Stickelberger elements and Kolyvagin systems

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ABSTRACT. In this paper we construct (many) Kolyvagin systems out of Stickelberger elements, utilizing ideas borrowed from our previous work on Kolyvagin systems of Stark elements. We show how to apply this construction to prove results on the *odd* parts of the ideal class groups of CM fields which are abelian over a totally real field, and deduce the main conjecture of Iwasawa theory for totally real fields (for totally odd characters). Although the main results of this paper have already been established by Wiles, our approach provides another example (which slightly differs from the case of Stark elements) on how to study *Kolyvagin systems of core rank* $r > 1$ (in the sense of Mazur and Rubin). The analogous (and in some sense complementary) results for *even* parts of the ideal class groups and main conjectures for totally even characters of totally real number fields have been previously obtained by the author using similar ideas.

INTRODUCTION

B. Howard, B. Mazur and K. Rubin show in [MR04] that the existence of Kolyvagin systems relies on a cohomological invariant, what they call the core Selmer rank. When the Selmer core rank is one, they determine the structure of the Selmer group completely in terms of a Kolyvagin system. However, when the Selmer core rank is greater than one, not much could be said.

The author [Büy07a, Büy07c] has studied the Kolyvagin system machinery for the Galois representation $\mathbb{Z}_p(1) \otimes \chi^{-1}$ of the absolute Galois group $G_k := \mathrm{Gal}(\overline{k}/k)$ of a totally real number field k (of degree $[k : \mathbb{Q}] = r$), where $\chi : G_k \to \mathbb{Z}_p^\times$ is a totally even Dirichlet character and $p > 2$ is a prime. This is one of the basic instances of Kolyvagin system theory of core rank $r > 1$. The main idea utilized by the author in this setting was to modify the relevant Selmer groups appropriately and construct Kolyvagin systems, out of Stark elements of

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Rubin [Rub96], to control these modified Selmer groups. In this paper, we are interested in the case when χ is totally odd, which in some sense is the opposite case of [Büy07a, Büy07c]. The G_k-representation in consideration now will be $T = \mathbb{Z}_p(\chi)$, which gives rise to a Kolyvagin system theory of core rank r. We will again use appropriately modified Selmer groups associated with the representation T, and the Euler system which will give rise to Kolyvagin systems that controls the modified Selmer groups will be obtained from Stickelberger elements.

Before we state the main results of this article, we set our notation which will be in effect throughout the paper. Let p, k, G_k, χ and r be as above. We assume that $\chi(\wp) \neq 1$ for any prime $\wp$ of k above p, and further that $\chi \neq \omega$, the Teichmüller character. Let L be the fixed field of $\ker(\chi)$ inside a fixed algebraic closure $\overline{k}$ of k, write $\Delta = \mathrm{Gal}(L/k)$. For any number field K containing L, let A_K be the p-part of the ideal class group of K, and A_K^χ its χ-isotypic part. We fix S as the set of places of k which consists of all infinite places of k, all places λ which divide the conductor $\mathfrak{f}_\chi$ of χ, as well as all the places of k above p. Finally, let $\theta_{L,S} = \theta_L \in \mathbb{Z}_p[\Delta]$ be the *Stickelberger element* (defined precisely in [Kur03, §1.2], see also §2 below) relative to S. The first application of our treatment is the following (Theorem 4.3 below):

Theorem A. *Under the assumptions above*

$$|A_L^\chi| \leq |\mathbb{Z}_p/\chi(\theta_L)\mathbb{Z}_p|.$$

With a bit more work, we can prove the Iwasawa theoretic version of Theorem A. Let k_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of k and assume that any prime of k above p totally ramifies in k_∞/k. This is true, for example, if $k/\mathbb{Q}$ is unramified. Set $\Gamma = \mathrm{Gal}(k_\infty/k)$ and $\Lambda = \mathbb{Z}_p[[\Gamma]]$, as usual. Let $\mathcal{L}_{\omega\chi^{-1}} \in \Lambda$ denote the Deligne-Ribet p-adic L-function (defined in [DR80]). The basic interpolation property which $\mathcal{L}_{\omega\chi^{-1}}$ satisfies is recalled in (4.1). Let $\mathrm{Tw}_{\langle\rho_{\mathrm{cyc}}\rangle}$ be a certain twisting operator on Λ (see §4.2 for its definition).

Theorem B. *Under the assumptions above*

$$\mathrm{char}\left(\varinjlim_n A_{L_n}^\chi)^\vee\right) = \mathrm{Tw}_{\langle\rho_{\mathrm{cyc}}\rangle}(\mathcal{L}_{\omega\chi^{-1}}).$$

As already remarked, these results have already been obtained by Wiles [Wil90b] *unconditionally.* The main objective of this paper is to set up an Euler system and Kolyvagin system machinery for Galois representations of *core rank r* (in the sense of [MR04]) which admit an Euler system of Stickelberger element-type. (For example, one could hope to utilize our approach in this paper for the Euler system obtained from Mazur-Tate elements attached to modular forms; we hope to get back to this point in a future paper.) The situation in this case, as far as the Kolyvagin system machinery (more precisely the Euler system to Kolyvagin system map of [MR04]) is concerned, seems to be slightly different from the case of Stark elements, studied in [Büy07a]. In fact, in a forthcoming paper [Büy08], the set up from the current paper and that from [Büy07b] are combined together to treat the theory of Kolyvagin systems of Euler systems for self dual Galois representations of core rank r[1].

Finally, we remark that thanks to (an appropriately variant of) Proposition 3.4 below, it seems that we may by-pass the need of appealing to Krasner's Lemma in [Büy07a, Büy07c] and hence we may remove the hypothesis that χ is unramified at all primes $\wp$ of k above p on the main results of [Büy07a, Büy07c].

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Notation: Besides what we have fixed above, the following notation will be in effect throughout. For any abelian group A, write

$$A^{\wedge} := \mathrm{Hom}(\mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)$$

for its p-adic completion and write

$$A^{\vee} := \mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$$

for its Pontryagin dual. Suppose in addition that Δ acts on A, we then write A^{χ} for the χ-isotypic component of $A^{\wedge}$.

For k_∞/k as above, let k_n/k be the unique subfield of degree p^n. We set $\Gamma_n = \mathrm{Gal}(k_n/k)$ and write $L_n = Lk_n$. For any prime $\mathfrak{q} \subset k$, let $k(\mathfrak{q})$ denote the p-part of the ray class field extension of k modulo $\mathfrak{q}$. For any square free integral ideal $\mathfrak{q}_1 \cdots \mathfrak{q}_n = \tau \subset k$, we set $k(\tau)$ as the composite

$$k(\tau) = k(\mathfrak{q}_1) \cdots k(\mathfrak{q}_n).$$

Set $\Delta_\tau = \mathrm{Gal}(k(\tau)/k)$. We let $L(\tau) = Lk(\tau)$, $k_n(\tau) = k_n k(\tau)$ and $L_n(\tau) = k_n L(\tau)$.

[1]These are called *Euler systems of rank r* in the terminology of [PR98].

1. Local conditions and Selmer groups

1.1. **Selmer structures on $T = \mathbb{Z}_p(\chi)$.** Below we use the notation that was set in the Introduction. Let $\Gamma := \mathrm{Gal}(k_\infty/k)$, and $\Lambda := \mathbb{Z}_p[[\Gamma]]$ be the cyclotomic Iwasawa algebra.

We first recall Mazur and Rubin's definition of a *Selmer structure*, in particular the *canonical Selmer structure* on T and $T \otimes \Lambda$.

1.1.1. *Local conditions.* Let R be a complete local noetherian ring, and let M be a $R[[G_k]]$-module which is free of finite rank over R. In this paper we will only be interested in the case when $R = \Lambda$ or its certain quotients, and M is $T \otimes \Lambda$ or its relevant quotients by an ideal of Λ. (For example, taking quotients by the augmentation ideal of Λ will give us $\mathbb{Z}_p$ and the representation T.)

For each place λ of k, a *local condition* $\mathcal{F}$ (at λ) on M is a choice of an R-submodule $H^1_{\mathcal{F}}(k_\lambda, M)$ of $H^1(k_\lambda, M)$. For the prime p, a local condition $\mathcal{F}$ at p will be a choice of an R-submodule $H^1_{\mathcal{F}}(k_p, M)$ of the semi-local cohomology group $H^1(k_p, M) := \oplus_{\wp|p} H^1(k_\wp, M)$, where the direct sum is over all the primes $\wp$ of k which lie above p.

For examples of local conditions see [MR04] Definitions 1.1.6 and 3.2.1.

Suppose that $\mathcal{F}$ is a local condition (at the prime λ of k) on M. If M' is a submodule of M (*resp.* M'' is a quotient module), then $\mathcal{F}$ induces local conditions (which we still denote by $\mathcal{F}$) on M' (*resp.* on M''), by taking $H^1_{\mathcal{F}}(k_\lambda, M')$ (*resp.* $H^1_{\mathcal{F}}(k_\lambda, M'')$) to be the inverse image (*resp.* the image) of $H^1_{\mathcal{F}}(k_\lambda, M)$ under the natural maps induced by

$$M' \hookrightarrow M, \qquad M \twoheadrightarrow M''.$$

Definition 1.1. *Propagation* of a local condition $\mathcal{F}$ on M to a submodule M' (and a quotient M'' of M is the local condition $\mathcal{F}$ on M' (and on M'') obtained following the above procedure.

For example, if I is an ideal of R, then a local condition on M induces local conditions on M/IM and $M[I]$, by *propagation*.

Definition 1.2. Define the *Cartier dual* of M to be the $R[[G_k]]$-module

$$M^* := \mathrm{Hom}(M, \mu_{p^\infty})$$

where μ_{p^∞} stands for the p-power roots of unity.

Let λ be a prime of k. There is the perfect local Tate pairing

$$<\,,\,>_\lambda : H^1(k_\lambda, M) \times H^1(k_\lambda, M^*) \longrightarrow H^2(k_\lambda, \mu_{p^\infty}) \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p$$

Definition 1.3. The *dual local condition* $\mathcal{F}^*$ on M^* of a local condition $\mathcal{F}$ on M is defined so that $H^1_{\mathcal{F}^*}(k_\lambda, M^*)$ is the orthogonal complement of $H^1_{\mathcal{F}}(k_\lambda, M)$ under the local Tate pairing $<\,,\,>_\lambda$.

1.1.2. *Selmer structures and Selmer groups.* Notation from §1.1.1 is in effect throughout this section. We will also denote $G_{k_\lambda} = \mathrm{Gal}(\overline{k_\lambda}/k_\lambda)$ by $\mathcal{D}_\lambda$, whenever we would like to identify this group by a closed subgroup of $G_k = \mathrm{Gal}(\overline{k}/k)$; namely with a particular decomposition group at λ in G_k. We further define $\mathcal{I}_\lambda \subset \mathcal{D}_\lambda$ to be the inertia group and $\mathrm{Fr}_\lambda \in \mathcal{D}_\lambda/\mathcal{I}_\lambda$ to be the arithmetic Frobenius element at λ.

Definition 1.4. A *Selmer structure* $\mathcal{F}$ on M is a collection of the following data:

- a finite set $\Sigma(\mathcal{F})$ of places of k, including all infinite places and primes above p, and all primes where M is ramified.
- for every $\lambda \in \Sigma(\mathcal{F})$ a local condition (in the sense of §1.1.1) on M (which we view now as a $R[[\mathcal{D}_\lambda]]$-module), i.e., a choice of R-submodule

$$H^1_{\mathcal{F}}(k_\lambda, M) \subset H^1(k_\lambda, M)$$

If $\lambda \notin \Sigma(\mathcal{F})$ we will also write $H^1_{\mathcal{F}}(k_\lambda, M) = H^1_{\mathrm{f}}(k_\lambda, M)$, where the module $H^1_{\mathrm{f}}(k_\lambda, M)$ is the *finite* part of $H^1(k_\lambda, M)$, defined as in [MR04] Definition 1.1.6.

For a Selmer structure $\mathcal{F}$ on M and for each prime λ of k, define $H^1_{\mathcal{F}^*}(k_\lambda, M^*) := H^1_{\mathcal{F}}(k_\lambda, M)^\perp$ as the orthogonal complement of $H^1_{\mathcal{F}}(k_\lambda, M)$ under the local Tate pairing. The Selmer structure $\mathcal{F}^*$ on M^* (with $\Sigma(\mathcal{F}) = \Sigma(\mathcal{F}^*)$) defined in this way will be called the *dual Selmer structure*.

Definition 1.5. If $\mathcal{F}$ is a Selmer structure on M, we define the *Selmer module* $H^1_{\mathcal{F}}(k, M)$ to be the kernel of the sum of the restriction maps

$$H^1(\mathrm{Gal}(k_{\Sigma(\mathcal{F})}/k), M) \longrightarrow \bigoplus_{\lambda \in \Sigma(\mathcal{F})} H^1(k_\lambda, M)/H^1_{\mathcal{F}}(k_\lambda, M)$$

where $k_{\Sigma(\mathcal{F})}$ is the maximal extension of k which is unramified outside $\Sigma(\mathcal{F})$. We also define the dual Selmer structure in a similar fashion; just replace M by M^* and $\mathcal{F}$ by $\mathcal{F}^*$ above.

