Yet Another Poincaré's Polyhedron Theorem

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Abstract. This work contains a new version of Poincaré's Polyhedron Theorem that also suits geometries of nonconstant curvature lacking concepts of convexity. Most conditions of the theorem, being as local as possible, are easy to verify in practice.

1. Introduction

Poincaré's Polyhedron Theorem (PPT) is a widely known, valuable tool in constructing manifolds endowed with prescribed geometric structure. It is one of very few criteria providing discreteness of groups of isometries. The theorem has a long history and plenty of versions (see [EPe], for instance).

Many higher dimensional versions of PPT, apart from those that are of no use, come from constant curvature (or even plane) geometries, where convexity arguments play an important role typically suited by polyhedra with totally geodesic faces whose angles of intersection are constant along common edges. In general, such approach is inapplicable to nonconstant curvature geometries, in particular to complex hyperbolic geometry. Usual requirements such as 'adjacent polyhedra intersect in a prescribed way' are difficult to check. Therefore, we look for a version where the conditions related to tessellation are as local as possible and provide global properties just *a posteriori.* In the present version of PPT, the local conditions concern the sum of angles at corresponding points of each cycle of edges and some uniform estimate for the behaviour of faces along common edges. If one seeks criteria easily verifiable in practice, the infinitesimal requirements are the most comfortable. We believe that this sort of approach will be payed off even for constant curvature.

As it stands, Theorem 3.2 looks a bit 'plane-like' since it does not deal with polyhedra containing elements of codimension > 2. However, it suits well the actual stages of study of nonconstant curvature hyperbolic 4-dimensional geometries because the manifolds being constructed are frequently fiber bundles. For instance, this theorem can be directly applied to constructing complex hyperbolic disc bundles in [AGG] and [AGu]. It is worthwhile mentioning that it is probably not difficult to elaborate a more general form of the theorem that takes into account the higher codimensional elements (see Final Remarks).

A serious defect of our version is the global requirement of simplicity of a polyhedron. In complex hyperbolic geometry, for example, it is already hard to check that bisectors (taken as faces) intersect properly, to say nothing about the attempts to use more complicate hypersurfaces. It seems that one should obtain a more local PPT making the verification of simplicity unnecessary and getting rid of polyhedra. Then there would be no more arbitrary choice involved.

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2. Preliminaries

This is essentially standard material (see, for instance, [Bea, §9.8, p. 242]).

Let M be a locally path-connected, connected, and simply-connected metric space. We regard a *polyhedron* in M as being a closed, locally path-connected, and connected subspace $P\subset M$ such that

• P is the closure of its nonempty interior: $\mathring{P}\neq\varnothing$ and $P=\operatorname{Cl}\mathring{P}$;

• the nonempty boundary of P is decomposed into the union of nonempty subsets $s\in S$ called *faces*: $\partial P:=P\setminus\mathring{P}=\bigcup\limits_{s\in S}s$.

A *face-pairing* of a polyhedron P is an involution $\overline{}:S\to S$ and a family of isometries $I_s\in\operatorname{Isom}M$ satisfying $I_ss=\overline{s}$ and $I_{\overline{s}}=I_s^{-1}$ for every $s\in S$.

Let P be a polyhedron with a given face-pairing and let G denote the group generated by the face-pairing isometries. We introduce a relation in $G\times P$ by putting $(g,x)\sim(h,y)$ exactly when $x\in s$ for some $s\in S$, $I_sx=y$, and $h^{-1}g=I_s$. Closing this symmetric relation with respect to transitivity (and reflexivity), we obtain an equivalence relation also denoted by $\sim$. Put $J:=G\times P/\sim$ and let $[g,x]$ denote the class of (g,x) in J. Taking the discrete topology on G and equipping P, $G\times P$, and J with their natural topologies, we have a commutative diagram of the continuous G-mappings $\psi(g,x):=gx$, $\pi(g,x):=[g,x]$, and $\varphi[g,x]:=gx$. (Actions of G by homeomorphisms are defined by $h(g,x):=(hg,x)$ and $h[g,x]:=[hg,x]$.) Put

$$\begin{array}{ccc} G\times P & \xrightarrow{\ \pi\ } & J\\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & M & \end{array}$$

$$[P]:=\big\{[1,x]\mid x\in P\big\}\quad\text{and}\quad[\mathring{P}]:=\big\{[1,x]\mid x\in\mathring{P}\big\}.$$

Clearly, $J=\bigcup\limits_{g\in G}g[P]$ and $g_1[\mathring{P}]\cap g_2[\mathring{P}]\neq\varnothing$ implies $g_1=g_2$. In other words, $[P]$ is a *fundamental region* for the action $G:J$.

