Topological String on $\mathcal{S}^2$ Revisited

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TOPOLOGICAL STRING ON $\mathcal{S}^2$ REVISITED

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We compute the genus zero prepotential of the topological string on a two dimensonal sphere $\mathbf{CP}^1$ with the twisted mass corresponding to the rotation isometry of the sphere. Our derivation is a head-on summation over the tree diagrams arising in the localization approach of M. Kontsevich. Our result generalizes the theory of the limit shape of a random Young diagram, which are known to describe the prepotentials of the low energy effective actions of the four dimensional $\mathcal{N} = 2$ theories, as well as the stationary sector of the Gromov-Witten theory on $\mathbf{CP}^1$. We extend the limit shape to the full phase space of the $\mathbf{CP}^1$ string as well as its the equivariant generalization and conjecture the relation to the geometry of a D4 brane suspended between two NS5 branes in IIA string theory.

1. Introduction.

Topological strings have recently been related to various statistical mechanical problems [1],[2],[3]. This is done by mapping the partition function of a topological string on (a typically) toric manifold to a sum over two or three dimensional partitions. Two and three dimensional partitions can be viewed as the configurations of a statistical mechanical system. The existence of these representations of the partition functions of the topological strings is a reflection of the non-trivial strong/weak coupling string dualities [4], or gauge/string dualities [5], adapted to the context of the topological string and gauge theories. The relation to the two dimensional partitions goes via the interpretation of the four dimensional gauge instanton counting [6], which becomes a sum over random two dimensional partitions upon the use of the localization technique. The four dimensional gauge theory can be engineered using a decoupling (large radius) limit of a type IIA string compactification [7]. In this interpretation the prepotential of the four dimensional $\mathcal{N} = 2$ supersymmetric gauge theory is mapped to the prepotential of the topological string on some local Calabi-Yau manifold.

One formulation of the problem views the boxes which form the diagram of the partition as the individual particles of the many-body system, and define the energy to be the sum of the pair-wise interaction terms for the boundary boxes, as if the boxes where dipoles.

For example, in two dimensions it is natural to define the Boltzman weight of the partition λ to be proportional to the square of the dimension of the representation (which we shall also denote by λ) of the symmetric group $\mathcal{S}_n$, where n is the total number of boxes:

$$\mathbf{m}_\lambda = \left(\frac{\mathrm{dim}\lambda}{n!}\right)^2 \tag{1.1}$$

In the three dimensional case the simplest reasonable Boltzmann weight is just a constant. In the problem where the number of boxes is not fixed, one sums over n with a fugacity q. In this way one gets the partition functions:

$$Z_2(q) = \sum_\lambda \mathbf{m}_\lambda\, q^{|\lambda|}, \qquad Z_3(q) = \sum_\pi q^{|\pi|} \tag{1.2}$$

A typical problem is to study the limit shape, a partition which dominates the partition sum (1.2) in the appropriately defined thermodynamic limit: in two dimensions it is the limit $q \to \infty$, in three dimensions it is the limit $q \to 1$. The two dimensional problem dates back to the late seventies [8], while the three dimensional problem was solved recently [9].

The statistical models (1.2) and their generalizations which we discuss below arise in applications of localization techniques to gauge theories in four and six dimensions. The partitions then correspond to the fixed points of the action of various symmetry groups on the moduli space of supersymmetric field configurations, e.g. instantons. The integral over the moduli space of instantons to which reduces the gauge theory correlation function becomes the sum over the fixed points.

The partitions λ can be viewed as the infinite-dimensional matrices X_λ with the eigenvalues $\lambda_i - i + \frac{1}{2}$, $i = 1, 2, 3, \ldots$. The random partition model is therefore an analogue of the large N matrix model, with two major differences: the integral over the matrices is replaced by the sum over the eigenvalues taken out of the discrete set, with some characteristic spacing $\hbar$, and the size of the matrix is already infinite. The large N expansion is replaced by the $\hbar$-expansion.

It has been shown in [1] that the partition function of the topological string on $\mathbf{CP}^1$, in the so-called stationary sector whose definition we recall below, is equivalent to a random partition model:

$$Z = \sum_{\lambda} \mathbf{m}_\lambda q^{|\lambda|} e^{-V(\lambda)} \tag{1.3}$$

where V_λ is a suitably regularized "singe-trace"symmetric function of the eigenvalues $\lambda_i - i + \frac{1}{2}$, a potential. On the other hand, in [10] a matrix model describing the full topological string on $\mathbf{CP}^1$ was proposed.

The $\mathbf{CP}^1$ topological string as well as the any other string model whose target space has isometries, has an equivariant generalization. For $\mathbf{CP}^1$ the isometry group is $SU(2)$ and its maximal torus is just a copy of $U(1)$ acting on $\mathbf{CP}^1$ by rotation. We shall study the $U(1)$-equivariant theory. If the topological string is defined by twisting the $\mathcal{N} = (2,2)$ supersymmetric sigma model, the equivariant generalization corresponds to turning on a twisted mass. The $\mathbf{CP}^1$-model can be studied with the help of a gauged linear sigma model, with the gauge group $U(1)$ and two charged chiral multiplets, of charge $+1$. This theory has a global $U(2)$ symmetry, of which the center $U(1)$ is gauged. The twisted mass corresponds to the element ϵ of the Cartan subalgebra of the remaining global group $SU(2)$.

The calculations in the theory with the twisted mass can be localized onto the integrals over the fixed loci of the global symmetry action on the moduli space of the supersymmetric field configurations. In the context of the topological string this moduli space is the space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of degree $\beta \in H_2(X,\mathbf{Z})$, genus g, with n marked points. As explained in [11] the fixed loci on the moduli spaces of stable maps correspond to certain graphs, whose contribution to the topological string free energy can be interpreted as a Feynmann graph contribution in some quantum field theory. Moreover, in [11] the vertices and propagators of this theory, at the tree level, were defined. In [12] the vertices at every loop level were defined in terms of the Hodge integrals over $\overline{\mathcal{M}}_{g,n}$. In this paper we shall only discuss the genus zero story and we shall not need these higher loop corrections.

Thus our problem, as in [11] , is to evaluate a certain sum over trees, which reduces to the calculation of a critical value of a certain functional. In section **2** we present this critical value problem for $\mathbf{CP}^1$ model. We manage to bring the functional to the form, which is quadratic in an infinite number of variables and non-linear in two variables only (these variables correspond to the small phase space of the $\mathbf{CP}^1$ model). Note that [11] also performs a map to the quasi-quadratic form. Our transformation has the same origin (it relies on the property of the genus zero free energy of pure topological gravity) yet it is somewhat simpler, ultimately leading us to the explicit solution of the problem. In section **3** we solve the extremization problem using a version of Krichever-Whitham integrable system. In particular, we demonstrate explicitly that the prepotential is given by the quasi-classical tau-function of 2-Toda hierarchy. We find the spectral curve for this solution to be a nontrivial generalization of the Seiberg-Witten spectral curve for the $U(1)$ gauge theory in four dimensions. Of course the emergence of 2-Toda is all but natural given the results of [13]. The new result here is the explicit (quasi-classical) characterization of the point in Sato Grassmanian corresponding to the equivariant $\mathbf{CP}^1$ tau-function. In section **4** we explore the non-equivariant limit of our solution. Our expression for the prepotential generalizes the one found in [14] for the stationary sector. In our approach, at finite value of the equivariant parameter ϵ, all the logarithms entering the differential dS are naturally regularized in variance with [14]. Our expression for the free energy satisfies the semi-classical Virasoro constraints. In section **5** we discuss various gauge theory aspects of our problem. First of all we present a two dimensional gauge theory which is conjectured to reproduce the equivariant string expansion to all orders in the string coupling constant. Motivated by this conjecture we relate our extremization problem to that of the limit shape of random partitions. We also discuss possible implications of our results for the four dimensional $\mathcal{N}=2$ gauge theory and its realization in IIA string theory using branes.

In this approach the extremization problem is related to the relaxation problem of the M-theory fivebrane subject to certain boundary conditions.

In the appendix **A** we present the critical value problem for $\mathbf{CP}^N$ model with arbitrary N without explicit solution. In the appendix **B** we give some technical details and prove the consistency of our expressions for non-equivariant model with one given by [15].

2. Equivariant Gromov-Witten theory

The equivariant Gromov-Witten theory [16] is the mathematical name for the topological string on a manifold with symmetries in the background with the twisted masses corresponding to these symmetries turned on. In particular for $X = \mathbf{CP}^1$ one can use the $\mathbf{C}^{\times}$-action on X.

It is perhaps useful to remind the Lagrangian formalism of the topological sigma model in the presence of the twisted mass terms. Consider the general Kähler target space X, with the holomorphic coordinates x^i. Let p_i be the bosonic dimension $(1,0)$ field, ψ^i a fermionic scalar and π_i the fermionic $(1,0)$ form on the worldsheet C. Let $X^m = (x^i, x^{\bar{i}})$ denote collectively the coordinates on X. In a similar fashion let $\Psi^m = (\psi^i, \psi^{\bar{i}})$ denote the scalar fermions. The nilpotent scalar supercharge $\mathcal{Q}$ of the topological sigma model acts on these fields via $\delta_0 \mathbf{\Phi} = \{\mathcal{Q}, \mathbf{\Phi}]$:

$$\begin{aligned} &\delta_0 X^m = \Psi^m \, , \ \delta_0 \Psi^m = 0 \\ &\delta_0 \pi_i = p_i \ \ \delta_0 \pi_{\bar{i}} = p_{\bar{i}} \\ &\delta_0 p_i = 0 \ \ \delta_0 p_{\bar{i}} = 0 \end{aligned} \tag{2.1}$$

Let G be a compact Lie group acting on X isometrically. Let $G_{\mathbf{C}}$ denote its complexification. It acts on X by holomorphic diffeomorphisms. Let $V^i(\epsilon)\partial_i = \epsilon^A V^i_A \partial_i$ denote the holomorphic vector field on X generating the action of the element $\epsilon = \epsilon^A \mathbf{t}_A$ of the Lie algebra of $G_{\mathbf{C}}$. In our conventions

$$\mathcal{V}(\epsilon) = \mathcal{V}^m(\epsilon)\frac{\partial}{\partial X^m} = V^i(\epsilon)\frac{\partial}{\partial x^i} + V^{\bar{i}}(\epsilon)\frac{\bar{\partial}}{\partial x^{\bar{i}}} \tag{2.2}$$

generates the action of G. The topological sigma model on X can be coupled to the twisted super-Yang-Mills theory on C with the gauge group G. The parameter ϵ of the infinitesimal G-transformations gets promoted to the scalar field of the vector multiplet. In addition, the twisted multiplet contains the gauge field A, the fermion one-form Υ, a pair of fermionic scalars χ and η, another scalar $\bar{\epsilon}$, and an auxiliary scalar boson H. For the theory coupled to the two dimensional gravity it is convenient to view $\bar{\epsilon}$ and η as two-forms. The presence of the background field ϵ modifies the supersymmetry transformations: $\delta_0 \to \delta_\epsilon$:

$$
\begin{aligned}
\delta_\epsilon X^m &= \Psi^m\,, & \delta_\epsilon \Psi^m &= \mathcal{V}^m(\epsilon) \\
\delta_\epsilon \pi_i &= p_i\,, & \delta_\epsilon p_i &= -\partial_i V^j(\epsilon)\pi_j \\
\delta_\epsilon \pi_{\bar{i}} &= p_{\bar{i}}\,, & \delta_\epsilon p_{\bar{i}} &= -\partial_{\bar{i}} V^{\bar{j}}(\epsilon)\pi_{\bar{j}} \\
\delta_\epsilon A &= \Upsilon\,, & \delta_\epsilon \Upsilon &= D_A \epsilon \\
\delta_\epsilon \bar{\epsilon} &= \eta\,, & \delta_\epsilon \eta &= [\epsilon, \bar{\epsilon}] \\
\delta_\epsilon \chi &= H\,, & \delta_\epsilon H &= [\epsilon, \chi]
\end{aligned}
\tag{2.3}
$$

The Lagrangian of the topological sigma model in the presence of the vector multiplet can be written as follows:

$$
\begin{aligned}
L = \delta_\epsilon \int_C \Big(& \pi_i \bar{\partial} x^i + \pi_{\bar{i}} \partial x^{\bar{i}} + g_{mn} \Psi^m \mathcal{V}^n(\bar{\epsilon}) + \\
& + g^{i\bar{j}}(X) \left(\pi_i \left(p_{\bar{j}} - \Gamma^{\bar{k}}_{\bar{i}\bar{j}} \pi_{\bar{k}} \psi^{\bar{j}} \right) - \pi_{\bar{j}} \left(p_i - \Gamma^k_{ij} \pi_k \psi^j \right) \right) + \\
& + \ \mathrm{tr}\, (\Upsilon \wedge \star D_A \bar{\epsilon}) + \mathrm{tr}\, (\eta [\epsilon, \bar{\epsilon}]) \\
& + \mathrm{tr}\, \chi \left(F_A + \mathbf{m}(X)\, \mathrm{vol}_g \right) \Big)
\end{aligned}
\tag{2.4}
$$

Here $\mathbf{m} : X \to \mathrm{Lie}^* G$ is the moment map.