Example 1.6. In this example we recall [MR04, Definitions 3.2.1 and 5.3.2].

(i) Let $R = \mathbb{Z}_p$ and let M be a free R-module endowed with a continuos action of G_k, which is unramified outside a finite set of places of k. We define a Selmer structure $\mathcal{F}_{\text{can}}$ on M by setting

$$\Sigma(\mathcal{F}_{\text{can}}) = \{\lambda : M \text{ is ramified at } \lambda\} \cup \{\wp \subset k : \wp | p\} \cup \{v|\infty\}$$

and

– if $\lambda \in \Sigma(\mathcal{F}_{\text{can}}) - \{\wp \subset k : \wp|p\} \cup \{v|\infty\}$, we set $H^1_{\mathcal{F}_{\text{can}}}(k_\lambda, M) = \ker[H^1(k_\lambda, M) \longrightarrow H^1(k_\lambda, M \otimes \mathbb{Q}_p)]$,

– if $\wp|p$, we set

$$H^1_{\mathcal{F}_{\text{can}}}(k_\wp, M) = H^1(k_\wp, M).$$

This is what we call the *canonical Selmer structure* on M.

(ii) Let now $R = \Lambda$ be the cyclotomic Iwasawa algebra, and let $\mathbb{M}$ be a free R-module endowed with a continuos action of G_k, which is unramified outside a finite set of places of k. We define a Selmer structure $\mathcal{F}_\Lambda$ on $\mathbb{M}$ by setting

$$\Sigma(\mathcal{F}_\Lambda) = \{\lambda : M \text{ is ramified at } \lambda\} \cup \{\wp \subset k : \wp|p\} \cup \{v|\infty\}$$

and $H^1_{\mathcal{F}_\Lambda}(k_\lambda, \mathbb{M}) = H^1(k_\lambda, \mathbb{M})$ for $\lambda \in \Sigma(\mathcal{F}_\Lambda)$. This is what we call the *canonical Selmer structure* on $\mathbb{M}$.

As in Definition 1.1, induced Selmer structure on the quotients $\mathbb{M}/I\mathbb{M}$ is still denoted by $\mathcal{F}_\Lambda$. Note that $H^1_{\mathcal{F}_\Lambda}(k_\lambda, \mathbb{M}/I\mathbb{M})$ will not usually be the same as $H^1(k_\lambda, \mathbb{M}/I\mathbb{M})$. In particular, when I is the augmentation ideal inside Λ, $\mathcal{F}_\Lambda$ on $\mathbb{M}$ will not always propagate to $\mathcal{F}_{\text{can}}$ on $M = \mathbb{M} \otimes \Lambda/I$.

However, when $M = T$ and $\mathbb{M} = M \otimes \Lambda$ as in the Introduction, $\mathcal{F}_\Lambda$ *will* propagate to $\mathcal{F}_{\text{can}}$.

Remark 1.7. When $R = \Lambda$ and $\mathbb{M} = T \otimes \Lambda$ (which is the case of interest in this paper), the Selmer structure $\mathcal{F}_{\text{can}}$ of [Büy07b] §2.1 on the quotients $T \otimes \Lambda/(f)$ may be identified[2], under our hypotheses on χ, with the propagation of $\mathcal{F}_\Lambda$ to the quotients $T \otimes \Lambda/(f)$, for every distinguished polynomial f inside Λ.

Definition 1.8. A *Selmer triple* is a triple $(M, \mathcal{F}, \mathcal{P})$ where $\mathcal{F}$ is a Selmer structure on M and $\mathcal{P}$ is a set of rational primes, disjoint from $\Sigma(\mathcal{F})$.

[2]For every prime λ of k, $H^1_{\mathcal{F}_{\text{can}}}(k_\lambda, T \otimes \Lambda/(f))$ is the image of $H^1(k_\lambda, T \otimes \Lambda)$ under the canonical map $H^1(k_\lambda, T \otimes \Lambda) \to H^1(k_\lambda, T \otimes \Lambda/(f))$ by the proofs of [Büy07b] Propositions 2.10 and 2.12. $H^1_{\mathcal{F}_\Lambda}(k_\lambda, T \otimes \Lambda/(f))$ is exactly the same thing by its very definition.

Remark 1.9. Although one might identify the cohomology groups in our setting (when the Galois module in question is $T \otimes \Lambda$ with $T = \mathbb{Z}_p(\chi)$, or its quotients by ideals of Λ) with certain groups of homomorphisms using inflation-restriction, we will insist on using the cohomological language for the sake of notational consistency with [MR04] from which we borrow the main technical results. We also hope that the similarity of ideas applied here and in [Büy07a, Büy07c] are more apparent this way.

1.2. **Computing Selmer groups explicitly.** In this section we compute the Selmer groups for the G_k-representations $T = \mathbb{Z}_p(\chi)$, $T^* = \boldsymbol{\mu}_{p^\infty} \otimes \chi^{-1}$ and for $T \otimes \Lambda$, $(T \otimes \Lambda)^*$; mostly following [Rub00, §I.6.2] and [MR04, §6.1].

1.2.1. *Selmer groups over k.* For any $m \in \mathbb{Z}^+$, it follows (as in [MR04, §6.1]) by inflation-restriction that (recall that L is the CM field cut by χ)

$$H^1(k, T/p^mT) = H^1(k, \mathbb{Z}/p^m\mathbb{Z}(\chi)) \cong \mathrm{Hom}(G_L, \mathbb{Z}/p^m\mathbb{Z})^{\chi^{-1}}, \tag{1.1}$$

and similarly for every prime λ of k

$$H^1(k_\lambda, T/p^mT) \cong \left(\bigoplus_{\mathfrak{q}|\lambda} \mathrm{Hom}(G_{L_\mathfrak{q}}, \mathbb{Z}/p^m\mathbb{Z}) \right)^{\chi^{-1}}, \tag{1.2}$$

and for the semi-local cohomology at a rational prime ℓ

$$H^1(k_\ell, T/p^mT) \cong \bigoplus_{\lambda|\ell} \left(\bigoplus_{\mathfrak{q}|\lambda} \mathrm{Hom}(G_{L_\mathfrak{q}}, \mathbb{Z}/p^m\mathbb{Z}) \right)^{\chi^{-1}}. \tag{1.3}$$

Taking inverse limits, we obtain

$$H^1(k, T) \cong \mathrm{Hom}(G_L, \mathbb{Z}_p)^{\chi^{-1}}, \tag{1.4}$$

and

$$H^1(k_\ell, T) \cong \bigoplus_{\lambda|\ell} \left(\bigoplus_{\mathfrak{q}|\lambda} \mathrm{Hom}(G_{L_\mathfrak{q}}, \mathbb{Z}_p) \right)^{\chi^{-1}}. \tag{1.5}$$

For the dual representation T^*, we have by inflation-restriction and Kummer theory

$$H^1(k, T^*[p^m]) = H^1(k, \boldsymbol{\mu}_{p^m} \otimes \chi^{-1}) \cong \left(L^\times/(L^\times)^{p^m}\right)^\chi, \tag{1.6}$$

and similarly for every prime λ of k

$$H^1(k_\lambda, T^*[p^m]) \cong \left(L_\lambda^\times/(L_\lambda^\times)^{p^m}\right)^\chi, \tag{1.7}$$

and for the semi-local cohomology

$$H^1(k_\ell, T^*[p^m]) \cong \left(L_\ell^\times/(L_\ell^\times)^{p^m}\right)^\chi, \tag{1.8}$$

where $L_\lambda := L \otimes_k k_\lambda$, the sum of completions of L at the primes above λ, and $L_\ell := L \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Taking direct limits, we have

$$H^1(k, T^*) \cong \left(L^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\chi, \tag{1.9}$$

and

$$H^1(k_\ell, T^*) \cong \left(L_\ell^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\chi, \tag{1.10}$$

Proposition 1.10. *The canonical Selmer structure $\mathcal{F}_{\mathrm{can}}$ on T (*resp. *$\mathcal{F}^*_{\mathrm{can}}$ on T^*) is given by taking $\Sigma(\mathcal{F}_{\mathrm{can}}) = \Sigma(\mathcal{F}^*_{\mathrm{can}})$ to be the set of primes where χ is ramified, together with places above p and ∞; and setting (using the identifications above):*

- $H^1_{\mathcal{F}_{\mathrm{can}}}(k_\ell, T) = \left(\bigoplus_{\mathfrak{q}|\ell} \mathrm{Hom}(G_{L_\mathfrak{q}}/\mathcal{I}_\mathfrak{q}, \mathbb{Z}_p)\right)^{\chi^{-1}}$,
  $H^1_{\mathcal{F}^*_{\mathrm{can}}}(k_\ell, T^*) = (\mathcal{O}^\times_{L,\ell} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\chi$, *if* $\ell \neq p$.
- $H^1_{\mathcal{F}_{\mathrm{can}}}(k_p, T) = H^1(k_p, T)$, $H^1_{\mathcal{F}^*_{\mathrm{can}}}(k_p, T^*) = 0$ *at* p.

Here, $\mathcal{I}_\mathfrak{q}$ stands for a fixed inertia group at $\mathfrak{q}$, and $\mathcal{O}_{L,\ell} := \mathcal{O}_L \otimes \mathbb{Z}_\ell$ is the sum of the local units of the completions of L at the primes above ℓ.

Proof. This is proved in [Rub00, §I.6.B & §I.6.C]. □

Definition 1.11. We define the classical Selmer structure $\mathcal{F}_{\mathrm{cl}}$ on T (and $\mathcal{F}^*_{\mathrm{cl}}$ on T^*) by setting $\Sigma(\mathcal{F}_{\mathrm{cl}}) = \Sigma(\mathcal{F}_{\mathrm{can}})$ and letting

- $H^1_{\mathcal{F}_{\mathrm{cl}}}(k_\ell, T) = H^1_{\mathcal{F}_{\mathrm{can}}}(k_\ell, T)$, and
  $H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k_\ell, T^*) = H^1_{\mathcal{F}^*_{\mathrm{can}}}(k_\ell, T^*)$, if $\ell \neq p$.
- $H^1_{\mathcal{F}_{\mathrm{cl}}}(k_p, T) = \left(\bigoplus_{\mathfrak{q}|p} \mathrm{Hom}(G_{L_\mathfrak{q}}/\mathcal{I}_\mathfrak{q}, \mathbb{Z}_p)\right)^{\chi^{-1}}$, and
  $H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k_p, T^*) = (\mathcal{O}^\times_{L,p} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\chi$.

Remark 1.12. Since we assumed $\chi(\wp) \neq 1$ for any prime $\wp \subset k$ above p, it follows from the proof of [Rub00, Proposition III.2.6] (see also [MR04, Lemma 6.1.2]) that $H^1_{\mathcal{F}_{\mathrm{cl}}}(k_p, T) = 0$ and $H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k_p, T^*) = H^1(k_p, T^*)$. We therefore have the following exact sequences

$$0 \longrightarrow H^1_{\mathcal{F}_{\mathrm{cl}}}(k, T) \longrightarrow H^1_{\mathcal{F}_{\mathrm{can}}}(k, T) \xrightarrow{\mathrm{loc}_p} H^1(k_p, T)$$

$$0 \longrightarrow H^1_{\mathcal{F}^*_{\mathrm{can}}}(k, T^*) \longrightarrow H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*) \xrightarrow{\mathrm{loc}^*_p} H^1(k_p, T^*)$$

such that the image of loc_p is the orthogonal complement of the image of loc^*_p. (The final statement is Poitou-Tate global duality.) We also remark that the classical Selmer group $H^1_{\mathcal{F}_{\mathrm{cl}}}(k, T)$ (*resp.* $H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*)$) is denoted by $\mathcal{S}(k, T)$ (*resp.* $\mathcal{S}(k, W^*)$) in [Rub00].

Proposition 1.13. *$H^1_{\mathcal{F}_{\mathrm{cl}}}(k, T) = 0$ and $H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*) \cong A_L^\chi$, where A_L denotes the p-part of the ideal class group of L.*

Proof. Proposition 6.1.3 of [MR04] gives

$$H^1_{\mathcal{F}_{\mathrm{cl}}}(k, T) = \varprojlim_m \mathrm{Hom}(A_L^\chi, \mathbb{Z}/p^m\mathbb{Z}) = \mathrm{Hom}(A_L^\chi, \mathbb{Z}_p);$$

we note that the propagation of $\mathcal{F}_{\mathrm{cl}}$ to $\mathbb{Z}/p^m\mathbb{Z}(\chi)$ coincides with the Selmer structure $\mathcal{F}^*$ of loc.cit. Since A_L^χ is finite, it follows that $H^1_{\mathcal{F}_{\mathrm{cl}}}(k, T) = 0$.

