry φ_0 if the φ_0-invariance $\ni_{\varphi_0}(R) = 0$ is valid. In this case, the brackets $[\,,\,]_R$ in the pre-images of R vanish.

Remark 2. The elements $\psi = \sum_{i=1}^m \psi^i \mathrm{d}_\mathcal{C} u^i$ of the space $\Omega^1(\pi) = \operatorname{Hom}_{C^\infty(M^n)}(\varkappa(\pi), \bar{\Lambda}^n(\pi))$ are $\mathbb{Z}$-graded objects as any differential forms are. Hence if π is a super-bundle with Grassmann-valued sections, then the theory is *bi-graded* [22]; the $\mathbb{Z}$- and $\mathbb{Z}_2$-gradings are different and must be never mixed.[5] Indeed, for any dependent variables u^i and u^j with parities $|u^i|_{\mathbb{Z}_2}$ and $|u^j|_{\mathbb{Z}_2}$, we have $|\mathrm{d}_\mathcal{C}|_\mathbb{Z} = 1$ and thus

$$\mathrm{d}_\mathcal{C} u^i\, \mathrm{d}_\mathcal{C} u^j = -(-1)^{|u^i|_{\mathbb{Z}_2}\cdot|u^j|_{\mathbb{Z}_2}} \mathrm{d}_\mathcal{C} u^j\, \mathrm{d}_\mathcal{C} u^i.$$

This implies that the Frobenius operators A are endowed with the proper grading $|A|_\mathbb{Z} = -1$ as $\ni_{A(\cdot)} \colon \Omega^1(\pi) \to \mathfrak{g}(\pi)$ produces 0-forms; the $\mathbb{Z}_2$-grading of A can be arbitrary.

Remark 3. In what follows, we confine ourselves to the local setting and thus consider Frobenius matrix-valued differential operators polynomial in total derivatives. Also, we consider the commutative case when the dependent variables in the jet bundle π are not $\mathbb{Z}_2$-graded and thus are permutable. However, the search for nonlocal Frobenius (super-)operators can be performed using standard techniques, see Remark 12 on p. 27.

Next, we say that a matrix operator A in total derivatives is *nondegenerate* if

$$\bigcap_\sigma \ker A_\sigma = \{0\}, \qquad A = \sum_\sigma A_\sigma \cdot D_\sigma. \tag{7}$$

From now on, we consider nondegenerate Frobenius operators. The coordinate-free definition of the nondegeneracy is that the linear operators A have trivial kernels on the ℓ^*-coverings over the jet bundle, see [14] for this construction.

Example 2.2. The scalar operator $A = u_x + \frac{1}{2}D_x$ is Frobenius such that the bracket induced in its inverse image is $\{\{p,q\}\}_A = p_x q - p q_x$. This operator factors higher

[4] The Frobenius recursion operators seem to provide a relevant extension of the R-matrix formalism to higher order differential operators; the parallel is as follows. Let $\mathfrak{g}$ be a Lie algebra with the Lie bracket $[\,,\,]$. The classical R-matrix [3, 13] is a linear map $R\colon \mathfrak{g} \to \mathfrak{g}$ that endows $\mathfrak{g}/\ker R$ with the second Lie product $[\,,\,]_R$; for any $a, b \in \mathfrak{g}$ one sets $[a,b]_R = [Ra, b] + [a, Rb]$. A sufficient condition for a zero order differential operator R to be an R-matrix is that R satisfies the Yang–Baxter equation $\mathrm{YB}(\alpha)$: $[Ra, Rb] = R\bigl([a,b]_R\bigr) - \alpha[a,b]$, $\alpha = 0, 1$. Now let R be a Frobenius recursion operator; then the YB(0)-type equation (5a$'$) is parameterized by the brackets $\{\{\,,\,\}\}_R$ rather than by the constants $\alpha \in \mathbb{R}$.

[5] We owe this remark to Yu. I. Manin (private communication).

symmetries in the commutative hierarchy of the potential modified KdV equation $u_t = -\frac{1}{2}u_{xxx}+u_x^3$ whose hierarchy consists of the Noether symmetries of the A_1-Toda equation $u_{xy} = \exp(2u)$, see section 5 and [16] for further details.

Example 2.3. A three-parametric class of scalar first order Frobenius operators $A = a(u,u_x)\,D_x + b(u,u_x)$ and their brackets $\{\{p,q\}\}_A = c(u,u_x)\cdot(pq_x - p_xq)$ are given through

$$a = \gamma(u), \quad b = \beta(u)\cdot u_x + \alpha\cdot\gamma(u), \quad c = -\beta(u), \tag{8}$$

where α is a constant and the functions β, γ are arbitrary. The proof is by direct calculation.

Example 2.4 (Hamiltonian operators)**.** Every Hamiltonian operator is Frobenius. Indeed, the Hamiltonianity criterion in Proposition 5 gives the formula for the bracket $\{\{\,,\,\}\}_A$ in the inverse image of a Hamiltonian operator A,

$$\{\{\psi_1,\psi_2\}\}_A = \ell^*_{A,\psi_1}(\psi_2). \tag{9}$$

In coordinates, for a Hamiltonian operator $A = \|\sum_\tau A^{\alpha\beta}_\tau D_\tau\|$ and $\psi_1,\psi_2 \in \Omega^1(\pi)$ we have

$$\{\{\psi_1,\psi_2\}\}^i_A = \sum_\sigma (-1)^\sigma \Bigl(D_\sigma \circ \Bigl[\sum_\tau D_\tau(\psi_1^\beta)\cdot\frac{\partial A^{\alpha\beta}_\tau}{\partial u^i_\sigma}\Bigr]\Bigr)(\psi_2^\alpha).$$

This demonstrates that the formalism of Frobenius operators is a true generalization of the Hamiltonian approach to nonlinear evolutionary PDE. The use of Frobenius operators can be a helpful intermediate step in the search and classification of the Hamiltonian structures. An advantage of this approach is that it is easier to solve first the equation (5) w.r.t. the operators A and filter out the skew-adjoint solutions rather than to solve the Jacobi identity in the form of (9). Thus, by a criterion contained in Proposition 5, a skew-adjoint Frobenius operator A is Hamiltonian iff the associated bracket $\{\{\,,\,\}\}_A$ is equal to the r.h.s. of (9). Also, we note that no restriction onto a differential equation $\mathcal{E} = \{u_t = A(\mathbf{E}(\mathcal{H}))\}$ is *a priori* required such that $A\colon \Omega^1(\mathcal{E}) \to \varkappa(\mathcal{E})$ is a pre-Hamiltonian operator in the terminology of [14].

Remark 4. A recent version [10, 26] of the Darboux theorem for $(1+1)$D evolutionary PDE implies that the Hamiltonian operators A for non-exceptional systems are transformed to $\mathrm{const}\cdot\mathrm{d}/\mathrm{d}x$. The bracket $\{\{\,,\,\}\}_A$ is then zero, which is readily seen from (9). Hence the actual statement of the Darboux theorem for PDE is that the "cocycle" $\{\{\,,\,\}\}_A$ can be trivialized for those Hamiltonian operators.

Formula (9) does not remain valid for arbitrary Frobenius operators, which are generally at least non-skew-adjoint. In appendix B we describe a procedure that reconstructs the bracket $\{\{\cdot,\cdot\}\}_A$ for an operator A whose closure of the image is established by exterior considerations; this happens, in particular, for the operators that factor symmetries of some ambient differential equations, see section 5.

Example 2.5 (Operators in the KdV weights)**.** Let us fix the weights $|u| = 2$, $|D_x| = 1$ that originate from the scaling invariance of the KdV equation $u_t = -\frac{1}{2}u_{xxx} + 3uu_x$ (we have $|D_t| = 3$). Using the method of undetermined coefficients, we performed the search for scalar Frobenius operators that are homogeneous w.r.t. the weights not greater than 7. We obtained the Poisson pencil $A_1^{\text{KdV}} = D_x$ and $A_2^{\text{KdV}} = -\frac{1}{2}D_x^3 + 2u\,D_x + u_x$, the generalizations D_x^{2n+1} of D_x, and the Hamiltonian operator

$$uu_{xxx} + 3uu_{xx}\,D_x + 3uu_x\,D_x^2 + u^2\,D_x^3.$$

Also, there are four non-skew-adjoint Frobenius operators

$$A_4^{(6)} = u^3 - u_x^2, \qquad A_5^{(6)} = 2u_x^2 - uu_{xx} - 2uu_x\,D_x + u^2\,D_x^2,$$
$$\{\{p,q\}\}_{A_4^{(6)}} = 2u_x\cdot(pq_x - p_xq), \qquad \{\{p,q\}\}_{A_5^{(6)}} = -2u_x\cdot(pq_x - p_xq) + u\cdot(pq_{xx} - p_{xx}q);$$
$$A_8^{(7)} = u_x^2\,D_x - 2uu_{xx}\,D_x - 4uu_xD_x^2 - 4u^2\,D_x^3, \qquad \{\{p,q\}\}_{A_8^{(7)}} = u^2\cdot(pq_x - p_xq);$$
$$A_9^{(7)} = -2u_xu_{xx} - u_x^2\,D_x, \qquad \{\{p,q\}\}_{A_9^{(7)}} = 8u_{xx}\cdot(pq_x - p_xq) + 2u_x\cdot(pq_{xx} - p_{xx}q).$$

Finally, we found the operators that contain arbitrary functions: $f(u)D_x^n$ and $f(u)u^2$ with the vanishing brackets and also

$$A_3 = f(u)u_x, \qquad \{\{p,q\}\}_{A_3} = f(u)\bigl(p_xq - pq_x\bigr);$$
$$A_4 = f(u)\bigl(u\,D_x - u_x\bigr), \qquad \{\{p,q\}\}_{A_4} = f(u)\bigl(pq_x - p_xq\bigr). \tag{10}$$

Formula (9) is not valid for any of these non-skew-adjoint operators.

Suppose a Frobenius operator is divisible by another operator; then their associated brackets are correlated; this property is valid in the Hamiltonian case although is not well known.

Proposition 8 (The chain rule, c.f. [31]). Suppose that the Frobenius operator A is nondegenerate, see (7), and the image of $A' = A\circ\square$ is closed w.r.t. the commutation. Then the brackets $\{\{\,,\,\}\}_A$ and $\{\{\,,\,\}\}_{A'}$ are related by the formula

$$\square(\{\{\xi_1,\xi_2\}\}_{A\circ\square}) = \partial_{A(\square(\xi_1))}(\square)(\xi_2) - \partial_{A(\square(\xi_2))}(\square)(\xi_1) + \{\{\square(\xi_1),\square(\xi_2)\}\}_A \tag{11}$$

for any ξ_1 and ξ_2.

This assertion provides a considerable simplification of the search for new Frobenius operators using known ones and allows to reconstruct the Koszul bracket in the inverse images of the Frobenius recursions factored by the Frobenius operators $A\colon \Omega^1(\pi) \to \varkappa(\pi)$. A weaker version of Proposition 8 was formulated in [31].