We assume that $\pi^{-1}[1,x]$ is **finite** for every $x\in\partial P$, hence, for every $x\in P$. Let $x\in P$. Then $\pi^{-1}[1,x]=\big\{(g_1,x_1),\dots,(g_n,x_n)\big\}$ for some $g_j\in G$ and $x_j\in P$. The polyhedra g_jP are the *formal neighbours* of P at x. For $\delta>0$, define

$$N_{x_j,\delta}:=\big\{y\in P\mid d(y,x_j)<\delta\big\}\subset P,\qquad N_{x,\delta}:=\bigcup_{j=1}^{n}(g_j,N_{x_j,\delta})\subset G\times P,\qquad W_{x,\delta}:=\pi N_{x,\delta}\subset J,$$

where d stands for the distance function on M. Using this notation, we state the

2.1. Tessellation Condition. A polyhedron P with a given face-pairing satisfies Tessellation Condition if

• for every $x\in P$, there exists some $\delta(x)>0$ such that $\pi^{-1}(W_{x,\delta})=N_{x,\delta}$ and $\varphi W_{x,\delta}=B\big(x,\delta\big)$ for all $0<\delta\le\delta(x)$;

• some open metric neighbourhood N of P in M is tessellated; this means that $N(P,\varepsilon)\subset N$ for some $\varepsilon>0$ and that there exists a function $f:P\to\mathbb{R}$ taking positive values such that $\varphi:W_{P,f}\to N$ is bijective, where $W_{P,f}:=\bigcup\limits_{x\in P}W_{x,f(x)}$.

2.2. Proposition. *Tessellation Condition implies that φ is a regular covering.*

Proof. Straightforward arguments show that J is Hausdorff and path-connected, that the family $\big\{gW_{x,\delta}\mid g\in G,\,x\in P,\,0<\delta\le\delta(x)\big\}$ is a base of the topology on J, and that φ is a local homeomorphism. Clearly, $\varphi:gW_{P,f}\to gN$ is a homeomorphism for all $g\in G$. We follow [AGG, Proof of Theorem 2.2.3] almost *verbatim*:

Being φ open, φJ is open in M. Let $x \in \mathrm{Cl}(\varphi J)$. Then $B(x,\varepsilon) \cap gP \neq \varnothing$ for some $g \in G$. It follows that $x \in N(gP,\varepsilon) \subset gN = \varphi(gW_{P,f}) \subset \varphi J$. Hence, φJ is closed in M. Being M connected, φ is surjective.

Take $x \in M$, define

$$G_x := \big\{g \in G \mid B(x,\varepsilon/2) \cap gP \neq \varnothing\big\},$$

and, for every $g \in G_x$, put

$$W_g := \varphi^{-1}\big(B(x,\varepsilon/2)\big) \cap gW_{P,f}.$$

Since $B(x,\varepsilon/2)\cap gP \neq \varnothing$ implies that $B(x,\varepsilon/2) \subset N(gP,\varepsilon) \subset gN$, we conclude that $\varphi : W_g \to B(x,\varepsilon/2)$ is a homeomorphism. Moreover,

$$\varphi^{-1}\big(B(x,\varepsilon/2)\big) = \bigcup_{g\in G_x} W_g.$$

It remains to show that the W_g's are disjoint. Suppose that $W_{g_1} \cap W_{g_2} \neq \varnothing$ for some $g_1, g_2 \in G_x$. The projection $W_{g_1} \times W_{g_2} \to W_{g_1}$ induces a homeomorphism between

$$X := \big\{(x_1,x_2) \in W_{g_1} \times W_{g_2} \mid \varphi x_1 = \varphi x_2\big\}$$

and W_{g_1}. The diagonal

$$\Delta_{W_{g_1}\cap W_{g_2}} = \Delta_J \cap (W_{g_1} \times W_{g_2}) \subset X$$

is closed in X since J is Hausdorff. Therefore, the image $W_{g_1} \cap W_{g_2}$ of $\Delta_{W_{g_1}\cap W_{g_2}}$ is closed in W_{g_1}. Being W_{g_1} connected, we obtain $W_{g_1} = W_{g_2}$ ■

Since M is simply-connected, $\varphi : J \to M$ is a homeomorphism. Hence, P is a fundamental region for the action $G : M$.