Similarly, the propagation of $\mathcal{F}^*_{\mathrm{cl}}$ to $\boldsymbol{\mu}_{p^m} \otimes \chi^{-1}$ coincides with the Selmer structure $\mathcal{F}$ of loc.cit. Hence, it follows from [MR04, Proposition 6.1.3] that there is an exact sequence

$$0 \longrightarrow \left(\mathcal{O}_L^\times/(\mathcal{O}_L^\times)^{p^m}\right)^\chi \longrightarrow H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*[p^m]) \longrightarrow A_L[p^m]^\chi \longrightarrow 0.$$

Taking direct limit, we obtain the following exact sequence:

$$0 \longrightarrow \left(\mathcal{O}_L^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\chi \longrightarrow H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*) \longrightarrow A_L^\chi \longrightarrow 0.$$

Since χ is totally odd, it follows from [Tat84, Proposition I.3.4] that $(\mathcal{O}_L^\times)^\chi$ is finite, hence $\left(\mathcal{O}_L^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\chi = 0$. This completes the proof of the Proposition. □

1.2.2. *Selmer groups over k_∞.* Let k_n denote the unique subfield k_∞ of degree p^n over k. We also set $L_n = L \cdot k_n$. Repeating the arguments of the previous section (replacing the totally real field k with the totally real field k_n), we prove

Lemma 1.14. $H^1_{\mathcal{F}_{\mathrm{cl}}}(k, (T \otimes \Lambda)^*) := \varinjlim_n H^1_{\mathcal{F}_{\mathrm{cl}}}(k_n, T^*) = \varinjlim_n A^\chi_{L_n}$.

1.3. **Modifying local conditions at p.** When the core Selmer rank of a Selmer structure (in the sense of [MR04], see also §1.5 below) is greater than one, it produces a Selmer group which is difficult to control using the Kolyvagin system machinery of [MR04]. As we will see in 1.5, $\mathcal{F}_{\text{can}}$ on T (*resp.* $\mathcal{F}_\Lambda$ on $T \otimes \Lambda$) will have core Selmer rank $r = [k : \mathbb{Q}]$. Hence, to be able to utilize Kolyvagin system machinery, we will need to modify $\mathcal{F}_{\text{can}}$ and $\mathcal{F}_\Lambda$ appropriately. This is what we do in this section.

1.3.1. *Local conditions at p over k.*

Lemma 1.15. *Under the assumptions above $H^1(k_p, T) := \oplus_{\wp|p} H^1(k_\wp, T)$ is a free $\mathbb{Z}_p$-module of rank $r = [k : \mathbb{Q}]$.*

Proof. We first prove this using the general structure theory of semi-local cohomology groups at p. All the references here are to [Büy07c, Appendix] and the results quoted here are due to Benois, Colmez, Herr and Perrin-Riou.

By Theorem A.8(i), Λ-torsion submodule $H^1(k_p, T \otimes \Lambda)_{\text{tors}}$ is isomorphic to $\oplus_{\wp|p} T^{H_{k_\wp}}$, where $H_{k_\wp} = \text{Gal}(\overline{k_\wp}/k_{\wp,\infty})$. Since we assumed $\chi(\wp) \neq 1$ for any prime $\wp$ of k above p, it follows that $H^1(k_p, T \otimes \Lambda)_{\text{tors}} = 0$. Now Theorem A.8(ii) concludes that Λ-module $H^1(k_p, T \otimes \Lambda)$ is free rank r. Further,

$$\text{coker}[H^1(k_p, T \otimes \Lambda) \longrightarrow H^1(k_p, T)] = H^2(k_p, T \otimes \Lambda)[\gamma - 1],$$

where γ is any topological generator of Γ. However, it follows from [Büy07b, Lemma 2.11] that $H^2(k_p, T \otimes \Lambda) = 0$, hence the map

$$H^1(k_p, T \otimes \Lambda) \longrightarrow H^1(k_p, T)$$

is surjective. Lemma now follows. $\square$

Remark 1.16. There is of course a more direct proof of Lemma 1.15. We also include this alternative proof of this Lemma. By the explicit description of the semi-local cohomology groups in (1.5)

$$H^1(k_p, T) \cong \bigoplus_{\wp|p} \left(\bigoplus_{\mathfrak{q}|\wp} \text{Hom}(G_{L_\mathfrak{q}}, \mathbb{Z}_p) \right)^{\chi^{-1}}.$$

It follows at once from this description that $H^1(k_p, T)$ is $\mathbb{Z}_p$-torsion free, hence free. Further, since $\mathbb{Z}_p$ is an abelian group we may rewrite above equality as

$$H^1(k_p,T)\cong\bigoplus_{\wp|p}\left(\bigoplus_{\mathfrak{q}|\wp}\mathrm{Hom}(G^{\mathrm{ab}}_{L_\mathfrak{q}},\mathbb{Z}_p)\right)^{\chi^{-1}},$$

where $G^{\mathrm{ab}}_{L_\mathfrak{q}}$ stands for the abelianization of $G_{L_\mathfrak{q}}$. By local class field theory $G^{\mathrm{ab}}_{L_\mathfrak{q}}\cong L^{\wedge}_\mathfrak{q}$, the p-adic completion of the multiplicative group of $L_\mathfrak{q}$. Further, via the valuation map

$$L^\times_\mathfrak{q}\xrightarrow{\mathrm{val}_\mathfrak{q}}\mathbb{Z}_p\oplus\mathcal{O}^{\times,\wedge}_{L_\mathfrak{q}}.$$

We therefore have

$$\begin{aligned}H^1(k_p,T)&\cong\mathrm{Hom}\left(\bigoplus_{\mathfrak{q}|p}(\mathbb{Z}_p\oplus\mathcal{O}^{\times,\wedge}_{L_\mathfrak{q}}),\mathbb{Z}_p\right)^{\chi^{-1}}\\&\cong\mathrm{Hom}\left(\left(\oplus_{\mathfrak{q}|p}\mathbb{Z}_p\right)^\chi\oplus\left(\oplus_{\mathfrak{q}|p}\mathcal{O}^{\times,\wedge}_{L_\mathfrak{q}}\right)^\chi,\mathbb{Z}_p\right)\end{aligned}$$

Since $\chi(\wp)\neq 1$ for any $\wp\subset k$ above p, it follows that $\left(\oplus_{\mathfrak{q}|p}\mathbb{Z}_p\right)^\chi=0$, hence

$$H^1(k_p,T)\cong\mathrm{Hom}\left(\left(\oplus_{\mathfrak{q}|p}\mathcal{O}^{\times,\wedge}_{L_\mathfrak{q}}\right)^\chi,\mathbb{Z}_p\right).$$

To prove the Lemma, it suffices to check that the $\mathbb{Q}_p$-dimension of $\left(\oplus_{\mathfrak{q}|p}\mathcal{O}^{\times,\wedge}_{L_\mathfrak{q}}\otimes\mathbb{Q}_p\right)^\chi$ is r. However, p-adic logarithm gives a homomorphism $\mathcal{O}^{\times,\wedge}_{L_\mathfrak{q}}\to\mathcal{O}_{L_\mathfrak{q}}$ with finite kernel and cokernel. Hence

$$\left(\oplus_{\mathfrak{q}|p}\mathcal{O}^{\times,\wedge}_{L_\mathfrak{q}}\otimes\mathbb{Q}_p\right)^\chi=\left(\oplus_{\mathfrak{q}|p}\mathcal{O}_{L_\mathfrak{q}}\otimes\mathbb{Q}_p\right)^\chi=(L\otimes\mathbb{Q}_p)^\chi$$

and therefore has $\mathbb{Q}_p$-dimension r by normal basis theorem.

Definition 1.17. Fix a $\mathbb{Z}_p$-rank one direct summand $\mathcal{L}\subset H^1(k_p,T)$ and a generator $\varphi=\varphi_\mathcal{L}$ of $\mathcal{L}$. Define the *$\mathcal{L}$-modified Selmer structure* $\mathcal{F}_\mathcal{L}$ on T as follows:

- $\Sigma(\mathcal{F}_\mathcal{L})=\Sigma(\mathcal{F}_{\mathrm{can}})$,
- if $\lambda\nmid p$, $H^1_{\mathcal{F}_\mathcal{L}}(k_\lambda,T)=H^1_{\mathcal{F}_{\mathrm{can}}}(k_\lambda,T)$,
- $H^1_{\mathcal{F}_\mathcal{L}}(k_p,T):=\mathcal{L}\subset H^1(k_p,T)=H^1_{\mathcal{F}_{\mathrm{can}}}(k_p,T)$.

1.3.2. *Local conditions at p over k_∞.* Set $\Gamma = \mathrm{Gal}(k_\infty/k)$, as before. Since we assumed that k_∞/k is totally ramified at all primes $\wp \subset k$ over p. Let $k_\wp$ denote the completion of k at $\wp$, and let $k_{\wp,\infty}$ denote the cyclotomic $\mathbb{Z}_p$-extension of $k_\wp$. We may identify $\mathrm{Gal}(k_{\wp,\infty}/k_\wp)$ by Γ for all $\wp|p$ and henceforth Γ will stand for any of these Galois groups. $\Lambda = \mathbb{Z}_p[[\Gamma]]$ is the cyclotomic Iwasawa algebra, as usual. We also fix a topological generator γ of Γ, and we set $\mathbf{X} = \gamma - 1$ (and occasionally we identify Λ by the power series ring $\mathbb{Z}_p[[\mathbf{X}]]$ in one variable).

Lemma 1.18. *Under the running assumptions*

$$H^1(k_p, T \otimes \Lambda) := \oplus_{\wp|p} H^1(k_\wp, T \otimes \Lambda)$$

is a free Λ-module of rank r.

Proof. This is already proved in the first part of the proof of Lemma 1.15. □

Definition 1.19. Fix a Λ-rank one direct summand $\mathbb{L} \subset H^1(k_p, T\otimes\Lambda)$ such that $\mathbb{L}$ maps onto $\mathcal{L}$ under the projection

$$H^1(k_p, T \otimes \Lambda) \twoheadrightarrow H^1(k_p, T);$$

and a generator $\Phi = \Phi_{\mathbb{L}}$ of $\mathbb{L}$ which maps to $\varphi = \varphi_{\mathcal{L}}$ under the projection above. Define the $\mathbb{L}$-*modified Selmer structure* $\mathcal{F}_{\mathbb{L}}$ on $T \otimes \Lambda$ as follows:

- $\Sigma(\mathcal{F}_{\mathbb{L}}) = \Sigma(\mathcal{F}_\Lambda)$,
- if $\lambda \nmid p$, $H^1_{\mathcal{F}_{\mathbb{L}}}(k_\lambda, T \otimes \Lambda) = H^1_{\mathcal{F}_\Lambda}(k_\lambda, T \otimes \Lambda)$,
- $H^1_{\mathcal{F}_{\mathbb{L}}}(k_p, T \otimes \Lambda) := \mathbb{L} \subset H^1(k_p, T \otimes \Lambda) = H^1_{\mathcal{F}_\Lambda}(k_p, T \otimes \Lambda)$.

Remark 1.20. By definition, the image of $H^1_{\mathcal{F}_{\mathbb{L}}}(k_p, T\otimes\Lambda)$ is $H^1_{\mathcal{F}_{\mathcal{L}}}(k_p, T)$ under the map $H^1(k_p, T \otimes \Lambda) \to H^1(k_p, T)$. Further, it follows from [MR04, Lemma 5.3.1(ii)] for $\ell \neq p$ that $H^1_{\mathcal{F}_{\mathbb{L}}}(k_\ell, T \otimes \Lambda)$ also maps to $H^1_{\mathcal{F}_{\mathcal{L}}}(k_\ell, T)$ under the natural map $H^1(k_\ell, T\otimes\Lambda) \to H^1(k_\ell, T)$. In other words, $\mathcal{F}_{\mathbb{L}}$ propagates to $\mathcal{F}_{\mathcal{L}}$, therefore we have an induced map

$$H^1_{\mathcal{F}_{\mathbb{L}}}(k, T \otimes \Lambda) \longrightarrow H^1_{\mathcal{F}_{\mathcal{L}}}(k, T).$$

1.4. **Global duality and a comparison of Selmer groups.** In this section, we compare classical Selmer groups (which we wish to relate to the L-values) to modified Selmer groups (for which we are able to apply the Kolyvagin system machinery and compute in terms of L-values). The necessary tool to accomplish this comparison is Poitou-Tate global duality.

1.4.1. *Comparison over k.* The definition of the modified Selmer structure $\mathcal{F}_{\mathcal{L}}$ and Remark 1.12 gives us the following exact sequences:

$$0 \longrightarrow H^1_{\mathcal{F}_{\mathrm{cl}}}(k,T) \longrightarrow H^1_{\mathcal{F}_{\mathcal{L}}}(k,T) \xrightarrow{\mathrm{loc}_p} \mathcal{L}$$

$$0 \longrightarrow H^1_{\mathcal{F}^*_{\mathcal{L}}}(k,T^*) \longrightarrow H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k,T^*) \xrightarrow{\mathrm{loc}^*_p} \frac{H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k_p,T^*)}{H^1_{\mathcal{F}^*_{\mathcal{L}}}(k_p,T^*)}$$

Poitou-Tate global duality (cf. [Rub00, Theorem I.7.3], [Mil86, Theorem I.4.10]) allows us to compare the image of loc_p to the image of loc^*_p, and together with Proposition 1.13 gives:

Proposition 1.21. *We have an exact sequence*

$$0 \to H^1_{\mathcal{F}_{\mathcal{L}}}(k,T) \xrightarrow{\mathrm{loc}_p} \mathcal{L} \xrightarrow{(\mathrm{loc}^*_p)^\vee} \Big(H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k,T^*)\Big)^\vee \longrightarrow \Big(H^1_{\mathcal{F}^*_{\mathcal{L}}}(k,T^*)\Big)^\vee \to 0,$$

where the map $(\mathrm{loc}^*_p)^\vee$ is induced from localization at p and the local Tate pairing between $H^1(k_p,T)$ and $H^1(k_p,T^*)$.