Finally, the topological string is obtained from the topological sigma model by coupling the latter to the topological gravity. There are two ingredients in the latter. First, one promotes the worldsheet derivatives $\bar{\partial}$ and ∂, which are defined with respect to some fixed complex structure on C to $\bar{\partial}^\mu \equiv \bar{\partial} - \mu \partial$ and $\partial^\mu = \partial - \bar{\mu} \bar{\partial}$, i.e. one makes the choice of a complex structure on C, represented e.g. by a Beltrami differential $\mu = \mu^w_{\bar{w}} \partial_w \mathrm{d}\bar{w}$, and its complex conjugate $\bar{\mu} = \bar{\mu}^{\bar{w}}_w \partial_{\bar{w}} \mathrm{d}w$, a dynamical variable. To preserve supersymmetry one also introduces a fermionic partner $\upsilon = \upsilon^w_{\bar{w}} \partial_w \mathrm{d}\bar{w}$, and its complex conjugate $\bar{\upsilon}$. Secondly, to preserve the number of degrees of freedom, one couples the enlarged space of fields $X, \Psi, A, \Upsilon, \mu, \upsilon, \bar{\mu}, \bar{\upsilon}$ to the $\mathrm{Diff}(C)$-equivariant cohomology multiplet. By analogy with the gauge multiplet, it contains a bosonic vector field $v = v^w \partial_w + v^{\bar{w}} \partial_{\bar{w}}$ (the analogue of ϵ), a bosonic one-form $\bar{v} = \bar{v}_w \mathrm{d}w + \bar{v}_{\bar{w}} \mathrm{d}\bar{w}$ (the analogue of $\bar{\epsilon}$), its fermionic partner $\nu = \nu_w \mathrm{d}w + \nu_{\bar{w}} \mathrm{d}\bar{w}$. The supersymmetry transformation δ_ϵ changes again $\delta_\epsilon \to \delta_{\epsilon,v}$. We shall not describe it here in detail. We also shall not describe here the coupling to the dynamical gauge multiplet, it can be found in [17]. It is well-known that one can turn on the background vector multiplet without breaking the supersymmetry by having (covariantly) constant ϵ, $D_A \epsilon = 0$, vanishing fermions, flat connection A, $F_A = 0$, and $\bar{\epsilon}$ commuting with ϵ. For our purposes we need to know the transformation properties of the fermions Ψ, v:

$$\begin{aligned}\delta_{\epsilon,v}\Psi^n &= v^w\partial_w X^n + v^{\overline{w}}\partial_{\overline{w}}X^n + \mathcal{V}^n(\epsilon)\\ \delta_{\epsilon,v}v &= \mathcal{L}_v\mu\end{aligned} \tag{2.5}$$

2.1. The moduli space

The path integral in the topological string on X in the background with non-vanishing ϵ localizes onto the field configurations obeying:

$$\begin{gathered}\partial_{\overline{w}}X^i - \mu^w_{\overline{w}}\partial_w X^i = 0\\ v^w\partial_w X^n + v^{\overline{w}}\partial_{\overline{w}}X^n + \mathcal{V}^n(\epsilon) = 0\\ \mathcal{L}_v\mu = 0\,, \qquad \mathcal{L}_v\overline{\mu} = 0\end{gathered} \tag{2.6}$$

The solutions to (2.6) which differ by a worldsheet diffeomorphism are identified. The solutions to the equations (2.6) are the instantons in the sigma model on X. The instantons carry a charge, a degree β of the map $\phi : C \to X$, $\beta = [\phi(C)] \in H_2(X, \mathbf{Z})$. In addition, we have am additional topological invariant of the problem, the genus g of the worldsheet C.

In this way we get a moduli space $\mathcal{M}_g(X, \beta)$ of holomorphic maps $C \to X$, of degree $\beta \in H_2(X, \mathbf{Z})$, of the genus g complex curves.

For example, when $C = \mathbf{CP}^1$, the deformations of its complex structure are all trivial, i.e. are related by some diffeomorphisms. Moreover, there are diffeomorphisms which preserve a given complex structure, the group $PGL_2(\mathbf{C})$ of them, generated by the vector fields $\partial_w, w\partial_w, w^2\partial_w, \partial_{\overline{w}}, \overline{w}\partial_{\overline{w}}, \overline{w}^2\partial_{\overline{w}}$.

In the problem of main interest for this paper the target space is also a two-sphere, $X = \mathbf{CP}^1$, and the degree β is just a non-negative integer. A typical holomorphic map $\mathbf{CP}^1 \to X$ is just a rational function of degree β:

$$\phi(w) = \frac{a_0 + a_1 w + \ldots + a_\beta w^\beta}{b_0 + b_1 w + \ldots + b_\beta w^\beta} \tag{2.7}$$

The moduli space $\mathcal{M}_0(\mathbf{CP}^1, \beta)$ is the space of coefficients

$$(a_0 : a_1 : \ldots : a_\beta : b_0 : b_1 : \ldots : b_\beta) \in \mathbf{CP}^{2\beta+1},$$

considered up to a common multiple, subject to certain restriction, divided by the action of the group $PGL_2(\mathbf{C})$:

$$\phi(w) \sim \phi\left(\frac{Aw + B}{Cw + D}\right), \; AD - BC \neq 0 \tag{2.8}$$

The restriction is that the numerator and the denominator in (2.7) have no common roots in $\mathbf{CP}^1$. The moduli space is, therefore, complex $2\beta - 2$ dimensional ($2\beta + 1$ coefficients a_i, b_i up to a common multiple minus 3 complex dimensions of A, B, C, D in (2.8) up to the irrelevant common multiple). Because of the restriction on the absence of common roots, the moduli space $\mathcal{M}_0(\mathbf{CP}^1, \beta)$ is non-compact in general. The exception is the case $\beta = 1$ where the moduli space consists of just one point.

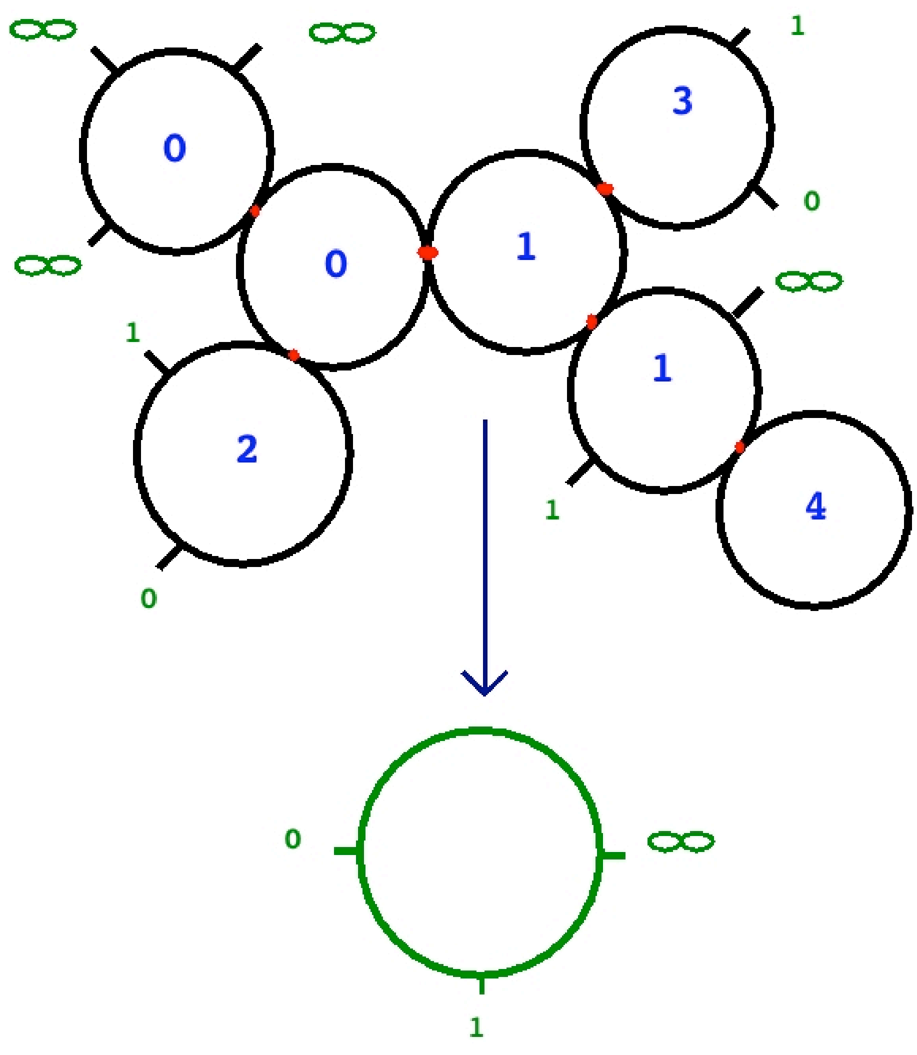

Fig.1 A stable map

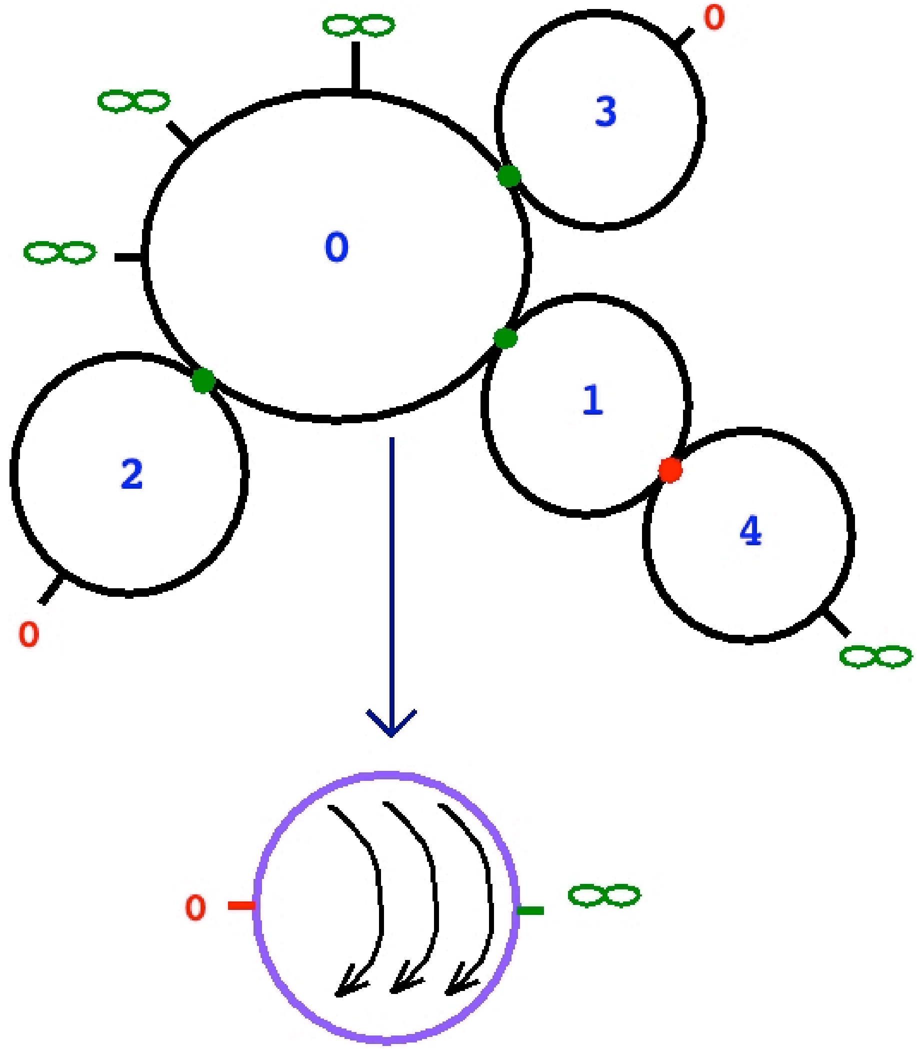

Fig.2 An example of a $\mathbf{C}^{\times}$-invariant stable map

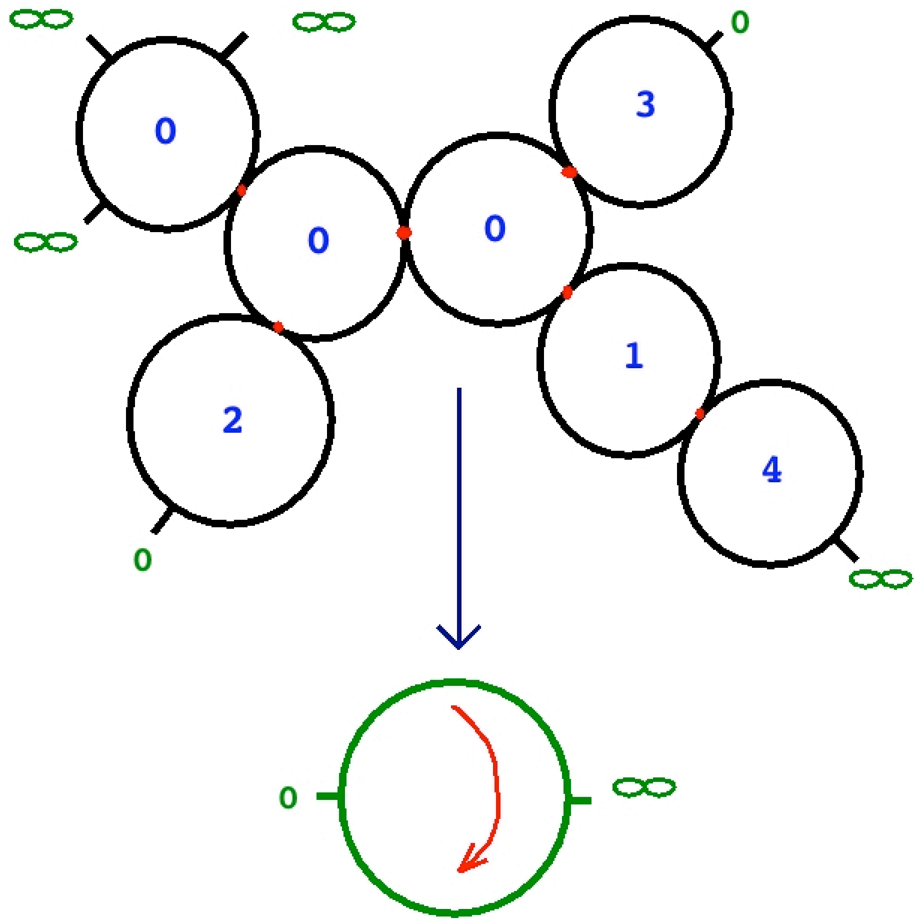

Fig.3 A connected component of the moduli space of $\mathbf{C}^{\times}$-invariant stable maps

2.2. Observables

The observables of the equivariant Gromov-Witten theory are the descendents of the equivariant cohomology classes of the target space X, i.e. the cohomology of the operator

$$D = \mathrm{d} + \iota_{\mathcal{V}(\epsilon)} \ . \tag{2.9}$$

In the case of our main interest, the group G acts on X with isolated fixed points. The equivariant cohomology of X is generated by the Poincare duals of the fixed points. Indeed, a top degree delta-form $\delta_f \in \Omega^{\mathrm{dim} X}(X)$ supported at some fixed point $f \in X$ is annihilated both by d (dimension count) and $\iota_{\mathcal{V}(\epsilon)}$ (the fixed point condition). Moreover, a Duistermaat-Heckman formula:

$$\int_X \alpha(\epsilon) = \sum_{f \in X^G} \frac{\alpha^{(0)}(\epsilon)|_f}{\prod_i w_{f,i}(\epsilon)} \tag{2.10}$$

expressing the integral of the equivariantly closed form $\alpha(\epsilon)$ (which is an inhomogeneous differential form on X, annihilated by D) in terms of the local data near the fixed points X^G shows that essentially (modulo $H^*(BG)$-torsion) everything comes from these fixed points.