3. A Plane-like Poincaré's Polyhedron Theorem

In what follows, M is a connected, oriented, and simply-connected Riemannian manifold. We regard a *cornerless polyhedron* $P \subset M$ with a face-pairing as a subspace satisfying the conditions stated in the beginning of the previous section as well as those below.

• The faces of P are topologically closed, oriented smooth connected submanifolds of codimension 1 in M with (possibly empty) boundary. Each face is oriented so that, out of boundary points, a normal vector points towards the interior of P.

• The boundary of every face $s \in S$ is a disjoint union $\partial s = \bigsqcup_{e\in E_s} e$ of nonempty connected *edges.* ($E_s = \varnothing$ is allowed.) We write $e \diamond s$ in place of $e \in E_s$. Clearly, $e \diamond s$ implies $\overline{s} \diamond I_s e$.

• P has a finite number of faces and edges. Each edge e belongs to exactly two distinct faces s_1 and s_2. In symbols: $s_1 \diamond e \diamond s_2$.

• (Strong Simplicity) The intersection of two distinct faces is included into the boundary of both faces and is a (possibly empty) union of entire edges. The distances between:
two distinct edges,
two distinct faces that do not share an edge,
a face and an edge not included into the face
are all greater than some $d > 0$.

Start with $\overline{s}_0 \diamond e \diamond s_1$. Applying I_{s_1} to s_1 and e, we obtain $\overline{s}_1 \diamond I_{s_1} e \diamond s_2$. Applying I_{s_2} to s_2 and $I_{s_1} e$, we obtain $\overline{s}_2 \diamond I_{s_2} I_{s_1} e \diamond s_3$, and so on. (Of course, we eventually arrive back at $\overline{s}_0 \diamond e \diamond s_1$.)

A cyclic sequence

$$\xrightarrow{I_{s_n}} \overline{s}_n=\overline{s}_0\diamond e\diamond s_1 \xrightarrow{I_{s_1}} \overline{s}_1\diamond I_{s_1}e\diamond s_2 \xrightarrow{I_{s_2}} \overline{s}_2\diamond I_{s_2}I_{s_1}e\diamond s_3 \xrightarrow{I_{s_3}} \dots \xrightarrow{I_{s_{n-1}}} \overline{s}_{n-1}\diamond I_{s_{n-1}}\cdots I_{s_1}e\diamond s_n \xrightarrow{I_{s_n}},$$

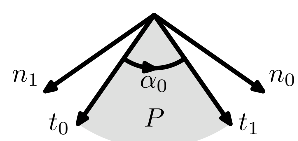
where each next term is obtained by the above rule, is called a *cycle of edges.* The number n is the length of the cycle and the isometry $I:=I_{s_n}\cdots I_{s_1}$ is the *cycle isometry.* A cycle can be read backwards, i.e., in opposite *orientation,* which inverts its isometry. If the cycle isometry is identical and if the cycle is the shortest one with such property, it is said to be *geometric* (see also Remark 3.1.) Clearly, every cycle is multiple of a shortest, combinatorial one. Notice that, in a geometric cycle, (every) edge may occur several times, which does not happen to a combinatorial one.

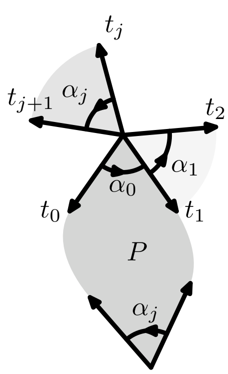
Assume that we are given a family of disjoint geometric cycles that include every edge of P. Fixing a term (say) $\overline{s}_0\diamond e\diamond s_1$ in some oriented cycle of the family, define $I_j:=I_{s_j}\cdots I_{s_1}$ for all $j=0,1,\dots,n$ (normally, we consider j modulo n) so that the cycle takes the form

$$\xrightarrow{I_{s_n}} \overline{s}_n=\overline{s}_0\diamond I_0e\diamond s_1 \xrightarrow{I_{s_1}} \overline{s}_1\diamond I_1e\diamond s_2 \xrightarrow{I_{s_2}} \overline{s}_2\diamond I_2e\diamond s_3 \xrightarrow{I_{s_3}} \dots \xrightarrow{I_{s_{n-1}}} \overline{s}_{n-1}\diamond I_{n-1}e\diamond s_n \xrightarrow{I_{s_n}} .$$