Suppose $c \in H^1_{\mathcal{F}_{\mathcal{L}}}(k,T)$ is any class. We still write c for the image of the class c inside $\mathcal{L} = H^1_{\mathcal{F}_{\mathcal{L}}}(k_p,T)$ under the (injective) map loc_p. It follows from Proposition 1.21 that

Corollary 1.22. *The following sequence is exact:*

$$0 \to \frac{H^1_{\mathcal{F}_{\mathcal{L}}}(k,T)}{\mathbb{Z}_p \cdot c} \xrightarrow{\mathrm{loc}_p} \frac{\mathcal{L}}{\mathbb{Z}_p \cdot c} \xrightarrow{(\mathrm{loc}^*_p)^\vee} \Big(H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k,T^*)\Big)^\vee \to \Big(H^1_{\mathcal{F}^*_{\mathcal{L}}}(k,T^*)\Big)^\vee \to 0$$

1.4.2. *Comparison over k_∞.* Repeating the argument of Proposition 1.21 for each field k_n (instead of k) and passing to inverse limit we obtain the following:

Proposition 1.23. *Both of the following sequences of Λ-modules are exact:*

(i) $0 \to H^1_{\mathcal{F}_{\mathbb{L}}}(k,T\otimes\Lambda) \xrightarrow{\mathrm{loc}_p} \mathbb{L} \longrightarrow \Big(\varinjlim_n A^\chi_{L_n}\Big)^\vee \to \Big(H^1_{\mathcal{F}^*_{\mathbb{L}}}(k,(T\otimes\Lambda)^*)\Big)^\vee \to 0,$

(ii) *For any class $c \in H^1(k,T\otimes\Lambda)$,*

$$0 \to \frac{H^1_{\mathcal{F}_{\mathbb{L}}}(k,T\otimes\Lambda)}{\Lambda\cdot c} \xrightarrow{\mathrm{loc}_p} \frac{\mathbb{L}}{\Lambda\cdot c} \longrightarrow \Big(\varinjlim_n A^\chi_{L_n}\Big)^\vee \to \Big(H^1_{\mathcal{F}^*_{\mathbb{L}}}(k,(T\otimes\Lambda)^*)\Big)^\vee \to 0.$$

Proof. We give a sketch. Thanks to [Rub00, Proposition B.1.1], we have an exact sequence

$$0 \to \varprojlim_n H^1_{\mathcal{F}_{\mathcal{L}_n}}(k_n,T) \longrightarrow \varprojlim_n \mathcal{L}_n \longrightarrow \Big(\varinjlim_n H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k_n,T^*)\Big)^\vee \to \Big(\varinjlim_n H^1_{\mathcal{F}^*_{\mathcal{L}_n}}(k_n,T^*)\Big)^\vee \to 0,$$

where $\mathcal{L}_n$ is the image of $\mathbb{L}$ under the natural map

$$H^1(k_p, T \otimes \Lambda) \longrightarrow H^1(k_p, T).$$

By definition $\varprojlim_n \mathcal{L}_n = \mathbb{L}$, and by [MR04, Lemma 5.3.1] (or rather by its proof) it follows that

$$\varprojlim_n H^1_{\mathcal{F}_{\mathcal{L}_n}}(k_n, T) \cong H^1_{\mathcal{F}_\Lambda}(k, T \otimes \Lambda),$$

canonically. Further, by Lemma 1.14 $\varinjlim_n H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k_n, T^*) = \varinjlim_n A^\chi_{L_n}$. Finally, by Shapiro's lemma

$$H^1(k_n, T^*) = H^1(k, T^* \otimes \mathbb{Z}_p[\Gamma_n]),$$

where $\Gamma_n = \mathrm{Gal}(k_n/k)$, hence

$$\varinjlim_n H^1(k_n, T^*) = H^1(k, \varinjlim_n T^* \otimes \mathbb{Z}_p[\Gamma_n]). \tag{1.11}$$

Now, using the fact that the functors $-\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\Gamma_n]$ and $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[\Gamma_n], -)$ are adjoint functors (we drop the subscripts below and write $\otimes$ and Hom for short), it follows that

$$\begin{aligned}(T \otimes \Lambda)^* &:= \mathrm{Hom}(\varprojlim_n T \otimes \mathbb{Z}_p[\Gamma_n], \mathbb{Q}_p/\mathbb{Z}_p)(1)\\ &\cong \varinjlim_n \mathrm{Hom}(T, \mathrm{Hom}(\mathbb{Z}_p[\Gamma_n], \mathbb{Q}_p/\mathbb{Z}_p))(1)\\ &\cong \varinjlim_n \mathrm{Hom}(T, \mathbb{Q}_p/\mathbb{Z}_p[\Gamma_n])(1)\\ &\cong \varinjlim_n \mathrm{Hom}(T, \mathbb{Q}_p/\mathbb{Z}_p)(1) \otimes \mathbb{Z}_p[\Gamma_n] =: \varinjlim_n T^* \otimes \mathbb{Z}_p[\Gamma_n],\end{aligned}$$

where the isomorphism of the second and the third line comes from

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}_p[\Gamma_n].\mathbb{Q}_p/\mathbb{Z}_p) &\xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p[\Gamma_n]\\ f &\longmapsto \textstyle\sum_{\gamma \in \Gamma_n} f(\gamma) \cdot \gamma\end{aligned}$$

of $\mathbb{Z}_p[\Gamma_n]$-modules. Now from this and (1.11) (together with its semi-local analogue) it follows at once that

$$\varinjlim_n H^1_{\mathcal{F}^*_{\mathcal{L}_n}}(k_n, T^*) = H^1_{\mathcal{F}^*_{\mathbb{L}}}(k, (T \otimes \Lambda)^*).$$

This completes the proof of (i) and (ii) follows trivially from (i). $\square$

1.5. **Kolyvagin systems for modified Selmer groups - I.** This section closely follows the exposition of [Büy07a, §1.2] and [Büy07c, §2.5].

One easily verifies that the classical Selmer structure $(T, \mathcal{F}_{cl})$ and the modified Selmer structure $(T, \mathcal{F}_{\mathcal{L}})$ satisfy the hypotheses H.0-H.5 and (by [MR04, Lemma 3.7.1]) H.6 of [MR04, §3.5] (with base field $\mathbb{Q}$ in their treatment replaced by k). Therefore, the existence of Kolyvagin systems for these Selmer structures will be decided by their *core Selmer ranks* (for a definition *cf*. [MR04, Definitions 4.1.8 and 4.1.11]). Let $\mathcal{X}(T, \mathcal{F})$ denote the core Selmer ranks of the Selmer structures $(T, \mathcal{F})$ for $\mathcal{F} = \mathcal{F}_{\mathrm{can}}$ or for $\mathcal{F} = \mathcal{F}_{\mathcal{L}}$. Since the hypotheses H.0-H.5 and (by Lemma 3.7.1 of [MR04]) H6 hold, $\mathcal{X}(T, \mathcal{F})$ will be (as in Definition 5.2.4 of [MR04], using Theorem 4.1.3 of [MR04]) the common value of $\mathcal{X}(T/p^nT, \mathcal{F})$.

Proposition 1.24. $\mathcal{X}(T, \mathcal{F}_{\mathrm{can}}) = r$ *(where we recall that* $r = [k : \mathbb{Q}]$*).*

Proof. This follows from [MR04, Theorem 5.2.15] (either following its proof with the base field $\mathbb{Q}$ replaced by k (therefore we have r real places instead of one), or applied directly with the $G_{\mathbb{Q}}$-representation $\mathrm{Ind}_{G_k}^{G_{\mathbb{Q}}} T$) and our assumption that χ is totally odd. □

Proposition 1.25. *The core Selmer rank of the Selmer structure* $(T, \mathcal{F}_{\mathcal{L}})$ *is one.*

Proof. By Proposition 1.6 of [Wil95]

$$\begin{gathered}
\mathrm{length}(H^1_{\mathcal{F}_{\mathcal{L}}}(k, T/p^nT)) - \mathrm{length}(H^1_{\mathcal{F}^*_{\mathcal{L}}}(k, T^*[p^n])) = \\
\mathrm{length}(H^0(k, T/p^nT)) - \mathrm{length}(H^0(k, T^*[p^n])) \\
- \sum_{\ell | f_\chi p} \{\mathrm{length}(H^0(k_\ell, T/p^nT) - \mathrm{length}(H^1_{\mathcal{F}_{\mathcal{L}}}(k_\ell, T/p^nT))\}
\end{gathered}$$

which is $p^n \mathcal{X}(T, \mathcal{F}_{\mathcal{L}})$. Applying the same formula to $(T, \mathcal{F}_{\mathrm{can}})$ we see that

$$p^n(\mathcal{X}(T, \mathcal{F}_{\mathrm{can}}) - \mathcal{X}(T, \mathcal{F}_{\mathcal{L}})) = \mathrm{length}(H^1_{\mathcal{F}_{\mathrm{can}}}(k_p, T/p^nT)) - \mathrm{length}(H^1_{\mathcal{F}_{\mathcal{L}}}(k_p, T/p^nT))$$

and this equals $(r-1)p^n$ by the very definition of the modified Selmer structure. We already know by Proposition 1.24 that $\mathcal{X}(T, \mathcal{F}_{\mathrm{can}}) = r$. The proof follows.

□

1.5.1. *Kolyvagin systems over k - I.* Let $\mathbf{KS}(T,\mathcal{F}_{\mathcal{L}})$ denote the $\mathbb{Z}_p$-module of Kolyvagin systems for the Selmer structure $(T,\mathcal{F}_{\mathcal{L}})$. See [MR04, Definition 3.1.3] for a precise definition.

Proposition 1.26. *The $\mathbb{Z}_p$-module $\mathbf{KS}(T,\mathcal{F}_{\mathcal{L}})$ is free of rank* one*, generated by a Kolyvagin system $\kappa \in \mathbf{KS}(T,\mathcal{F}_{\mathcal{L}})$ whose image (under the canonical map induced from reduction mod p) in $\mathbf{KS}(T/pT,\mathcal{F}_{\mathcal{L}})$ is nonzero.*

Proof. This is immediate after Proposition 1.25 and Theorem 5.2.10 of [MR04]. □

Remark 1.27. Note that the *choice* of a rank one direct summand $\mathcal{L} \subset H^1(k_p,T)$ somewhat makes our approach unnatural. We address this issue in this Remark. Put

$$H^1(k_p,T) = \bigoplus_{i=1}^{r} \mathcal{L}_i \tag{1.12}$$

(where each $\mathcal{L}_i$ is a free $\mathbb{Z}_p$-submodule of $H^1(k_p,T)$ of rank one) and consider

$$\sum_{i=1}^{r} \mathbf{KS}(T,\mathcal{F}_{\mathcal{L}_i}) \subset \mathbf{KS}(T,\mathcal{F}_{\text{can}}). \tag{1.13}$$

Claim. *The sum in (1.13) is in fact a direct sum.*

Proof. Assume contrary: Suppose $0 \neq \boldsymbol{\kappa}^i \in \mathbf{KS}(T,\mathcal{F}_{\mathcal{L}_i})$ (for $i = 1,\ldots,r$) is such that

$$\sum_{i=1}^{r} a_i \boldsymbol{\kappa}^i = 0$$

for some $a_i \in \mathbb{Z}_p$, and $a_{i_0} \neq 0$ for a certain $1 \leq i_0 \leq r$. This means

$$a_{i_0}\boldsymbol{\kappa}^{i_0} = -\sum_{\substack{i=1\\ i\neq i_0}}^{r} a_i \boldsymbol{\kappa}^i \in \sum_{\substack{i=1\\ i\neq i_0}}^{r} \mathbf{KS}(T,\mathcal{F}_{\mathcal{L}_i}). \tag{1.14}$$

Write $\boldsymbol{\kappa}^{i_0} = \{\kappa_n^{i_0}\}$ (see [MR04, §3] for a precise definition of a Kolyvagin system to clarify this notation). (1.14) therefore shows that

$$\text{loc}_p(a_{i_0}\kappa_1^{i_0}) \in \bigoplus_{\substack{i=1\\ i\neq i_0}}^{r} \mathcal{L}_i.$$

On the other hand, by definition, $\text{loc}_p(a_{i_0}\kappa_1^{i_0}) \in \mathcal{L}_{i_0}$, from which we see that $\text{loc}_p(a_{i_0}\kappa_1^{i_0}) = 0$. Further, injectivity of loc_p (see §1.4.1) gives $a_{i_0}\kappa_1^{i_0} = 0$.