In the case of $X = \mathbf{CP}^1$, $G = U(1)$ acting by rotations, we have two such points 0, and ∞. The Poincare duals $\delta_{\mathbf{0}}$, δ_∞ (which are represented by the delta two-forms on $\mathbf{CP}^1$) of the two fixed points $\mathbf{0}$, ∞ in $\mathbf{CP}^1$ are our basic observables. The conventional cohomology classes $\mathbf{1} \in H^0(\mathbf{CP}^1)$ and $\omega \in H^2(\mathbf{CP}^1)$ (the Kähler form, normalized so that $\int_X \omega = 1$) are expressed through $\mathbf{0}$ and ∞ via:

$$\mathbf{1} = \frac{1}{\epsilon}\left(\delta_{\mathbf{0}} - \delta_\infty\right) , \qquad \omega = \delta_{\mathbf{0}} \tag{2.11}$$

In addition, the gravitational dressing makes two infinite sequences of observables out of these two equvariant cohomology classes:

$$\tau_k(\delta_{\mathbf{0}}) \ , \ \tau_k(\delta_\infty) \qquad k = 0, 1, 2, \ldots \tag{2.12}$$

We shall not describe the Lagrangian formalism of the dressing here. The mathematical definition of the descendents τ_k is well-known [18], [11], [19]. The marked point on the worldsheet reduces the group of diffeomorphisms to the one which preserves this point. In this way a puncture comes with the operator $\delta(v^w)\delta(v^{\overline{w}})$ of dimension $(1,1)$, which can be viewed as a topological descendent of the operator ϕ of dimension zero. By taking the product of the observables coming from the cohomology classes α of the target space and the powers of ϕ one gets the gravitational descendents $\tau_k(\alpha) \sim \phi^k \cdot \alpha$.

2.3. String partition function

The string partition function is the function of $3 + 2 \times \infty$ variables:

$$Z\left(\varepsilon, \hbar, q; x_k, x_k^*\right) = \exp \sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g\left(\varepsilon, q; x_k, x_k^*\right) \tag{2.13}$$

The calculation of the prepotential using localization onto the fixed points [11] of the $\mathbf{C}^*$-action on the moduli space $\overline{\mathcal{M}_{0,n}\left(\mathbf{CP}^1, d\right)}$ of stable maps can be recast in the form of calculating the critical value of the following action functional:

$$\begin{aligned}
\mathcal{S}\left(\phi_d,\phi_d^*;\xi,\xi^*;\mathbf{x},\mathbf{x}^*\right) &= \sum_{d=1}^{\infty}(-1)^{d+1}\phi_d\phi_d^*\,\frac{d!^2}{d^{2d-1}}\left(\frac{\varepsilon^2}{q}\right)^d + \\
&\quad \frac{1}{2\varepsilon}\left[\int_0^{\xi}(t-\Upsilon(t))^2\,dt+\int_{\xi^*}^{0}(t-\Upsilon^*(t))^2\,dt\right] \\
&\quad + \text{“unstable“ contributions}: \\
&\quad \frac{1}{2}\sum_{d_1,d_2=1}^{\infty}\frac{d_1d_2}{d_1+d_2}\left(\phi_{d_1}\phi_{d_2}+\phi_{d_1}^*\phi_{d_2}^*\right) \\
&\quad +\varepsilon\sum_{k=0}^{\infty}\sum_{d=1}^{\infty}\left(-\frac{\varepsilon}{d}\right)^k\left(x_k\phi_d+(-1)^{k+1}x_k^*\phi_d^*\right) \\
&\quad +\varepsilon\sum_{d=1}^{\infty}\frac{1}{d}\left(\phi_d-\phi_d^*\right)
\end{aligned} \tag{2.14}$$

where we denote by

$$\begin{aligned}
\Upsilon(t) = \varepsilon\mathbf{x}(t)+\phi(t)\ ,\ \Upsilon^*(t) = -\varepsilon\mathbf{x}^*(t)+\phi^*(t) \\
\phi(t)=\sum_{p=1}^{\infty}p\,\phi_p\exp\left(\frac{tp}{\varepsilon}\right)\ ,\ \phi^*(t)=\sum_{p=1}^{\infty}p\,\phi_p^*\exp\left(-\frac{tp}{\varepsilon}\right) \\
\mathbf{x}(t)=\sum_{k=0}^{\infty}\frac{x_k}{k!}t^k\ ,\ \mathbf{x}^*(t)=\sum_{k=0}^{\infty}\frac{x_k^*}{k!}t^k
\end{aligned} \tag{2.15}$$

We extremize $\mathcal{S}$ w.r.t. $\phi_p,\phi_p^*,\xi,\xi^*$.

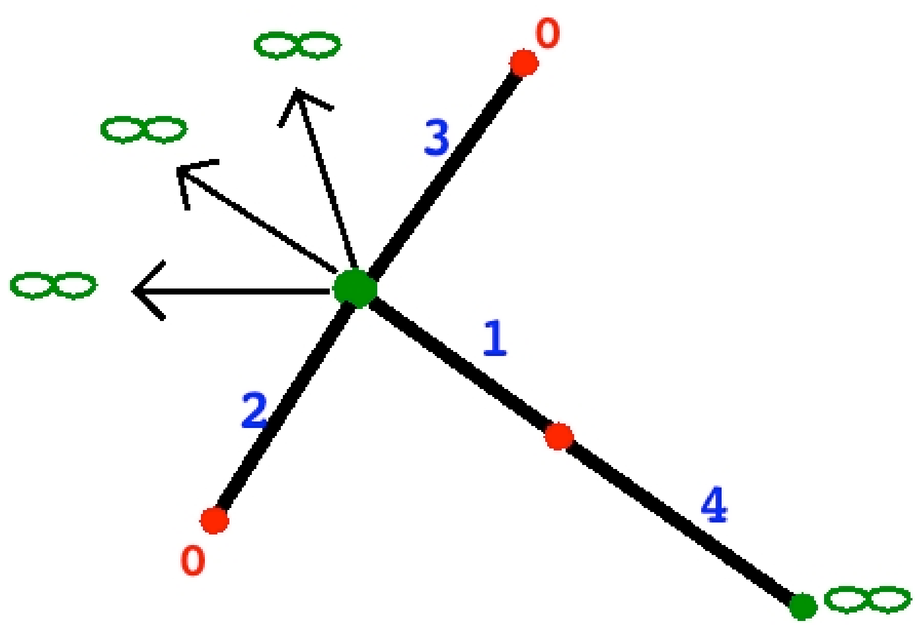

Fig.4 Tree diagram corresponding to the $\mathbf{C}^{\times}$-fixed locus

In deriving (2.14) we had to use the following expression for the free energy of the pure topological gravity in genus zero:

$$\mathbf{F}_0(t_0, t_1, \ldots) = \mathrm{Crit}_\xi \frac{1}{2} \int_0^\xi (s - \mathbf{t}(s))^2 \, \mathrm{d}s \tag{2.16}$$

where:

$$\mathbf{t}(s) = \sum_{k=0}^{\infty} t_k \frac{s^k}{k!} \tag{2.17}$$

and

$$\mathbf{F}(t_0, t_1, \ldots) = \sum_{N=3}^{\infty} \frac{1}{N!} \sum_{k_1, \ldots, k_N \geq 0} \prod_{i=1}^{N} t_{k_i} \int_{\overline{\mathcal{M}}_{0,N}} \psi_1^{k_1} \wedge \ldots \wedge \psi_N^{k_N} \tag{2.18}$$

(the elementary derivation of (2.16) is presented in the appendix)

2.4. Unstable contributions

By unstable contributions in (2.14) (the last three lines) we mean the contribution of the components of the worldsheet which would have been unstable were not there a non-trivial holomorphic map attached to them. The $\phi_{d_1}\phi_{d_2}$ structure, for example, corresponds to the double point, mapped to a point $\mathbf{0}$ in the target, which connects two spheres, which are in turn mapped non-trivially with degrees d_1 and d_2. □

The structure $x_k \phi_d$

The structure $\frac{1}{d}\phi_d$....

□

The last three lines in (2.14) can be written in the integral form:

$$\frac{1}{2\varepsilon} \left[\int_{-\infty}^{0} \left\{ (t - \Upsilon(t))^2 - (t - \varepsilon \mathbf{x}(t))^2 \right\} \mathrm{d}t + \int_0^{\infty} \left\{ (t - \Upsilon^*(t))^2 - (t + \varepsilon \mathbf{x}^*(t))^2 \right\} \mathrm{d}t \right] \tag{2.19}$$

Thus, the interaction part of (2.14) simplifies to:

$$\begin{aligned} S^{\mathrm{in}} = & \frac{1}{2\varepsilon} \left[\int_{-\infty}^{\xi} \left((t - \Upsilon(t))^2 - (t - \varepsilon \mathbf{x}(t))^2 \right) \mathrm{d}t \right. \\ & \left. + \int_{\xi^*}^{+\infty} \left((t - \Upsilon^*(t))^2 - (t + \varepsilon \mathbf{x}^*(t))^2 \right) \mathrm{d}t \right] \\ & + \frac{1}{2\varepsilon} \left[\int_0^{\xi} (t - \varepsilon \mathbf{x}(t))^2 \, \mathrm{d}t + \int_{\xi^*}^{0} (t + \varepsilon \mathbf{x}^*(t))^2 \, \mathrm{d}t \right] \end{aligned} \tag{2.20}$$

2.5. Prepotential

We are after the prepotential:

$$\boxed{\mathcal{F}(\mathbf{x}, \mathbf{x}^*, \varepsilon) = \mathrm{Crit}_{\xi, \xi^*, \phi_p, \phi_p^*}\ \mathcal{S}\left(\phi_p, \phi_p^*; \xi, \xi^*; \mathbf{x}, \mathbf{x}^*\right)} \tag{2.21}$$

Variation of $\mathcal{S}$ with respect to ϕ_p, ϕ_p^* gives the equations:

$$\begin{aligned}
(-1)^{p+1}\phi_p^* &= \frac{q^p}{\varepsilon}\left(\frac{p^p}{p!\,\varepsilon^p}\right)^2 \int_{-\infty}^{\xi} (t - \Upsilon(t)) \exp\left(\frac{t\,p}{\varepsilon}\right) \mathrm{d}t \\
(-1)^{p+1}\phi_p &= \frac{q^p}{\varepsilon}\left(\frac{p^p}{p!\,\varepsilon^p}\right)^2 \int_{\xi^*}^{+\infty} (t - \Upsilon^*(t)) \exp\left(-\frac{t\,p}{\varepsilon}\right) \mathrm{d}t \\
p &= 1, 2, 3, \ldots
\end{aligned} \tag{2.22}$$

$$\xi = \Upsilon(\xi)\ ,\ \xi^* = \Upsilon^*(\xi^*) \tag{2.23}$$

and evaluate (2.14) on the solution to (2.22). We are interested in the solutions perturbative in q.

2.6. Derivatives of the prepotential

In what follows we shall need the expressions for the derivatives of the prepotential with respect to the times x_k, x_k^*. Since the variation of a critical value of a functional is the value of the variation of the functional at the critical point, we have:

$$\begin{aligned}
\frac{\partial \mathcal{F}}{\partial x_k} &= -\int_0^{\xi} \mathrm{d}t\ \left(t - \varepsilon\mathbf{x}(t)\right)\frac{t^k}{k!} + \int_{-\infty}^{\xi} \phi(t)\frac{t^k}{k!}\ \mathrm{d}t \\
\frac{\partial \mathcal{F}}{\partial x_k^*} &= -\int_0^{\xi^*} \mathrm{d}t\ \left(t + \varepsilon\mathbf{x}^*(t)\right)\frac{t^k}{k!} + \int_{+\infty}^{\xi^*} \phi^*(t)\frac{t^k}{k!}\ \mathrm{d}t
\end{aligned} \tag{2.24}$$

2.7. Mirror map

The couplings x_k, x_k^* of the topological string are not the most convenient parameters to describe the solution. This is the well-known phenomenon of the *mirror map* where the naive couplings, the "times" T_n, T_n^*, to be defined below, which are convenient to describe the B model side of the story, are not the canonical coupling constants of the A model.