Proof. Suppose $\xi_1,\xi_2 \in \Omega^1(\pi)$ and $\psi_i = \square(\xi_i)$, $\varphi_i = A(\psi_1)$ for $i = 1,2$. We have

$$[\varphi_1,\varphi_2] = (A\circ\square)\bigl(\partial_{\varphi_1}(\xi_2) - \partial_{\varphi_2}(\xi_1) + \{\{\xi_1,\xi_2\}\}_{A\circ\square}\bigr). \tag{12a}$$

On the other hand, we recall that $\psi_i = \square(\xi_i)$ and deduce

$$\begin{aligned}[\varphi_1,\varphi_2] &= A\bigl(\partial_{\varphi_1}(\psi_2) - \partial_{\varphi_2}(\psi_1) + \{\{\psi_1,\psi_2\}\}_A\bigr)\\ &= (A\circ\square)\bigl(\partial_{\varphi_1}(\xi_2) - \partial_{\varphi_2}(\xi_1)\bigr) + A\bigl(\partial_{\varphi_1}(\square)(\xi_2) - \partial_{\varphi_2}(\square)(\xi_1) + \{\{\psi_1,\psi_2\}\}_A\bigr).\end{aligned} \tag{12b}$$

Now substract (12a) from (12b) and, in view of the nondegeneracy (7), cancel by the operator A, whence the assertion follows. $\square$

Example 2.6. Historically, the class of scalar Frobenius operators first regarded in view of Definition 1 was studied in [34]. In [31], Gelfand's symbolic method [9] was applied and it was argued that the class is infinite and contains the Frobenius operators

$$A_1^{(1)} = D_x;\quad A_1^{(2)} = D_x \circ (D_x + u);\quad A_1^{(3)} = A_1^{(2)} \circ (D_x + u);$$
$$A_1^{(4)} = A_1^{(3)} \circ (D_x + u),\qquad A_2^{(4)} = A_1^{(3)} \circ (D_x + 2u);$$

here the superscripts denote the differential order and the subscripts enumerate operators of equal order. Further, there are two operators

$$A_1^{(n)} = A_1^{(n-1)} \circ (D_x + u),\qquad A_2^{(n)} = A_2^{(n-1)} \circ (D_x + (n-2)u)$$

for any odd $n \geq 5$ and there are four operators

$$\begin{aligned} A_1^{(n)} &= A_1^{(n-1)} \circ (D_x + u), & A_3^{(n)} &= A_1^{(n-1)} \circ (D_x + 2u),\\ A_4^{(n)} &= A_2^{(n-1)} \circ (D_x + (n-3)u), & A_2^{(n)} &= A_2^{(n-1)} \circ (D_x + (n-2)u) \end{aligned} \tag{13}$$

for any even $n \geq 6$. These operators are homogeneous w.r.t. the weights $|u| = |D_x| \equiv 1$, hence the weights coincide with the differential orders.

None of these operators is Hamiltonian, so the brackets $\{\{\,,\,\}\}_{A_i^{(n)}}$ are reconstructed from the vanishing bracket for $A_1^{(1)}$ by using the chain rule (see Proposition 8); the first four of them are

$$\{\{p,q\}\}_{A_1^{(2)}} = p_x q - p q_x,$$
$$\{\{p,q\}\}_{A_1^{(3)}} = 2u(p_x q - p q_x) + p_{xx} q - p q_{xx},$$
$$\{\{p,q\}\}_{A_1^{(4)}} = 3(u^2 + u_x)(p_x q - p q_x) + 3u(p_{xx} q - p q_{xx}) + p_{xxx} q - p q_{xxx},$$
$$\{\{p,q\}\}_{A_2^{(4)}} = 6(u^2 + 2u_x)(p_x q - p q_x) + 6u(p_{xx} q - p q_{xx}) + 2(p_{xxx} q - p q_{xxx}) + p_x q_{xx} - p_{xx} q_x.$$

It remains unknown whether operators (13), which are factored to products of primitive first order operators with integer coefficients, exhaust all Frobenius operators with differential polynomial coefficients and homogeneous w.r.t. the weight $|u| = |D_x| = 1$.

Remark 5. One bracket $\{\{\,,\,\}\}_A$ can correspond to several Frobenius operators A. For example, the second structure $A_2^{\text{KdV}} = -\frac{1}{2}D_x^3 + 2uD_x + u_x$ for the KdV equation determines the bracket $\{\{p,q\}\}_{A_2^{\text{KdV}}} = pq_x - p_x q$, which is also induced by the operators (10) and by $A_1^{(2)} = D_x \circ (D_x + u)$, see Example 2.6. Actually, this Wronskian-based bracket (c.f. [18]) is scattered through the text, see Example 5.1. Hence there are fewer brackets $\{\{\,,\,\}\}_A$ than there are Frobenius operators A.

3. Compatibility of Frobenius operators

We analyze two types of compatibility of the Frobenius operators. The weak compatibility of the operators means that their linear combinations remain Frobenius and hence their 'individual' Koszul brackets are correlated. The strong compatibility of N operators determines involutive distributions of finite reduced dimension N in the Lie algebra of evolutionary vector fields; this extends the geometry of affine connections on super-manifids to the bi-differential Christoffel symbols, and a flat connection is then defined on the vector fields that belong to the distributions. Let us consider these aspects in more detail.

3.1. **The weak compatibility.** First recall that any linear combination of two compatible Hamiltonian operators is Hamiltonian by definition and hence Frobenius. Example 2.5 shows that a Hamiltonian operator can be decomposed to a sum of the Frobenius operators, e.g., $A_2^{\text{KdV}} = -\frac{1}{2}\cdot D_x^3 + 2\cdot uD_x + u_x$. The decomposition may not be unique owing to the existence of several linear dependent Frobenius operators that appear in the splittiong; indeed, A_2^{KdV} can be also obtained using $A_4 = uD_x - u_x$, see (10).

Definition 2. The Frobenius operators $A_1, \ldots, A_N$ are *weakly compatible* if their arbitrary linear combinations $A_{\vec{\lambda}} = \sum_{i=1}^N \lambda_i A_i$ are Frobenius for any $\vec{\lambda} \in \mathbb{RP}^N$. The operators are weakly compatible *at a point* $\vec{\lambda}_0 \in \mathbb{RP}^N$ if $A_{\vec{\lambda}_0}$ is Frobenius for the fixed linear combination.

Example 3.1. There are two classes of pairwise weakly compatible Frobenius scalar operators (8) of first order. The first type of pairs is $\gamma_2(u) = \text{const}\cdot\gamma_1(u)$ with any α_1, $\alpha_2 \in \mathbb{R}$ and arbitrary functions $\beta_1(u)$, $\beta_2(u)$. The second class is given by coinciding $\alpha_1 = \alpha_2 \in \mathbb{R}$ and arbitrary functions β_1, β_2, γ_1, and γ_2.

Proposition 9. The bracket induced in the pre-image of a linear combination $A_{\vec{\lambda}} = \sum_{i=1}^N \lambda_i A_i$ of weakly compatible Frobenius operators is

$$\{\![\,,\,]\!\}_{\sum_i \lambda_i A_i} = \sum_{i=1}^N \lambda_i \cdot \{\![\,,\,]\!\}_{A_i}.$$

The proof is by inspecting the coefficients of λ_i^2 in the quadratic polynomials in λ_i that appear in both sides of the equality $\bigl[A_{\vec{\lambda}}(p), A_{\vec{\lambda}}(q)\bigr] = A_{\vec{\lambda}}\bigl([p,q]_{A_{\vec{\lambda}}}\bigr)$ upon the Koszul bracket.

Corollary 10 (Infinitesimal deformations of Frobenius operators)**.** Two Frobenius operators are weakly compatible iff for any p, $q \in \Omega^1(\pi)$ one has

$$\bigl[B(p), A(q)\bigr] + \bigl[A(q), B(p)\bigr] = A\bigl([p,q]_B\bigr) + B\bigl([p,q]_A\bigr),$$

which is equivalent to the relation

$$\partial_{A(p)}(B)(q) + \partial_{B(p)}(A)(q) - \partial_{A(q)}(B)(p) - \partial_{B(q)}(A)(p) = A\bigl(\{\![p,q]\!\}_B\bigr) + B\bigl(\{\![p,q]\!\}_A\bigr).$$

Remark 6. The operators remain Frobenius when multiplied by a constant, therefore pass to the projective frame $\lambda \in \mathbb{RP}^N$ of $N \in \mathbb{N}$ Frobenius operators. Then in $\mathcal{C}\mathrm{Diff}(\Omega^1(\pi), \varkappa(\pi))$ there is a basis of the Frobenius operators which either are isolated points or which generate Frobenius cells with a nontrivial topology of attaching the simplexes together.

An illustration is given by Example 2.5: for $\vec{e}_1 = D_x^3$, $\vec{e}_2 = uD_x$, and $\vec{e}_3 = u_x$ the curve $A_2^{\text{KdV}} = (\lambda : 2 : 1)$ is Hamiltonian and the ray $A_4 = f(u)\cdot(0 : 1 : -1)$ is Frobenius.

3.2. **The strong compatibility.** Starting with a unique Frobenius operator A_1 and using the twisting construction (see Remark 10 on p. 20), one can easily obtain *two* Frobenius operators A_1, A_2 such that $[\operatorname{im} A_1, \operatorname{im} A_2] \subseteq \operatorname{im} A_1$ and hence the sum of the images is an involutive distribution. In the meantime, we impose further specification on the structure of the commutators for images of several Frobenius operators.

Definition 3. The Frobenius operators $A_1, \ldots, A_N$ are *strongly compatible* if the commutators of evolutionary fields in the images of any two of them belong to the sum of the images of all the N operators such that

$$[A_i(p), A_j(q)] = A_j\bigl(\ni_{A_i(p)}(q)\bigr) - A_i\bigl(\ni_{A_j(q)}(p)\bigr) + \sum_{k=1}^{N} A_k\bigl(\Gamma^k_{ij}(p,q)\bigr) \in \sum_{\ell=1}^{N} \operatorname{im} A_\ell. \quad (14)$$

In view of the functional arbitrariness of $p, q \in \Omega^1(\pi)$ in the pre-images of the operators, we say that the involutive distribution of evolutionary vector fields in the images of linearly independent operators $A_1, \ldots, A_N$ has *reduced dimension* (*rank*) N.

Let us formulate some properties of the bi-differential symbols Γ^k_{ij} determined by strongly compatible operators $A_1, \ldots, A_N$:

(1) For any $i, j, k \in [1, N]$ and $\psi_1, \psi_2 \in \Omega^1(\pi)$ of $\mathbb{Z}$-grading 1, we have
$$\Gamma^k_{ij}(\psi_1, \psi_2) = (-1)^{|\psi_1|_{\mathbb{Z}}\cdot|\psi_2|_{\mathbb{Z}}} \cdot (-1)^{|\psi_1|_{\mathbb{Z}_2}\cdot|\psi_2|_{\mathbb{Z}_2}} \Gamma^k_{ji}(\psi_2, \psi_1) \quad (15)$$
owing to the $\mathbb{Z}_2$-graded skew-symmetry of the commutators. Hence the symbols Γ^k_{ij} constitute a symmetric bi-differential affine connection on a graded manifold.

(2) The symbols Γ^k_{ij} are not uniquely defined; indeed, they are restricted by the conditions
$$\sum_k A_k\Bigl(\ni_{A_j(q)}(p)\delta^k_i - \ni_{A_i(p)}(q)\delta^k_j + \Gamma^k_{ij}(p,q)\Bigr) = 0. \quad (16)$$

(3) By construction, $\{\{\ ,\ \}\}_{A_i} = \Gamma^i_{ii}$ for any N and hence a Frobenius operator spans the distribution of reduced dimension one.

(4) If, additionally, two operators A_i and A_j are weakly compatible, then their 'individual' brackets are
$$\{\{p,q\}\}_{A_i} = \Gamma^j_{ij}(p,q) + \Gamma^j_{ji}(p,q) \text{ and } \{\{p,q\}\}_{A_j} = \Gamma^i_{ij}(p,q) + \Gamma^i_{ji}(p,q).$$

By (15), the bi-differential operators Γ^k_{ij} in (14) incorporate the symmetric Christoffel symbols of standard affine geometry as the zero order terms. Moreover, put

$$\nabla_{A_i(p)}\bigl(A_j(q)\bigr) := \bigl[A_i(p), A_j(q)\bigr];$$

then the Jacobi identity

$$\bigl[A_i(p), \bigl[A_j(q), A_k(\psi)\bigr]\bigr] + \bigl[A_j(q), \bigl[A_k(\psi), A_i(p)\bigr]\bigr] + \bigl[A_k(\psi), \bigl[A_i(p), A_j(q)\bigr]\bigr] = 0$$

for the vector fields in the sum of images of the stronly compatible operators is the flatness condition

$$\Bigl(\nabla_{A_i(p)} \circ \nabla_{A_j(q)} - \nabla_{A_j(q)} \circ \nabla_{A_i(p)} - \nabla_{[A_i(p), A_j(q)]}\Bigr)\bigl(A_k(\psi)\bigr) = 0 \quad \forall \psi \in \Omega^1(\pi).$$

Therefore we obtain a new non-Cartan flat connection of zero curvature on the fibre bundle of evolutionary derivations. The commutative hierarchies are the geodesics w.r.t. this connection as p, q move along the gradients $\mathbf{E}(\mathbf{c})$ of the Hamiltonians from the linear spans of descendants of the Casimirs $\mathbf{c}$.