On the other hand, Proposition 1.21 (applied with $\mathcal{L}=\mathcal{L}_{i_0}$) shows that $H^1_{\mathcal{F}^*_{\mathcal{L}_{i_0}}}(k,T^*)$ is finite (as the finite group $H^1_{\mathcal{F}^*_{\rm cl}}(k,T^*)^\vee=(A_L^\chi)^\vee$ maps onto its Pontryagin dual), which in return shows, by [MR04, 5.2.12(v)], that for any $\boldsymbol{\kappa}=\{\kappa_n\}\in\mathbf{KS}(T,\mathcal{F}_{\mathcal{L}_{i_0}})$ we have $\kappa_1\neq 0$. Therefore, $a_{i_0}\kappa_1^{i_0}=0$ implies that $a_{i_0}\kappa^{i_0}=0$, a contradiction. □

Note that, in order to prove the Claim above, we used the facts that loc_p is injective (on $H^1_{\mathcal{F}_{\mathcal{L}_{i_0}}}(k,T)$) and that $H^1_{\mathcal{F}^*_{\mathcal{L}_{i_0}}}(k,T^*)$ is finite in our current setting. With a bit more work, it is possible to prove this Claim without having either of these conditions.

It would be very interesting to have an answer for the following question:

Question: Is the direct sum

$$\bigoplus_{i=1}^{r}\mathbf{KS}(T,\mathcal{F}_{\mathcal{L}_i})\subset\mathbf{KS}(T,\mathcal{F}_{\rm can})$$

independent of the choice of the decomposition (1.12)?

When the answer to this question is affirmative, we would have a *canonical* rank r submodule of $\mathbf{KS}(T,\mathcal{F}_{\rm can})$. It would be even more interesting to see if this rank r submodule descends from Euler systems. Below, we construct such a (rank r) submodule of $\mathbf{KS}(T,\mathcal{F}_{\rm can})$ out of Stickelberger elements; which still does depend on the decomposition (1.12).

1.5.2. *Kolyvagin systems over k_∞ - I.* The analogue of [MR04, Theorem 5.2.10] for the big Galois representation $T\otimes\Lambda$ has been proved by the author in [Büy07b, Theorem 3.23]. Using this result together with Proposition 1.26 we prove:

Proposition 1.28. *he Λ-module of Kolyvagin Systems $\overline{\mathbf{KS}}(T\otimes\Lambda,\mathcal{F}_{\mathbb{L}})$ for the Selmer structure $\mathcal{F}_{\mathbb{L}}$ on $T\otimes\Lambda$ is free of rank one. Further, the map*

$$\overline{\mathbf{KS}}(T\otimes\Lambda,\mathcal{F}_{\mathbb{L}})\longrightarrow\overline{\mathbf{KS}}(T,\mathcal{F}_{\mathcal{L}})$$

is surjective.

See [Büy07b] §§3.1-3.2 for a precise definition of the Λ-module of *Λ-adic Kolyvagin systems* $\overline{\mathbf{KS}}(T\otimes\Lambda,\mathcal{F}_{\mathbb{L}})$.

The proof of this Proposition is identical to the proof of [Büy07c, Theorem 2.15]. We refer the reader to §2.5 of loc.cit. for details.

In §3.2 below we explain how to obtain these Kolyvagin systems out of Stickelberger elements, *assuming Brumer's conjecture.*

2. Euler systems from Stickelberger elements

We begin with recalling the definition of Stickelberger elements. We first set some notation. Assume $k, \chi, \mathfrak{f} = \mathfrak{f}_\chi$ and L are as above. For a (square free) cycle $\tau = \mathfrak{q}_1 \dots \mathfrak{q}_m$ of the number field k let $k(\tau)$ be the compositum

$$k(\tau) = k(\mathfrak{q}_1) \cdots k(\mathfrak{q}_m)$$

where $k(\mathfrak{q})$ denotes the maximal p-extension inside the ray class field of k modulo the prime ideal $\mathfrak{q}$. For any field K, define $K(\tau)$ as the composite of $k(\tau)$ and K. Let

$$\mathcal{K} = \{L_n(\tau) : \tau \text{ is a (finite) square free cycle of } k \text{ prime to } \mathfrak{f}p\}$$

be a collection of abelian extensions of k, where $k(\tau)$, L, $L(\tau)$ and $\mathfrak{f}$ are defined above. Note that any $L_n(\tau) \in \mathcal{K}$ is CM and abelian over the totally real field k. Let S be the set of places of k, consisting all places above p, all places dividing $\mathfrak{f}$ and all infinite places. For any $K \in \mathcal{K}$, write S_K for the set of all places of the field K lying above the places in S. When there is no confusion, we will only write S for S_K as well.

For any $K \in \mathcal{K}$, the partial zeta function for $\sigma \in \mathrm{Gal}(K/k)$ is defined as usual by

$$\zeta_S(s, \sigma) := \sum_{\substack{(\mathfrak{a}, K/k) = \sigma \\ \mathfrak{a} \text{ is prime to } S}} \mathbf{N}\mathfrak{a}^{-s}$$

for $\mathrm{Re}(s) > 1$, here $\mathbf{N}\mathfrak{a}$ is the norm of the ideal $\mathfrak{a} \in k$, and $(\mathfrak{a}, K/k)$ is the Artin symbol. The partial zeta functions admit a meromorphic continuation to the whole complex plane, and holomorphic everywhere except at $s = 1$. We may therefore set

$$\theta_K = \theta_{K,S} := \sum_{\sigma \in \mathrm{Gal}(K/k)} \zeta_S(0, \sigma)\sigma^{-1}$$

which is an element of $\mathbb{Q}[\mathrm{Gal}(K/k)]$ thanks to [Sie70]. Further, it is known, thanks to Deligne and Ribet [DR80], that $\theta_K^\chi \in \mathbb{Z}_p[\mathrm{Gal}(K/k)]^\chi$.

Lemma 2.1. *For any $L_n(\tau) = K \subset K' = L_n(\tau')$ inside $\mathcal{K}$*

$$\theta_{K'}|_K = \prod_{\mathfrak{q} | \tau', \mathfrak{q} \nmid \tau} (1 - \mathrm{Fr}_{\mathfrak{q}}^{-1})\theta_K.$$

Proof. This follows from [Tat84, Proposition IV.1.8]. □

As before, let A_K denote the p-part of the ideal class group of $K \in \mathcal{K}$, and A_K^χ its χ-isotypic part. Until the end of this section we assume the χ-part of the Brumer's conjecture:

Assumption 2.2. θ_K^χ annihilates A_K^χ.

Remark 2.3. Greither [Gre04, Corollary 2.7] and Kurihara [Kur03, Corollary 2.4] have proved that this assumption follows from the main conjectures of Iwasawa theory in this setting (which holds thanks to [Wil90b]) and the vanishing of Iwasawa μ-invariant for K. However, we wish not to assume main conjectures; in fact rather assume 2.2 and deduce main conjectures.

Having said that, we should warn the reader about one minor point: If a prime $\wp \subset k$ above p is unramified in K/k, then Kurihara's Stickelberger element $\tilde{\theta}_K^\chi$ differs from our θ_K^χ by a factor of $(1-\mathrm{Fr}_\wp)^\chi$, where $\mathrm{Fr}_\wp$ is the Frobenius of $\wp$ for the unramified extension K/k. Since we assumed $\chi(\wp) \neq 1$ for any $\wp \subset k$ above p, it follows that $(1-\mathrm{Fr}_\wp)^\chi$ is a unit of $\mathbb{Z}_p[\mathrm{Gal}(K/k)]^\chi$. Therefore the statement of 2.2 is still equivalent to the statement $\tilde{\theta}_K^\chi \cdot A_K^\chi = 0$.

Suppose F is any finite abelian extension of k, and $K = FL$. Then by the inflation-restriction sequence and class field theory one has

$$H^1(F, \mathbb{Z}_p(\chi)) \cong H^1(K, \mathbb{Z}_p)^{\chi^{-1}} = \mathrm{Hom}(\mathbb{A}_K^\times/K^\times, \mathbb{Z}_p)^{\chi^{-1}} \tag{2.1}$$

where $\mathbb{A}_K^\times$ denotes the ideles of K. Since any continuous homomorphism of $\mathbb{A}_K^\times$ into $\mathbb{Z}_p$ should vanish on

$$B_K := \prod_{w|\infty} K_w^\times \times \prod_{w|p} \{1\} \times \prod_{w\nmid p\infty} \mathcal{O}_{K_w}^\times \subset \mathbb{A}_K^\times,$$

(2.1) gives

$$H^1(F, \mathbb{Z}_p(\chi)) \cong \mathrm{Hom}(\mathbb{A}_K^\times/K^\times B_K, \mathbb{Z}_p)^{\chi^{-1}} = \mathrm{Hom}\left(\left(\mathbb{A}_K^\times/K^\times B_K\right)^\chi, \mathbb{Z}_p\right). \tag{2.2}$$

Further, there is an exact sequence

$$0 \longrightarrow U_K/\overline{\mathcal{O}_K^\times} \longrightarrow \mathbb{A}_K^\times/K^\times B_K \longrightarrow A_K \longrightarrow 0$$

which is induced from the map that sends an idele to the corresponding ideal class. Here $\overline{\mathcal{O}_K^\times}$ is the closure of the global units $\mathcal{O}_K^\times$ in the local units U_K inside $K \otimes \mathbb{Q}_p$. Since taking χ-parts is exact (as the order of χ is prime to p) we obtain an exact sequence

$$0 \longrightarrow U_K^\chi/(\overline{\mathcal{O}_K^\times})^\chi \longrightarrow (\mathbb{A}_K^\times/K^\times B_K)^\chi \longrightarrow A_K^\chi \longrightarrow 0.$$

Thus, by Assumption 2.2, multiplication by θ_K^χ gives a map

$$(\mathbb{A}_K^\times/K^\times B_K)^\chi \xrightarrow{\theta_K^\chi} U_K^\chi/(\overline{\mathcal{O}_K^\times})^\chi,$$

and since we assumed χ is totally odd $(\overline{\mathcal{O}_K^\times})^\chi$ is finite (see the final paragraph of the proof of Proposition 1.13), therefore there is an induced map

$$(\mathbb{A}_K^\times/K^\times B_K)^\chi \xrightarrow{\theta_K^\chi} U_K^\chi/(U_K^\chi)_{\mathrm{tors}}. \tag{2.3}$$

Suppose we are given a collection of homomorphisms $\boldsymbol{\lambda} = \{\lambda_n^\tau\}$ with $\lambda_n^\tau \in \mathrm{Hom}(U_{L_n(\tau)}^\chi, \mathbb{Z}_p)$ such that for all $L_n(\tau), L_{n'}(\tau\mathfrak{q}) \in \mathcal{K}$ the following diagram

$$\begin{array}{ccc} U^\chi_{L_{n'}(\tau\mathfrak{q})} & \xrightarrow{\lambda_{n'}^{\tau\mathfrak{q}}} & \\ {\scriptstyle -\mathrm{Fr}_\mathfrak{q}}\big\uparrow & & \mathbb{Z}_p \\ U^\chi_{L_n(\tau)} & \xrightarrow{\lambda_n^\tau} & \end{array}$$

commutes and $\lambda_{n'}^\tau|_{U^\chi_{L_n(\tau)}} = \lambda_n^\tau$ for $n' \geq n$. Define

$$\tilde{c}_{k_n(\tau)} \in \mathrm{Hom}\left(\left(\mathbb{A}_{L_n(\tau)}^\times/(L_n(\tau))^\times B_{L_n(\tau)}\right)^\chi, \mathbb{Z}_p\right)$$

(which we view also as an element of $H^1(k_n(\tau), T)$ via the identification (2.2)) as the composition[3]

$$\tilde{c}_{k_n(\tau)} : \left(\mathbb{A}_{L_n(\tau)}^\times/(L_n(\tau))^\times B_{L_n(\tau)}\right)^\chi \xrightarrow{\theta^\chi_{L_n(\tau)}} U^\chi_{L_n(\tau)}/(U_{L_n(\tau)})^\chi_{\mathrm{tors}} \xrightarrow{\lambda_n^\tau} \mathbb{Z}_p.$$

Set $\tilde{\mathbf{c}} = \{\tilde{c}_{k_n(\tau)}\}$.

Theorem 2.4. *There is an Euler system* $\mathbf{c} = \{c_{k_n(\tau)}\}$ *for the Galois representation* T *(in the sense of* [Rub00, Definition II.1.1 and Remark II.1.4][4]*) such that* $c_{k_n} = \tilde{c}_{k_n}$ *for all* n.

Proof. Since the proof of this Theorem follows line by line [Rub00, §III.3.4] we only give a sketch. First, one checks (in an identical fashion to the proof of Proposition III.3.4 of loc.cit.) that the collection $\tilde{\mathbf{c}}$ (which should be compared with the collection $\tilde{\mathbf{c}}'$ of Rubin) satisfies a distribution relation with *wrong* Euler factors. This could be remedied, as in the paragraph following Remark III.4.4 of loc.cit., using Lemma IX.6.1 of loc.cit. to obtain a new collection $\mathbf{c}$ (which corresponds to what Rubin calls $\tilde{\mathbf{c}}$) as desired. □

We close this section with a final remark which will be referenced in what follows:

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[3]We remark that any homomorphism $\lambda \in \mathrm{Hom}(U^\chi_{L_n(\tau)}, \mathbb{Z}_p)$ necessarily factors through the quotient $U^\chi_{L_n(\tau)}/(U_{L_n(\tau)})^\chi_{\mathrm{tors}}$; this is how we make sense of the right most map in the diagram above.