Let $\widetilde{x}_k = x_k - \delta_{k,1}\varepsilon^{-1}$, $\widetilde{x}_k^* = x_k^* + \delta_{k,1}\varepsilon^{-1}$ denote the "dilaton-shifted" couplings. Following [13] we introduce the set of times T_n, T_n^*, where $n = 1, 2, \ldots$ (the zero time(s) should be discussed separately), which are related by the transformation (conjectured by E. Getzler [20])

$$T_n = \sum_{k=0}^{\infty} \widetilde{x}_k \ \mathrm{Coeff}_{z^{k+1}} \ \frac{z^n}{(1+\varepsilon z)(2+\varepsilon z)\dots(n+\varepsilon z)} \tag{2.25}$$

The converse relation is:

$$\sum_{k=0}^{\infty} \widetilde{x}_k w^k = \sum_{n=1}^{\infty} n T_n \ (\varepsilon + w)(\varepsilon + 2w)\dots(\varepsilon + (n-1)w) \tag{2.26}$$

Similarly:

$$\begin{gathered} T_n^* = \sum_{k=0}^{\infty} \widetilde{x}_k^* \ \mathrm{Coeff}_{z^{k+1}} \ \frac{z^n}{(1-\varepsilon z)(2-\varepsilon z)\dots(n-\varepsilon z)} \\ \sum_{k=0}^{\infty} \widetilde{x}_k^* w^k = \sum_{n=1}^{\infty} n T_n^* \ (-\varepsilon + w)(-\varepsilon + 2w)\dots(-\varepsilon + (n-1)w) \end{gathered} \tag{2.27}$$

We shall also need the relation between $x(t)$ and T_n's:

$$\varepsilon x(t) - t = \sum_n n \varepsilon^{n-1} T_n P_n'(t/\varepsilon) \tag{2.28}$$

where Getzler-Okounkov-Pandharipande (GOP) polynomials $P_n(t)$ are given by:

$$\begin{aligned} P_n(t) &= \\ &\frac{1}{2\pi i} \oint_{w=\infty} e^{t/w} \, \mathrm{d}w \, (\varepsilon + w)(\varepsilon + 2w)\dots(\varepsilon + (n-1)w) = \\ &\qquad = \frac{1}{n} \prod_{l=1}^{n-1} (1 + \frac{\varepsilon}{l} \partial_t) t^n = \\ &\qquad = \sum_{l=0}^{n-1} \frac{(n-1)!}{(n-l)!} h_{n-1}^{(l)} \ \varepsilon^l \ t^{n-l} \\ P_1(t) &= t \, , \quad P_2(t) = \frac{1}{2} t^2 + \varepsilon t \, , \dots \end{aligned} \tag{2.29}$$

where $h_{n-1}^{(l)}$ are the generalized harmonic numbers:

$$\begin{gathered} h_k^{(l)} = \sum_{1 \le k_1 < k_2 < \dots < k_l \le k} \frac{1}{k_1 k_2 \dots k_l} \\ \prod_{m=1}^{k} \left(1 + \frac{x}{m}\right) = \sum_{l=0}^{k} h_k^{(l)} \, x^l \end{gathered} \tag{2.30}$$

The GOP polynomials $P_n(t)$ have the following nice property. Define the generating function

$$Y(z,t) = z - t - \sum_{k=1}^{\infty} P_k(t) z^{-k} \tag{2.31}$$

Then,

$$\boxed{Y \exp(-Y) = z \exp(-z+t)} \tag{2.32}$$

We also need to compute:

$$R_n(\rho) = \rho \int_0^1 P_n' \left(\rho(1-u) + \log u\right) \mathrm{d}u = \frac{\rho^n}{n} \tag{2.33}$$

The proof of the last equality is given in Appendix **B**.

3. The solution

In this section the solution to the problem of calculating the genus zero prepotential of the $\mathbf{CP}^1$ model is presented. It uses the formalism of Krichever-Whitham hierarchies. We describe a family of analytic (non-algebraic) curves, and meromorphic differentials on them, whose periods capture the free energy of the topological string.

3.1. The curve

Introduce the following analytic curve:

$$\boxed{\begin{aligned} \rho\ \exp\left(-\frac{\rho}{\varepsilon}\right) &= \Lambda w\ \exp\left(-\frac{1}{\varepsilon}\left(\Lambda\left(w+\frac{1}{w}\right)+v_+\right)\right) \\ \overline{\rho}\ \ \exp\left(\frac{\overline{\rho}}{\varepsilon}\right) &= \frac{\Lambda}{w}\ \ \exp\left(\frac{1}{\varepsilon}\left(\Lambda\left(w+\frac{1}{w}\right)+v_-\right)\right) \end{aligned}} \tag{3.1}$$

We can think of (3.1) as of a curve $\mathcal{C}_{v,\Lambda}$ sitting in a three dimensional complex manifold

$$\mathcal{C}_{v,\Lambda} \subset \mathbf{C}\times\mathbf{C}\times\mathbf{C}^{\times}$$

with the coordinates $(\rho,\overline{\rho},w)$. The coordinate $\overline{\rho}$ is not the complex conjugate of ρ. The parameters $\varepsilon,\Lambda,v_{\pm}$ are fixed.

The curve (3.1) has some similarity with the curves one

3.2. The differentials

On the curve (3.1) we define functions Ω_k, Ω_k^*, $k=1,2,3,\ldots$, as follows. Near $w=\infty$ choose the following branch of ρ:

$$\rho = \Lambda w + v_+ + \sum_{k=1} w^{-k}\mathcal{P}_k(\Lambda, v_+) \tag{3.2}$$

where $\mathcal{P}_k$ are the polynomials in v_+ to be defined below. Define Ω_k as a polynomial in w:

$$\Omega_k = \left(\rho^k\right)_+ \tag{3.3}$$

In other words, Ω_k is a meromorphic function on the curve which has no singularities everwhere except for $w=\infty$ where it has the asymptotics $\rho^k + O(1)$. Similarly, Ω_k^* is a polynomial in w^{-1},

$$\Omega_k^* = \left(\overline{\rho}^k\right)_- \tag{3.4}$$

with the branch of $\overline{\rho}$ near $w = 0$ given by:

$$\overline{\rho} = \Lambda w^{-1} + v_- + \sum_{k=1} w^k \overline{\mathcal{P}}_k(\Lambda, v_-) \tag{3.5}$$

Define the functions S, S^* by:

$$\begin{aligned} S &= \sum_{k=1}^{\infty} T_k \Omega_k \\ S^* &= \sum_{k=1}^{\infty} T_k^* \Omega_k^* \end{aligned} \tag{3.6}$$

Thus S is non-singular near $w = 0$, while S^* is non-singular near $w = \infty$. Finally, define the meromorphic one-form:

$$\lambda = dS - dS^* \tag{3.7}$$

3.3. The solution for ϕ, ϕ^*

Then*:

$$\begin{aligned} \phi(t) &= \frac{1}{2\pi i} \oint \frac{\partial \rho^{\vee}(t)}{\partial t} \lambda \\ \phi^*(t) &= \frac{1}{2\pi i} \oint \frac{\partial \overline{\rho}^{\vee}(t)}{\partial t} \lambda \end{aligned} \tag{3.8}$$

where $\rho^{\vee}(t), \overline{\rho}^{\vee}(t)$ are given by:

$$\begin{aligned} \rho^{\vee}(t) &= \varepsilon \sum_{p=1}^{\infty} \frac{p^{p-1}}{p!} \left(\frac{\Lambda w}{\varepsilon} e^{\frac{1}{\varepsilon}\left(-\Lambda\left(w + \frac{1}{w}\right) - v_+ + t\right)} \right)^p \\ \overline{\rho}^{\vee}(t) &= -\varepsilon \sum_{p=1}^{\infty} \frac{p^{p-1}}{p!} \left(-\frac{\Lambda}{\varepsilon} w^{-1} e^{\frac{1}{\varepsilon}\left(\Lambda\left(w + \frac{1}{w}\right) + v_- - t\right)} \right)^p \end{aligned} \tag{3.9}$$

In terms of the components ϕ_p, ϕ_p^*:

$$\begin{aligned} \phi_p &= \frac{p^{p-1} \Lambda^p}{p! \, \varepsilon^p} \frac{1}{2\pi i} \oint w^p e^{-\frac{p}{\varepsilon}\left(\Lambda\left(w + \frac{1}{w}\right) + v_+\right)} \lambda \\ \phi_p^* &= -\frac{p^{p-1} \Lambda^p}{p! \, (-\varepsilon)^p} \frac{1}{2\pi i} \oint w^{-p} e^{\frac{p}{\varepsilon}\left(\Lambda\left(w + \frac{1}{w}\right) + v_-\right)} \lambda \end{aligned} \tag{3.10}$$

Note that

$$\varepsilon x(t) - t = -\frac{1}{2\pi i} \oint \frac{\partial \rho^{\wedge}(t)}{\partial t} dS = -\frac{1}{2\pi i} \oint \frac{\partial \rho^{\wedge}(t)}{\partial t} \lambda \tag{3.11}$$

---

* We put $\frac{1}{2\pi i} \oint_0 \frac{dw}{w} = \frac{1}{2\pi i} \oint_\infty \frac{dw}{w} = 1$

where $\rho^{\wedge}(t)$ denotes the branch

$$\rho^{\wedge}(t) = \rho - t - \varepsilon \sum_{k=1}^{\infty} \varepsilon^k P_k(t/\varepsilon)\, \rho^{-k} \tag{3.12}$$

of the function $\varepsilon Y(\rho/\varepsilon, t/\varepsilon)$ defined by (2.31). $\rho^{\wedge}(t)$ and $\rho^{\vee}(t)$ are two different branches of the same one-parametric family of curves, which is t-deformation of the curve (3.1):

$$\rho e^{-\frac{\rho}{\varepsilon}} = \Lambda w\, e^{-\frac{1}{\varepsilon}\left(\Lambda(w+\frac{1}{w})+v_+-t\right)} \tag{3.13}$$

Analogously there are $\overline{\rho}^{\wedge}(t)$ and $\overline{\rho}^{\vee}(t)$ defined by the equation

$$\overline{\rho} e^{\frac{\overline{\rho}}{\varepsilon}} = \frac{\Lambda}{w}\, e^{\frac{1}{\varepsilon}\left(\Lambda(w+\frac{1}{w})+v_--t\right)} \tag{3.14}$$

Only two of three parameters Λ and $v_\pm$ are independent:

$$v_\pm = v \pm \epsilon \log\left(\frac{\Lambda}{\sqrt{q}}\right) \tag{3.15}$$

They are defined from the equations

$$\frac{d\mathcal{S}}{dw}\Big|_{w_\pm} = 0 \tag{3.16}$$

where $w_\pm$ are solutions of the equation $\frac{\partial \rho}{\partial w} = 0$, which is equivalent (for almost all values of t) to quadratic one:

$$\Lambda w_\pm^2 - \epsilon w_\pm - \Lambda = 0 \tag{3.17}$$

3.4. The solution for ξ, ξ^*

Parameters ξ and ξ^* are given by

$$\xi = v_+ - t_0, \quad \xi^* = v_- + t_0 \tag{3.18}$$

where

$$t_0 = \varepsilon\left(\log\left(\frac{f_+}{2}\right) - f_-\right) \tag{3.19}$$

and

$$f_\pm = \sqrt{1 + \frac{4\Lambda^2}{\varepsilon^2}} \pm 1 \tag{3.20}$$

Among all representatives of the families (3.13) and (3.14) curves with t equal to ξ and ξ^* correspondingly are very special. An important property of these representatives is that they correspond to singular curves. But we use another property, namely:

$$\begin{aligned} \rho^{\wedge}(\xi) &= \rho^{\vee}(\xi) \\ \overline{\rho}^{\wedge}(\xi^*) &= \overline{\rho}^{\vee}(\xi^*) \end{aligned} \tag{3.21}$$

The proof of the first equality is given in the Appendix **B**. Proof of the second one is absolutely analogous.

3.5. Baker-Akhiezer functions

Let us now show that $\mathcal{S} = S - S^*, \mathcal{S}^* = S^* - S = -\mathcal{S}$ defined by (3.6) are the quasiclassical Baker-Akhiezer (BA) functions, in the sense of [21]:

$$\begin{aligned} \mathcal{S} &= \sum_{n=1}^{\infty} T_n \rho^n - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial \mathcal{F}}{\partial T_n} \rho^{-n} \\ \mathcal{S}^* &= \sum_{n=1}^{\infty} T_n^* \overline{\rho}^n - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial \mathcal{F}}{\partial T_n^*} \overline{\rho}^{-n} \end{aligned} \tag{3.22}$$

where the expansions (3.22) are understood near the points $w = \infty$ and $w = 0$ respectively.