Remark 7. Although the presence of the flat connection introduced above is self-evident, it allows to note that the Frobenius operators A determine a flat connection in the triple $\bigl(\Omega^1(\pi), \bigwedge^\bullet \mathfrak{g}(\pi), A\bigr)$ of two algebras and the morphism. The algebra structure on $\bigwedge^\bullet \mathfrak{g}(\pi)$ is the Schouten bracket; the multiplication on $\Omega^1(\pi)$ is the Lie algebra structure defined by the Koszul bracket, see (5). Using A, the action of $\Omega^1(\pi)$ on $\bigwedge^\bullet \mathfrak{g}(\pi)$ can be further extended to the action of $\bigwedge^\bullet \Omega^1(\pi)$.

A straightforward calculation shows that two operators (8) are strongly compatible iff they are proportional. We argue that this negative result as well as the apparent absence of the strong compatibility for the three Frobenius operators for the dispersionless 3-component Boussinesq equation (see Example 4.1 on p. 15) have a very deep motivation.

Indeed, by definition, the commutator (14) always contains the standard first two terms in the r.h.s., and they take values in the entire sum of images $\sum_k \operatorname{im} A_k$ for generic $p, q \in \Omega^1(\pi)$. Hence the commutation relations that determine the Lie-algebraic type of the involutive distribution in the sum of images of $A_1, \ldots, A_N$ depend on the linear subspace $S \subset \Omega^1(\pi)$ which we may choose ourselves such that some of the operators become restricted on their kernels (up to the condition (16)).

Example 3.2 (The Magri scheme)**.** The strong compatibility *can be* achieved on linear subspaces of the inverse images of Frobenius operators. This is precisely the case of the linear span of the 'gradients' of the Hamiltonians which descend from the Casimirs in the Magri scheme, see Theorem 7. Indeed, one has $\operatorname{im} A_2 \subset \operatorname{im} A_1$ whenever both Hamiltonian operators are restricted onto the descendants of the Casimirs for A_1, and hence the commutators (however, which vanish[6] by the same theorem) belong to the image of A_1. Thus the resolution (4) of the Magri scheme corresponds to the involutive distributions of reduced dimension two which are regarded as the operator-valued generalizations of the solvable two-dimensional Lie algebra with the relation $[a_1, a_2] = a_1$.

We conclude that the entire space $\Omega^1(\pi)$ corresponds to the flows in general position while the completely integrable bi-Hamiltonian hierarchies determine solvable (and moreover, commutative) distributions of reduced dimension $N = 2$.

Conjecture 11. There are Frobenius operators $A_1, \ldots, A_N$ such that, by choosing appropriate subspaces $S \subset \Omega^1(\pi)$ in their pre-images, the classical structural theory of (nilpotent, semi-simple, or solvable) Lie algebras of reduced dimension N can be recovered for the sum of images of the operators whenever they are restricted on S.

Remark 8 (Frobenius–Lie groups and algebras). The Frobenius recursions are strongly compatible if the commutators of evolutionary fields in their images belong to the sum of the images; the definition (14) of the Christoffel symbols remains the same. The main distinction of the recursions from the Frobenius operators $A\colon \Omega^1(\pi) \to \varkappa(\pi)$ is that the recursions always constitute an algebra (which is always a monoid but may not

[6]For example, the first and second Hamiltonian structures for KdV are not strongly compatible unless restricted onto some subspaces of their arguments. On the linear subspace of descendants of the Casimir $\int u\,dx$, we have $\operatorname{im} A_2 \subset \operatorname{im} A_1$ and, since the image of the Hamiltonian operator A_1 is closed, we have $[\operatorname{im} A_1, \operatorname{im} A_2] \subset \operatorname{im} A_1$. We emphasize that we do not exploit the commutativity of the flows.

be a group); the associative composition $\circ\colon \mathrm{Rec}\times\mathrm{Rec}\to\mathrm{Rec}$ determines the structural relations.

The two multiplications, the composition $\circ$ and the commutation (14) for strongly compatible Frobenius recursions are higher differential order extensions of the Lie group and algebra structures: one can either multiply or commute the same elements of the linear space Rec.

This conceptual step succeeds the replacement of partial derivatives w.r.t. coordinates and momenta by the Euler operator $\delta/\delta u = \partial/\partial u + \cdots$ in the Hamiltonian evolutionary PDE. In this sense, Proposition 3 proves the existence of the one-dimensional Frobenius–Lie algebras.

4. The bracket of conservation laws

Contrarily to the Hamiltonian case (see Proposition 6), Frobenius operators do not generally map $\Omega^1(\mathcal{E})$ to $\operatorname{sym}\mathcal{E}$ for any explicitly defined equation $\mathcal{E}$; a class of Frobenius operators that map $\Omega^1(\mathcal{E})$ to symmetries $\operatorname{sym}\mathcal{E}'$ of *different* equations related by substitutions $\mathcal{E}'\to\mathcal{E}$ will be considered in sec. 5. In this section we demonstrate that the operators $A\colon\Omega^1(\mathcal{E})\to\operatorname{sym}\mathcal{E}$ which do map the appropriate spaces for evolutionary systems induce the skew-symmetric brackets between the conservation laws $\eta\in\bar{H}^{n-1}(\mathcal{E})$ and between their generating sections $\Omega^1(\mathcal{E})$. If the operator A is not Hamiltonian (c.f. equation (1) and Lemma 6), the new bracket $[\![\,,\,]\!]_A$ on $\Omega^1(\mathcal{E})$ is different from the Koszul bracket defined in (5) because $[\![\,,\,]\!]_A$ may not satisfy the Jacobi identity. In Example 4.1 we consider weakly compatible *Frobenius* operators that map $\Omega^1(\mathcal{E})\to\operatorname{sym}\mathcal{E}$ for the dispersionless 3-component Boussinesq equation (18).

Let $\mathcal{E}$ be an evolutionary system and $\eta=\rho\,\mathrm{d}x+\cdots\in\bar{\Lambda}^{n-1}(\mathcal{E})$ be a conserved current. Then its generating section $\psi_\eta=\mathbf{E}(\rho)\in\Omega^1(\mathcal{E})$ is the Euler derivative by Lemma 1. For any $\varphi\in\operatorname{sym}\mathcal{E}$, the current $\partial\!\!\!\!\ni_\varphi(\eta)$ is obviously conserved on $\mathcal{E}$ and its generating section $\mathbf{E}(\ni_\varphi(\rho))$ equals $\ni_\varphi(\psi)+\ell^*_\varphi(\psi)$ by Lemma 2.

Suppose further that there is an operator $A\colon\Omega^1(\mathcal{E})\to\operatorname{sym}\mathcal{E}$; operators of this type are referred as *pre-Hamiltonian* in [14, 15]. Then for any two conserved currents $\eta_1,\eta_2\in\bar{\Lambda}^{n-1}(\mathcal{E})$ we set[7]

$$\langle\!\langle\eta_1,\eta_2\rangle\!\rangle_A \stackrel{\text{def}}{=} \ni_{A(\psi_1)}(\eta_2)-\ni_{A(\psi_2)}(\eta_1),$$
$$\langle\!\langle\,,\,\rangle\!\rangle_A\colon\bar{\Lambda}^{n-1}(\mathcal{E})\times\bar{\Lambda}^{n-1}(\mathcal{E})\to\bar{\Lambda}^{n-1}(\mathcal{E}),\tag{17a}$$

and

$$[\![\psi_1,\psi_2]\!]_A \stackrel{\text{def}}{=} \ni_{A(\psi_1)}(\psi_2)-\ni_{A(\psi_2)}(\psi_1)+\ell^*_{A(\psi_1)}(\psi_2)-\ell^*_{A(\psi_2)}(\psi_1),$$
$$[\![\,,\,]\!]_A\colon\Omega^1(\mathcal{E})\times\Omega^1(\mathcal{E})\to\Omega^1(\mathcal{E}).\tag{17b}$$

Recall that $\{\mathcal{H}_1,\mathcal{H}_2\}_A=-\ni_{A(\mathbf{E}(\mathcal{H}_1))}(\mathcal{H}_2)$; for a Hamiltonian operator A we have $\langle\!\langle\mathcal{H}_1,\mathcal{H}_2\rangle\!\rangle_A=-2\{\mathcal{H}_1,\mathcal{H}_2\}_A$. Therefore formula (17b) gives an alternative way to calculate the bracket $\{\{\,,\,\}\}_A$, see (9); the equivalence between (9) and (17b) implies nontrivial identities even for the simplest scalar Hamiltonian operator D_x.

[7] The arising algebra $(\bar{H}^{n-1}(\mathcal{E}),[\![\,,\,]\!]_A)$ of conservation laws is a deformation of the Lie algebra of the Hamiltonians through a pre-Hamiltonian operator $A\colon\Omega^1(\mathcal{E})\to\operatorname{sym}\mathcal{E}$. This algebra could be called the "current algebra" if the term were not already in use.

However, for a non-Hamiltonian operator A the bilinear bracket $[\![\,,\,]\!]_A$ on $\Omega^1(\mathcal{E})$ remains skew-symmetric but, generally, the Jacobi identity is lost for it. So, both the Koszul bracket $[\,,\,]_A$ and the new bracket $[\![\,,\,]\!]_A$ lift to the Poisson bracket $\{\,,\,\}_A$ on the space $\bar{H}^n(\pi)$ in the Hamiltonian case, while for a non-Hamiltonian Frobenius operator that maps $\Omega^1(\mathcal{E}) \to \operatorname{sym}\mathcal{E}$ the two brackets on $\Omega^1(\mathcal{E})$ are different.

Example 4.1. Consider the dispersionless 3-component Boussinesq system [11]

$$w_t = u_x, \quad u_t = ww_x + v_x, \quad v_t = -uw_x - 3u_x w, \tag{18}$$

which is equivalent to the Benney–Lax equation. In [15], two pre-Hamiltonian matrix operators $A_1, A_2\colon \Omega^1(\mathcal{E}) \to \operatorname{sym}\mathcal{E}$ were found for (18) using a calculation on the ℓ^*-covering and in view of the scaling invariance of the system. These operators are

$$A_1 = \begin{pmatrix} ww_x + v_x & -3u_x w - uw_x & u_x \\ -3u_x w - uw_x & -3w^2 w_x - 4v_x w - uu_x & v_x \\ u_x & v_x & w_x \end{pmatrix} \tag{19a}$$

and

$$A_2 = \frac{1}{2}\begin{pmatrix} (2w^2 + 4v)\,D_x + ww_x + v_x & -11uw\,D_x - (2u_x w + 8uw_x) & 3u\,D_x \\ -11uw\,D_x - 3u_x w - uw_x & h_1\,D_x + h_0 & 4v\,D_x \\ 3u\,D_x + u_x & 4v\,D_x + 2v_x & 2w\,D_x \end{pmatrix}, \tag{19b}$$

here $h_0 = -(2uu_x + 8vw_x + 4v_x w + 6w^2 w_x)$ and $h_1 = -(3u^2 + 16vw + 6w^3)$. The right-hand side of system (18) belongs to the image of each operator: one has $(u_t,\, v_t,\, w_t) = A_1(1,0,0)^T = A_2(1,0,0)^T$.

We note that these two operators are Frobenius.

