Lifetime of a massive particle in a de Sitter universe

Jacques BROS, Henri EPSTEIN and Ugo MOSCHELLA

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

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Jacques Bros[1], Henri Epstein[2] and Ugo Moschella[3]
[1] *Service de Physique théorique - CEA. Saclay. 91191 Gif-sur Yvette.*
[2] *Institut des Hautes Études Scientifiques, 91440 Bures-sur-Yvette.*
[3] *Università dell'Insubria, Como and INFN Milano*

We study particle decay in de Sitter space-time as given by first order perturbation theory in an interacting quantum field theory. We show that for fields with masses above a critical mass m_c there is no such thing as particle stability, so that decays forbidden in flat space-time do occur there. The lifetime of such a particle also turns out to be independent of its velocity when that lifetime is comparable with de Sitter radius. Particles with lower mass are even stranger: The masses of their decay products must obey quantification rules, and their lifetime is zero.

Some important progress in the astronomical observations of the last ten years [1, 2] have led to the surprising conclusion that the recent universe is dominated by an exotic form of energy density with a negative pressure that acts repulsively at large scales, opposing itself to the gravitational attraction. It has become customary to characterize such energy density by the term "dark". The simplest and best known candidate for the "dark energy" is the cosmological constant. As of today, the ΛCDM (Cold Dark Matter) model, which is obtained by adding a cosmological constant to the standard model, is the one in best agreement with the cosmological observations. In addition, if the description provided by the ΛCDM model is correct, the other energy components must in the future progressively thin out and eventually vanish thus letting the cosmological constant term alone survive. In this context, the de Sitter geometry, which is the homogeneous and isotropic solution of the vacuum Einstein equations with cosmological term, appears to take the double role of reference geometry of the universe, namely the geometry of spacetime deprived of its matter and radiation content and of the geometry that the universe approaches asymptotically.

One might think that the presence of a cosmological constant, while having a huge impact on our understanding of the universe as a whole, would not influence microphysics in its quantum aspects. This is also the viewpoint taken in the context of inflationary models [3], where the effective cosmological constant is many orders of magnitude larger than the one observed today. However this conclusion may have to be reassessed. Indeed, in the presence of a cosmological constant, however small, it is the notion of elementary particle itself which has to be reconsidered, since the usual asymptotic theory is based on concepts which refer closely to Minkowski spacetime and to its Fourier representation, and do not apply to the de Sitter universe which is not asymptotically flat. A possible approach is perturbation theory. Unfortunately, calculations of perturbative amplitudes which in the Minkowskian case would be simple or even trivial become rapidly prohibitive or impossible in the de Sitter case: this in spite of the fact that one is dealing with a maximally symmetric manifold.

In this letter we have tackled one such calculation, namely that of the mean lifetime of de Sitterian unstable scalar particles. This task already presents considerable mathematical difficulties. The results exhibit significant differences compared to the Minkowskian case and processes which are forbidden in the absence of curvature become possible and, viceversa, processes that are possible become forbidden. For the Poincaré group, the tensor product of two unitary irreducible representations of masses m_1 and m_2 decomposes into a direct integral of unitary irreducible representations of masses $m \geq m_1 + m_2$. For the de Sitter group this is not always so: all representations of mass larger than a critical value (principal series) appear in the decomposition of the tensor product of any two unitary irreducible representations of positive mass. This fact was shown in [4] for the two-dimensional case and will be established here in general. This means that the de Sitter symmetry does not prevent a particle with mass in the principal series from decaying into e.g. pairs of heavier particles. This phenomenon also implies that there can be nothing like a mass gap in that range. This is a major obstruction to attempts at constructing a de Sitter S-matrix; the Minkowskian asymptotic theory makes essential use of an isolated point in the spectrum of the mass operator, and this will generally not occur in the de Sitter case. In fact a true asymptotic theory does not exist at present for de Sitter space, although it still makes sense to follow Wigner [5] in associating a particle with a unitary irreducible representation of the de Sitter group, labeled by a mass parameter, as we do here. We will also show that the tensor products of two representations of sufficiently small mass below the critical value (complementary series) contains an additional finite sum of discrete terms in the complementary series itself (at most one term in dimension four). This implies a form of particle stability, but the new phenomenon is that a particle of this kind cannot disintegrate unless the masses of the decay products have certain quantized values.