Indeed, the formulae (3.22) are equivalent to the residue formulae:

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial T_n} &= \frac{1}{2\pi i} \oint_{w=\infty} \rho^n d\mathcal{S} \\ \frac{\partial \mathcal{F}}{\partial T_n^*} &= -\frac{1}{2\pi i} \oint_{w=0} \overline{\rho}^n d\mathcal{S}^* \end{aligned} \tag{3.23}$$

Using (2.24) and (2.28) we can write:

$$\begin{aligned} \frac{1}{n} \frac{\partial \mathcal{F}}{\partial T_n} &= - \int_0^{\xi} (t - \varepsilon x(t)) P_n'(t) dt + \int_{-\infty}^{\xi} \phi(t) P_n'(t) dt \\ &= \frac{1}{2\pi i} \oint d\mathcal{S} \left[- \int_0^{\xi} \frac{\partial \rho^{\wedge}}{\partial t} P_n'(t) dt + \int_{-\infty}^{\xi} \frac{\partial \rho^{\vee}}{\partial t} P_n'(t) dt \right] \end{aligned} \tag{3.24}$$

Now the idea is to change the integration variable from t to $\widetilde{\rho} = Y(\rho, t)$ in a smart way so that to employ the integration formula (2.33). Indeed, if $\widetilde{\rho} = Y(\rho, t)$, then:

$$t = \rho - \widetilde{\rho} + \log(\widetilde{\rho}/\rho) = \rho(1 - e^{-s}) - s \tag{3.25}$$

where we define s via: $\widetilde{\rho} = e^{-s}\rho$. As s changes from 0 to $+\infty$, $\widetilde{\rho}$ changes from ρ to 0, while t goes from 0 to ξ, with $\widetilde{\rho}$ following the $\rho^{\wedge}$ branch, then t goes through the point $t = \xi$ where $\rho^{\wedge}(t)$ joins $\rho^{\vee}(t)$ (see above), and then t goes from ξ all the way to $-\infty$, with $\widetilde{\rho}$ following the $\rho^{\vee}(t)$ branch. Therefore the expression in the square brackets in (3.24) is equal to ρ^n/n.

3.6. Prepotential

Prepotential (2.21) can be represented in Whitham type form [22] :

$$\mathcal{F} = \frac{1}{2}\left(\frac{1}{2\pi i}\oint_{\infty} S_+ d\mathcal{S} - \frac{1}{2\pi i}\oint_{0} S_+^* d\mathcal{S}^*\right) = \frac{1}{8\pi i}\oint (S+S^*)_+ d(S-S^*) \tag{3.26}$$

where + in first and second integrals are with respect to coordinates ρ and $\overline{\rho}$ respectively. It first derivatives

$$\frac{\partial \mathcal{F}}{\partial T_k} = \frac{1}{2\pi i}\oint \rho^k d\mathcal{S}\,, \; \frac{\partial \mathcal{F}}{\partial T_k^*} = \frac{1}{2\pi i}\oint \overline{\rho}^k d\mathcal{S} \tag{3.27}$$

and second derivatives

$$\begin{aligned} \frac{\partial^2 \mathcal{F}}{\partial T_k \partial T_m} &= \frac{1}{2\pi i}\oint \rho^k d\Omega_m\,, \; \frac{\partial^2 \mathcal{F}}{\partial T_k^* \partial T_m^*} = -\frac{1}{2\pi i}\oint \overline{\rho}^k d\overline{\Omega}_m \\ \frac{\partial^2 \mathcal{F}}{\partial T_k \partial T_m^*} &= -\frac{1}{2\pi i}\oint \rho^k d\overline{\Omega}_m = \frac{1}{2\pi i}\oint \overline{\rho}^k d\Omega_m \end{aligned} \tag{3.28}$$

come explicit quadratic dependence on times. In particular:

$$\frac{\partial^2 \mathcal{F}}{\partial x_0^2} = \Lambda^2 + v_+\varepsilon\,, \; \frac{\partial^2 \mathcal{F}}{\partial x_0^{*2}} = \Lambda^2 - v_-\varepsilon \tag{3.29}$$

$$\frac{\partial^2 \mathcal{F}}{\partial x_0 \partial x_0^*} = -\Lambda^2 \tag{3.30}$$

3.7. The proof

Let us now prove that formulas (3.8), (3.18)actually solve equations of motion (2.22), (2.23). First of all, from (3.8)and (3.11)it follows that an equation for ξ is equivalent to an equation

$$\oint \frac{\partial \rho^{\vee}(t)}{\partial t}\Big|_{t=\xi} d(S - S*) = \oint \frac{\partial \rho^{\wedge}(t)}{\partial t}\Big|_{t=\xi} d(S - S*) \tag{3.31}$$

t In difference with values of functions $\rho^{\vee}(t)$ and $\rho^{\wedge}(t)$ their differentials w.r.t. t at $t=\xi$ do not coincide. Both of them can be represented as

$$\frac{\partial \rho(t)}{\partial w}\Big|_{t=\xi} = -\frac{w^2}{\Lambda w^2 - \varepsilon w - \Lambda}\frac{\partial \rho(\xi)}{\partial w} \tag{3.32}$$

but decomposition rules for the fraction in r.h.s. are different and as formal powers in w expressions do not coincide. This non-coincidence is compensated by differential $d(S-S^*)$ which, by definition of Λ and v (3.16), is proportional to denominator in (3.32). Thus integrals in (3.31)are equal to each other.

To prove an equation (2.22)let us represent r.h.s. of it with help of (3.8) and (3.11):

$$\begin{aligned}
&\int_{-\infty}^{\xi} (t-\Upsilon(t)) \exp\left(\frac{t\,p}{\varepsilon}\right) \mathrm{d}t = \\
&\frac{q^p}{\varepsilon}\left(\frac{p^p}{p!\,\varepsilon^p}\right)^2 \int_{-\infty}^{\xi} \frac{1}{2\pi i}\oint\left(\frac{\partial\rho^\wedge(t)}{\partial t}-\frac{\partial\rho^\vee(t)}{\partial t}\right) d\mathcal{S} \exp\left(\frac{t\,p}{\varepsilon}\right) \mathrm{d}t = \\
&\frac{1}{2\pi i}\oint\left(\int_{-\infty}^{\xi} \exp\left(\frac{t\,p}{\varepsilon}\right)\left(d\rho^\wedge(t)-d\rho^\vee(t)\right)\right) d\mathcal{S}
\end{aligned} \tag{3.33}$$

Then, similar to (3.24), we change integration variable. Since $\rho^\wedge(t)$ and $\rho^\vee(t)$ goes to ∞ and 0 respectively when $t\to\infty$, one has:

$$\begin{aligned}
&\int_{-\infty}^{\xi} \exp\left(\frac{t\,p}{\varepsilon}\right)\left(d\rho^\wedge(t)-d\rho^\vee(t)\right) = \\
&\left(\frac{1}{\Lambda w} e^{\frac{1}{\varepsilon}\left(\Lambda\left(w+\frac{1}{w}\right)+v_+\right)}\right)^p \left(\int_0^\infty \widetilde{\rho}^p e^{\frac{-p\widetilde{\rho}}{\varepsilon}} d\widetilde{\rho}\right)
\end{aligned} \tag{3.34}$$

Then (2.22) follows from this and relation between $v_\pm$ and Λ (3.15).

3.8. The first times and small phase space

Let us describe the solution in more detail for only first few times switched on, i.e. when $x_k = x_k^* = 0$ for $k>1$. In this case:

$$\begin{aligned}
T_2 &= \frac{\widetilde{x}_1}{2}\,,\ T_1 = x_0 - \varepsilon\widetilde{x}_1 \\
T_2^* &= \frac{\widetilde{x}_1^*}{2}\,,\ T_1^* = x_0^* + \varepsilon\widetilde{x}_1^*
\end{aligned} \tag{3.35}$$

To find an expression for free energy one needs explicit formulas for ρ and $\overline{\rho}$:

$$\begin{aligned}
\rho = \Lambda w + v_+ + \left(\frac{\epsilon v_+}{\Lambda}+\Lambda\right)\frac{1}{w} + \left(\frac{v_+\epsilon}{\Lambda^2}+1-\frac{1}{2}\left(\frac{v_+}{\Lambda}\right)^2\right)\frac{\epsilon}{w^2} + \\
+\frac{1}{6}\frac{2v_+^3-9\epsilon v_+^2+6(\epsilon^2-\Lambda^2)v_+ + 6\Lambda^2\epsilon}{\Lambda^3}\frac{\epsilon}{w^3} + O\left(\frac{1}{w^4}\right)
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
\overline{\rho} = \frac{\Lambda}{w} + v_- + \left(-\frac{\epsilon v_-}{\Lambda}+\Lambda\right) w + \left(\frac{v_-\epsilon}{\Lambda^2}-1+\frac{1}{2}\left(\frac{v_-}{\Lambda}\right)^2\right)\epsilon w^2 + \\
+\frac{1}{6}\frac{-2v_-^3-9\epsilon v_-^2-6(\epsilon^2-\Lambda^2)v_- + 6\Lambda^2\epsilon}{\Lambda^3}\epsilon w^3 + O\left(w^4\right)
\end{aligned} \tag{3.37}$$

and polynomials Ω:

$$\begin{aligned}
\Omega_1 &= \Lambda w\,,\ \Omega_2 = \Lambda^2 w^2 + 2v_+\Lambda w \\
\Omega_1^* &= \frac{\Lambda}{w}\,,\ \Omega_2^* = \frac{\Lambda^2}{w^2} + \frac{2v_-\Lambda}{w}
\end{aligned} \tag{3.38}$$

Then differential $d\mathcal{S}$ is:

$$d\mathcal{S} = \Lambda \left(\widetilde{x}_1 \Lambda w + x_0 + (v_+ - \varepsilon)\widetilde{x}_1 + (x_0^* + (v_- + \varepsilon)\widetilde{x}_1^*) \frac{1}{w^2} + \frac{\Lambda \widetilde{x}_1^*}{w^3} \right) dw \tag{3.39}$$

To satisfy (3.16) this differential should have the following decomposition:

$$\frac{d\mathcal{S}}{dw} = (\Lambda w^2 - \varepsilon w - \Lambda) \left(\frac{a}{w} + \frac{b}{w^2} + \frac{c}{w^3} \right) \tag{3.40}$$

From this equation one can get expressions for a,b and c

$$a = \widetilde{x}_1 \Lambda \, , \, b = -\frac{(\widetilde{x}_1 + \widetilde{x}_1^*)\Lambda^2}{\varepsilon} \, , \, c = -\widetilde{x}_1^* \Lambda \tag{3.41}$$

as well as

$$\Lambda^2 \text{exp} \left[-\frac{(x_1 + x_1^*)^2}{(1 - \varepsilon x_1)(1 + \varepsilon x_1^*)} \Lambda^2 \right] = q \text{ exp} \left[\frac{x_0}{1 - \varepsilon x_1} + \frac{x_0^*}{1 + \varepsilon x_1^*} \right] \tag{3.42}$$

and

$$\begin{aligned} v_+ &= \frac{\varepsilon x_0 + (x_1 + x_1^*)\Lambda^2}{1 - \varepsilon x_1} \\ v_- &= -\frac{\varepsilon x_0^* - (x_1 + x_1^*)\Lambda^2}{1 + \varepsilon x_1^*} \end{aligned} \tag{3.43}$$

An explicit expression for free energy is as follows

$$\mathcal{F} = \frac{\varepsilon^2}{3!} \left(\frac{x_0^3}{1 - \varepsilon x_1} + \frac{x_0^{*3}}{1 + \varepsilon x_1^*} \right) + \frac{(1 - \varepsilon x_1)^2 (1 + \varepsilon x_1^*)^2}{(x_1 + x_1^*)^2} f \left(\frac{q (x_1 + x_1^*)^2}{(1 - \varepsilon x_1)(1 + \varepsilon x_1^*)} e^{\frac{x_0}{1 - \varepsilon x_1} + \frac{x_0^*}{1 + \varepsilon x_1^*}} \right) \tag{3.44}$$

where

$$f(x) = \sum_{k=1}^{\infty} \frac{k^{k-3}}{k!} x^k \tag{3.45}$$

Expressions for $\phi(t)$ and $\phi^*(t)$ can be represented as follows:

$$\begin{aligned} \phi(t) &= \frac{1}{2\pi i} \oint \frac{\partial \rho^\vee(t)}{\partial t} d\mathcal{S} = \frac{1}{2\pi i} \oint \frac{\partial \rho^\vee(t)}{\partial w} \left(\frac{\widetilde{x}_1^* \Lambda}{w} + \frac{(\widetilde{x}_1 + \widetilde{x}_1^*)\Lambda^2}{\varepsilon} - \widetilde{x}_1 \Lambda w \right) = \\ &- \frac{1}{2\pi i} \oint \rho^\vee(t) \left(\varepsilon \widetilde{x}_1^* - (x_1 + x_1^*)\Lambda w \right) \frac{dw}{w} \\ \phi^*(t) &= \frac{1}{2\pi i} \oint \overline{\rho}^\vee(t) \left(\varepsilon \widetilde{x}_1 + \frac{(x_1 + x_1^*)\Lambda}{w} \right) \frac{dw}{w} \end{aligned} \tag{3.46}$$

On the small phase space previous formulas simplifies to:

$$\begin{aligned}\phi(t) &= \frac{1}{2\pi i}\oint \rho^\vee(t)\frac{dw}{w}\\ \phi^*(t) &= \frac{1}{2\pi i}\oint \overline{\rho}^\vee(t)\frac{dw}{w}\end{aligned} \tag{3.47}$$

$$\Lambda = \sqrt{q}\exp\frac{x_0 + x_0^*}{2}\ ,\ v = \frac{\epsilon(x_0 - x_0^*)}{2} \tag{3.48}$$

$$\mathcal{F} = \frac{\epsilon^2(x_0^3 + x_0^{*3})}{3!} + q\exp(x_0 + x_0^*) \tag{3.49}$$