The respective u-, v-, and w-components of the skew-symmetric bracket $\{\{\,,\,\}\}_{A_1}$ in the inverse image of A_1 are the bi-differential operators (see Example B.1 on p. 26):

$$\begin{pmatrix} 0 & 3w\cdot(\mathbf{1}\otimes D_x - D_x\otimes\mathbf{1}) + 2w_x\cdot\mathbf{1}\otimes\mathbf{1} & D_x\otimes\mathbf{1} - \mathbf{1}\otimes D_x \\ 3w\cdot(\mathbf{1}\otimes D_x - D_x\otimes\mathbf{1}) - 2w_x\cdot\mathbf{1}\otimes\mathbf{1} & u\cdot(\mathbf{1}\otimes D_x - D_x\otimes\mathbf{1}) & 0 \\ D_x\otimes\mathbf{1} - \mathbf{1}\otimes D_x & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} D_x\otimes\mathbf{1} - \mathbf{1}\otimes D_x & 0 & 0 \\ 0 & 4w\cdot(\mathbf{1}\otimes D_x - D_x\otimes\mathbf{1}) & D_x\otimes\mathbf{1} - \mathbf{1}\otimes D_x \\ 0 & D_x\otimes\mathbf{1} - \mathbf{1}\otimes D_x & 0 \end{pmatrix},$$

$$\begin{pmatrix} w\cdot(D_x\otimes\mathbf{1} - \mathbf{1}\otimes D_x) & u\cdot(\mathbf{1}\otimes D_x - D_x\otimes\mathbf{1}) - 2u_x\cdot\mathbf{1}\otimes\mathbf{1} & 0 \\ u\cdot(\mathbf{1}\otimes D_x - D_x\otimes\mathbf{1}) + 2u_x\cdot\mathbf{1}\otimes\mathbf{1} & 3w^2\cdot(\mathbf{1}\otimes D_x - D_x\otimes\mathbf{1}) & 0 \\ 0 & 0 & D_x\otimes\mathbf{1} - \mathbf{1}\otimes D_x \end{pmatrix}.$$

The notation means that the differential operators standing in the first and second tensor factors act on the respective arguments of the coupling $\langle\psi_1 \mid \{\{\,,\,\}\}_A \mid \psi_2\rangle$.

The components of the bracket $\{\!\{\,,\,\}\!\}_{A_2}$ associated with the operator A_2 are

$$\begin{pmatrix} 0 & 3w\cdot\mathbf{1}\otimes D_x - 2w\,D_x\otimes\mathbf{1} + 2w_x\cdot\mathbf{1}\otimes\mathbf{1} & -\mathbf{1}\otimes D_x \\ 2w\cdot\mathbf{1}\otimes D_x - 3w\,D_x\otimes\mathbf{1} - 2w_x\cdot\mathbf{1}\otimes\mathbf{1} & 2u\cdot(\mathbf{1}\otimes D_x - D_x\otimes\mathbf{1}) & 0 \\ D_x\otimes\mathbf{1} & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} D_x\otimes\mathbf{1} - \mathbf{1}\otimes D_x & 0 & 0 \\ 0 & 4w\cdot(\mathbf{1}\otimes D_x - D_x\otimes\mathbf{1}) & -2\cdot\mathbf{1}\otimes D_x \\ 0 & 2D_x\otimes\mathbf{1} & 0 \end{pmatrix},$$

$$\begin{pmatrix} w\cdot(D_x\otimes\mathbf{1} - \mathbf{1}\otimes D_x) & u\cdot\mathbf{1}\otimes D_x - 8u\,D_x\otimes\mathbf{1} - 2u_x\cdot\mathbf{1}\otimes\mathbf{1} & 0 \\ 2u_x\cdot\mathbf{1}\otimes\mathbf{1} + 8u\cdot\mathbf{1}\otimes D_x - u\,D_x\otimes\mathbf{1} & (8v+6w^2)\cdot(\mathbf{1}\otimes D_x - D_x\otimes\mathbf{1}) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The sums of the third and fourth terms in each component of the non-Koszul bracket (17b) for the operator A_1 are equal to the standard difference of the first two and do not produce other addends. The same terms for the bracket $[\![\vec{p},\vec{q}]\!]_{A_2}$ are

$$\begin{aligned} [\![\vec{p},\vec{q}]\!]^u_{A_2} &= \ldots + 3u(p^v q^v_x - p^v_x q^v) + 5w(q^v_x p^u - p^v_x q^u) + 6w(p^v q^u_x - q^v p^u_x) \\ &\quad + 3w_x(q^v p^u - p^v q^u) + 2(p^u_x q^w - 2q^u_x p^w) + q^u p^w_x - p^u q^w_x, \\ [\![\vec{p},\vec{q}]\!]^v_{A_2} &= \ldots + 8w(p^v q^v_x - p^v_x q^v) + 3(p^v_x q^w - 3q^v_x p^w) + q^v p^w_x - p^v q^w_x, \\ [\![\vec{p},\vec{q}]\!]^w_{A_2} &= \ldots + 9w^2(q^v_x p^v - q^v p^v_x) + 9u(q^v_x p^u - q^u p^v_x) + 8v(q^v_x p^v - q^v p^v_x) \\ &\quad + 3u_x(q^v p^u - q^u p^v) + 2w(q^u p^u_x - q^u_x p^u) + 2u(q^u_x p^v - q^v u p^u_x) \\ &\quad + 2(p^u_x q^u - p^u q^u_x) + p^w_x q^w - q^w_x p^w. \end{aligned}$$

Finally, we recall that system (18) is bi-Hamiltonian. Its first Hamiltonian structure is given by the operator

$$A_0 = \begin{pmatrix} D_x & 0 & 0 \\ 0 & -4wD_x - 2w_x & D_x \\ 0 & D_x & 0 \end{pmatrix},$$

which is of course Frobenius. The second Hamiltonian operator for (18) is the linear combination $\frac{1}{2}A_1 + A_2$. The brackets in the inverse images of A_0 and $\frac{1}{2}A_1 + A_2$ are obtained using formula (9), which is not valid for each of the operators A_1 and A_2 separately; for A_0, the u- and v- components of $\{\!\{\,,\,\}\!\}_{A_0}$ equal zero and $\{\!\{\vec{p},\vec{q}\}\!\}^w_{A_0} = 2\bigl(p^v q^v_x - p^v_x q^v\bigr)$.

The Hamiltonian pair $(A_0, \frac{1}{2}A_1 + A_2)$ is the Poisson pencil. We discover that the three Frobenius operators A_0, A_1, and A_2 are weakly compatible such that any linear combination $\sum_{i=0}^{2}\lambda_i A_i$ is Frobenius again. If no restriction is made upon the subspace of $\Omega^1(\mathcal{E})$ and thus the arguments of A_k are generic, the commutators $[\operatorname{im} A_i, \operatorname{im} A_j]$, $0 \le i < j \le 2$, are *not* decomposed using (14) w.r.t. the images of the *two* operators A_i and A_j themselves. The problem of decomposition of the commutators w.r.t. the whole set of the mappings $\Omega^1(\mathcal{E}) \to \operatorname{sym}\mathcal{E}$ for (18) is huge and will be discussed elsewhere.

The recursion[8]

$$R = A_2 \circ A_1^{-1} \tag{20}$$

[8] The operator A_2 is 'first' and A_1 is 'second' with the lexicographical order $w \prec u \prec v$ of their components in [15]. The Koszul bracket for the nonlocal Frobenius recursion $R^{-1} = A_1 \circ A_2^{-1}$ is nonlocal unless R^{-1} is restricted onto the image of the first-order operator A_2.

for (18) is Frobenius. Indeed, its image is contained in the image of the Frobenius operator A_2 whose induced Koszul bracket $[\,,\,]_{A_2}$ is pushed forward by (11) to $[\,,\,]_R$ on $\operatorname{sym}\mathcal{E}$ using the zero-order operator A_1 with the local inverse.

5. Factorizations of symmetries of Liouville-type systems

In this section we describe an infinite class of Frobenius operators $\square$ and the brackets $\{\{\,,\,\}\}_\square$ induced by them. These operators appear in factorizations of symmetries of the hyperbolic Liouville-type Euler–Lagrange nonlinear systems [4, 32, 33, 34].

To start with, we extend the collection of known Frobenius operators with the one that factors point symmetries of the non-evolutionary $(2+1)$-dimensional A_∞-Toda equation.

Example 5.1 ([19])**.** The generators of the point symmetry algebra for the 'heavenly' Toda equation $u_{xy} = \exp(-u_{zz})$ have the form $\varphi^x = \hat{\square}^x\bigl(\phi(x)\bigr)$ or $\varphi^y = \hat{\square}^y\bigl(\phi(y)\bigr)$, where $\phi \in C^\infty(\mathbb{R})$ and

$$\hat{\square}^x = u_x + \tfrac{1}{2}z^2\,D_x,\quad \hat{\square}^y = u_y + \tfrac{1}{2}z^2\,D_y = (x\leftrightarrow y)\bigl(\hat{\square}^x\bigr). \tag{21}$$

Clearly, the commutator of any two point symmetries of x- or y-type is a point symmetry again such that the action of $\hat{\square}$'s on the spaces of the free functional parameters is given by the Wronskian, $\{\{\phi_1,\phi_2\}\}_{\hat{\square}} = \phi_1\cdot(\phi_2)_x - (\phi_1)_x\cdot\phi_2$.

The operators (21) factor the symmetry flows $u_t = \hat{\square}(\phi)$ on the heavenly equation ambient with respect to them. This approach to constructing Frobenius operators is very productive.

Definition 4 ([34])**.** A *Liouville-type system* $\mathcal{E}_{\mathrm{L}}$ is a system $\{u_{xy} = F(u, u_x, u_y; x, y)\}$ of hyperbolic equations which possesses the *integrals* $w_1, \ldots, w_r$; $\bar{w}_1, \ldots, \bar{w}_{\bar{r}} \in C^\infty(\mathcal{E}_{\mathrm{L}})$ such that the relations $D_y\big|_{\mathcal{E}_{\mathrm{L}}}(w_i) \doteq 0$ and $D_x\big|_{\mathcal{E}_{\mathrm{L}}}(\bar{w}_j) \doteq 0$ hold by virtue ($\doteq$) of $\mathcal{E}_{\mathrm{L}}$ and all conservation laws for $\mathcal{E}_{\mathrm{L}}$ are of the form $\int f(x,[w])\,\mathrm{d}x \oplus \int g(y,[\bar{w}])\,\mathrm{d}y$.

Example 5.2. The 2D Toda lattices associated with semi-simple Lie algebras [24] constitute an important class $u^i_{xy} = \exp\bigl[(Ku)^i\bigr]$ of Liouville-type systems, here $u = (u^1,\ldots,u^m)$. They possess [32] the complete sets of $2m$ integrals $w_1,\ldots,w_m$; $\bar{w}_1,\ldots,\bar{w}_m$ and are integrable iff K is the Cartan matrix. This class is covered by the anzats in Proposition 12 below.

Remark 9. Consider the Liouville-type systems that possess the complete sets of $2m$ integrals: $r = m$ and $\bar{r} = m$. By sligtly narrowing the class of these equations, let us consider the systems $\mathcal{E}_{\mathrm{L}}$ whose general solutions are parameterized by arbitrary functions $f^1(x)$, …, $f^m(x)$ and $g^1(y)$, …, $g^m(y)$. These equations are represented as the diagrams [17]

$$\bigoplus_{i=1}^{m} J^\infty(\pi^x)\oplus J^\infty(\pi^y) \xrightarrow{\text{sol}} \mathcal{E}_{\mathrm{L}} \xrightarrow{\text{int}} \bigoplus_{i=1}^{m} J^\infty(\pi^x)\oplus J^\infty(\pi^y),$$

where π^x and π^y are the trivial fibre bundles $\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that f, w and g, $\bar{w}$ determine their sections, respectively; the first arrow is given by the formulas for exact solutions and the second arrow is determined by the integrals.

The (higher) symmetries φ of the Liouville-type systems $\mathcal{E}_{\mathrm{L}}$ are factored by differential operators $\square$, $\bar{\square}$ that originate from the integrals for $\mathcal{E}_{\mathrm{L}}$: we have

$$\varphi = \square\bigl(\phi(x,[w])\bigr), \qquad \bar{\varphi} = \bar{\square}\bigl(\bar{\phi}(y,[\bar{w}])\bigr), \tag{22}$$

where the sections ϕ and $\bar{\phi}$ are arbitrary. In what follows, we show that some of these factoring operators are Frobenius.

Example 5.3 (e.g., [34])**.** The integrals of the scalar Liouville equation $\mathcal{U}_{xy} = \exp(2\mathcal{U})$ are

$$w = \mathcal{U}_{xx} - \mathcal{U}_x^2 \quad \text{and} \quad \bar{w} = \mathcal{U}_{yy} - \mathcal{U}_y^2. \tag{23}$$

The Frobenius operators

$$\square = \mathcal{U}_x + \tfrac{1}{2}D_x = \tfrac{1}{2}D_x^{-1} \circ \bigl(\ell_w^{(\mathcal{U})}\bigr)^* \tag{24}$$

and $\bar{\square} = \mathcal{U}_y + \frac{1}{2}D_y$ factor (higher and Noether's) symmetries of this Euler–Lagrange equation. The bracket induced in the inverse image of $\square$ is $\{\{p,q\}\}_\square = p_x q - p q_x$ and similarly for $\bar{\square}$.