We can describe all formal neighbours of P at a point $x\in\partial P$. If x does not belong to any edge, then there is a unique face s containing x, $\pi^{-1}[1,x]=\big\{(1,x),(I_{\overline{s}},I_sx)\big\}$, and the only formal neighbours of P at x are P and $I_{\overline{s}}P$. If x belongs to an edge $\overline{s}_0\diamond e\diamond s_1$, then $\pi^{-1}[1,x]=\big\{(I_j^{-1},I_jx)\mid j=0,1,\dots,n-1\big\}$ and the $I_j^{-1}P$ are the formal neighbours of P at x. Indeed, suppose that $(I_j^{-1},I_jx)\sim(h,y)$. This means that $I_jx\in s'$, $I_{s'}I_jx=y$, and $h^{-1}I_j^{-1}=I_{s'}$ for some $s'\in S$. In particular, I_je and s' intersect. It follows from Strong Simplicity that $I_je\diamond s'$. Hence, either $s'=\overline{s}_j$ or $s'=s_{j+1}$. Therefore, either $(h,y)=(I_{j-1}^{-1},I_{j-1}x)$ or $(h,y)=(I_{j+1}^{-1},I_{j+1}x)$. It remains to observe that I_j, $j=0,1,\dots,n-1$, are all distinct because we could otherwise take a shorter cycle with identical isometry.

Pick a point x in some edge $\overline{s}_0\diamond e\diamond s_1$ of an oriented cycle. Let $\mathrm{N}_x\,e:=(\mathrm{T}_x\,e)^\perp$ and let n_0,n_1 stand respectively for the unitary normal vectors to $\overline{s}_0,s_1$ at x that point towards the interior of P, as above. Denote by $t_0\in\mathrm{T}_x\,\overline{s}_0\cap\mathrm{N}_x\,e$ and $t_1\in\mathrm{T}_x\,s_1\cap\mathrm{N}_x\,e$ the unitary vectors that point respectively towards the interiors of $\overline{s}_0$ and s_1. The basis t_0,n_0 orients $\mathrm{N}_x\,e$. This orientation is related to the orientation of the cycle. The oriented *interior angle* α_0 from $\overline{s}_0$ to s_1 at x is the angle from t_0 to t_1. It takes values in $[0,2\pi]$. We define similarly the interior angle α_j from $\overline{s}_j$ to s_{j+1} at I_jx. The sum $\sum_{j=0}^{n-1}\alpha_j$ is the *total interior angle* of the cycle at x. It is easy to see that altering the orientation of the cycle alters the orientation of corresponding $\mathrm{N}_x\,e$ and keeps the same values of the α_j's.





Suppose that the face-pairing isometries *send interior into exterior.* By definition, this means that $I_sn_s=-n_{\overline{s}}$ for every face $s\in S$, where n_s stands for the unitary normal vector to s at some $x\in s$. This property implies the following. Take a point x in some edge $\overline{s}_0\diamond e\diamond s_1$ of an oriented geometric cycle. Let $t_1\in\mathrm{T}_x\,s_1\cap\mathrm{N}_x\,e$ be the unitary vector that points towards the interior of $I_1^{-1}\overline{s}_1=s_1$ and let $t_2\in\mathrm{T}_x\,I_1^{-1}s_2\cap\mathrm{N}_x\,e$ be the unitary vector that points towards the interior of $I_1^{-1}s_2$. Then the oriented angle from t_1 to t_2 equals α_1. In the same way, denoting by $t_j\in\mathrm{T}_x\,I_j^{-1}s_j\cap\mathrm{N}_x\,e$ the unitary vector that points towards the interior of $I_1^{-1}s_j$, we can see that the oriented angle from t_j to t_{j+1} equals α_j. This implies immediately that $\sum_{j=0}^{n-1}\alpha_j\equiv0\mod 2\pi$. In particular, the total interior angle of a geometric cycle is **constant**: it does not depend on the choice of $x\in e$.