4. Non-equivariant limit

The prepotential of the non-equivariant theory can be obtained in the limit $\varepsilon \to 0$ from the formulas of the previous section. Times of the non-equivariant theory are given by

$$x_i = \frac{1}{\varepsilon} t_i^1 \, , \; x_i^* = t_i^\omega - \frac{1}{\varepsilon} t_i^1 \tag{4.1}$$

In the non-equivariant limit both ρ and $\overline{\rho}$ goes to the same expression:

$$\rho_0 = \Lambda \left(w + \frac{1}{w} \right) + v \tag{4.2}$$

It can be shown that limit of (3.26) is regular, namely

$$2\mathcal{F}_0 = \frac{1}{2\pi i} \oint (A + B) \, d \left(A_+ - B_- \right) + \frac{1}{2\pi i} \oint C d\mathbf{v}_1 \tag{4.3}$$

where

$$\begin{aligned} A &= \mathbf{v}_1' \Delta\rho^- - \mathbf{v}_c \\ B &= -\mathbf{v}_1' \Delta\rho^+ + \mathbf{v}_\omega - \mathbf{v}_c \\ C &= (\mathbf{v}_\omega' - \mathbf{v}_c')\Delta\rho^+ - \mathbf{v}_c'\Delta\rho^- + \frac{\mathbf{v}_1'}{\rho_0} \left(\Delta\rho^- + \Delta\rho^+ \right) + \mathbf{v}_1'' \left(\frac{(\Delta\rho^-)^2}{2} - \frac{(\Delta\rho^+)^2}{2} \right) \end{aligned} \tag{4.4}$$

and

$$\mathbf{v}_1 = v_1(\rho_0) = \sum_{n=1}^{\infty} \frac{t_{n-1}^1}{n!} \rho_0^n \, , \; \mathbf{v}_c = v_c(\rho_0) = \sum_{n=1}^{\infty} \frac{c_n t_n^1}{n!} \rho_0^n \, , \; \mathbf{v}_\omega = v_\omega(\rho_0) = \sum_{n=1}^{\infty} \frac{t_{n-1}^\omega}{n!} x^n \tag{4.5}$$

$\Delta\rho^\pm$ are logarithmic corrections coming from ε decomposition of (3.1)

$$\Delta\rho^\pm = \pm \left(\log \left(\frac{\sqrt{q} w^{\mp 1}}{\rho_0} \right) \right)_\pm \tag{4.6}$$

and

$$c_n = \sum_{k=1}^{n} \frac{1}{k} \tag{4.7}$$

harmonic numbers. Prepotential (4.3) can be represented as

$$\mathcal{F}_0 = \frac{1}{8\pi i} \oint S dR \tag{4.8}$$

where

$$S = \sum_{k=0}^{\infty} \frac{t_k^\omega}{(k+1)!} \rho_0^{k+1} + \sum_{k=0}^{\infty} \frac{t_k^1}{k!} \rho_0^k \left(\Delta\rho^- - \Delta\rho^+ - 2c_k \right) \tag{4.9}$$

and

$$dR = dS_+ - dS_- + dQ \tag{4.10}$$

with

$$dQ = \mathbf{v}'_1 d(\Delta\rho^- + \Delta\rho^+ + 2\text{log}w) \tag{4.11}$$

These formulas generalize expressions for stable sector found in [14]. Derivatives of the prepotential are

$$\begin{aligned} \frac{\partial \mathcal{F}_0}{\partial t^\omega_n} &= \frac{1}{4\pi i}\oint \frac{\rho_0^{n+1}}{(n+1)!}dR \\ \frac{\partial \mathcal{F}_0}{\partial t^1_n} &= \frac{1}{4\pi i}\oint \frac{\rho_0^{n}}{n!}\left(\Delta\rho_- - \Delta\rho_+ - 2c_n\right)dR \end{aligned} \tag{4.12}$$

Furthermore in this section we omit subscript 0 for non-equivariant Lax ρ and prepotential $\mathcal{F}$. For example, from (3.44) it follows that

$$\mathcal{F}(t^1_0, t^1_1, t^\omega_0, t^\omega_1) = \frac{t^\omega_0 (t^1_0)^2}{2!(1-t^1_1)} + \frac{t^\omega_1 (t^1_0)^3}{3!(1-t^1_1)^2} + \frac{(1-t^1_1)^4}{(t^\omega_1)^2} f\left(\frac{q(t^\omega_1)^2}{(1-t^1_1)^2} e^{\frac{t^\omega_0}{1-t^1_1} + \frac{t^1_0 t^\omega_1}{(1-t^1_1)^2}}\right) \tag{4.13}$$

with function $f(x)$ defined by (3.45). In Appendix **B** we show that our expression (4.8) is compatible with one given by B, Dubrovin and Y. Zhang [15].

4.1. Virasoro constraints

As usual [22] , Virasoro constraints for the prepotential (4.3) follow from the invariance of the integral with respect to change of variables

$$\rho \to \rho + \epsilon_k \rho^{k+1} \tag{4.14}$$

namely

$$0 = \delta\mathcal{F} = \frac{1}{8\pi i}\oint (\delta S)dR + S(\delta dR) \tag{4.15}$$

Function ρ is symmetric under transformation $w \to \frac{1}{w}$, hence δdR is antisymmetric with respect to this transformation.

At $w \to \infty$

$$dR = d\mathbf{v}_\omega - 2d\mathbf{v}_c + (\Delta\rho^- - \Delta\rho^+)d\mathbf{v}'_1 + 2\left(\frac{1}{\rho}\right)_- d\mathbf{v}_1 - 2\sum_{n=0}^{\infty}\frac{\partial\mathcal{F}}{\partial t^\omega_n} d\frac{n!}{\rho^{n+1}} \tag{4.16}$$

It can be shown that $\mathcal{F}$ satisfy quasiclassical Virasoro constraints (cf. [23] , [15])

$$\begin{aligned}
&\sum_{k=1}^{\infty} t_k^{\alpha} \frac{\partial \mathcal{F}}{\partial t_{k-1}^{\alpha}} + t_0^1 t_0^{\omega} = 0 \\
&\sum_{k=1}^{\infty} k \left(t_k^1 \frac{\partial \mathcal{F}}{\partial t_k^1} + t_{k-1}^{\omega} \frac{\partial \mathcal{F}}{\partial t_{k-1}^{\omega}} \right) + 2 \sum_{k=1}^{\infty} t_k^1 \frac{\partial \mathcal{F}}{\partial t_{k-1}^{\omega}} + (t_0^1)^2 = 0 \\
&\sum_{k=1}^{\infty} k!(m-k)! \frac{\partial \mathcal{F}}{\partial t_{k-1}^{\omega}} \frac{\partial \mathcal{F}}{\partial t_{m-k-1}} + \sum_{k=1}^{\infty} \frac{(m+k)!}{(k-1)!} \left(t_k^1 \frac{\partial \mathcal{F}}{\partial t_{m+k}^1} + t_{k-1}^{\omega} \frac{\partial \mathcal{F}}{t_{m+k+1}^{\omega}} \right) + \\
&+ 2 \sum_{k=0}^{\infty} \alpha_m(k) t_k^1 \frac{\partial \mathcal{F}}{t_{m+k-1}^{\omega}}, \quad m \geq 1
\end{aligned} \tag{4.17}$$

where

$$\alpha_m(0) = m!, \quad \alpha_m(k) = \frac{(m+k)!}{(k-1)!} \sum_{j=k}^{m+k} \frac{1}{j}, \quad k > 0 \tag{4.18}$$

4.2. Random partitions and limit shape

The prepotential in the non-equivariant stationary sector is also a critical value of a quadratic-looking functional:

$$\mathcal{F}(\mathbf{t}^{\omega}; a) = \mathrm{Crit}_{f(x)} \mathcal{S} \left[f(x); \mathbf{t}^{\omega}(x); a \right] \tag{4.19}$$

where we minimize w.r.t to a function $f : \mathbf{R} \to \mathbf{R}$, obeying the following restrictions:

$$\begin{aligned}
&f \quad \text{is} \quad \text{Lipschitz}: \ |f(x) - f(y)| \leq |x-y| \\
&f(x) = |x-a| \qquad \text{for} \qquad |x-a| \qquad \text{sufficiently large}
\end{aligned} \tag{4.20}$$

and

$$\mathcal{S} \left[f(x); \mathbf{t}^{\omega}(x); a \right] = \tfrac{1}{2} \int \int \mathrm{d}x \mathrm{d}y \ f''(x) f''(y) K(x-y) + \tfrac{1}{2} \int \mathrm{d}x \ f''(x) \mathbf{t}^{\omega}(x) \tag{4.21}$$

with

$$\mathbf{t}^{\omega}(x) = \sum_{k=0}^{\infty} T_k \frac{x^{k+1}}{k+1} \tag{4.22}$$

where

$$a = \frac{\partial \mathcal{F}}{\partial T_0} \ . \tag{4.23}$$

The critical point $f_*(x)$ obeys:

$$\int_{\mathbf{R}} \mathrm{d}y \ K(x-y) f_*''(y) + \mathbf{t}^{\omega}(x) = \mathrm{const} \ , \ x \in \mathrm{supp} f_*'' \tag{4.24}$$

where const is the Lagrange multiplier for the condition:

$$\int_{\mathbf{R}} f_*''(x)\mathrm{d}x = 2 \tag{4.25}$$

Note that

$$\int_{\mathbf{R}} x f_*''(x)\mathrm{d}x = 2a \quad . \tag{4.26}$$

Let us define the Baker-Akhiezer function:

$$\Theta(\lambda) = \sum_{k=0}^{\infty} T_k \lambda^k + \frac{\partial \mathcal{F}}{\partial T_0}\log(\lambda) - \sum_{k=1}^{\infty} \frac{1}{k}\frac{\partial \mathcal{F}}{\partial T_k}\lambda^{-k} \tag{4.27}$$

We have:

$$\begin{aligned}\Theta_-(\lambda) &= \tfrac{1}{2}\int_{\mathbf{R}} \mathrm{d}x\ f''(x)\ \left[x\log(\lambda) - \sum_{k=1}^{\infty}\frac{x^{k+1}\lambda^{-k}}{k(k+1)}\right] \\ &= \lambda\left(\log\lambda - 1\right) + \\ &\quad \tfrac{1}{2}\int_{\mathbf{R}} \mathrm{d}x\ f''(x)\left(x-\lambda\right)\left(\log\left(\lambda - x\right) - 1\right)\end{aligned} \tag{4.28}$$

By differentiating (4.24) w.r.t x we can reformulate the e.o.m. as the conditions on $\Theta(\lambda)$: define $\Sigma(\lambda) = -\lambda\left(\log\lambda - 1\right) + \Theta(\lambda)$, then:

$$\begin{aligned}\Sigma(x \pm i0) &= \sum_k T_k x^k + \int_{\mathbf{R}} \mathrm{d}y\ f''(y)\ K'(x-y) = \\ &\sum_k T_k x^k + \tfrac{1}{2}\int \mathrm{d}y\ f''(y)\ (y-x)\left(\log|x-y| + i\mathrm{Arg}(x \pm i0 - y) - 1\right)\end{aligned} \tag{4.29}$$

for

$$K(s) = -\frac{s^2}{4}\left(\log(s) - \frac{3}{2}\right) \tag{4.30}$$

Note that the non-equivariant limit of the OP change of times in the stationary sector is :

$$t_k^{\omega} = (k+1)! T_{k+1} \implies \mathbf{t}^{\omega}(x) = \sum_{k=1}^{\infty} t_k^{\omega}\frac{x^k}{k!} \tag{4.31}$$

In equivariant setup one has two functions $\phi(t)$ and $\phi^*(t)$ and both of them are generalizations of limit shape of stationary sector . We conjecture that after passage (4.1)to non-equivariant times and turning off non-stationary coupling constants except for t_0^1 in the limit $\varepsilon \to 0$ one gets

$$\begin{aligned}\phi(t) &\to \frac{f_*(t)}{2} - \frac{t-v}{2\Lambda} \\ \phi^*(t) &\to \frac{f_*(t)}{2} + \frac{t-v}{2\Lambda}\end{aligned} \tag{4.32}$$

We checked this conjecture for t_0^1 and t_k^ω with $k = 0..2$ swithed on.

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Appendix A. Equivariant Gromov-Witten theory of $\mathbf{CP}^N$

In this section we present an expression for the action functional whose critical values give the equivariant genus zero GW prepotential of $\mathbf{CP}^N$. We also discuss the J-function, the quantum differential equation, and its solution in the case of $N = 1$, expressed in terms of our data: the curve and the differential.