We will resort to first order perturbation theory in our calculations. These are made trivial in the Minkowski case by the use of momentum-space, but this is not so in the de Sitter space. We restrict at first our attention to the principal series and start by deriving a general formula expressing the decay probability of a particle which applies to both Minkowski and de Sitter space-times in d dimensions where the de Sitter manifold is identified with the hyperboloid $\{x \in \mathbf{R}^{d+1} \ : \ x^2 = x_0^2 - x_1^2 - \ldots - x_d^2 = -R^2\}$ in the $(d+1)$-dimensional Minkowski space. Differences will come in later.

A Klein-Gordon neutral scalar field ϕ with mass $m \geq 0$, in d-dimensional Minkowski or de Sitter space-time, is fully characterized by its two-point vacuum expectation value

$$w_m(x,\ y) = (\Omega,\ \phi(x)\,\phi(y)\Omega). \tag{1}$$

$w_m(x,\ y)$ is uniquely specified up to a constant factor by requiring invariance, locality, and a suitable spectral condition that can be formulated both in Minkowski and de Sitter spaces by imposing that $w_m(x,\ y)$ be the boundary value of an analytic function of $(x-y)^2$ in the complex plane cut along the positive real axis (for the physical interpretation of this property see [6, 7]). The knowledge of w_m allows the reconstruction of the Fock space of the theory and of a representation of the invariance group that is irreducible when restricted to the one-particle space. In the de Sitter case, it can be labeled by a dimensionless parameter ν as follows

$$m^2R^2 \ = \ \left(\frac{d-1}{2}\right)^2 + \nu^2. \tag{2}$$

The range $m \geq m_c = (d-1)/2R$ corresponds to the principal series (ν real) while $0 \leq m < m_c$ corresponds to the complementary series (ν imaginary). These restrictions ensure that w_m is positive definite and therefore a quantum theoretical interpretation is available. Two properties are crucial in our derivation:

a) The *projector identity* (dy denotes either the Poincaré or the de Sitter invariant measure):

$$\int w_m(x,y)\, w_{m'}(y,\ z)\, dy = C(m)\delta(m^2 - m'^2) w_m(x,z). \tag{3}$$

The proof of this property is easy in the flat case and not trivial for de Sitter theories; it holds as such only for fields of the principal series. $C(m) = 2\pi$ in the Minkowskian case, while in the de Sitter case $C(\nu) = 2\pi|\coth(\pi\nu)|$.

b) The *Källén-Lehmann type representation* for the product of n two-point functions:

$$\prod_{j=1}^n w_{m_j}(x,\ y) = \int da^2 \rho(a^2;\ m_1,\ldots,\ m_n)\, w_a(x,y). \tag{4}$$

Again, the proof of the existence of ρ is non-trivial in de Sitter space. An explicit calculation of the weights ρ is easy in the Minkowski case for $n = 2$, while difficult or impossible in other cases.

Consider now $N+1$ independent neutral Klein-Gordon scalar fields $\phi_0,\ \phi_1,\ \ldots,\ \phi_N$ with masses $m_0,\ m_1,\ \ldots,\ m_N$ respectively, operating in a Fock space $\mathcal{H}$, and an interaction term of the form

$$\int \gamma\, g(x)\, \mathcal{L}(x)\, dx, \quad \mathcal{L}(x) = :\phi_0(x)\phi_1(x)^{q_1} \ldots \phi_N(x)^{q_N}:$$

where g is a smooth spacetime dependent "switching-on factor" which, in the end, should be made to tend to the constant 1. Self-interactions $\mathcal{L}(x) = :\phi(x)^n:$ are a special case of this coupling. At first order the transition amplitude between two orthogonal states ψ_0 and ψ_1 is

$$(\psi_0,\ iT_1(\gamma g)\psi_1), \qquad T_1(\gamma g) = \int \gamma g(x)\, \mathcal{L}(x)\, dx\ . \tag{5}$$