Example 5.4. Consider the parametric extension of the scalar Liouville equation,

$$u_{xy} = \exp(2u)\cdot\sqrt{1+4\varepsilon^2 u_x^2}. \tag{25}$$

This equation is ambient w.r.t. the hierarchy of Gardner's deformation of the potential modified KdV equation, see [17]. The contraction $\mathcal{U} = \mathcal{U}(\varepsilon,[u(\varepsilon)])$ from (25) to the non-extended equation $\mathcal{U}_{xy} = \exp(2\mathcal{U})$ is $\mathcal{U} = u + \frac{1}{2}\operatorname{arcsinh}(2\varepsilon u_x)$; it determines the third order integral for (25) using the one at $\varepsilon = 0$, see Example 5.3. However, the regularized minimal integral of second order for (25) is

$$w = \bigl(1 - \sqrt{1+4\varepsilon^2u_x^2}\bigr)/2\varepsilon^2 + u_{xx}\big/\sqrt{1+4\varepsilon^2u_x^2}; \tag{26}$$

such that all other x-integrals for (25) are differential functions of (26). The second integral for (25) is $\bar{w} = u_{yy} - u_y^2 - \varepsilon^2\cdot\exp(4u) \in \ker D_x\big|_{\mathcal{E}}$. The operators $\bar{\square} = u_y + \frac{1}{2}D_y$ and

$$\square = \tfrac{1}{2}(1+4\varepsilon^2u_x^2 - 2\varepsilon^2u_{xx})\cdot D_x + u_x + 4\varepsilon^2u_x^3 - 2\varepsilon^2u_{xxx} + \frac{12\varepsilon^4u_xu_{xx}^2}{1+4\varepsilon^2u_x^2} \tag{27}$$

factor symmetries (22) of (25). We emphasize that operators in the family (27) assign *higher* symmetries $\varphi = \square\bigl(\phi(x)\bigr)$ of (25) to functions on the base of the jet bundle whenever $\varepsilon \neq 0$, while the operator $\bar{\square}$ always determines point symmetries $\bar{\varphi} = \bar{\square}\bigl(\bar{\phi}(y)\bigr)$.

Both operators $\square$ and $\bar{\square}$ are Frobenius. The bracket $\{\{p,q\}\}_{\bar{\square}} = p_y q - p q_y$ for $\bar{\square}$ is familiar; the bracket induced in the inverse image of $\square$ is calculated in appendix B. The surprisingly high differential orders of $\{\{\,,\,\}\}_\square$ with respect to its arguments and coefficients is motivated by the presence of higher order derivatives of u in (27).

The scalar Liouville-type equations were studied in [34] and their symmetries have been further analyzed in [4]. The operators $\square$ that factor Noether's symmetries of the Euler–Lagrange Liouville-type systems $u_{xy} = F(u;x,y)$ and the brackets $\{\{\,,\,\}\}_\square$ induced by the commutation in their inverse images are constructed in [16]; in what follows, we discuss this procedure in more detail. The general case $u_{xy} = F(u,u_x,u_y;x,y)$ of the Euler–Lagrange Liouville-type systems is considered in [33], where a version of

Theorem 13 below is formulated; however, no method for reconstructing the brackets $\{\{\,,\,\}\}_\Box$ is described there.

Now we construct a class of Frobenius operators $\Box$ associated with the factorizations of symmetries of the Euler–Lagrange Liouville-type systems; moreover, we obtain explicitly the brackets $\{\{\,,\,\}\}_\Box$ in the inverse images of $\Box$. This is done by the following argument.

Proposition 12 ([16])**.** Let κ be a nondegenerate symmetric constant $(m\times m)$-matrix. Suppose that $\mathcal{L} = \iint L\,\mathrm{d}x\mathrm{d}y$ with the density $L = -\frac{1}{2}\sum_{i,j}\kappa_{ij}u^i_x u^j_y - H_{\mathrm{L}}(u;x,y)$ is the Lagrangian of a Liouville-type equation $\mathcal{E}_{\mathrm{L}} = \{\mathbf{E}(\mathcal{L}) = 0\}$. Let $\mathfrak{m} = \partial L/\partial u_y$ be Dirac's momenta [5] and $w(\mathfrak{m}) = (w^1,\ldots,w^r)$ be the integrals for $\mathcal{E}_{\mathrm{L}}$ that belong to the kernel of $D_y\big|_{\mathcal{E}_{\mathrm{L}}}$. Then the adjoint linearization

$$\Box = \left(\ell^{\mathfrak{m}}_w\right)^* \tag{28}$$

of the integrals w.r.t. the momenta factors Noether's symmetries $\varphi_{\mathcal{L}}$ of $\mathcal{E}_{\mathrm{L}}$, which are given by

$$\varphi_{\mathcal{L}} = \Box\bigl(\delta\mathcal{H}/\delta w\bigr) \tag{29}$$

for any $\mathcal{H} = \int H(x,[w])\,\mathrm{d}x$.

Sketch of the proof. The assertion follows from

- the structure $\psi = -D_x^{-1}\bigl(\mathbf{E}_u(\mathcal{H})\bigr)$ of the generating sections of conservation laws for hyperbolic systems,
- the correlation
$$\frac{\delta}{\delta u} = \left(\ell^{(u)}_{\mathfrak{m}}\right)^* \circ \left(\ell^{(\mathfrak{m})}_w\right)^* \circ \frac{\delta}{\delta w}$$
between the variational derivatives w.r.t. u and w, and
- the correlation $\psi = \kappa\varphi_{\mathcal{L}}$ between the generating sections of conservation laws and the Noether symmetries.

□

Corollary 13. Under the assumptions and notation of Proposition 12, the section

$$\varphi = \Box\bigl(\vec{\phi}(x,[w])\bigr)$$

is a (higher) symmetry of the Liouville-type equation $\mathcal{E}_{\mathrm{L}}$ for any $\vec{\phi} = {}^t(\phi^1,\ldots,\phi^r)$.

Proof. The proof is standard and analogous to the one for Lemma 4 with the only alteration in the jet space at hand. Consider the jet bundle over the fiber bundle with the base $\mathbb{R}\ni x$ and the fibers $\mathbb{R}^r$ with coordinates $w^1,\ldots,w^r$. By Proposition 12, the statement of the theorem is valid for any $\vec{\phi}$ in the image of the variational derivative; obviously, the image contains all r-tuples $\vec{\phi}$ whose components $\psi^i(x)\in C^\infty(\mathbb{R})$ are functions on the base of the new jet bundle. Now recall that $\Box$ is an operator in total derivatives whose action on functions on the jet space is defined through their restriction onto the jets of sections $w = \vec{\phi}(x)$, whence the assertion follows. □

Theorem 14. *The operators* (28) *are Frobenius if the integrals* w *are minimal.*

Proof. The commutator of two Noether's symmetries is a Noether symmetry $\varphi_{\mathcal{L}}$ and hence the conservation law corresponds to it. The geometry of the Euler–Lagrange Liouville-type equations $\mathcal{E}_{\mathrm{L}} \simeq \{F \equiv \kappa^{-1}\mathbf{E}_u(\mathcal{L}) = 0\}$ is such that the conservation law is represented by an integral, $D_y(H) \doteq 0$ on $\mathcal{E}_{\mathrm{L}}$. By assumption, the integrals w that specify the symmetries $\varphi' = \square\big(\vec{\phi}'[w]\big)$ and $\varphi'' = \square\big(\vec{\phi}''[w]\big)$ are minimal meaning that *any* integral is a differential function of them, hence $H = H(x, [w])$. Let the gauge of the minimal integrals be fixed. The the factorization (29) for the new symmetry $\varphi_{\mathcal{L}}$ follows from Proposition 12. $\square$

Theorem 14 assigns Frobenius operators to the Euler–Lagrange Liouville-type systems. By Remark 9, the Frobenius operators $\square$ associated with the exactly solvable Liouville-type systems $\mathcal{E}_{\mathrm{L}}$ can be understood as operators on the direct sums of the infinite jet spaces.

Remark 10 (The twisting construction). Let $\square$ be the Frobenius operator (28) determined by the minimal integrals w of an Euler–Lagrange Liouville-type system. Then one can produce infinitely many Frobenius operators ∇ such that $[\operatorname{im}\square, \operatorname{im}\nabla] \subseteq \operatorname{im}\square$. Indeed, let $\tilde{w}(w)$ be an equivalent basis of the integrals obtained by a non-degenerate transformation (the twisting). Then the relation

$$\frac{\delta}{\delta w} = \big(\ell_{\tilde{w}}^{(w)}\big)^* \circ \frac{\delta}{\delta \tilde{w}}$$

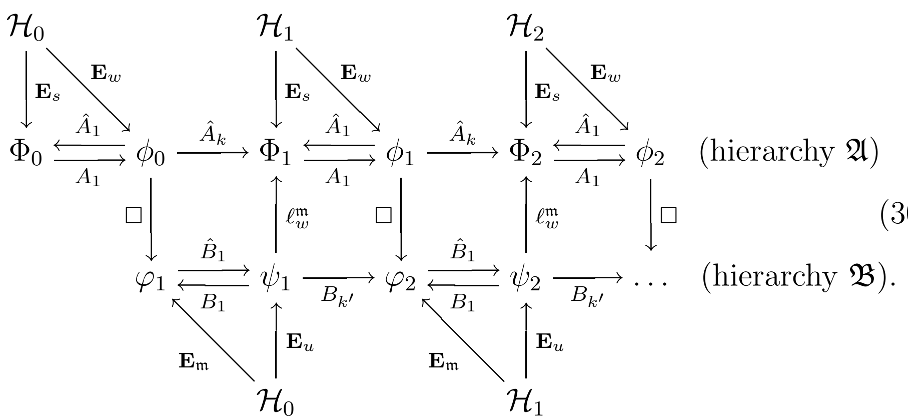
implies that $\nabla = \square \circ \big(\ell_{\tilde{w}}^{(w)}\big)^*$ is Frobenius, acting on $\big(\delta/\delta\tilde{w}\big)\mathcal{H}\big[w[\tilde{w}]\big]$, and also $\operatorname{im}\nabla \subseteq \operatorname{im}\square$ (actually, the images are isomorphic). Therefore,

$$[\square(\psi), \nabla(\chi)] = \square\Big(\ni_{\square(\psi)}\big(\big(\ell_{\tilde{w}}^{(w)}\big)^*(\chi)\big) - \ni_{\nabla(\chi)}(\psi) + \{\{\psi, \big(\ell_{\tilde{w}}^{(w)}\big)^*(\chi)\}\}_{\square}\Big),$$

although the structure of the three terms in the r.h.s. of (14) is not preserved.

The integrals $w[\mathfrak{m}]$ of the Euler–Lagrange Liouville-type systems $\mathcal{E}_{\mathrm{L}}$ determine the Miura substitutions from the commutative Hamiltonian hierarchies $\mathfrak{B}$ of the Noether symmetry flows on $\mathcal{E}_{\mathrm{L}}$ to the completely integrable hierarchies $\mathfrak{A}$ of higher symmetries of the multi-component wave equations; the potential modified KdV transformed to KdV by (23) gives a natural example, which is discussed in detail in [16].

The hierarchies share the Hamiltonians $\mathcal{H}_i[\mathfrak{m}] = \mathcal{H}\big[w[\mathfrak{m}]\big]$ through the Miura substitution. The Hamiltonian structures for the Magri schemes of $\mathfrak{A}$ and $\mathfrak{B}$ are correlated by the diagram



(30)

The first Hamiltonian structure $\hat{B}_1 = (\ell^u_{\mathfrak{m}})^*$ for $\mathfrak{B}$ originates from the differential constraint $\mathfrak{m} = \partial L/\partial u_y$ upon the coordinates u and the momenta $\mathfrak{m}$ for $\mathcal{E}_{\mathrm{L}}$. The higher Hamiltonian operator $\hat{A}_k$ for $\mathfrak{A}$ is factored by the Frobenius operator $\square$,

$$\hat{A}_k = \square^* \circ \hat{B}_1 \circ \square, \qquad k = k(\square, \mathfrak{m}) \geq 2.$$

The bracket $\{\{\,,\,\}\}_\square$ in the inverse image of $\square$ is equal to the bracket $\{\{\,,\,\}\}_{\hat{A}_k}$ induced by the Hamiltonian operator $\hat{A}_k$ and which is hence calculated by formula (9).