A.1. The $\mathbf{CP}^N$ functional

The prepotenial is a functional on the positive part of the loop space $H^*_{\mathbf{T}}(\mathbf{CP}^N)\otimes\mathbf{C}[z]$, where

$$H^*_{\mathbf{T}}(\mathbf{CP}^N) = \mathbf{C}[p, \mu_1, \ldots, \mu_{N+1}]/Q(p) \tag{A.1}$$

is the equivariant cohomology of $\mathbf{CP}^N$. Here the degree $N + 1$ polynomial Q is given in terms of the twisted masses of the $SU(N + 1)$ action on $\mathbf{CP}^N$:

$$Q(x) = \prod_{i=1}^{N+1} (x - \mu_i) \tag{A.2}$$

Our conventions are: let $\mathbf{x}(z) = \sum_{k=0}^{\infty} \mathbf{x}_k \frac{z^k}{k!}$, where $\mathbf{x}_k \in H^*_{\mathbf{T}}(\mathbf{CP}^N)$. The latter cohomology group is generated by the equivariant Chern classes $p - \mu_i$ of the line bundles L_i, $i = 1, \ldots, N + 1$. From the results of [11], [12] we know that the genus zero prepotential is given by the sum over the trees. The vertices are labelled by the fixed points of the $\mathbf{T}$ action on $\mathbf{CP}^N$, while the edges are labelled by the integers $d \geq 1$. Each edge comes with the factor:

$$\mathrm{Edge}_{ij,d} = \frac{q^d}{d} \prod_{k=1}^{N+1} \prod_{a=0}^{d} \left[\frac{1}{\left(a\frac{\mu_{ik}}{d} + b\frac{\mu_{jk}}{d}\right)} \right]_{(a,k)\neq(0,j),\,(d,i)} \tag{A.3}$$

where

$$\mu_{ij} = \mu_j - \mu_i,$$

while the vertex is:

$$\mathrm{Vertex}_v = \int_{\overline{\mathcal{M}}_{0,\mathrm{val}(v)}} \prod_{l=1}^{n_1} \frac{1}{1 - \frac{d}{\mu_{i(v)j(v)}}\psi_l} \prod_{m=1}^{n_2} x_{i(m)}(\psi_m) \tag{A.4}$$

Thus the prepotential, $\mathcal{F}(\mathbf{x}; q, \vec{\lambda})$, is the critical point of the following functional:

$$\mathcal{S}[\mathbf{x}, \vec{\phi}] = \mathcal{S}_2[\mathbf{x}, \vec{\phi}] + \mathcal{S}_{\mathrm{int}}[\mathbf{x}, \vec{\phi}]$$

where

$$\mathcal{S}_2[\mathbf{x},\vec{\phi}] = \sum_{1\le i<j\le N+1}\sum_{d=1}^{\infty}(-1)^d\frac{(d!)^2}{q^d}\left(\frac{\mu_{ij}}{d}\right)^{2d-1}\epsilon_j\prod_{k\neq i,j}\left[\left(\frac{\mu_{ij}}{d}\right)^d\frac{\left(d\frac{\mu_{ik}}{\mu_{ij}}\right)!}{\left(d\frac{\mu_{jk}}{\mu_{ij}}\right)!}\right]\phi_{ij,d}\phi_{ji,d}$$

$$\mathcal{S}_{\text{int}}[\mathbf{x},\vec{\phi}] = \sum_{i=1}^{N+1}\frac{1}{\epsilon_i}\mathbf{F}\left[\epsilon_i\mathbf{x}_i(z)+\phi_i(z)\right] + \text{ unstable contributions}$$

$$\epsilon_i = \prod_{j\neq i}\mu_{ij}\ ,\ \phi_i(z) = \epsilon_i\sum_{j\neq i}\sum_{d=1}^{\infty}\frac{d}{\mu_{ij}}\phi_{ij,d}\,e^{\frac{d}{\mu_{ij}}z} \tag{A.5}$$

where $\mathbf{F}[\mathbf{t}(z)]$ is the prepotential of the pure topological gravity:

$$\mathbf{F}[\mathbf{t}(z)] = \text{Crit}_{\xi}\ \tfrac{1}{2}\int_0^{\xi}(z-\mathbf{t}(z))^2\,dz \tag{A.6}$$

and *the unstable contributions* are given by:

$$\begin{aligned}\text{unstable} = \sum_{d=1}^{\infty}q^d\sum_{i=1}^{N+1}\epsilon_i\ \Bigg(&\frac{1}{2}\sum_{d'=1}^{d-1}\sum_{j,k\neq i}\frac{d'(d-d')}{d'\mu_{ik}+(d-d')\mu_{ij}}\phi_{ij,d'}\phi_{ik,d-d'}\\ &+\sum_{j\neq i}\phi_{ij,d}\sum_{k=0}^{\infty}\mathbf{x}_k^{(i)}\left(-\frac{\mu_{ij}}{d}\right)^k\\ &+\sum_{j\neq i}\phi_{ij,d}\ \frac{1}{\epsilon_i}\ \frac{\mu_{ij}}{d}\Bigg)\end{aligned} \tag{A.7}$$

Finally,

$$\mathcal{F}[\mathbf{x}] = \text{Crit}_{\vec{\phi}}\ \mathcal{S}[\mathbf{x},\vec{\phi}] \tag{A.8}$$

Critical point equations. At the critical point we have:

$$\begin{aligned}&\epsilon_j\phi_{ji,d}\times d!\left(\frac{\mu_{ji}}{d}\right)^{d-1}\prod_{k\neq i}\psi_{ik}\left(\frac{d}{\mu_{ij}}\right)\\ &\qquad = -q^d\prod_{k\neq i,j}\psi_{jk}\left(\frac{d}{\mu_{ij}}\right)\int^{\xi_i}\text{d}t\ \left(t-\epsilon_i\mathbf{x}_i(t)-\phi_i(t)\right)e^{\frac{d}{\mu_{ij}}t}\end{aligned} \tag{A.9}$$

(note in the left hand side the product is over $k\neq i$ only), where

$$\psi_{ik}(\delta) = (\delta\mu_{ik})!\delta^{-1-\delta\mu_{ik}} = \int_0^{\infty}\text{d}s\ \left(s^{\mu_{ik}}e^{-s}\right)^{\delta} \tag{A.10}$$

A.2. Quantum cohomology of $\mathbf{CP}^N$ and the J-function

The J-function is a solution to

$$\hbar \nabla J = p \star J \tag{A.11}$$

where

$$\nabla = q \frac{d}{dq} \tag{A.12}$$

where p is the generator of the quantum cohomology ring. In this ring there is a polynomial relation:

$$Q(p) = q \tag{A.13}$$

Expand:

$$J = \sum_{i=1}^{N+1} \varepsilon_i(p)\, J_i \tag{A.14}$$

where

$$\varepsilon_i(x) = \frac{Q(x)}{(x-\mu_i)Q'(\mu_i)} = \prod_{j \neq i} \frac{x-\mu_j}{\mu_i - \mu_j} \tag{A.15}$$

Note that:

$$\sum_{i=1}^{N+1} \varepsilon_i(x) \equiv 1 \tag{A.16}$$

in the ring $\mathbf{C}[x]$ of polynomials,

$$p \star \varepsilon_i(p) = \mu_i \varepsilon_i(p) = \frac{q}{Q'(\mu_i)} \tag{A.17}$$

in the quantum cohomology algebra. The equation (A.11) reads:

$$\hbar \nabla J_i = \mu_i J_i + q \sum_{j=1}^{N+1} \frac{J_j}{Q'(\mu_j)} \tag{A.18}$$

and can be solved via:

$$J_i = Q'(\mu_i)\varepsilon_i(\hbar\nabla)\Psi \ , \tag{A.19}$$

where Ψ is any solution to

$$Q(\hbar\nabla)\Psi = q\Psi \tag{A.20}$$

Note that (A.19) implies

$$\Psi = \sum_{i=1}^{N+1} \frac{J_i}{Q'(\mu_i)}$$

Now, the $N+1$ independent solution $\Psi_1, \ldots, \Psi_{N+1}$ to (A.20) are found as:

$$\Psi(y) = \sum_{d=0}^{\infty} q^{\frac{y}{\hbar}+d} \prod_{m=1}^{d} \frac{1}{Q(y+\hbar m)} \qquad \mathrm{mod} Q(y)$$

$$\Psi(y) = \sum_{i=1}^{N+1} \varepsilon_i(y)\, \Psi^i \tag{A.21}$$

Explicitly:

$$J_i(y) = Q'(\mu_i)\varepsilon_i(y) + \sum_{d=1}^{\infty} \frac{q^{\frac{y}{\hbar}+d}}{y+\hbar d-\mu_i} \prod_{m=1}^{d-1} \frac{1}{Q(y+\hbar m)} \qquad \mathrm{mod} Q(y)$$

The fundamental matrix of the solutions to (A.11) is given by:

$$J_i^k = Q'(\mu_i)\delta_i^k + \sum_{d=1}^{\infty} \frac{q^{\frac{\mu_k}{\hbar}}}{\hbar^{d-1}\,(\hbar d+\mu_k-\mu_i)} \frac{q^d}{(d-1)!} \prod_{j\neq k} \prod_{m=1}^{d-1} \frac{1}{\mu_k-\mu_j+\hbar m} \tag{A.22}$$

In terms of polynomials:

$$J(x,y) = \frac{Q(x)-Q(\hbar\nabla)}{x-\hbar\nabla} \Psi(y) \tag{A.23}$$

A.3. Quantum cohomology of $\mathbf{CP}^1$

For $X = \mathbf{CP}^1$ we have:

$$Q(x) = (x-\mu_0)(x-\mu_1) \tag{A.24}$$

and

$$J_0^0 = Q'(\mu_0)e^{\frac{\mu_0}{\hbar}} + \sum_{d=1}^{\infty} \frac{q^{d+\frac{\mu_0}{\hbar}}}{\hbar^d d!} \prod_{m=1}^{d-1} \frac{1}{\mu_0-\mu_1+\hbar m}$$

$$J_0^1 = \sum_{d=1}^{\infty} \frac{q^{d+\frac{\mu_0}{\hbar}}}{\hbar^{d-1}(d-1)!} \prod_{m=1}^{d} \frac{1}{\mu_0-\mu_1+\hbar m} \tag{A.25}$$

with $J_1^1 = J_0^0(\mu_0 \leftrightarrow \mu_1)$, $J_1^0 = J_0^1(\mu_0 \leftrightarrow \mu_1)$. According to [16], another fundamental solution to the quantum differential equation (A.11) is given by the set of the two-point functions:

$$S_\alpha^\beta = \eta_\alpha^\beta + \langle \phi_\alpha, \frac{\phi_\beta}{\hbar - c} \rangle \tag{A.26}$$

For $\mathbf{CP}^1$ we have, on the large phase space:

$$\begin{aligned}
\langle\phi_0,\frac{\phi_0}{\hbar-c}\rangle &= \sum_{k=0}^{\infty}\frac{1}{\hbar^{k+1}}\frac{\partial^2\mathcal{F}}{\partial x_0\partial x_k} = \sum_{k=0}^{\infty}\frac{\Lambda}{\hbar^{k+1}}\oint_0 dw\sum_{n=1}^{\infty}\oint_0\frac{dz}{z^{k+2}}\frac{z^n\rho^n}{\prod_{m=1}^{n}(m+\epsilon z)}\\
\langle\phi_\infty,\frac{\phi_0}{\hbar-c}\rangle &= \sum_{k=0}^{\infty}\frac{1}{\hbar^{k+1}}\frac{\partial^2\mathcal{F}}{\partial x_0^*\partial x_k} = \sum_{k=0}^{\infty}\frac{\Lambda}{\hbar^{k+1}}\oint_0\frac{dw}{w^2}\sum_{n=1}^{\infty}\oint_0\frac{dz}{z^{k+2}}\frac{z^n\rho^n}{\prod_{m=1}^{n}(m+\epsilon z)}
\end{aligned} \tag{A.27}$$

with second solution given by the same formulas with ρ substituted by $\overline{\rho}$. To compare two matrices one should properly identify parameters. Namely, we put $\mu_0=0$, $\mu_1=-\epsilon$ and get:

$$\begin{aligned}
J_0^0 &= \epsilon+\sum_{d=1}^{\infty}\frac{q^d}{\hbar^d d!}\prod_{m=1}^{d-1}\frac{1}{\hbar m+\epsilon}\\
J_0^1 &= \sum_{d=1}^{\infty}\frac{q^{d-\frac{\epsilon}{\hbar}}}{\hbar^{d-1}(d-1)!}\prod_{m=1}^{d}\frac{1}{\hbar m-\epsilon}\\
J_1^1 &= -\epsilon q^{-\frac{\epsilon}{\hbar}}+\sum_{d=1}^{\infty}\frac{q^{d-\frac{\epsilon}{\hbar}}}{\hbar^d d!}\prod_{m=1}^{d-1}\frac{1}{\hbar m-\epsilon}\\
J_1^0 &= \sum_{d=1}^{\infty}\frac{q^d}{\hbar^{d-1}(d-1)!}\prod_{m=1}^{d}\frac{1}{\hbar m+\epsilon}
\end{aligned} \tag{A.28}$$

In the two point functions one should put $q=1$ and then $t_0^1=0$, $t_0^\omega=\log(q)$, so that $\Lambda=\sqrt{q}$ and $v_+=0$, $v_-=-\epsilon\log(q)$. Explicit check shows that at least up to $O(\frac{1}{\hbar^9})$:

$$J_0^0=S_0^0,\ J_1^0=S_\infty^0,\ J_1^1=S_\infty^\infty,\ J_0^1=S_0^\infty \tag{A.29}$$

with $\eta_0^0=\epsilon,\ ,\eta^{\infty\infty}=-\epsilon$

Appendix B. Technical details

B.0.1. *Proof of (2.33)*

Write:

$$P_n'(t) = \sum_{l=0}^{n-1} t^{n-l-1} \frac{(n-1)!}{(n-l-1)!} h_{n-1}^{(l)} \tag{B.1}$$

and

$$\begin{aligned} R_n(\rho) &= \rho \int_0^\infty ds e^{-s} P_n'(\rho(1-e^{-s}) - s) = \\ &(-1)^n (n-1)! \sum_{a=0}^{n-1} (-\rho)^{a+1} \left[\sum_{b=0}^{a} \frac{(-1)^b (b+1)^{a-n}}{b!(a-b)!} \left\{ \sum_{l=0}^{n-a-1} (-1)^l h_{n-1}^{(l)} (b+1)^l \right\} \right] \end{aligned} \tag{B.2}$$

Using the identity:

$$\frac{1}{c!} \sum_{b=0}^{a} \binom{a}{b} (-1)^b (b+1)^c = \mathrm{Coeff}_{y^c} e^y (1-e^y)^a = 0 \quad , \quad 0 \le c \le a-1 \tag{B.3}$$

we can set the upper limit in the sum over l in (B.2) to $n-1$ instead of $n-a-1$, since the error term will be proportional to (B.3) with $c = a+l-n \le a-1$. Then it suffices to note that

$$\sum_{l=0}^{n-1} (-1)^l h_{n-1}^{(l)} (b+1)^l = \prod_{k=1}^{n-1} \left(1 - \frac{b+1}{k}\right) \tag{B.4}$$

which vanishes for $0 \le b \le n-2$. Thus the sum over b in (B.2) only gets contribution from $b = n-1$, which implies that $a = n-1$. Thus (B.4) is equal to $(-1)^{n-1}$, and (B.2) is equal to

$$R_n(\rho) = \frac{\rho^n}{n} \tag{B.5}$$

as promised.