Let ψ_0 be a one-particle state of the form $\int f(x)\,\phi_0(x)\Omega\, dx$; the smooth wave-function f contains the physical details about the quantum state of the unstable particle whose disintegration we aim to study. Let $\mathcal{H}_{0,q}$ be the space of all states containing q_1 particles of type 1,..., q_N particles of type N, and $P_{0,q}$ be the projector onto this space. If ψ_0 has norm 1, Wick's theorem gives the probability of its transition to any possible q-particle state of $\mathcal{H}_{0,q}$:

$$\begin{aligned}\Gamma(1_0; q_1, \ldots, q_N) &= (\psi_0,\ T_1(\gamma g) P_{0,q} T_1(\gamma g)^* \psi_0) = \\ &= \gamma^2 \int dx\, dy\, du\, dv\, \overline{f(x)}\, f(y)\, g(u)\, g(v) \times \\ &\times w_{m_0}(x,\ u) \textstyle\prod_{j=1}^N q_j!\ w_{m_j}(u,\ v)^{q_j}\, w_{m_0}(v,\ y).\end{aligned} \tag{6}$$

We now replace one of the switching-on factors by 1 in the above expression. By using Eqs. (4) and (3) we find the following general formula for the transition probability:

$$\Gamma(1_0; q_1, \ldots, q_N) = \frac{\gamma^2\, C(m_0) \int g(x)\, |F(x)|^2\, dx}{\int \overline{f(x)} w_{m_0}(x,\ y)\, f(y)\, dx\, dy} \times \left(\prod_{j=1}^N q_j!\right) \rho(m_0^2; m_1, \ldots, m_1, \ldots, m_N, \ldots, m_N). \tag{7}$$

Here $F(x) = \int w_{m_0}(x,\ y)\, f(y)\, dy$; the denominator is the squared norm of ψ_0 which is no longer assumed to be one. This formula has an interesting simple structure: the first factor does not depend on the number or nature of the decay particles but only on the wavefunction of the incoming unstable particle. The infrared problem is contained in this factor and has to be overcome when letting the remaining $g(x)$ tend to 1 (adiabatic limit). The second factor is the relevant Källén-Lehmann weight times the right combinatorial factor. Let us now focus on the decay of a particle of mass m_0 into two identical particles of mass m_1.

Case of Minkowski space

Here the availability of momentum space renders simple both the adiabatic limit and the evaluation of the weight ρ. For the adiabatic limit, the common choice is to let $g(x)$ be the characteristic function of some time interval T. It is then found that the transition probability

(7) is proportional to T and thus diverges when $T \to \infty$. Fermi's golden rule tells us that the transition probability per unit time (see e.g. [8]) has a finite limit as follows ($\widetilde{f}(p)$ is the Fourier transform of the wavepacket $f(x)$):

$$\frac{1}{\tau} = \lim_{T\to\infty} \frac{\Gamma(1_0; 2_1)}{T} = 2\rho(m_0^2;\ m_1,\ m_1)\times \frac{(2\pi)\gamma^2 \int (2p^0)^{-1} |\widetilde{f}(p)|^2\, \delta(p^2 - m_0^2)\theta(p^0)\, dp}{\int |\widetilde{f}(p)|^2\, \delta(p^2 - m_0^2)\theta(p^0)\, dp} \quad (8)$$

The weight ρ can be computed by Fourier transforming the squared v.e.v. $w_{m_1}^2(x,y)$. The (well-known) result is

$$\rho = \frac{1}{2^d\, \pi m_0\, \Gamma(\frac{d-1}{2})} \left(\frac{m_0^2 - 4m_1^2}{4\pi}\right)^{\frac{d-3}{2}} \theta(m_0^2 - 4m_1^2). \quad (9)$$

It is seen here that the decay of one particle into two that are globally heavier is forbidden. This is a familiar consequence of the Poincaré invariance of the theory.