The multi-component wave equation $s_{xy} = 0$ whose symmetries contain the hierarchy $\mathfrak{A}$ and such that $\hat{A}_1 = (\ell^s_w)^*$ potentiates the image of the Miura substitution is not *a priori* unique. Again, the constraint between the coordinates s and the momenta w for $\mathfrak{A}$ determines its first Hamiltonian operator $\hat{A}_1$ but appears apparently from nowhere[9]; the shift of the field or the frozen point argument are customary procedures here. Our paradoxal conclusion is that the first structure $\hat{A}_1 = A_1^{-1}$ for $\mathfrak{A}$ is chosen such that A_1 factors the higher Hamiltonian structure for $\mathfrak{B}$! We thus have

$$B_{k'} = \square \circ A_1 \circ \square^*, \quad k' = k'\bigl(\square, (\ell^s_w)^*\bigr) \geq 2,$$

which specifies the required nonlocalities.

This means that Frobenius operators and the factorizations they provide are helpful in the bi-Hamiltonianity tests for integrable systems. It is likely that one can reveal a similar origin of the nonlocal first Hamiltonian structures for the Drinfeld–Sokolov hierarchies [7] associated with the Kac–Moody algebras whose Cartan matrices are degenerate.

Example 5.5 (The modified Kaup–Boussinesq equation)**.** Consider an Euler–Lagrange extension of the scalar Liouville equation [17],

$$A_{xy} = -\tfrac{1}{8} A \exp\bigl(-\tfrac{1}{4}B\bigr), \qquad B_{xy} = \tfrac{1}{2}\exp\bigl(-\tfrac{1}{4}B\bigr). \tag{31}$$

Denote the momenta by

$$a = \tfrac{1}{2} B_x \quad \text{and} \quad b = \tfrac{1}{2} A_x. \tag{32}$$

The minimal integrals of system (31) are

$$w_1 = -\tfrac{1}{4}a^2 - a_x, \qquad w_2 = ab + 2b_x$$

such that $\bar{D}_y(w_i) \doteq 0$ on (31), $i = 1, 2$. Hence the operator

$$\square = \Bigl(\ell^{(a,b)}_{w_1,w_2}\Bigr)^* = \begin{pmatrix} -\frac{1}{4}B_x + D_x & \frac{1}{2}A_x \\ 0 & \frac{1}{2}B_x - 2\,D_x \end{pmatrix}$$

factors the (Noether) symmetries of (31). The bracket $\{\{\,,\,\}\}_\square$ induced in the inverse image of the Frobenius operator $\square$ is

$$\{\{\vec{\psi}, \vec{\chi}\}\}_\square = \tfrac{1}{2} \cdot \begin{pmatrix} \psi^B_x\,\chi^A - \psi^A\,\chi^B_x + \psi^A_x\,\chi^B - \psi^B\,\chi^A_x \\ \psi^B_x\,\chi^B - \psi^B\,\chi^B_x \end{pmatrix},$$

where $\vec{\psi} = (\psi^A, \psi^B)$ and $\vec{\chi} = (\chi^A, \chi^B)$. Thus we have obtained an extension of the Wronskian-based bracket for the second Hamiltonian structure of KdV, see Remark 5.

[9]This difficulty of the theory was pointed to us by B. A. Dubrovin (private communication).

Next, let us choose an equivalent pair of integrals $u = w_2$, $v = w_1 + \frac{1}{4}w_2^2$ and consider a symmetry of (31),

$$A_t = \tfrac{1}{2}A_x A_{xx} + \tfrac{1}{2}\left(\tfrac{1}{4}A_x^2 - 1\right) B_x, \qquad B_t = -2A_{xxx} + \tfrac{1}{8}A_x B_x^2 - \tfrac{1}{2}A_x B_{xx}. \tag{33}$$

Using the constraint (32) between the coordinates A, B and the momenta a, b, we cast (33) to the canonical form

$$A_t = \frac{\delta\mathcal{H}}{\delta a}, \quad a_t = -\frac{\delta\mathcal{H}}{\delta A}; \qquad B_t = \frac{\delta\mathcal{H}}{\delta b}, \quad b_t = -\frac{\delta\mathcal{H}}{\delta B},$$

where $\mathcal{H} = \int\left[\frac{1}{32}A_x^2B_x^2 + \frac{1}{4}A_xA_{xx}B_x + \frac{1}{2}A_{xx}^2 - \frac{1}{8}B_x^2\right]\mathrm{d}x$.

It is remarkable that the evolution of u and v along (33),

$$u_t = uu_x + v_x, \qquad v_t = (uv)_x + u_{xxx}, \tag{34}$$

is the Kaup–Boussinesq system and (33) is actually the potential twice-modified Kaup–Boussinesq equation. The right hand side of the integrable system (33) belongs to the image of the adjoint linearization $\tilde{\square} = \left(\ell_{(u,v)}^{(a,b)}\right)^*$. The Frobenius operator $\tilde{\square}$ factors the *third* Hamiltonian structure $\hat{A}_3^{\text{KB}} = \tilde{\square}^* \circ \left(\ell_{(a,b)}^{(A,B)}\right)^* \circ \tilde{\square}$ for (34); we have $k = 3$ and

$$\hat{A}_3^{\text{KB}} = \begin{pmatrix} u\,D_x + \frac{1}{2}u_x & D_x^3 + (\frac{1}{4}u^2 + v)\,D_x + \frac{1}{4}(u^2+2v)_x \\ D_x^3 + (\frac{1}{4}u^2 + v)\,D_x + \frac{1}{2}v_x & \frac{1}{2}(2u\,D_x^3 + 3u_x\,D_x^2 + (3u_{xx} + 2uv)D_x + u_{xxx} + (uv)_x) \end{pmatrix}.$$

The bracket $\{\{\,,\,\}\}_{\tilde{\square}}$ is equal to $\{\{\,,\,\}\}_{\hat{A}_3^{\text{KB}}}$, which is given by formula (9):

$$\{\{\vec{\psi},\vec{\chi}\}\}_{\tilde{\square}} = \{\{\vec{\psi},\vec{\chi}\}\}_{\hat{A}_3^{\text{KB}}} = \begin{pmatrix} \vec{\psi}\cdot\nabla_1(\vec{\chi}) - \nabla_1(\vec{\psi})\cdot\vec{\chi} \\ \vec{\psi}\cdot\nabla_2(\vec{\chi}) - \nabla_2(\vec{\psi})\cdot\vec{\chi} \end{pmatrix},$$

where $\nabla_1 = -\frac{1}{2}\left(\begin{smallmatrix} D_x & 0 \\ u\,D_x & D_x^3 + v\,D_x \end{smallmatrix}\right)$, $\nabla_2 = -\frac{1}{2}\left(\begin{smallmatrix} 0 & D_x \\ D_x & u\,D_x \end{smallmatrix}\right)$; here we use an alternative notation for the components of the bracket. Finally, we recall that $\hat{A}_1 = \left(\begin{smallmatrix} 0 & D_x \\ D_x & 0 \end{smallmatrix}\right)$ is the first Hamiltonian structure for (34); its inverse $A_1 = \hat{A}_1^{-1}$ factors the second Hamiltonian structure $B_2 = \tilde{\square} \circ A_1 \circ \tilde{\square}$ for (33)

Remark 11. A very general construction appears by virtue of the Gardner deformations, see Example 5.4 and [17]. Let $\mathcal{E}(\mu)$ and $\mathcal{E}(\nu)$ be the extensions of an equation $\mathcal{E}(0)$ and let $\mathfrak{m}_\varepsilon\colon \mathcal{E}(\varepsilon) \to \mathcal{E}(0)$ be the Miura contraction for $\varepsilon \in I \subset \mathbb{R}$. The symmetries $\varphi_\mu \in \operatorname{sym}\mathcal{E}(\mu)$ and $\varphi_\nu \in \operatorname{sym}\mathcal{E}(\nu)$ induce the symmetries of $\mathcal{E}(0)$ through the contraction, although the induced flows can be formal sums of infinitely many generators of higher orders. However, the commutator of the (formal) symmetries is well defined. Hence for all $\varepsilon(\mu,\nu)$ such that the commutator is lifted to a true symmetry of $\mathcal{E}(\varepsilon)$ we define the product $\diamond\colon (\mu,\nu) \mapsto \mu\diamond\nu = \varepsilon(\mu,\nu)$. Thus we obtain the multiplication $M\diamond N = \{\varepsilon\} \subset \mathbb{R}$ on the sets $M = \{\mu\}$, $N = \{\nu\} \subset \mathbb{R}$. For the Liouville-type Gardner's extension (25), we further obtain the product $\diamond\colon \big(\square(\mu), \square(\nu)\big) \mapsto \square(\mu\diamond\nu)$ of the Frobenius operators (27) together with the bracket on their arguments.

Table 1. Hamiltonian ODE and PDE.

• No internal structure of a time point $t \in \mathbb{R}$;	Smooth base manifold M^n and a fibre bundle $\pi\colon E^{n+m} \to M^n$.
• Symplectic manifold $M^{2n} \ni (p,q)$;	Infinite jet space $J^\infty(\pi)$.
∗ Components X^i of vector fields $X \in \Gamma(TM^{2n})$ on M^{2n};	Sections $\varphi = (\varphi^1[u], \ldots, \varphi^m[u]) \in \varkappa(\pi) = \Gamma(\pi_\infty^*(\pi))$ of the induced fibre bundle.
⋆ Vector fields $X \in \Gamma(TM^{2n})$;	Evolutionary vector fields $\reflectbox{$\partial$}_\varphi \in \mathrm{D}^v(J^\infty(\pi))$.
• Integral trajectories $\bigl(p(t), q(t)\bigr) \subset M^{2n}$ of the field X;	Solutions of the autonomous equation $u_t = \varphi$.
• The Lie algebra $\bigl(TM^{2n}, [\,,\,]\bigr)$ of vector fields on M^{2n};	The Lie algebra $\mathfrak{g}(\pi)$ of evolutionary derivations.
• The de Rham differential d;	The de Rham differential $\mathrm{d} = \mathrm{d}_h + \mathrm{d}_{\mathcal{C}}$ split to horizontal and vertical parts w.r.t. π.
• The de Rham complex;	The variational bi-complex and the $\mathcal{C}$-spectral sequence.
• The space of Hamiltonians $\mathcal{H} \in C^\infty(M^{2n})$;	The senior horizontal cohomology $\bar{H}^n(\pi) \ni \mathcal{H}$.
† The cotangent bundle $T^*M^{2n} \ni \psi$ and its sections $\psi\colon TM^{2n} \to C^\infty(M^{2n})$;	The module $\hat{\varkappa} = \mathrm{Hom}_{C^\infty(M^n)}\bigl(\varkappa(\pi), \bar{H}^n(\pi)\bigr)$.
• The differential $\mathrm{d}\mathcal{H}$ of a Hamiltonian;	The Euler operator $\mathbf{E}$ as a restriction of $\mathrm{d}_{\mathcal{C}}$ onto $\bar{H}^n(\pi)$, the variational derivative $\mathbf{E}(\mathcal{H}) \in \hat{\varkappa}$.
• Symplectic 2-form $\omega \in \Omega^2(M^{2n})$;	Hamiltonian operator $A \in \mathcal{C}\mathrm{Diff}(\hat{\varkappa}, \varkappa)$ in total derivatives.
‡ Hamiltonian vector field $X_{\mathcal{H}}$ such that $X_{\mathcal{H}} \lrcorner\, \omega = \mathrm{d}\mathcal{H}$;	The section $\varphi = A(\psi)$, $\psi \in \hat{\varkappa}(\pi)$.
• The Poisson bracket $\{\mathcal{H}_1, \mathcal{H}_2\} = \omega\bigl(X_{\mathcal{H}_1}, X_{\mathcal{H}_2}\bigr) = X_{\mathcal{H}_1} \lrcorner\, \mathrm{d}\mathcal{H}_2$;	The Poisson bracket $\{\mathcal{H}_1, \mathcal{H}_2\}_A = \bigl\langle \mathbf{E}(\mathcal{H}_1), A\bigl(\mathbf{E}(\mathcal{H}_2)\bigr)\bigr\rangle = \reflectbox{∂}_{A(\mathbf{E}(\mathcal{H}_2))}(\mathcal{H}_1)$.