Obviously, the distinct formal neighbours of P at a point in an edge overlap when the total interior angle of a cycle is different from 2π. In terms of Proposition 2.2, this corresponds to a ramification of φ.

3.1. Remark. For some geometries, the nature of edges allows to (formally) weaken the condition that the cycle isometry is identical. This happens in the case when every isometry I that fixes pointwise some edge e is completely determined by the rotation angle about some $x\in e$, that is, by the image $In\in\mathrm{N}_x\,e$ of some $0\ne n\in\mathrm{N}_x\,e$. In this case, it suffices only to require that $I|_e=1_e$ and that the total interior angle at x vanishes modulo 2π.

3.2. Theorem. *Let P be a cornerless polyhedron with a face-pairing providing a family of geometric cycles that include every edge of P. Suppose that*

(1) *the face-pairing isometries send interior into exterior*;

(2) *the total interior angle equals 2π at some point of an edge for every cycle of the family*;

(3) *for every two distinct faces s,s' such that $s\cap s'\ne\varnothing$ and for every $\vartheta>0$, there exists $\varepsilon=\varepsilon(s,s',\vartheta)>0$ such that $s'\cap N(s,\varepsilon)\subset\bigcup\limits_{s\diamond e\diamond s'}N(e,\vartheta)$.*

Then Tessellation Condition 2.1 *is satisfied.*

Proof. In what follows, we denote $\widetilde X:=X\cap P$ for $X\subset M$.

First step. We will show that there exists a sufficiently small tessellated open ball centred in x for every $x\in P$. In other words, for every $x\in P$, we will find some $\delta(x)>0$ such that the first part of Tessellation Condition 2.1 is valid and, additionally, $\varphi:W_{x,\delta}\to B(x,\delta)$ is injective for all $0<\delta\le\delta(x)$. We distinguish the cases $x\in s\setminus\partial s$ for some $s\in S$ and $x\in e$ for some edge e.

• In the first case, choose $\delta_1>0$ such that $B(x,\delta_1)$ does not intersect the edges of s and such that $B(x,\delta_1)\cap\partial P=B(x,\delta_1)\cap s$. Choose $\delta_2>0$ analogously with respect to $I_sx\in\overline s$. Put

$$N_{x,\delta}:=\big(1,\widetilde B(x,\delta)\big)\bigcup\big(I_{\overline s},\widetilde B(I_sx,\delta)\big),\qquad W_{x,\delta}:=\pi N_{x,\delta},$$

where $0<\delta\le\delta(x):=\min(\delta_1,\delta_2)$. Clearly, $\pi^{-1}W_{x,\delta}=N_{x,\delta}$. We need to show that $\varphi:W_{x,\delta}\to B(x,\delta)$ is a bijection.

Notice that $s\cap B(x,\delta)\subset\widetilde B(x,\delta)\cap I_{\overline s}\widetilde B(I_sx,\delta)$. Also, $\widetilde B(x,\delta)\ne I_{\overline s}\widetilde B(I_sx,\delta)$ by condition (1). Pick a point $q_0\in\widetilde B(x,\delta)\setminus I_{\overline s}\widetilde B(I_sx,\delta)$ such that $q_0\notin s$. Due to $\delta\le\delta(x)$, a smooth oriented curve $\gamma\subset B(x,\delta)$ connecting q_0 and $q\in B(x,\delta)\setminus s$ can intersect ∂P and $\partial I_{\overline s}P$ only in $(s\setminus\partial s)\cap B(x,\delta)$. We can assume that such intersections are transversal. According to (1), when intersecting s, the curve γ leaves $\widetilde B(x,\delta)$ and enters $I_{\overline s}\widetilde B(I_sx,\delta)$ or *vice-versa.* Hence, q belongs to exactly one of $\widetilde B(x,\delta)$ and $I_{\overline s}\widetilde B(I_sx,\delta)$. The result follows $\square$