Let us examine the dependence of the result (8) on the wavepacket f. The decay rate of a particle at rest (zero momentum) in our frame can be obtained by letting $|\widetilde{f}(p)|^2$ tend to $\delta(\vec{p})$. Any wavepacket of the form $\varepsilon^{(1-d)/2}\widetilde{\varphi}(\vec{p}/\varepsilon)$ will do the job. In the limit $\varepsilon \to 0$ the factor $(2p^0)^{-1}$ in the numerator of (8) becomes $(2m_0)^{-1}$ and everything else cancels:

$$\frac{1}{\tau_0} = \frac{2^{1-d}\gamma^2}{\Gamma(\frac{d-1}{2})} \frac{1}{m_0^2} \left(\frac{m_0^2 - 4m_1^2}{4\pi}\right)^{\frac{d-3}{2}} \theta(m_0^2 - 4m_1^2). \quad (10)$$

Had we chosen a particle of sharp momentum $\vec{p}$, we would have obtained the Lorentz factor with the corresponding velocity $\vec{v} = c\vec{p}/p^0$:

$$\tau(\vec{v}) = \frac{\tau_0}{\sqrt{1 - v^2/c^2}}. \quad (11)$$

Case of de Sitter space

We use the dimensionless parameter ν (see Eq. 2) to label the two-point functions; they can be expressed in terms of Legendre functions of the first kind as follows:

$$w_\nu(z, z') = \frac{\Gamma\left(\frac{d-1}{2} + i\nu\right)\Gamma\left(\frac{d-1}{2} - i\nu\right)}{2(2\pi)^{\frac{d}{2}} R^{d-2}} \times (\zeta^2 - 1)^{-\frac{d-2}{4}}\, P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(\zeta); \quad (12)$$

z, z' are events belonging to the complex de Sitter spacetime; $\operatorname{Im} z$ belongs to the past cone of the ambient spacetime whereas $\operatorname{Im} z'$ belongs to the future cone; the scalar product $\zeta = z \cdot z'/R^2$ is in the ambient spacetime sense (see [6] for details).

The first task is to compute the Källén-Lehmann weight $\rho(\kappa^2; \nu, \nu) \equiv \rho_\nu(\kappa)$. To this purpose we use the following (suitably normalized) generalized Mehler-Fock transform of the squared two-point function:

$$\rho_\nu(\kappa) = \frac{\left(\Gamma\left(\frac{d-1}{2} + i\nu\right)\Gamma\left(\frac{d-1}{2} - i\nu\right)\right)^2 \sinh \pi\kappa}{2(2\pi)^{1+\frac{d}{2}} R^{d-2}} \times \int_1^\infty P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\kappa}(x)\, [P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(x)]^2 (x^2 - 1)^{-\frac{d-2}{4}}\ dx. \quad (13)$$

This integral is well defined for masses such that $|\operatorname{Im}\nu| < \frac{d-1}{4}$; this includes the principal series and a portion of the complementary series. Inversion [9] gives precisely

$$w_\nu^2 = \int_0^\infty d\kappa^2 \rho_\nu(\kappa) w_\kappa(x,y) = \int_{-\infty}^\infty \kappa d\kappa \rho_\nu(\kappa) w_\kappa. \quad (14)$$