Appendix A. Analogy between the Hamiltonian ODE and PDE.

In Table 1 we track the geometric correspondence between Hamiltonian ODEs and PDEs; we adapt the parallel to further material and therefore it is forced to remain incomplete. The distinction between the coordinates and momenta in the PDE framework, which is implemented in section 5 to the (symmetries of) Euler–Lagrange systems, is addressed in [5, 16]. The concept of Δ-coveings over PDE, which is convenient in practical calculations exposed here, is developed in [14], see Remark 12 on p. 27.

The comments to Table 1 are as follows:

∗ The sections $\varphi = (\varphi^1, \ldots, \varphi^m) \in \Gamma(\pi_\infty^*(\pi))$ "look like" the sections $s(x) = (s^1(x), \ldots, s^m(s))$ of the bundle π but their components $\varphi^i[u]$ are functions on the jet space over π.

⋆ The evolutionary derivations

$$\reflectbox{∂}_\varphi = \sum_\sigma D_\sigma(\varphi) \cdot \frac{\partial}{\partial u_\sigma}$$

and the linearizations (the Frechét derivatives) $\ell_\psi = \sum_\sigma \frac{\partial\psi}{\partial u_\sigma} \cdot D_\sigma$ are correlated by $\partial_\varphi(\psi) = \ell_\psi(\varphi)$.

† We use the notation $\Omega^1(\pi) \equiv \hat{\varkappa}(\pi)$ in agreement with [6].

‡ A Hamiltonian evolutionary vector field $\partial_{A(\psi)}$ may not possess a Hamiltonian $\mathcal{H}$ such that $\psi = \mathbf{E}(\mathcal{H})$!

Now we recall the proofs of auxiliary statements.

Proof of Lemma 2. Suppose $\Delta \in \mathcal{C}\mathrm{Diff}(\varkappa(\pi), \bar{\Lambda}^n(\pi))$. By multiple integrating by parts, we transform the expression $\ell_{\Delta(\varphi)}$ to $\ell_{\Delta_0(\varphi)} + D_x \circ \Delta'(\varphi)$, where the order of Δ_0 is zero and $\Delta'(\varphi) \in \mathcal{C}\mathrm{Diff}(\varkappa(\pi), \bar{\Lambda}^n(\pi))$. Then, using the Leibnitz rule $\mathbf{E}(\langle\psi, F\rangle) = \ell^*_{\langle\psi,F\rangle}(1) = \ell^*_F(\psi) + \ell^*_\psi(F)$, we obtain $\mathbf{E}(\Delta(\varphi)) = \ell^*_\varphi(\Delta^*(1)) + \ell^*_{\Delta^*(1)}(\varphi)$ for any section $\varphi \in \varkappa(\pi)$. Now let $\rho \in \bar{\Lambda}^n(\pi)$ be a form and put $\Delta = \ell_\rho \colon \varkappa(\pi) \to \bar{\Lambda}^n(\pi)$. The linearization

$$\ell_{\mathbf{E}(\rho)} = \ell^*_{\mathbf{E}(\rho)} \tag{35}$$

of the image of the Euler operator is self-adjoint, hence we obtain the equality $\mathbf{E}(\ell_\rho(\varphi)) = \ell^*_\varphi(\mathbf{E}(\rho)) + \ell_{\mathbf{E}(\rho)}(\varphi)$, whence Lemma 2 follows. □

Recall that we put $\ell_{A,\psi}(\varphi) = \partial_\varphi(A)(\psi)$ by definition for any $\varphi \in \mathfrak{g}(\pi)$ and $\psi \in \Omega^1(\pi)$.

Lemma 15 ([23]). Let A be a matrix operator in total derivatives. Then one has

$$\ell^*_{A,\psi_1}(\psi_2) = \ell^*_{A^*,\psi_2}(\psi_1)$$

for any ψ_1, ψ_2.

Proof of Proposition 5. By construction of the Poisson bracket $\{\,,\,\}_A$, we have $\{\mathcal{H}_1, \mathcal{H}_2\}_A = \langle A(\psi_1), \mathbf{E}(\mathcal{H}_2)\rangle \approx \partial_{A(\psi_1)}(\mathcal{H}_2) \approx -\partial_{A(\psi_2)}(\mathcal{H}_1)$ modulo the exact terms, here $\psi_i = \mathbf{E}(\mathcal{H}_i)$.

Let $\mathcal{H}_\alpha$, $\mathcal{H}_\beta$, and $\mathcal{H}_\gamma$ be the Hamiltonians. The Jacobi identity is

$$\{\{\mathcal{H}_\alpha, \mathcal{H}_\beta\}_A, \mathcal{H}_\gamma\}_A + \{\{\mathcal{H}_\beta, \mathcal{H}_\gamma\}_A, \mathcal{H}_\alpha\}_A + \{\{\mathcal{H}_\gamma, \mathcal{H}_\alpha\}_A, \mathcal{H}_\beta\}_A = -\sum_{\circlearrowleft} \partial_{A(\psi_\gamma)}(\langle A(\psi_\alpha), \psi_\beta\rangle)$$
$$= -\sum_{\circlearrowleft} \Big[\langle \partial_{A(\psi_\gamma)}(A)(\psi_\alpha), \psi_\beta\rangle + \langle A(\partial_{A(\psi_\gamma)}(\psi_\alpha)), \psi_\beta\rangle + \langle A(\psi_\alpha), \partial_{A(\psi_\gamma)}(\psi_\beta)\rangle\Big] = 0. \tag{36}$$

Consider the elements of the second sum,

$$\begin{aligned}\langle A(\partial_{A(\psi_\gamma)}(\psi_\alpha)), \psi_\beta\rangle &= \langle \psi_\beta, A(\partial_{A(\psi_\gamma)}(\psi_\alpha))\rangle = -\langle A(\psi_\beta), \partial_{A(\psi_\gamma)}(\psi_\alpha)\rangle \\ &= -\langle A(\psi_\beta), \ell_{\psi_\alpha}(A(\psi_\gamma))\rangle = -\langle \ell^*_{\psi_\alpha}(A(\psi_\beta)), A(\psi_\gamma)\rangle = (\text{by (35)}) \\ &= -\langle \ell_{\psi_\alpha}(A(\psi_\beta)), A(\psi_\gamma)\rangle = -\langle A(\psi_\gamma), \ell_{\psi_\alpha} A(\psi_\beta)\rangle.\end{aligned}$$

Substituting this back in (36), we obtain

$$\begin{aligned}0 &= -\sum_{\circlearrowleft} \langle (\partial_{A(\psi_\gamma)}(A))(\psi_\alpha), \psi_\beta\rangle + \Big[\sum_{\circlearrowleft} \langle A(\psi_\gamma), \ell_{\psi_\alpha} A(\psi_\beta)\rangle - \sum_{\circlearrowleft} \langle A(\psi_\alpha), \ell_{\psi_\beta} A(\psi_\gamma)\rangle\Big] \\ &= -\langle (\partial_{A(\psi_\gamma)}(A))(\psi_\alpha), \psi_\beta\rangle - \langle (\partial_{A(\psi_\alpha)}(A))(\psi_\beta), \psi_\gamma\rangle - \langle (\partial_{A(\psi_\beta)}(A))(\psi_\gamma), \psi_\alpha\rangle.\end{aligned}$$

Now set $\alpha = 3$, $\beta = 2$, $\gamma = 1$; thence we have

$$0 = -\langle (\partial_{A(\psi_1)}(A))(\psi_3), \psi_2\rangle - \langle (\partial_{A(\psi_3)}(A))(\psi_2), \psi_1\rangle - \langle (\partial_{A(\psi_2)}(A))(\psi_1), \psi_3\rangle. \tag{37}$$

Consider the first summand,

$$\begin{aligned}\langle(\ni_{A(\psi_1)}(A))(\psi_3),\psi_2\rangle &= \langle(\ell_{A,\psi_3}(A(\psi_1))),\psi_2\rangle = \langle A(\psi_1),\ell^*_{A,\psi_3}(\psi_2)\rangle = \text{(by Lemma 15)}\\ &= \langle A(\psi_1),\ell^*_{A^*,\psi_2}(\psi_3)\rangle = \langle \ell_{A^*,\psi_2}(A(\psi_1)),\psi_3\rangle\\ &= -\langle \ell_{A,\psi_2}(A(\psi_1)),\psi_3\rangle. \qquad (38a)\end{aligned}$$

Next, the second summand in (37) is

$$\langle(\ni_{A(\psi_3)}(A))(\psi_2),\psi_1\rangle = \langle\psi_1,\ell_{A,\psi_2}(A(\psi_3))\rangle = \langle \ell^*_{A,\psi_2}(\psi_1),A(\psi_3)\rangle = -\langle A(\ell^*_{A,\psi_2}(\psi_1),\psi_3\rangle. \qquad (38b)$$

Now consider the third term in the r.h.s. of (37),

$$\langle(\ni_{A(\psi_2)}(A))(\psi_1),\psi_3\rangle = \langle(\ell_{A,\psi_1}(A(\psi_2)),\psi_3\rangle. \qquad (38c)$$

Substituting (38) in (37), we finally obtain

$$\langle \ell_{A,\psi_2}(A(\psi_1)),\psi_3\rangle + \langle A(\ell^*_{A,\psi_2}(\psi_1),\psi_3\rangle - \langle(\ell_{A,\psi_1}(A(\psi_2)),\psi_3\rangle = 0,$$

whence follows

$$\ell_{A,\psi_1}(A(\psi_2)) - \ell_{A,\psi_2}(A(\psi_1)) = A(\ell^*_{A,\psi_2}(\psi_1). \qquad (2)$$

The proof is complete. □

Proof of Theorem 7. The main homological equality (4) is established by induction on k. Starting with a Casimir $\mathcal{H}_0$, we obtain

$$0 = [\![A_2,0]\!] = [\![A_2,[\![A_1,\mathcal{H}_0]\!]]\!] = -[\![A_1,[\![A_2,\mathcal{H}_0]\!]]\!] \mod [\![A_1,A_2]\!] = 0$$

using the Jacobi identity (3). Now recall that the first Poisson cohomology $H^1_{A_1}(\pi) = 0$ is trivial by assumption of the theorem, and hence the closed element $[\![A_2,\mathcal{H}_0]\!]$ in the kernel of $[\![A_1,\cdot]\!]$ is exact: $[\![A_2,\mathcal{H}_0]\!] = [\![A_1,\mathcal{H}_1]\!]$ for some $\mathcal{H}_1$.

For $k \geq 1$, we have

$$[\![A_1,[\![A_2,\mathcal{H}_k]\!]]\!] = -[\![A_2,[\![A_1,\mathcal{H}_k]\!]]\!] = -[\![A_2,[\![A_2,\mathcal{H}_{k-1}]\!]]\!] = 0$$

in view of the Jacobi identity (3) and $[\![A_2,A_2]\!] = 0$. Hence $[\![A_2,\mathcal{H}_k]\!] = [\![A_1,\mathcal{H}_{k+1}]\!]$ by $H^1_{A_1} = 0$, and thus we proceed infinitely. □

Appendix B. Reconstruction of the brackets $\{\{\cdot,\cdot\}\}_A$

In this appendix we describe an inductive procedure that assigns the bracket $\{\{\cdot,\cdot\}\}_A$ to a Frobenius operator A, see (5). The bracket may be not contained in our knowledge that A is Frobenius if, e.g., the operator determines the factorization of symmetries of a Liouville-type system and has minimal differential order. This is precisely the case of operator (27), which is used in Example B.2 as an illustration.