B.0.2. *Proof of (3.21)*

Let us show that series (3.9) for $\rho^\wedge(\xi)$ is of the form (3.12), that is it does not contain positive powers of w except for linear one Λw:

$$\begin{aligned} \rho^\wedge(\xi) &= \varepsilon \sum_{p=1}^{\infty} \frac{p^{p-1}}{p!} \left(\frac{\Lambda w}{\varepsilon} e^{-\frac{\Lambda}{\varepsilon}(w+\frac{1}{w}) - \frac{t_0}{\varepsilon}} \right)^p = \varepsilon \sum_{p=1}^{\infty} \frac{p^{p-1}}{p!} \left(\widetilde{w} e^{-\widetilde{w} - \frac{f_-}{2} \frac{(\widetilde{w}-1)^2}{\widetilde{w}}} \right)^p = \\ &\varepsilon \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(-\frac{f_-}{2} \frac{(\widetilde{w}-1)^2}{\widetilde{w}} \right)^k \end{aligned} \tag{B.6}$$

where

$$\widetilde{w} = \frac{\varepsilon f\_w}{2\Lambda} \tag{B.7}$$

and

$$a_k = \sum_{p=1}^{\infty} \frac{p^{p-1+k}}{p!} \left(\widetilde{w} e^{-\widetilde{w}}\right) \tag{B.8}$$

Now all terms except the first one in the last summation in (B.6) do not contain positive powers of w. $a_0 = \widetilde{w}$ and

$$a_k = \left(\frac{\widetilde{w}}{1-\widetilde{w}} \frac{\partial}{\partial \widetilde{w}}\right)^k \widetilde{w} \tag{B.9}$$

It is straightforward to show that all negative powers of $\widetilde{w} - 1$ in $\left(\frac{(\widetilde{w}-1)^2}{\widetilde{w}}\right)^k \left(\frac{\widetilde{w}}{1-\widetilde{w}} \frac{\partial}{\partial \widetilde{w}}\right)^k \widetilde{w}$ are canceled and result does not contain positive powers of $\widetilde{w}$. Easy check shows that ξ is the only expression which satisfy this property.

B.0.3. *Consistency with Dubrovin-Zhang*

In [15] B. Dubrovin and Y. Zhang give explicit expression for prepotential, namely

$$\sum_{k,m=0}^{\infty} \frac{\partial \mathcal{F}}{\partial t_k^\alpha \partial t_n^\beta} z^k x^n = \frac{1}{z+x} \left(\frac{\partial \theta_\alpha(z)}{\partial v} \frac{\partial \theta_\beta(x)}{\partial u} + \frac{\partial \theta_\alpha(z)}{\partial u} \frac{\partial \theta_\beta(x)}{\partial v} - \eta_{\alpha\beta} \right) \tag{B.10}$$

where

$$\begin{aligned} \theta_1(z) &= \sum_{k=0}^{\infty} z^k \frac{1}{2\pi i} \oint \frac{\rho^k}{k!} L_k \frac{dw}{w} \\ \theta_\omega(z) &= \sum_{k=0}^{\infty} z^k \frac{1}{2\pi i} \oint \frac{\rho^{k+1}}{(k+1)!} \frac{dw}{w} \end{aligned} \tag{B.11}$$

and in our notations $\Lambda = e^{u/2}$ and $L_k = (\Delta\rho^- - \Delta\rho^+ - 2c_k)$. Then derivatives with respect to v and u are given by

$$\begin{aligned} \frac{\partial \theta_\omega(z)}{\partial v} &= \sum_{k=0}^{\infty} \theta_{\omega,k}^{(v)} z^k = \sum_{k=0}^{\infty} z^k \frac{1}{2\pi i} \oint \frac{\rho^k}{k!} \frac{dw}{w} \\ \frac{\partial \theta_\omega(z)}{\partial u} &= \sum_{k=0}^{\infty} \theta_{\omega,k}^{(u)} z^k = \frac{\Lambda}{2} \sum_{k=0}^{\infty} z^k \frac{1}{2\pi i} \oint \frac{\rho^k}{k!} \left(w + \frac{1}{w}\right) \frac{dw}{w} \\ \frac{\partial \theta_1(z)}{\partial v} &= \sum_{k=0}^{\infty} \theta_{1,k}^{(v)} z^k = \sum_{k=1}^{\infty} z^k \frac{1}{2\pi i} \oint \frac{\rho^{k-1}}{(k-1)!} L_{k-1} \frac{dw}{w} \\ \frac{\partial \theta_1(z)}{\partial u} &= \sum_{k=0}^{\infty} \theta_{1,k}^{(u)} z^k = 1 + \frac{\Lambda}{2} \sum_{k=1}^{\infty} z^k \frac{1}{2\pi i} \oint \frac{\rho^{k-1}}{(k-1)!} L_{k-1} \left(w + \frac{1}{w}\right) \frac{dw}{w} \end{aligned} \tag{B.12}$$

An expression (B.10) for $\alpha = \beta = \omega$ follows from an equality

$$\sum_{k,m=0}^{\infty} \frac{1}{2\pi i} \oint \left(\frac{\rho^m}{m!} d \frac{\rho_+^{k+1}}{(k+1)!} + \frac{\rho^{m+1}}{(m+1)!} d \frac{\rho_+^k}{k!} \right) z^m x^k = \left(\frac{\partial \theta_\omega(z)}{\partial v} \frac{\partial \theta_\omega(x)}{\partial u} + \frac{\partial \theta_\omega(z)}{\partial u} \frac{\partial \theta_\omega(x)}{\partial v} \right) \tag{B.13}$$

To prove it one can note that

$$\frac{d\rho_+^{k+1}}{(k+1)!} = \frac{(\rho_+^k d\rho)_+}{k!} = \frac{\left(\rho_+^k d\rho - (\rho_+^k d\rho)_-\right)}{k!} = \left(\frac{\rho_+^k d\rho}{k!} + \Lambda \frac{dw}{w^2} \theta_{\omega,k}^{(v)} + \frac{dw}{w} \theta_{\omega,k}^{(u)} \right) \tag{B.14}$$

Then

$$\frac{1}{2\pi i} \oint \left(\frac{\rho^m}{m!} d \frac{\rho_+^{k+1}}{(k+1)!} + \frac{\rho^{m+1}}{(m+1)!} d \frac{\rho_+^k}{k!} \right) = \frac{1}{2\pi i} \oint \left(\frac{\rho^m}{m!} d \frac{\rho_+^{k+1}}{(k+1)!} - \frac{\rho^m}{m!} \frac{\rho_+^k}{k!} d\rho \right) =$$
$$= \frac{1}{2\pi i} \oint \frac{\rho^m}{m!} \left(\Lambda \frac{dw}{w^2} \theta_{\omega,k}^{(v)} + \frac{dw}{w} \theta_{\omega,k}^{(u)} \right) = \theta_{\omega,k}^{(v)} \theta_{\omega,m}^{(u)} + \theta_{\omega,m}^{(v)} \theta_{\omega,k}^{(u)} \tag{B.15}$$

For $\alpha = \omega$, $\beta = 1$ one should prove that r.h.s. of (B.10) times $x + z$ is equal to

$$\sum_{m=1,k=0}^{\infty} \frac{1}{2\pi i} \oint \left(\frac{\rho^m}{m!} L_k d \frac{\rho_+^k}{k!} + \frac{\rho^{m-1}}{(m-1)!} L_{m-1} d \frac{\rho_+^{k+1}}{(k+1)!} \right) z^m x^k + \sum_{k=1} x^k \frac{1}{2\pi i} \oint L_0 d \frac{\rho_+^k}{k!} \tag{B.16}$$

For $m \geq 1$

$$d \frac{\rho^m}{m!} L_m = \frac{\rho^{m-1}}{(m-1)!} L_{m-1} d\rho \tag{B.17}$$

and thus, using (B.14)

$$\frac{1}{2\pi i} \oint \left(\frac{\rho^m}{m!} L_m d \frac{\rho_+^k}{k!} + \frac{\rho^{m-1}}{(m-1)!} L_{m-1} d \frac{\rho_+^{k+1}}{(k+1)!} \right) = \frac{1}{2\pi i} \oint \frac{\rho^{m-1}}{(m-1)!} L_{m-1} \left(\Lambda \frac{dw}{w^2} \theta_{\omega,k}^{(v)} + \frac{dw}{w} \theta_{\omega,k}^{(u)} \right) =$$
$$= \theta_{\omega,k}^{(v)} \theta_{1,m}^{(u)} + \theta_{1,m}^{(v)} \theta_{\omega,k}^{(u)} \tag{B.18}$$

For $k > 0$

$$\frac{1}{2\pi i} \oint L_0 d\rho_+^k = -\frac{1}{2\pi i} \oint \rho_+^k d\Delta\rho^- = -\frac{1}{2\pi i} \oint \rho^k d\Delta\rho^- = \frac{1}{2\pi i} \oint \rho^k \frac{dw}{w} = k! \theta_{\omega,k}^{(v)} \tag{B.19}$$

Substituting this and (B.18) into (B.16) and using $\theta_{1,0}^{(u)} = \theta_{\omega,0}^{(v)} = 1$, $\theta_{1,0}^{(v)} = 0$ one finally gets

$$\sum_{m=1,k=0}^{\infty} \left(\theta_{\omega,k}^{(v)} \theta_{1,m}^{(u)} + \theta_{1,m}^{(v)} \theta_{\omega,k}^{(u)} \right) z^m x^k + \sum_{k=1}^{\infty} x^k \theta_{\omega,k}^{(v)} =$$
$$= \sum_{m=0,k=0}^{\infty} \left(\theta_{\omega,k}^{(v)} \theta_{1,m}^{(u)} + \theta_{1,m}^{(v)} \theta_{\omega,k}^{(u)} \right) z^m x^k - 1 \tag{B.20}$$

Appendix C. Low instanton charges

Let us assume that $x_k, x_k^* = 0$ for $k > 1$, and $x_2^3 = x_2^{*3} = 0$. Then, the contribution of the instantons of the charges 0 and 1 to the prepotential is given by:

$$\mathcal{F}_0 = \mathcal{F}_0^{(0)} + q\mathcal{F}_0^{(1)} + O(q^2; x_2^3; x_2^{*3})$$

$$\begin{aligned}
\mathcal{F}_0^{(0)} &= \frac{\epsilon}{3!}\left(\frac{x_0^{*3}}{\widetilde{x}_1^*} - \frac{x_0^3}{\widetilde{x}_1}\right) + \frac{\epsilon}{4!}\left(\frac{x_0^{*4}x_2^*}{\widetilde{x}_1^{*3}} - \frac{x_0^4 x_2}{\widetilde{x}_1^3}\right) + \frac{\epsilon}{40}\left(\frac{x_0^{*5}x_2^{*2}}{\widetilde{x}_1^{*5}} - \frac{x_0^5 x_2^2}{\widetilde{x}_1^5}\right) \\
\mathcal{F}_0^{(1)} &= -\epsilon^2\, \widetilde{x}_1 \widetilde{x}_1^* \exp\left(-t\right) \times \\
&\left[1 - \frac{\epsilon x_2}{\widetilde{x}_1}\left(1 + \frac{x_0}{\epsilon \widetilde{x}_1} + \frac{x_0^2}{2\epsilon^2 \widetilde{x}_1^2}\right) + \frac{\epsilon x_2^*}{\widetilde{x}_1^*}\left(1 - \frac{x_0^*}{\epsilon \widetilde{x}_1^*} + \frac{x_0^{*2}}{2\epsilon^2 \widetilde{x}_1^{*2}}\right)\right. \\
&\quad + \frac{1}{8\epsilon^2}\left(\frac{x_0^{*4}x_2^{*2}}{\widetilde{x}_1^{*6}} + \frac{x_0^4 x_2^2}{\widetilde{x}_1^6}\right) + \\
&\quad \left.\frac{\epsilon^2 x_2 x_2^*}{\widetilde{x}_1 \widetilde{x}_1^*}\left(1 + t\left(1 - \frac{x_0 x_0^*}{2\epsilon^2 \widetilde{x}_1 \widetilde{x}_1^*}\right) + \frac{t^2}{2} + \left(\frac{x_0 x_0^*}{2\epsilon \widetilde{x}_1 \widetilde{x}_1^*}\right)^2\right)\right]
\end{aligned} \tag{C.1}$$

$$t = \frac{1}{\epsilon}\left(\frac{x_0}{\widetilde{x}_1} - \frac{x_0^*}{\widetilde{x}_1^*}\right)$$

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