[4]and not in the sense of [MR04, Definition 3.2.2].

Remark 2.5. The argument in Remark 1.16 shows, under our running assumptions,

$$H^1(k_n(\tau)_p, T) \cong \mathrm{Hom}(U^{\chi}_{L_n(\tau)}, \mathbb{Z}_p).$$

3. THE EULER SYSTEMS TO KOLYVAGIN SYSTEMS MAP

We first recall what Mazur and Rubin call *Euler system to Kolyvagin system map*. Suppose T and $\mathcal{K}$ is as above, and $\mathcal{P}$ is the set of primes of k which does not divide $p\mathfrak{f}_\chi$. Let $\mathbf{ES}(T) = \mathbf{ES}(T, \mathcal{K})$ denote the collection of Euler systems for $(T, \mathcal{K})$ in the sense of [Rub00, §3][5]. Recall also *the generalized module of Kolyvagin systems* (in the sense of [MR04, Definition 3.1.6], see also §3.2 below).

Theorem 3.1. ([MR04, Theorem 3.2.4 & Theorem 5.3.3; see also Remark 3.2.3]) *There are canonical maps*

- $\mathbf{ES}(T) \longrightarrow \overline{\mathbf{KS}}(\mathcal{F}_{\mathrm{can}}, T, \mathcal{P})$,
- $\mathbf{ES}(T) \longrightarrow \overline{\mathbf{KS}}(\mathcal{F}_{\Lambda}, T \otimes \Lambda, \mathcal{P})$

  with the properties that
- *if* $\mathbf{c}$ *maps to* $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(\mathcal{F}_{\mathrm{can}}, T, \mathcal{P})$ *then* $\kappa_1 = c_k$,
- *if* $\mathbf{c}$ *maps to* $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(\mathcal{F}_{\Lambda}, T \otimes \Lambda, \mathcal{P})$ *then*

$$\kappa_1 = \{c_{k_n}\} \in \varprojlim_n H^1(k_n, T) = H^1(k, T \otimes \Lambda).$$

We would like to apply this map on the Euler systems we have constructed in §2. Note however that Theorem 3.1 will give rise to Kolyvagin systems only for the coarser Selmer structures $\mathcal{F}_\Lambda$ and $\mathcal{F}_{\mathrm{can}}$ (rather than finer Selmer structures $\mathcal{F}_{\mathbb{L}}$ and $\mathcal{F}_{\mathcal{L}}$). To be able to obtain Kolyvagin systems for the modified Selmer structures $\mathcal{F}_{\mathbb{L}}$ and $\mathcal{F}_{\mathcal{L}}$, we need to analyze the structure of semi-local cohomology groups for $T \otimes \Lambda$ and T at p, over various ray class fields of k. This is carried out in §3.1. We then apply the results of §3.1 to construct the desired Kolyvagin systems for the modified Selmer structures in §3.2.

Remark 3.2. In effect, one only needs *weak Kolyvagin systems* (in the sense of [MR04, Definition 3.1.8]) for the main applications of the Kolyvagin system machinery: Bounding the dual Selmer group. Weak Kolyvagin systems are essentially the derivative classes of Kolyvagin (cf. [Rub00, §IV]) which are obtained directly applying the derivative operators, without the necessity of any alterations carried out in [MR04, Appendix A].

3.1. **A good choice of homomorphisms.**

Recall that k_∞ is the cyclotomic $\mathbb{Z}_p$-extension of k, and $\Gamma = \mathrm{Gal}(k_\infty/k)$. k_n denotes the unique sub-extension of k_∞/k with $[k_n : k] = p^n$ and $\Gamma_n := \mathrm{Gal}(k_n/k)$.

[5] and *not* in the sense of [MR04, Definition 3.2.2].

Lemma 3.3. *For every $n \in \mathbb{Z}_{\geq 0}$ and τ as above, the corestriction maps*

(i) $H^1(k_n(\tau)_p, T) \longrightarrow H^1(k(\tau)_p, T)$,
(ii) $H^1(k(\tau)_p, T) \longrightarrow H^1(k_p, T)$,
(iii) $H^1(k_n(\tau)_p, T) \longrightarrow H^1(k_p, T)$

on the semi-local cohomology at p are all surjective.

Proof. The cokernel of the map

$$H^1(k(\tau), T \otimes \Lambda) = \varprojlim_n H^1(k_n(\tau)_p, T) \longrightarrow H^1(k(\tau)_p, T)$$

is given by $H^2(k(\tau)_p, T\otimes\Lambda)[\gamma-1]$, where γ is any topological generator of $\Gamma = \mathrm{Gal}(k_\infty/k)$. Since it is known that (cf. [PR94]) $H^2(k(\tau)_p, T\otimes\Lambda)$ is a finitely generated $\mathbb{Z}_p$-module, it follows that

$$H^2(k(\tau)_p, T \otimes \Lambda)[\gamma - 1] = 0 \iff H^2(k(\tau)_p, T \otimes \Lambda)/(\gamma - 1) = 0.$$

Since the cohomological dimension of the absolute Galois group of any local field is 2,

$$H^2(k(\tau)_p, T \otimes \Lambda)/(\gamma - 1) \cong H^2(k(\tau)_p, T \otimes \Lambda/(\gamma - 1)) = H^2(k(\tau)_p, T).$$

It therefore suffices to check that

$$H^2(k(\tau)_p, T) := \bigoplus_{v|p} H^2(k(\tau)_v, T) = 0,$$

which, via local duality is equivalent to checking that $(T^*)^{G_{k(\tau)_v}} = 0$ for each $v|p$.

Write $\mathcal{D}_v$ for the decomposition group of v inside $\mathrm{Gal}(k(\tau)/k) := \Delta_\tau$. We may identify $\mathcal{D}_v \subset \Delta_\tau$ by the local Galois group $\mathrm{Gal}(k(\tau)_v/k_\wp)$ where $\wp \subset k$ is the prime below v. Since Δ_τ is generated by inertia groups at the primes dividing τ, all of which act trivially on T^* (by the choice of τ's). Hence, it follows that

$$(T^*)^{G_{k(\tau)_v}} = (T^*)^{G_{k_\wp}}.$$

Note that $T^* = \boldsymbol{\mu}_{p^\infty} \otimes \chi^{-1}$, it follows at once that[6] $(T^*)^{G_{k_\wp}} = 0$, and thus (i) is proved.

Now set $T_\tau := \mathrm{Ind}_{k(\tau)}^k T$. The semi-local version of Shapiro's lemma (which is explained in [Rub00, §A.5]) gives

$$H^1(k(\tau)_p, T) \cong H^1(k_p, T_\tau)$$

and the map

$$\mathbf{N}_\tau : H^1(k_p, T_\tau) \cong H^1(k(\tau)_p, T) \longrightarrow H^1(k_p, T)$$

[6]Because χ is a finite character and $G_{k_\wp}$ acts on $\boldsymbol{\mu}_{p^\infty}$ via the cyclotomic character which is of infinite order.

is simply induced from the augmentation sequence

$$0 \longrightarrow \mathcal{A}_\tau \cdot T_\tau \longrightarrow T_\tau \longrightarrow T \longrightarrow 0,$$

where $\mathcal{A}_\tau$ is the augmentation ideal of the local ring $\mathbb{Z}_p[\Delta_\tau]$. The argument above shows that the cokernel of $\mathbf{N}_\tau$ is dual to

$$H^0(k_p, (\mathcal{A}_\tau \cdot T_\tau)^*).$$

Further,

$$(\mathcal{A}_\tau \cdot T_\tau)^* := \mathrm{Hom}(\mathcal{A}_\tau \cdot T_\tau, \boldsymbol{\mu}_{p^\infty}) = \mathrm{Hom}(\mathcal{A}_\tau \cdot T_\tau, \mathbb{Q}_p/\mathbb{Z}_p) \otimes \mathbb{Z}_p(1),$$

and $\mathrm{Hom}(\mathcal{A}_\tau \cdot T_\tau, \mathbb{Q}_p/\mathbb{Z}_p) = \mathcal{A}_\tau \cdot \mathrm{Hom}(T_\tau, \mathbb{Q}_p/\mathbb{Z}_p)$, we therefore see that

$$H^0(k_p, (\mathcal{A}_\tau \cdot T_\tau)^*) \hookrightarrow H^0(k_p, T_\tau^*),$$

hence it suffices to show that $H^0(k_p, T_\tau^*) = 0$. By local duality this is equivalent to proving $H^2(k_p, T_\tau) = 0$, which by the (semi-local) Shapiro's Lemma equivalent to show $H^2(k(\tau)_p, T) = 0$, which again by local duality equivalent to the statement $H^0(k(\tau)_p, T^*) = 0$; and this we have verified in the paragraph above. This completes the proof of (ii).

(iii) clearly follows from (i) and (ii). □

Recall that $\Delta_\tau := \mathrm{Gal}(k(\tau)/k)$ and $\Gamma_n = \mathrm{Gal}(k_n/k)$.

Proposition 3.4. *For every τ as above,*

(i) *the semi-local cohomology group $H^1(k(\tau)_p, T)$ is a free $\mathbb{Z}_p[\Delta_\tau]$-module of rank r, and,*
(ii) *for every $n \in \mathbb{Z}_{\geq 0}$, the semi-local cohomology group $H^1(k_n(\tau)_p, T)$ is a free $\mathbb{Z}_p[\Gamma_n \times \Delta_\tau]$-module of rank r.*

Proof. We start with the remark that $H^1(k(\tau)_p, T)$ is a free $\mathbb{Z}_p$-module of rank $r \cdot |\Delta_\tau|$. Indeed, this may be proved following the proof of Lemma 1.15 (or alternatively, and more directly, following the argument of Remark 1.16). Further, we know thanks to Lemma 3.3 that the map

$$H^1(k(\tau)_p, T) \longrightarrow H^1(k_p, T)$$

(which could be thought of as reduction modulo the augmentation ideal $\mathcal{A}_\tau$) is surjective. Therefore Nakayama's lemma and Lemma 1.15 implies that $H^1(k(\tau)_p, T)$ is generated by (at most) r elements over $\mathbb{Z}_p[\Delta_\tau]$. Let $\mathfrak{B} = \{x_1, x_2, \ldots, x_r\}$ be any set of such generators. To prove (i), it suffices to check that the x_i's do not admit any $\mathbb{Z}_p[\Delta(\tau)]$-linear relation. Assume contrary, and suppose there is a relation

$$\sum_{i=1}^{r} \alpha_i x_i = 0, \ \ \alpha_i \in \mathbb{Z}_p[\Delta_\tau]. \tag{3.1}$$

Write

$$S = \{\delta x_j : \delta \in \Delta_\tau, 1 \le j \le r\},$$

note that by our assumption on the set $\mathfrak{B}$, S generates $H^1(k(\tau)_p, T)$ as a $\mathbb{Z}_p$-module, and further $|S| = r \cdot |\Delta_\tau|$. The equation (3.1) can be rewritten as

$$\sum_{\delta,j} a_{\delta,j} \cdot \delta x_j = 0$$

with $a_{\delta,j} \in \mathbb{Z}_p$. Since we already know that $H^1(k(\tau)_p, T)$ is $\mathbb{Z}_p$-torsion free, we may assume without loss of generality that $a_{\delta_0,j_0} \in \mathbb{Z}_p^\times$ for some δ_0, j_0. This in return implies that

$$\delta_0 x_{j_0} \in \mathrm{span}_{\mathbb{Z}_p}(S - \{\delta_0 x_{j_0}\}),$$

hence $H^1(k(\tau)_p, T)$ is generated by $S - \{\delta_0 x_{j_0}\}$. This, however, is a contradiction since we already checked that the $\mathbb{Z}_p$-rank of $H^1(k(\tau)_p, T)$ is $r \cdot |\Delta_\tau| = |S|$, hence cannot be generated by $|S| - 1$ elements over $\mathbb{Z}_p$. (i) is now proved.

(ii) is proved in an identical fashion, now considering the *augmentation map*

$$H^1(k_n(\tau)_p, T) \longrightarrow H^1(k(\tau)_p, T)$$

which is surjective thanks to Lemma 3.3. □

Let $\mathbb{F}$ be the composite of all fields $k(\tau)$ where τ runs through the set $\mathcal{N}$ of all square free integral ideals of k which are prime $p\mathfrak{f}_\chi$. Set $\boldsymbol{\Delta} := \mathrm{Gal}(\mathbb{F}/k)$.

Corollary 3.5. *$\varprojlim_{n,\tau} H^1(k_n(\tau)_p, T)$ is a free $\mathbb{Z}_p[[\Gamma \times \boldsymbol{\Delta}]]$-module of rank r and the natural projections*

$$\varprojlim_{n,\tau} H^1(k_n(\tau)_p, T) \longrightarrow H^1(k_m(\eta)_p, T)$$

are surjective for all $m \in \mathbb{Z}_{\ge 0}$ and $\eta \in \mathcal{N}$.