For brevity, let us technically assume that $A = \|\sum_k A^{ij}_k\, D^k_x\|$ is a matrix operator in D_x, where $A^{ij}_k \in C^\infty(J^\infty(\pi)) =: \mathcal{F}(\pi)$. Recall that the elements of $\Omega^1(\pi)$ are naturally represented as the Cartan 1-forms $\omega = \sum_{i=1}^m \psi^i \cdot \mathrm{d}_{\mathcal{C}}u^i$, $\psi^i \in \mathcal{F}(\pi)$; consequently, we use the notation $\mathbf{1}_i$ for $\psi = (0,\ldots,0,1,0,\ldots,0)^T$ such that $\omega = \mathrm{d}_{\mathcal{C}}u^i$. Suppose further that

$$\bigcap_k \ker A_k = \{0\} \quad \text{for } A = \sum_k A_k \cdot D^k_x, \qquad (7)$$

and hence we encounter no difficulties when resolving inhomogeneous equations $A\bigl(\{\{\psi_1,\psi_2\}\}_A\bigr)=\varphi$ w.r.t. $\{\{\,,\,\}\}_A\in\mathcal{C}\mathrm{Diff}_{(2)}(\Omega^1(\pi),\varkappa(\pi))$ bilinear in generic $\psi_1,\psi_2\in\Omega^1(\pi)$.

Let the (yet unknown) bracket be

$$\{\{\psi_1,\psi_2\}\}_A=\sum_{i,j,k=1}^{m} c_{ijk}^{\alpha\beta}\cdot(\psi_1^i)_\alpha(\psi_2^j)_\beta\cdot\mathbf{1}_k,$$

where $c_{ijk}^{\alpha\beta}\in\mathcal{F}(\pi)$ and the condition $c_{ijk}^{\alpha\beta}=-c_{jik}^{\beta\alpha}$ follows from the skew-symmetry of the bracket. The coefficients c_{ijk}^{00} can be nontrivial if the number of unknowns is $m>1$, see Example 4.1.

The base of the algorithm is given by the Jacobi bracket of the Cartan forms $\mathrm{d}_{\mathcal{C}}u^i$, $\mathrm{d}_{\mathcal{C}}u^j$:

$$\bigl[A(\mathbf{1}_i),A(\mathbf{1}_j)\bigr]=A(c_{ijk}^{00}\cdot\mathbf{1}_k). \tag{39}$$

The choice $1\leq i,j\leq m$ yields $m(m-1)/2$ compatible systems of m equations enumerated by k, which are resolved in view of the nondegeneracy (7) of A. Actually, these systems are overdetermined whenever the differential order of A is positive and hence the left-hand sides of (39) and (40), see below, contain higher order derivatives of u that are not present among the arguments of $c_{ijk}^{\alpha\beta}[u]$.

The inductive step is made by using the sections $x^\alpha\cdot\mathbf{1}_i$ and $x^\beta\cdot\mathbf{1}_j$. We obviously have

$$\bigl[A(x^\alpha\cdot\mathbf{1}_i),A(x^\beta\cdot\mathbf{1}_j)\bigr]-A\Bigl(\sum_{0\leq\alpha'+\beta'<\alpha+\beta}(x^\alpha)^{(\alpha')}(x^\beta)^{(\beta')}c_{ijk}^{\alpha'\beta'}\cdot\mathbf{1}_k\Bigr)=A\bigl(\alpha!\beta!c_{ijk}^{\alpha\beta}\cdot\mathbf{1}_k\bigr), \tag{40}$$

whence the coefficients $c_{ijk}^{\alpha\beta}$ on the diagonal $\alpha+\beta=\text{const}$ are obtained one by one. Having passed through the diagonal $0\leq\alpha+\beta=\text{const}$, $\alpha\geq\beta$ or $\alpha\leq\beta$, we check the condition

$$\bigl[A(\psi),A(\chi)\bigr]=A\Bigl(\sum_{\alpha+\beta\leq\text{const}}c_{ijk}^{\alpha\beta}\cdot\psi_\alpha^i\chi_\beta^j\Bigr) \tag{41}$$

that terminates the algorithm when holds. The differential order of the bracket $\{\{\,,\,\}\}_A$ with respect to its arguments is estimated by calculating the Lie bracket $[A(\psi),A(\chi)]$ and taking into account the Leibnitz rule in the right-hand side of (5a).

Example B.1. The operator A_1 for the dispersionless 3-component Boussinesq system (18) has order zero and its matrix (19a) is nondegenerate; hence the condition (7) is valid, We reconstruct the bracket $\{\{\,,\,\}\}_{A_1}$ for this operator performing only one step of the above algorithm; the terminal check (41) is fulfilled for the arguments $\psi=\mathbf{1}_i$ and $\chi=\mathbf{1}_j$. The components of the bracket are

$$\begin{aligned}
\{\{p,q\}\}_{A_1}^u&=p_x^wq^u-p^uq_x^w+3w(p^uq_x^v-p_x^vq^u)+3w(p^vq_x^u-p_x^uq^v)\\
&\quad+p_x^uq^w-p^wq_x^u+2w_x(p^uq_v-p^vq^u)+u(p^vq_x^v-p_x^vq^v),\\
\{\{p,q\}\}_{A_1}^v&=+p_x^uq^u-p^uq_x^u+4w(p^vq_x^v-p_x^vq^v)+p_x^vq^w-p^wq_x^v+p_x^wq^v-p^vq_x^w,\\
\{\{p,q\}\}_{A_1}^w&=u(p^uq_x^v-p_x^vq^u)+3w^2(p^vq_x^v-p_x^vq^v)+2u_x(p^vq^u-p^uq^v)\\
&\quad+w(p_x^uq^u-p^uq_x^u)+u(p^vq_x^u-p_x^uq^v)+p_x^wq^w-p^wq_x^w.
\end{aligned}$$

The first order operator (19b) is also nondegenerate, and we obtain

$$\begin{aligned}
\{\{p,q\}\}^u_{A_2} &= 3w(p^u q^v_x - p^v_x q^u) + 2w(p^v q^u_x - p^u_x q^v) + 2w_x(p^u q^v - p^v q^u)\\
&\quad + p^w_x q^u - p^u q^w_x + 2u(p^v q^v_x - p^v_x q^v),\\
\{\{p,q\}\}^v_{A_2} &= 4w(p^v q^v_x - p^v_x q^v) + p^u_x q^u - p^u q^u_x + 2(p^w_x q^v - p^v q^w_x),\\
\{\{p,q\}\}^w_{A_2} &= 8v(p^v q^v_x - p^v_x q^v) + 8u(p^v q^u_x - p^u_x q^v) + 6w^2(p^v q^v_x - p^v_x q^v)\\
&\quad + w(p^u_x q^u - p^u q^u_x) + 2u_x(p^v q^u - 2u_x p^u q^v) + u(p^u q^v_x - u p^v_x q^u).
\end{aligned}$$

Example B.2. Following the above algorithm and using the package [27], we obtain the bracket in the inverse image of the Frobenius operator (27): for any $p,q \in \Omega^1(\mathcal{E}^\varepsilon_{\mathrm{L}})$ we have

$$\begin{aligned}
\{\{p,q\}\}_\Box &= \varepsilon^2\cdot\big(p_{xx}q_x - p_x q_{xx}\big) - 2\varepsilon^2\cdot\big(p_{xxx}q - p q_{xxx}\big)\\
&\quad - 12\varepsilon^4\cdot\big(8\varepsilon^2 u_x^3 u_{xx} - 4\varepsilon^2 u_x^2 u_{xxx} + 4\varepsilon^2 u_x u_{xx}^2 + 2u_x u_{xx} - u_{xxx}\big)\\
&\qquad \times\big[1 + 8\varepsilon^2 u_x^2 + 16\varepsilon^4 u_x^4 - 2\varepsilon^2 u_{xx} - 8\varepsilon^4 u_x^2 u_{xx}\big]^{-1}\cdot\big(p_{xx}q - p q_{xx}\big)\\
&\quad + \big(\underline{1} + 288\varepsilon^4 u_x^4 - 288\varepsilon^4 u_x^2 u_{xx} + 28\varepsilon^2 u_x^2 - 16\varepsilon^2 u_{xx} - 288\varepsilon^6 u_x u_{xx} u_{xxx}\\
&\qquad - 96\varepsilon^6 u_{xx}^3 + 3072\varepsilon^{10} u_x^{10} + 24\varepsilon^6 u_{xxx}^2 + 24\varepsilon^4 u_{4x} + 1408\varepsilon^6 u_x^6 + 3328\varepsilon^8 u_x^8\\
&\qquad - 768\varepsilon^{10} u_{4x} u_{xx} u_x^4 - 384\varepsilon^8 u_{4x} u_x^2 u_{xx} - 2304\varepsilon^8 u_x^3 u_{xx} u_{xxx} + 384\varepsilon^8 u_{xx}^2 u_x u_{xxx}\\
&\qquad - 4608\varepsilon^{10} u_x^5 u_{xx} u_{xxx} + 16\varepsilon^4 u_{xx}^2 - 5632\varepsilon^8 u_x^6 u_{xx} - 1920\varepsilon^6 u_{xx} u_x^4 + 3328\varepsilon^8 u_x^4 u_{xx}^2\\
&\qquad + 512\varepsilon^6 u_{xx}^2 u_x^2 + 384\varepsilon^{10} u_x^4 u_{xxx}^2 - 960\varepsilon^{10} u_{xx}^4 u_x^2 - 48\varepsilon^4 u_x u_{xxx} - 3072\varepsilon^{10} u_x^7 u_{xxx}\\
&\qquad + 3072\varepsilon^{10} u_{xx}^3 u_x^4 - 2304\varepsilon^8 u_x^5 u_{xxx} - 576\varepsilon^6 u_x^3 u_{xxx} + 288\varepsilon^6 u_{4x} u_x^2 + 384\varepsilon^8 u_x^2 u_{xx}^3\\
&\qquad + 6144\varepsilon^{10} u_{xx}^2 u_x^6 - 6144\varepsilon^{10} u_{xx} u_x^8 + 1152\varepsilon^8 u_{4x} u_x^4 + 1536\varepsilon^{10} u_{4x} u_x^6 + 192\varepsilon^8 u_{xxx}^2 u_x^2\\
&\qquad + 240\varepsilon^8 u_{xx}^4 + 1536\varepsilon^{10} u_{xx}^2 u_x^3 u_{xxx} - 48\varepsilon^6 u_{4x} u_{xx}\big)\\
&\qquad \times\big[\underline{1} + 96\varepsilon^4 u_x^4 + 256\varepsilon^6 u_x^6 + 256\varepsilon^8 u_x^8 + 4\varepsilon^4 u_{xx}^2 - 48\varepsilon^4 u_x^2 u_{xx} + 32\varepsilon^6 u_{xx}^2 u_x^2\\
&\qquad\quad - 4\varepsilon^2 u_{xx} - 256\varepsilon^8 u_x^6 u_{xx} + 64\varepsilon^8 u_x^4 u_{xx}^2 - 192\varepsilon^6 u_{xx} u_x^4 + 16\varepsilon^2 u_x^2\big]^{-1}\cdot\big(p_x q - p q_x\big).
\end{aligned}$$

The two underlined units correspond to the bracket $p_x q - p q_x$ in the inverse image of the operator $\Box = \mathcal{U}_x + \frac{1}{2}D_x$ that factors symmetries of the Liouville equation $\mathcal{U}_{xy} = \exp(2\mathcal{U})$ at $\varepsilon = 0$.

Remark 12. The classification problem for the Frobenius operators A and the task of reconstruction of the associated brackets $\{\{\,,\,\}\}_A$ can be performed using any software capable for calculation of the commutators, e.g., [20, 27] designed for symmetry analysis of evolutionary (super-)PDE. The implementation of technique of the Δ-coverings [14] is extremely productive here; see [20] for numerous examples. In our case, it amounts to 'forgetting' the functional structure of $\Omega^1(\pi)$ and treating this module as a jet (super-)bundle over $J^\infty(\pi)$, see [14] for details. Hence, instead of calculating $D_x(\psi[u])$ for $\psi \in \Omega^1(\pi)$, one introduces the variable ψ_x and so on, setting the derivatives $\ni_\varphi(\psi)$ in the Koszul bracket to zero, see (5). Consequently, only the derivatives of coeficients of the operator A contribute to the left-hand side of (5a) while its right-hand side becomes $A(\{\{\psi_1,\psi_2\}\}_A)$. This is the structure of equations (39–40). Also, the nature of the assumption (7) becomes clear: indeed, thus we require the nondegeneracy of the operator A on bi-differential functions of ψ_1, ψ_2, and of their derivatives instead of differential functions of the dependent variables u.

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