• The second case is similar. Let the I_j's be related to the geometric cycle including e. Choose $\delta_j>0$ such that $B(I_jx,\delta_j)$ does not intersect other edges of $\overline s_j$ and s_{j+1} except of I_je and such that $B(I_jx,\delta_j)\cap\partial P=\big(B(I_jx,\delta_j)\cap\overline s_j\big)\cup\big(B(I_jx,\delta_j)\cap s_{j+1}\big)$. Put

$$N_{x,\delta}:=\bigcup_j\big(I_j^{-1},\widetilde B(I_jx,\delta)\big),\qquad W_{x,\delta}:=\pi N_{x,\delta},$$

where $0<\delta\le\delta(x):=\min\delta_j$. The above description of formal neighbours implies $\pi^{-1}(W_{x,\delta})=N_{x,\delta}$. We have

$$I_j^{-1}s_{j+1}\cap B(x,\delta)\subset I_j^{-1}\widetilde B(I_jx,\delta)\cap I_{j+1}^{-1}\widetilde B(I_{j+1}x,\delta).$$

Put $F:=\bigcup\limits_j I_j^{-1}s_{j+1}\cap B(x,\delta)$ and let $q_0\in\widetilde B(x,\delta)\setminus F$. Due to $\delta\le\delta(x)$, a smooth oriented $\gamma\subset B(x,\delta)$ connecting q_0 and $q\in B(x,\delta)\setminus F$ may intersect $\bigcup\limits_j\partial I_j^{-1}P$ only in F. We assume that γ does not intersect e and is transversal to F. Condition (1) implies that, when intersecting $I_j^{-1}s_{j+1}$, the curve γ

leaves $I_j^{-1}\widetilde{B}(I_jx,\delta)$ and enters $I_{j+1}^{-1}\widetilde{B}(I_{j+1}x,\delta)$ or *vice-versa.* Hence, $\varphi: W_{x,\delta}\to B(x,\delta)$ is surjective. Following the discussion concerning the total angle of the cycle at x, we consider the closed sectors $T_j\subset \mathrm{N}_x\,e$ containing the oriented interior angle of $I_j^{-1}P$ at x. Conditions (1) and (2) imply that $\bigcup\limits_j T_j=N_xe$ and $\mathring{T}_{j_1}\cap\mathring{T}_{j_2}=\varnothing$ if $j_1\not\equiv j_2\mod n$. Hence, distinct formal neighbours $I_j^{-1}P$ cannot be equal.

Suppose that $\varphi: W_{x,\delta}\to B(x,\delta)$ is not injective at some $q\in B(x,\delta)$. It follows from the above description of formal neighbours that $q\notin F$. Pick a point q_0 living in exactly one of the $I_j^{-1}\widetilde{B}(I_jx,\delta)\setminus F$ and connect q_0 and q with a smooth oriented curve $\gamma\subset B(x,\delta)$ that does not intersect e and is transversal to F. By the properties of δ and by the arguments similar to those above, we arrive at a contradiction $\square$

Second step. We are going to use condition (3) in order to 'integrate' the above 'infinitesimal' tessellation and to get the tessellation of a metric neighbourhood of P.

Fix some $\vartheta<d/2$, where d is guaranteed by Strong Simplicity, and fix some $\varepsilon>0$ such that $\varepsilon<\frac12\min\limits_{s\cap s'\neq\varnothing}\varepsilon(s,s',\vartheta/2)$ and $\varepsilon<\vartheta/2$, where $\varepsilon(s,s',\vartheta/2)$ is given by condition (3).

Given an edge e, we put $N_{e,r}:=\bigcup\limits_j\big(I_j^{-1},\widetilde{N}(I_je,r)\big)$ with respect to the geometric cycle including e. For $s\in S$, define

$$N_{s,r}:=\big(1,\widetilde{N}(s,r)\big)\bigcup\big(I_{\overline{s}},\widetilde{N}(\overline{s},r)\big),\qquad W_s:=\pi N_{s,\varepsilon}\bigcup_{e\in E_s}\pi N_{e,\vartheta}.$$

- Let us show that $\varphi: W_s\to N(s,\varepsilon)\bigcup\limits_{e\in E_s}N(e,\vartheta)$ is a bijection.