Proof. Immediate after Proposition 3.4. □

Definition 3.6. Fix a $\mathbb{Z}_p[[\Gamma \times \boldsymbol{\Delta}]]$-rank one direct summand $\boldsymbol{\mathcal{L}}$ of $\varprojlim_{n,\tau} H^1(k_n(\tau)_p, T)$. Denote its image under the (surjective) map

$$\varprojlim_{n,\tau} H^1(k_n(\tau)_p, T) \longrightarrow H^1(k_m(\eta)_p, T)$$

by $\mathcal{L}_m^\eta$. When $\eta = 1$, we simply write $\mathcal{L}_m$ instead of $\mathcal{L}_m^1$; and when $m = 0$ we write $\mathcal{L}$ for $\mathcal{L}_0$. Finally, let $\mathbb{L}$ denote the image of $\boldsymbol{\mathcal{L}}$ under the projection

$$\varprojlim_{n,\tau} H^1(k_n(\tau)_p, T) \longrightarrow \varprojlim_{n} H^1((k_n)_p, T) = H^1(k_p, T\otimes\Lambda).$$

We also fix generators $\boldsymbol{\varphi}, \varphi_m^\eta, \varphi_m, \varphi$ and Φ of $\boldsymbol{\mathcal{L}}, \mathcal{L}_m^\eta, \mathcal{L}_m, \mathcal{L}$ and $\mathbb{L}$ respectively; such that

$$\boldsymbol{\varphi} \mapsto \varphi_m^\eta \mapsto \varphi_m, \text{ and,}$$
$$\boldsymbol{\varphi} \mapsto \Phi \mapsto \varphi$$

under the projection maps mentioned above.

As in Definition 3.6, we could start with a choice of $\boldsymbol{\mathcal{L}}$ which in return fixes $\mathbb{L}$ and $\mathcal{L}$. Alternatively, we could start with arbitrary $\mathcal{L}$ (and $\mathbb{L}$) as we did in §1.3 and show (using linear algebra) that there is a rank one direct summand $\boldsymbol{\mathcal{L}} \subset \varprojlim_{n,\tau} H^1(k_n(\tau)_p, T)$ which projects down to $\mathcal{L}$ (and $\mathbb{L}$), as in Definition 3.6.

By Remark 2.5 we may (and we will) identify $\varprojlim_{n,\tau} H^1(k_n(\tau)_p, T)$ by $\varprojlim_{n,\tau} \mathrm{Hom}\left(U^\chi_{L_n(\tau)}, \mathbb{Z}_p\right)$ where we recall that $U_{L_n(\tau)}$ stands for the local units inside $L_n(\tau)\otimes\mathbb{Q}_p$. We define, for each $m \geq 0$ and $\eta \in \mathcal{N}$ a homomorphism $\lambda_m^\eta \in \mathrm{Hom}\left(U^\chi_{L_m(\eta)}, \mathbb{Z}_p\right)$ as the composite

$$\lambda_m^\eta := \varphi_m^\eta \circ \prod_{\mathfrak{q}|\eta}(-\mathrm{Fr}_{\mathfrak{q}}).$$

We further set $\daleth_m^\eta$ for the (free of rank one) $\mathbb{Z}_p[\Gamma_n\times\Delta_\tau]$-module generated by λ_m^η. Clearly, $\daleth_m^\eta$ is related to $\mathcal{L}_m^\eta$ by

$$\daleth_m^\eta = \prod_{\mathfrak{q}|\eta}(-\mathrm{Fr}_{\mathfrak{q}}^{-1})\mathcal{L}_m^\eta = \mathcal{L}_m^\eta. \tag{3.2}$$

where the final equality is because $\mathcal{L}_m^\eta$ is a $\mathbb{Z}_p[\Gamma_n\times\Delta_n]$-stable submodule of

$$H^1(k_n(\tau), T) \cong \mathrm{Hom}\left(U^\chi_{L_n(\tau)}, \mathbb{Z}_p\right).$$

When η is fixed and m varies, note that the collection $\{\lambda_m^\eta\}_m$ forms a projective system with respect to norm maps[7]. When $\eta = 1$, we write λ_m (*resp.* $\daleth_m$) instead of λ_m^η (*resp.* $\daleth_m^\eta$). Also when $m = 0$, we simply write λ (*resp.* $\daleth$) for λ_0 (*resp.* $\daleth_0$).

We finally remark that $\lambda_m = \varphi_m$ for all m.

Proposition 3.7. *For $\eta, \eta\mathfrak{q} \in \mathcal{N}$, and any $m' \geq m$,*

[7] which we remind the reader that the norm maps are induced from the inclusions $U^\chi_{L_m(\eta)} \hookrightarrow U^\chi_{L_{m'}(\eta)}$, for $m' \geq m$.

(i) $\lambda_{m'}^{\eta\mathfrak{q}}|_{U^{\chi}_{L_m(\eta)}} = \lambda_m^{\eta} \circ (-\mathrm{Fr}_{\mathfrak{q}})$,

(ii) $\lambda_{m'}^{\eta}|_{U^{\chi}_{L_m(\eta)}} = \lambda_m^{\eta}$.

Proof. This is evident, since by construction

$$\begin{aligned}\lambda_{m'}^{\eta\mathfrak{q}}|_{U^{\chi}_{L_m(\eta)}} &= \varphi_{m'}^{\eta\mathfrak{q}} \circ \prod_{\varpi|\eta}(-\mathrm{Fr}_{\varpi})(-\mathrm{Fr}_{\mathfrak{q}})|_{U^{\chi}_{L_m(\eta)}}\\ &= \varphi_{m}^{\eta} \circ \prod_{\varpi|\eta}(-\mathrm{Fr}_{\varpi})(-\mathrm{Fr}_{\mathfrak{q}})|_{U^{\chi}_{L_m(\eta)}}\\ &= \lambda_m^{\eta} \circ (-\mathrm{Fr}_{\mathfrak{q}})|_{U^{\chi}_{L_m(\eta)}}\end{aligned}$$

where the second equality is because

$$\varphi_{m'}^{\eta\mathfrak{q}}|_{U^{\chi}_{L_m(\eta)}} = \varphi_{m}^{\eta}|_{U^{\chi}_{L_m(\eta)}}$$

by the norm coherence property of the collection $\{\varphi_m^{\eta}\}_{\eta,m}$. This completes (i), and (ii) is proved similarly.

□

Next, we will use the collection $\{\lambda_m^{\eta}\}_{m,\eta}$ we constructed above to obtain first an Euler system, via Theorem 2.4 and then a (weak) Kolyvagin system for the Selmer structure $\mathcal{F}_{\mathcal{L}}$ (*resp.* $\mathcal{F}_{\mathbb{L}}$) on the Galois representation T (*resp.* $T \otimes \Lambda$).

3.2. **Kolyvagin systems for modified Selmer groups - II.** Suppose the collection of homomorphisms $\{\lambda_m^{\eta}\}_{m,\eta}$ is as in §3.1. Let $\mathbf{c}^{\mathrm{St}} = \{c^{\mathrm{St}}_{k_n(\tau)}\} \in \mathbf{ES}(T)$ be the Euler system constructed in Theorem 2.4 using this particular collection $\{\lambda_m^{\eta}\}$ and Stickelberger elements.

Before, we set some notation. Recall that $\mathcal{P}$ denotes the set primes of k whose elements do not divide pf_{χ}. For each positive integer m and n, let

$$\mathcal{P}_{m+n} = \{\mathfrak{q} \in \mathcal{P} : \mathfrak{q} \text{ splits completely in } L(\mu_{p^{m+n+1}})/k\}$$

be a subset of $\mathcal{P}$. Note that $\mathcal{P}_{m+n}$ is exactly the set of primes being determined by [Rub00, Definition IV.1.1] when $T = \mathbb{Z}_p(\chi)$. Let $\mathcal{N}$ (resp. $\mathcal{N}_j$) denote the square free products of primes $\mathfrak{q}$ in $\mathcal{P}$ (*resp.* $\mathcal{P}_j$). We also include 1 in $\mathcal{N}$ (and $\mathcal{N}_j$). For notational simplicity, we also write $\mathbb{T} := T \otimes \Lambda$ from now on, and for a fixed topological generator γ of $\Gamma = \mathrm{Gal}(k_\infty/k)$ we set $\gamma_n = \gamma^{p^n}$.

Define for $\mathcal{F} = \mathcal{F}_{\mathrm{can}}$ or $\mathcal{F}_{\mathcal{L}}$ (*resp.* $\mathbb{F} = \mathcal{F}_{\Lambda}$ or $\mathcal{F}_{\mathbb{L}}$)

$$\overline{\mathbf{KS}}(T, \mathcal{F}, \mathcal{P}) := \varprojlim_{m}(\varinjlim_{j} \mathbf{KS}(T/p^mT, \mathcal{F}, \mathcal{P} \cap \mathcal{P}_j)),$$

the (generalized) $\mathbb{Z}_p$-module of Kolyvagin systems for the triple $(T, \mathcal{F}, \mathcal{P})$ (see [MR04, Definition 3.1.6]) (*resp.*

$$\overline{\mathbf{KS}}(\mathbb{T}, \mathbb{F}, \mathcal{P}) := \varprojlim_{m,n} (\varinjlim_{j} \mathbf{KS}(\mathbb{T}/(p^m, \gamma_n - 1)\mathbb{T}, \mathbb{F}, \mathcal{P} \cap \mathcal{P}_j)),$$

the (generalized) Λ-module of Kolyvagin systems for the triple $(\mathbb{T}, \mathbb{F}, \mathcal{P})$.)

Write

$$\left\{\{\kappa^{\mathrm{St}}_{\tau,m}\}_{\tau\in\mathcal{N}_m}\right\}_m = \boldsymbol{\kappa}^{\mathrm{St}} \in \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$$

(*resp.*

$$\left\{\left\{\kappa^{\mathrm{St}\infty}_{\tau}(m,n)\right\}_{\tau\in\mathcal{N}_{m+n}}\right\}_{m,n} = \boldsymbol{\kappa}^{\mathrm{St}\infty} \in \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\Lambda}, \mathcal{P}))$$

for the Kolyvagin systems obtained via the descent procedure of [Rub00, §4] applied on the Euler system $\mathbf{c}^{\mathrm{St}} = \{c^{\mathrm{St}}_{k_n(\tau)}\}$. We further know that

$$\begin{array}{l} \kappa^{\mathrm{St}}_1 \overset{\mathrm{def}}{=\!=\!=} \varprojlim_m \kappa^{\mathrm{St}}_{1,m} \in \varprojlim_m H^1(k, T/p^mT) = H^1(k,T) \\ \| \\ c^{\mathrm{St}}_k \overset{\mathrm{def}}{=} \lambda \circ \theta^{\chi}_L = \varphi \circ \theta^{\chi}_L \in \mathrm{Hom}\left(\left(\mathbb{A}^{\times}_L/L^{\times}\right)^{\chi}, \mathbb{Z}_p\right) = H^1(k,T) \end{array}$$

and

$$\begin{array}{l} \kappa^{\mathrm{St}\infty}_1 \overset{\mathrm{def}}{=} \varprojlim_{m,n} \kappa_1(m,n) \in \varprojlim_{m,n} H^1(k, \mathbb{T}/(p^m, \gamma_n - 1)\mathbb{T}) = H^1(k, \mathbb{T}) \\ \| \\ \{c^{\mathrm{St}}_{k_n}\}_n \overset{\mathrm{def}}{=\!=} \{\lambda_n \circ \theta^{\chi}_{L_n}\}_n = \{\varphi_n \circ \theta^{\chi}_{L_n}\}_n \in \varprojlim_n H^1(k_n, T) = H^1(k, \mathbb{T}). \end{array}$$

Remark 3.8. Using Shapiro's lemma one easily checks that

$$H^1(k(\tau), \mathbb{T}/(p^m, \gamma_n - 1)\mathbb{T}) \cong H^1(k_n(\tau), T/p^mT), \text{ and}$$

$$H^1(k(\tau)_p, \mathbb{T}/(p^m, \gamma_n - 1)\mathbb{T}) \cong H^1(k_n(\tau)_p, T/p^mT).$$

See [MR04] Lemma 5.3.1 and [Rub00] Appendix B.5 for the semi-local version. We have already used these identifications in the paragraph above (with $\tau = 1$).