The integral (13) can be directly computed for odd d [10]. For even spacetime dimensions computing (13) is far from obvious. We have devised a method based on Mellin transform techniques [11] that allows the computation for any dimension d (real or complex). Similar techniques are at present used to evaluate quite complicated Feynman integrals in flat spacetime [12]. Here is the result:

$$\rho_\nu(\kappa) = \frac{R^{2-d}\sinh \pi\kappa}{(4\pi)^{\frac{d+2}{2}} \sqrt{\pi}\Gamma(\frac{d-1}{2})} \frac{\Gamma\left(\frac{d-1}{4} + \frac{i\kappa}{2}\right)\Gamma\left(\frac{d-1}{4} - \frac{i\kappa}{2}\right)}{\Gamma\left(\frac{d+1}{4} + \frac{i\kappa}{2}\right)\Gamma\left(\frac{d+1}{4} - \frac{i\kappa}{2}\right)} \times \prod_{\epsilon,\epsilon'=\pm} \Gamma\left(\frac{d-1}{4} + i\epsilon\nu + \frac{i\epsilon'\kappa}{2}\right) \quad (15)$$

The weight ρ never vanishes. This means that for $m > m_c$ decay processes into heavier particles are always possible. In particular in that range of masses one is not allowed to draw conclusions about the stability of a certain particle just from its being the lightest in a hierarchy. This result has nothing to do with the standard thermal interpretation of the de Sitter "vacuum". A similar computation in flat thermal field theory does not exhibit this phenomenon in two-particle decays. The Minkowskian result (9) is recovered in the limit of zero curvature that is achieved by setting $\kappa = m_0 R$ and $\nu = m_1 R$:

$$\lim_{R\to\infty} \rho(\kappa^2; \nu, \nu)\ d\kappa^2 = \rho(m_0^2; m_1, m_1)\ dm_0^2. \quad (16)$$

Lowest order corrections to the flat case give:

$$R^2 \rho_{m_1 R}(m_0 R) \sim \frac{|\Delta m|^{\frac{d-3}{2}}}{2^d \pi^{\frac{d-1}{2}} \Gamma\left(\frac{d-1}{2}\right) m_0} \left(\frac{m_0 + 2m_1}{4}\right)^{\frac{d-3}{2}} \times \left(1 + \frac{A}{R^2}\right) \left[\theta(\Delta m) + e^{-|\Delta m| R}\theta(-\Delta m)\right], \quad (17)$$

$\Delta m = m_0 - 2m_1$. The lack of particle stability ($\Delta m < 0$) is exponentially small in R. If $\Delta m > 0$ there is a correction to the flat case of the order of the cosmological constant $\Lambda = \frac{(d-1)(d-2)}{2R^2}$. In the four dimensional case

$$A = \frac{17}{64\left(m_1 + \frac{m_0}{2}\right)^2} - \frac{107}{24 m_0^2} + \frac{17}{64\left(\frac{m_0}{2} - m_1\right)^2} \quad (18)$$

All these effects are of course extremely small with the current value of the cosmological constant. What about particle physics at inflation? At that epoch $mR \sim m \times 10^{-15}\text{GeV}^{-1} \ll \frac{3}{4}$ for every particle of reasonable mass. Our results should therefore be extended to the remaining portion of the complementary series $|\operatorname{Im}\nu| > \frac{d-1}{4}$ where all scalar particles lie at the inflation era (but: there is no complementary series in the Fermionic case). By analytic continuation of Eq. (15) in ν,

$$w_\nu^2 = \int_{-\infty}^{\infty} \kappa\, d\kappa\ \rho_\nu(\kappa) w_\kappa \ + \sum_{n=0}^{N-1} A_n(\nu)\, w_{i(\mu+2i\nu+2n)}$$

$$A_n(\nu) \ = \ \frac{8\pi(-1)^n}{n! 2^d \pi^{\frac{1+d}{2}} R^{d-2}\Gamma(\mu)} \frac{\Gamma(\mu+2i\nu+n)\Gamma(-2i\nu-n)}{\Gamma(\mu+2i\nu+2n)\Gamma(-\mu-2i\nu-2n)} \times \frac{\Gamma(\mu+n)\Gamma(-i\nu-n)\Gamma(\mu+i\nu+n)}{\Gamma(-i\nu-n+\frac{1}{2})\Gamma(\mu+i\nu+n+\frac{1}{2})}$$