Let $e\in E_s$. As above, define $F:=\bigcup\limits_j I_j^{-1}s_{j+1}\cap N(e,\vartheta)$ with respect to the geometric cycle including e. Pick a point $x\in e$. Due to the 'infinitesimal' tessellation of a small open ball B centred in x, we can choose $q_0\in B$ living in exactly one of the $I_j^{-1}\widetilde{N}(I_je,\vartheta)$. Clearly, $F\subset\varphi\pi N_{e,\vartheta}$. It follows from the description of formal neighbours that $\varphi:\pi N_{e,\vartheta}\to N(e,\vartheta)$ is injective at the points of F. Let $q\in N(e,\vartheta)\setminus F$. As above, connecting q_0 and q with a smooth oriented curve $\gamma\subset N(e,\vartheta)$ that does not intersect e and is transversal to F, we can see that γ intersects only the prescribed faces because $\vartheta<d$. We conclude that $\varphi:\pi N_{e,\vartheta}\to N(e,\vartheta)$ is surjective and injective. Since $\vartheta<d/2$, the $N(e,\vartheta)$ are disjoint. Therefore, $\varphi:\bigcup\limits_{e\in E_s}\pi N_{e,\vartheta}\to\bigcup\limits_{e\in E_s}N(e,\vartheta)$ is a bijection.

It is easy to see that

$$s\setminus\bigcup_{e\in E_s}N(e,\vartheta)\subset\varphi\Big(W_s\setminus\bigcup_{e\in E_s}\pi N_{e,\vartheta}\Big)\subset N(s,\varepsilon)\setminus\bigcup_{e\in E_s}N(e,\vartheta).$$

The description of formal neighbours implies that $\varphi: W_s\setminus\bigcup\limits_{e\in E_s}\pi N_{e,\vartheta}\to N(s,\varepsilon)\setminus\bigcup\limits_{e\in E_s}N(e,\vartheta)$ is injective at the points of $s\setminus\bigcup\limits_{e\in E_s}N(e,\vartheta)$. Pick a point $q\in N(s,\varepsilon)\setminus\bigcup\limits_{e\in E_s}N(e,\vartheta)$ such that $q\notin s$. There exist $x\in s$ and an oriented smooth curve $\gamma\subset N(s,\varepsilon)$ of length $\ell(\gamma)<\varepsilon$ that connects x and q.

We claim that γ can intersect ∂P and $\partial I_{\overline{s}}P$ only in $s\setminus\partial s$. Indeed, γ cannot intersect the faces of P or of $I_{\overline{s}}P$ that are disjoint with s because $\varepsilon<d$. Let s' be a face of P or of $I_{\overline{s}}P$ that intersects γ and such that $s\cap s'\neq\varnothing$. By condition (3) and by the choice of ε, we have

$$s'\cap\gamma\subset s'\cap N(s,\varepsilon)\subset\bigcup_{s\diamond e\diamond s'}N(e,\vartheta/2)\subset\bigcup_{e\in E_s}N(e,\vartheta/2),$$

which implies $q \in \bigcup\limits_{e\in E_s} N(e,\vartheta)$ because $\varepsilon < \vartheta/2$. A contradiction. The inequality $\varepsilon < \vartheta$ implies that γ does not intersect ∂s.

We can assume that γ is transversal to s. Due to the 'infinitesimal' tessellation of a small ball centred in x, at the very beginning, γ enters into the interior of P or of $I_{\overline{s}}P$. When γ intersects $s \setminus \partial s$, it leaves P and enters $I_{\overline{s}}P$ or *vice-versa.* As above, $\varphi : W_s \setminus \bigcup\limits_{e\in E_s} \pi N_{e,\vartheta} \to N(s,\varepsilon) \setminus \bigcup\limits_{e\in E_s} N(e,\vartheta)$ is surjective and injective $\square$