Theorem 3.9.

$$\boldsymbol{\kappa}^{\mathrm{St}} \in \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P}) \textit{ and } \boldsymbol{\kappa}^{\mathrm{St}\infty} \in \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\Lambda}, \mathcal{P}).$$

Proof. Identical to the proofs of [Büy07a, Theorem 2.19] and [Büy07c, Theorem 3.18]. We remark that the only essential point beyond [Rub00, MR04] is to verify that

$$\mathrm{loc}_p\left(\kappa^{\mathrm{St}}_{\tau,m}\right) \in H^1_{\mathcal{F}_{\mathcal{L}}}(k_p, T/p^mT) \cong \mathcal{L}/p^m\mathcal{L}$$

for each $\tau \in \mathcal{N}_m$ and for each m; and that

$$\begin{aligned}\mathrm{loc}_p\left(\kappa_\tau^{\mathrm{St}_\infty}(m,n)\right) &\in H^1_{\mathcal{F}_{\mathbb{L}}}(k_p, \mathbb{T}/(p^m, \gamma_n - 1)\mathbb{T}) \\ &\cong \mathbb{L}/(p^m, \gamma_n - 1)\mathbb{L} := \mathcal{L}_n/p^m\mathcal{L}_n\end{aligned}$$

for every $\tau \in \mathcal{N}_{m+n}$ and every m, n. □

We give the main applications of this Theorem in the next Section. Of course these will be somewhat standard and will not involve much surprise: We will bound (the modified dual) Selmer groups.

Remark 3.10. As remarked earlier, one only needs weak Kolyvagin systems (in the sense of [MR04, Definition 3.1.8]; in the context of [Rub00], Kolyvagin's descent [Rub00, §4] applied on an Euler system gives weak Kolyvagin systems) in order to deduce the applications (which we give below in §4) of the Euler system $\mathbf{c}^{\mathrm{St}}$ we have constructed above using Stickelberger elements and the 'line' $\boldsymbol{\mathcal{L}} \in \varprojlim_{n,\tau} H^1(k_n(\tau), T)$. The weak Kolyvagin system could be applied following the formalism of [Rub00, §5 and §7] with slight alterations, to deduce identical results to what we present below.

4. Applications

Before we state our main applications of Theorem 3.9, we remind our running hypotheses. χ is a totally odd character of $G_k := \mathrm{Gal}(\overline{k}/k)$ which is *not* the Teichmüller character ω (giving the action of G_k on the p-th roots of unity $\boldsymbol{\mu}_p$). We assume further that $\chi(\wp) \neq 1$ for any prime $\wp \subset k$ above p.

4.1. **Main Theorem over k.** We first prove a bound on the size of the dual Selmer group $H^1_{\mathcal{F}^*_{\mathcal{L}}}(k, T^*)$, which we use, via the comparison Theorem established in §§1.4.1-1.4.2, we obtain bounds on the classical (dual) Selmer groups.

Theorem 4.1. $\mathrm{length}_{\mathbb{Z}_p}(H^1_{\mathcal{F}^*_{\mathcal{L}}}(k, T^*)) \leq \mathrm{length}_{\mathbb{Z}_p}(H^1_{\mathcal{F}_{\mathcal{L}}}(k, T)/\mathbb{Z}_p \cdot \kappa_1^{\mathrm{St}})$.

Proof. This is the standard application of $\boldsymbol{\kappa}^{\mathrm{St}} \in \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P})$. □

Corollary 4.2. $\mathrm{length}_{\mathbb{Z}_p}(H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*)) \leq \mathrm{length}_{\mathbb{Z}_p}(\mathcal{L}/\mathbb{Z}_p \cdot c_k^{\mathrm{St}})$.

Proof. This follows from Theorem 4.1 and Corollary 1.22 with $c = c_k^{\mathrm{St}} = \kappa_1^{\mathrm{St}}$. □

Let $\theta_L^\chi \in \mathbb{Z}_p[\Delta]^\chi$ be as in §2. The evaluation map

$$\chi : \mathbb{Z}_p[\Delta]^\chi \longrightarrow \mathbb{Z}_p$$

is an isomorphism; we write $\chi(\theta_L)$ for the image of θ_L^χ under this map. Recall also the definition of φ and λ, which was used to define c_k^{St}, and that $\lambda = \varphi$ by definition.

Theorem 4.3. *Under the assumptions above*

$$|A_L^\chi| \leq |\mathbb{Z}_p/\chi(\theta_L)\mathbb{Z}_p|.$$

Proof. By Proposition 1.13 $H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*) \cong A_L^\chi$, and by construction $c_k^{\mathrm{St}} = \chi(\theta_L)\lambda := \chi(\theta_L)\varphi$. Since φ (by definition) is a $\mathbb{Z}_p$-generator of $\mathcal{L}$, it follows that $\mathcal{L}/\mathbb{Z}_p \cdot c_k^{\mathrm{St}} = \mathbb{Z}_p/\chi(\theta_L)\mathbb{Z}_p$. Theorem now follows from Corollary 4.2. □

Unless $\boldsymbol{\mu}_p \subset L$, we may turn the inequality of Theorem 4.3 into an equality using a standard argument involving the class number formula; see [Rub92, §5] and[Büy07a, §3] for details. We need (a priori) to assume $\boldsymbol{\mu}_p \subset L$ is because otherwise we would need the inequality of Theorem 4.3 also for the Teichmüller character ω. Although this escapes the methods of this paper, we may appeal to [Wil90a, Theorem 3] (*only* for the character ω) and still deduce the following strengthening of Theorem 4.3:

Theorem 4.4.

$$|A_L^\chi| = |\mathbb{Z}_p/\chi(\theta_L)\mathbb{Z}_p|.$$

Remark 4.5. Note that we excluded all the Euler factors at the primes $\wp \subset k$ above p from the definition of θ_K^χ, for any $K \in \mathcal{K}$ (recall $\mathcal{K}$ from §2), contrary to the standard definition of Stickelberger elements when K/k is unramified at a certain prime above p. Still, Theorem 4.4 is equivalent to [Wil90a, Theorem 3], since we assumed $\chi(\wp) \neq 1$ for any $\wp \subset k$ above p, and therefore our Stickelberger elements agree with that of [Wil90a, Kur03] up to units.

4.2. **Main Theorem over k_∞.** Along with the assumptions we recalled above, suppose also that all the primes $\wp \subset k$ above p are totally ramified in the cyclotomic $\mathbb{Z}_p$-extension k_∞/k. Note that this assumption is satisfied if $k/\mathbb{Q}$ is unramified above p. Also, write $\mathrm{char}(M)$ for the characteristic ideal of a torsion Λ-module M.

We again proceed as in the previous section: We first prove a bound for the characteristic ideal of the dual Selmer group $H^1_{\mathcal{F}^*_{\mathbb{L}}}(k, \mathbb{T}^*)^\vee$, which we use, via the comparison Theorem established in §§1.4.1-1.4.2, we obtain bounds on the characteristic ideal of the (Pontryagin duals of the) classical (dual) Selmer groups.

Theorem 4.6. $\mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathbb{L}}}(k, \mathbb{T}^*)^\vee\right) \mid \mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}}}(k, \mathbb{T})/\Lambda \cdot \kappa_1^{\mathrm{St}_\infty}\right).$

Proof. This is the standard application of the Kolyvagin system $\boldsymbol{\kappa}^{\mathrm{St}_\infty}$. □

Set $c_{k_\infty}^{\mathrm{St}} := \{c_{k_n}^{\mathrm{St}}\}_n \in \varprojlim_n H^1(k_n, T) = H^1(k, \mathbb{T})$.

Corollary 4.7. $\mathrm{char}\left(\varinjlim_n A_{L_n}^\chi)^\vee\right) \mid \mathrm{char}\left(\mathbb{L}/\Lambda \cdot c_{k_\infty}^{\mathrm{St}_\infty}\right).$

Proof. This follows from Theorem 4.6 and Proposition 1.23(ii) applied with $c = c_{k_\infty}^{\mathrm{St}_\infty}$; together with the fact that $\kappa_1^{\mathrm{St}_\infty} = c_{k_\infty}^{\mathrm{St}_\infty}$. □

Recall $\theta_{L_n}^\chi \in \mathbb{Z}_p[\Delta \times \Gamma_n]^\chi = \mathbb{Z}_p[\Delta]^\chi[\Gamma_n]$ (where we remind the reader that $\Gamma_n = \mathrm{Gal}(k_n/k)$) from §2. As in the previous section, we denote the image of $\theta_{L_n}^\chi$ under the map

$$\chi_{\Lambda} : \mathbb{Z}_p[\Delta]^\chi[\Gamma_n] \longrightarrow \mathbb{Z}_p[\Gamma_n],$$

(which extends χ from the previous section to Γ_n by letting $\chi_{\Lambda}(\gamma) = \gamma$ for $\gamma \in \Gamma_n$) still by $\chi_{\Lambda}(\theta_{L_n})$. Lemma 2.1 shows that $\{\chi_{\Lambda}(\theta_{L_n})\}$ is a projective system with respect to natural surjections $\mathbb{Z}_p[\Gamma_{n'}] \to \mathbb{Z}_p[\Gamma_n]$. We define

$$\chi_\Lambda(\Theta_{L_\infty}) := \{\chi_\Lambda(\theta_{L_n})\} \in \varprojlim_n \mathbb{Z}_p[\Gamma_n] = \Lambda.$$

Finally, let $x \mapsto x^\bullet$ be the involution on Λ induced from $\gamma \mapsto \gamma^{-1}$ for $\gamma \in \Gamma$.

Theorem 4.8. *Under the assumptions above*

$$\operatorname{char}\left(\varinjlim_n A_{L_n}^{\chi})^\vee\right) \mid \chi_\Lambda(\Theta_{L_\infty})^\bullet.$$

Proof. By construction (of $c_{k_n}^{\mathrm{St}}$ and λ_n)

$$c_{k_n}^{\mathrm{St}} = \lambda_n \circ \theta_{L_n}^{\chi} = \chi_\Lambda(\theta_{L_n})^\bullet \lambda_n = \chi_\Lambda(\theta_{L_n})^\bullet \varphi_n.$$

It follows that

$$c_{k_\infty}^{\mathrm{St}} = \{\chi_\Lambda(\theta_{L_n})^\bullet \varphi_n\} = \chi_\Lambda(\Theta_{L_\infty})^\bullet \Phi,$$

with $\Phi = \{\varphi_n\}$ as in §3.1. Since Φ is a generator of $\mathbb{L}$, by construction, Theorem follows from Corollary 4.7. ☐

Once again by a standard class number argument (and yet again the case $\boldsymbol{\mu}_p \subset L$ requires more care as above) shows that this equality may be turned into an equality:

Theorem 4.9. $\operatorname{char}\left(\varinjlim_n A_{L_n}^{\chi})^\vee\right) = \chi_\Lambda(\Theta_{L_\infty})^\bullet.$

Let $\rho_{\mathrm{cyc}} : G_k \to \mathbb{Z}_p^\times$ be the cyclotomic character (giving the action of G_k on $\boldsymbol{\mu}_{p^\infty}$), and set $\langle\rho_{\mathrm{cyc}}\rangle = \omega^{-1}\rho_{\mathrm{cyc}} : \Gamma \to 1 + p\mathbb{Z}_p$. We define a twisting map $\mathrm{Tw}_{\langle\rho_{\mathrm{cyc}}\rangle} : \Lambda \to \Lambda$ by setting

$$\gamma \mapsto \langle\rho_{\mathrm{cyc}}\rangle(\gamma)\gamma \text{ for } \gamma \in \Gamma,$$

and extending to Λ by linearity and continuity. Finally, let $\mathcal{L}_{\omega\chi^{-1}} \in \Lambda$ denote the *Deligne-Ribet p-adic L-function* for the character $\omega\chi^{-1}$. We won't say much about $\mathcal{L}_{\omega\chi^{-1}}$; we will loosely mention that this $\mathcal{L}_{\omega\chi^{-1}}$ is determined by the following interpolation property:

$$\langle\rho_{\mathrm{cyc}}\rangle^k \rho(\mathcal{L}_{\omega\chi^{-1}}) = \prod_{\wp \mid p} (1 - \omega^{-k}\rho\chi(\wp)\mathbf{N}\wp^{k-1}) L(1-k, \omega^{-k}\rho\chi), \tag{4.1}$$

for every $k \geq 1$ and every character ρ of Γ of finite order. Here $L(s,\psi)$ is the (abelian) Artin L-function attached to the character ψ (of finite order) of G_k.

Lemma 4.10. $\chi_\Lambda(\Theta_{L_\infty})^\bullet = \mathrm{Tw}_{\langle\rho_{\mathrm{cyc}}\rangle}(\mathcal{L}_{\omega\chi^{-1}}).$

Proof. For every character ρ of Γ of finite order, it follows from definitions that

$$\rho(\chi_{\Lambda}(\Theta_{L_\infty})^{\bullet}) = \rho^{-1}(\chi_{\Lambda}(\Theta_{L_\infty}) = \prod_{\wp|p}(1-\chi^{-1}\rho(\wp))L(0,\chi^{-1}\rho)$$
$$= \langle\rho_{\mathrm{cyc}}\rangle\rho(\mathcal{L}_{\omega\chi^{-1}}) = \rho(\langle\rho_{\mathrm{cyc}}\rangle\mathcal{L}_{\omega\chi^{-1}}).$$

Since this is true for every ρ, Lemma follows. □

Corollary 4.11. *Under the assumptions above*

$$\mathrm{char}\left(\varinjlim_{n} A_{L_n}^{\chi})^{\vee}\right) = \mathrm{Tw}_{\langle\rho_{\mathrm{cyc}}\rangle}(\mathcal{L}_{\omega\chi^{-1}}).$$

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