where $\mu = (d-1)/2$. The number of discrete terms is the largest N satisfying $N < 1 + |\operatorname{Im}\nu| - \mu/2$, or 0 if this is negative. A particle of the complementary series with parameter $\kappa = i\beta$ can only decay into two particles with parameter $\nu = \frac{i}{2}(|\beta| + \mu + 2n)$, where n is any integer such that $0 \le 2n < \mu - |\beta|$, and the decay is instantaneous. A particle with mass $m \ll m_c$ can only decay into two particles of mass $m_1 \sim m/\sqrt{2}$.

Even if the geometry of the universe at inflation was not exactly de Sitterian, this example indicates that quantum field theoretical arguments concerning particle physics at inflation might need revision.

We now turn to the adiabatic limit and its meaning in the de Sitter context, in the case when all particles are in the principal series. A first complication is the existence of several choices of cosmic time, having different physical implications and the result might depend on one's preferred choice. In the closed model, the cosmic time t is related to the ambient space coordinates as follows: $x^0 = R \sinh(t/R)$. In strict analogy to the Minkowski case, $g(x)$ can be chosen as the indicator function of some cosmic time interval T, say $g(x) = g_T(x) = \theta(T/2 - |t|)$.

In the flat model the situation is a bit more tricky. Cosmic time is now defined by the relation $x^0 + x^d = R\exp(t/R)$; flat coordinates cover only half of the de Sitter manifold, namely all the events such that $x^0 + x^d > 0$. If we introduce the characteristic function $h_T(x) = \theta(Re^{T/2R} - x^0 - x^d)\, \theta(x^0 + x^d - Re^{-T/2R})$ then we have to add the contribution coming from the other half, i.e. $g(x) = g_T(x) = h_T(x) + h_T(-x)$. With these premises we have found that in both models the first factor in (7) diverges like T; thus it has to be divided by T to extract a finite result which is the same in both models:

$$\lim_{T\to\infty} \frac{\gamma^2\, C(\kappa) \int g(x)\, |F(x)|^2\, dx}{T \int \overline{f(x)} w_\kappa(x,\ y)\, f(y)\, dx\, dy} = \frac{\gamma^2 \pi \coth(\pi\kappa)^2}{|\kappa|} \quad (19)$$

Here the second (unforeseen) result comes in: in contrast to the Minkowskian case the limiting probability per unit of time does not depend on the wavepacket! This result seems to contradict what we see everyday in laboratory experiments, a well known effect of special relativity (Eq. 11). Furthermore, in contrast with the violation of particle stability that is exponentially small in the de Sitter radius, this phenomenon does not depend on how small is the cosmological constant. How can we solve this paradox and reconcile the result with everyday experience? The point is that the idea of probability per unit time (Fermi's golden rule) has no scale-invariant meaning in de Sitter: if we use the limiting probability to evaluate amplitudes of processes that take place in a short time we get a grossly wrong result. This is in strong disagreement with what happens in the Minkowski case where the limiting probability is attained almost immediately (i.e. already for finite T). Therefore to describe what we are really doing in a laboratory we should not take the limit $T \to \infty$ and rather use the probability per unit of time relative to a laboratory consistent scale of time. In that case we will recover all the standard wisdom even in presence of a cosmological constant. But, if an unstable particle lives a very long time ($>> R$) and we can accumulate observations then a nonvanishing cosmological constant would radically modify the Minkowski result and de Sitter invariant result will emerge. This result should not be shocking: after all erasing any inhomogeneity is precisely what the quasi de Sitter phase is supposed to do at the epoch of inflation; in the same way, from the viewpoint of an accelerating universe all the long-lived particles look as if they were at rest and so their lifetime would not depend on their peculiar motion.

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