• Finally, let us show that the open metric neighbourhood $N := \mathring{P} \bigcup\limits_{s\in S} N(s,\varepsilon) \bigcup\limits_{e\in E_s} N(e,\vartheta)$ of P is tessellated. (Notice that $N(P,\varepsilon) \subset N$.) Define $f(x) = \vartheta$ if $x \in e$ for some edge e of P and $f(x) = \varepsilon$ if $x \in s \setminus \partial s$ for some $s \in S$. If $x \in \mathring{P}$, we take an arbitrary $f(x) > 0$ such that $B\big(x, f(x)\big) \subset \mathring{P}$. It is immediate that $\varphi W_{P,f} \subset N$ and that $W_{P,f} = [\mathring{P}] \bigcup\limits_{s\in S} W_s$. Hence, $\varphi : W_{P,f} \to N$ is surjective. If $\varphi w = \varphi w'$, where $w \in [\mathring{P}]$ and $w' \in W_s$, then $w = [1,x]$, $x \in \mathring{P}$, and $x \in N(s,\varepsilon) \bigcup\limits_{e\in E_s} N(e,\vartheta)$, implying $w \in W_s$. If $x := \varphi w = \varphi w'$, where $w \in W_s$ and $w' \in W_{s'}$, then $s \neq s'$ and we have two cases: $s \cap s' = \varnothing$ and $s \cap s' \neq \varnothing$. The first case is impossible because $N(s,\varepsilon) \bigcup\limits_{e\in E_s} N(e,\vartheta) \subset N(s,\vartheta)$, $N(s',\varepsilon) \bigcup\limits_{e\in E_{s'}} N(e,\vartheta) \subset N(s',\vartheta)$, and $N(s,\vartheta) \cap N(s',\vartheta) = \varnothing$ due to $\vartheta < d/2$.

In the second case, suppose that $x \in N(e,\vartheta)$ for some $e \in E_s$. Then $w \in \pi N_{e,\varepsilon}$ because the bijection $\varphi : \pi N_{e,\varepsilon} \to N(e,\vartheta)$ is a part of the bijection $\varphi : W_s \to N(s,\varepsilon) \bigcup\limits_{e\in E_s} N(e,\vartheta)$ as shown above. For the reason of $\varepsilon < \vartheta < d/2$, the inclusion $x \in N(s',\varepsilon) \bigcup\limits_{e\in E_{s'}} N(e,\vartheta)$ implies $e \in E_{s'}$ due to Strong Simplicity. So, $w' \in \pi N_{e,\varepsilon}$ and $w = w'$. The same arguments work if $x \in N(e,\vartheta)$ for some $e \in E_{s'}$.

Therefore, we can assume that $x \in N(s,\varepsilon) \cap N(s',\varepsilon)$ and $x \notin \bigcup\limits_{s\diamond e\diamond s'} N(e,\vartheta)$. We can find some $p \in s' \cap B(x,\varepsilon) \subset s' \cap N(s,2\varepsilon)$. It follows from $p \in B(x,\varepsilon)$, $x \notin \bigcup\limits_{s\diamond e\diamond s'} N(e,\vartheta)$, and $\varepsilon < \vartheta/2$ that $p \notin \bigcup\limits_{s\diamond e\diamond s'} N(e,\vartheta/2)$. This contradicts the choice of ε and ϑ $\blacksquare$

3.3. Final Remarks. The present version of PPT originates from [AGG]. We gave it a more general form that is applicable, as we believe, to many other cases. Probably, the first version of PPT with nonidentical combinatorial cycle isometry can be found in [Kui, Subsection 3.1, p. 60], although in an implicit form and in the easy real hyperbolic 4-dimensional context of constant curvature. (It is related to Remark 3.1.) Our version may be the first dealing with nonconstant angle along a common edge of two faces.

The second part of Tessellation Condition 2.1 (or condition (3) in Theorem 3.2 which has a similar nature) is almost necessary for the discreteness of G. It is violated in all counter-examples to PPT that we are familiar with. Therefore, Proposition 2.2 may be used in any version of PPT involving polyhedra.

Condition (3) in Theorem 3.2 is not trivial in the case of nonconstant curvature. In complex hyperbolic geometry, even in such a simple situation where the faces are bisectors intersecting in a common slice, the proof of this condition requires some analytic effort [AGG, Lemma 5.3.4]. So, in general, an attempt to use more complicate hypersurfaces may face serious difficulties.

In spite of the 'plane-like' character of our version, no extra conditions seem to be necessary if one wishes to deal with polyhedra possessing elements of codimension > 2. Notice that Theorem 3.2 does not require the completeness of the metric. Since a polyhedron is expected to be, *a posteriori,* a fundamental region for the action $G : M$, one can exclude all conjugates to the elements of codimension > 2 and simply apply Theorem 3.2 to what is left. Of course, this requires some effort. Instead of formulating yet another version of PPT, we quote:

'... *the condition can be verified by checking that the dihedral angles add up to* 360° *around each edge, and the corners fit together exactly to form a spherical neighborhood of a point in the model space. (Actually, the second condition follows from the first.)*' [Thu, p. 127, 11th line from the bottom]

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