# The QED $\beta$-function from global solutions to Dyson-Schwinger equations

**G. VAN BAALEN, D. KREIMER, D. UMINSKY and K. YEATS**

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

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# The QED $\beta$-function from global solutions to Dyson-Schwinger equations

Guillaume van Baalen[*], Dirk Kreimer[†], David Uminsky[*], and Karen Yeats[*]

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**Abstract**

We discuss the structure of beta functions as determined by the recursive nature of Dyson–Schwinger equations turned into an analysis of ordinary differential equations, with particular emphasis given to quantum electrodynamics. In particular we determine when a separatrix for solutions to such ODEs exists and clarify the existence of Landau poles beyond perturbation theory. Both are determined in terms of explicit conditions on the asymptotics for the growth of skeleton graphs.

## Acknowledgments

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## 1 Introduction

### 1.1 The method

Results on the structure of amplitudes in the theory of local interacting quantum fields are notoriously hard to come by beyond perturbation theory. We refrain from discussing the various approaches de-

[*]Department of Mathematics and Statistics, Boston University, 111 Cummington Street, Boston MA, 02215, USA

[†]CNRS-IHES 91440 Bures sur Yvette, France; and Center for Mathematical Physics Boston University, 111 Cummington Street, Boston MA, 02215, USA

veloped in the past and shortly summarize our approach here, which has been developed by one of us (D.K.) in the last decade [8, 13, 14, 9, 7, 5, 18, 19, 16]. It lead already to progress at very high orders [6, 3, 4] and all orders of perturbation theory [7] (see also the $P = x$, $s = 2$ case in the examples below).

It is a pleasure to emphasize that our approach connects to old attempts [20] in quantum field theory to use the soft breaking of conformal symmetry by renormalizable quantum fields for non-perturbative results. The recent developments which allow us to understand the notion of locality mathematically combine rather nicely with such ideas. A crucial ingredient is that the mathematical structure of the quantum equations of motion remains form-invariant under inclusion of more and more skeleton graphs, and this fact allows the development of an approximation to these equations in terms of periods of increasing complexity, without ever changing the structure of these equations. This is very different from, for example, any truncation of high frequency modes in the path-integral. Whilst the approach used here can re-derive results of such constructive methods [19], we here go beyond what is possible by such truncations of the path-integral.

In particular, for theories which are non-asymptotically free, a study of low orders of perturbation theory indicates the presence of a Landau pole (the invariant charge approaching infinity at a finite scale $q^2/\mu^2$), which is also believed to exist for such theories in the constructive approach, if one attempts to remove the cut-off which necessarily has to be introduced in such theories, as well as in perturbation theory. In our approach, we only choose a boundary condition for the equation of motion, the Dyson–Schwinger equations. We approximate the full theory by the choice of a function $P(x)$ which describes the growth of the skeleton expansion, and carefully make that choice to maintain the Lie- and Hopf-algebraic structure of the forest formula and the equations of motion at the same time. We thus do not need to introduce a cut-off, a familiar phenomenon when scaling dimensions of Green functions are taken into account in those equations [20]. Perturbative approximations to $P(x)$ lead to a non-perturbative behavior for $\beta$ functions in such theories which reconfirms the existence of a Landau pole. Rather mild assumptions on the non-perturbative behavior of $P(x)$ allow for solutions though which avoid such a pole, as discussed below, with the charge going to infinity only at infinite scale, and hence realizing a possibility already discussed in [25] section 18.3. Finally, we emphasize that we assume below that $P(x)$ is a nowhere vanishing function, and hence that we do not have a non-trivial zero for the $\beta$-function for a non-asymptotically free theory: so we are not assuming an eigenvalue condition [2], but much to the contrary, analyze the structure of the theory under the assumption that such an eigenvalue does not exist.

We will not attempt any serious discussion of the asymptotics of $P(x)$, though that the work of [11] emphasizes the need of such a discussion. Here, we are content with the classifications of the behavior of the $\beta$-function as a function of the possible asymptotics of $P(x)$, emphasizing the possibility of the absence of Landau poles in well-specified conditions. Also, we re-emphasize that Dyson–Schwinger equations do not demand the introduction of a cut-off, but rather demand the specification of a finite number of conditions to fix the amplitudes needing renormalization as initial conditions for the renormalization group flow.

So our approach is based on two main ingredients: the existence of quantum equations of motion — Dyson–Schwinger equations —, and the consequences of the renormalization group for such local field theories. The latter guarantee that amplitudes develop anomalous scaling exponents under the action of the dilatation group which re-scales parameters in the theory, the former guarantees sufficient recursive structure in the theory such that a non-perturbative approach becomes feasible. The rich Hopf algebraic foundations of these phenomena make our approach possible.

We consider Green functions as functions of two variables, a 'running coupling constant' $x$ and a single kinematical variable $L = \ln q^2/\mu^2$ (in the deep Euclidean regime, or suitably continued to physical regions). This implies that vertex functions are considered only at zero momentum transfer or for symmetric external momenta.

We define Green functions as the scalar coefficient functions which provide quantum corrections to tree-level amplitudes $r \in \mathcal{R}$, where $r$ denotes the chosen amplitude. We store all the information on parameters which determine the amplitude under consideration in its lowest order contribution, the tree level form-factor $f(r)$. The Lagrangian is then given as $\mathcal{L} = \sum_{r\in\mathcal{R}} mr$, for monomials $m$ in fields for each amplitude which needs renormalization. Green functions modify this amplitude $f(r)$ in a multiplicative manner: $f(r) \to \Phi(r)(1 + \mathcal{O}(\hbar))$, and hence start with one.

The equivalent expansions (taking the negative sign for propagators and the positive for vertices)

$$G^r(x, L) = 1 \pm \sum_{j=1}^{\infty} \gamma_j^r(x) L^j = 1 \pm \sum_{j=1}^{\infty} c_j^r(L) x^j \ ,$$

where $c_j^r(L)$ is a polynomial in $L$ bounded in degree by $j$ and $\gamma_j^r$ a series in $x$, are triangular and recursive for $\gamma_j^r$: the renormalization group determines the $\gamma_j^r$, $j > 1$ in terms of all the series $\gamma_1^r$. We denote $\gamma_1^r$ as the anomalous dimensions of the amplitude $r$, even if $r$ is a vertex function. Any $r$ for a vertex amplitude corresponds to a field monomial $m(r) = \prod_i \eta_i$ in the Lagrangian, and the $\eta_i$ are fields which we assume to have kinetic energy. There are then corresponding monomials $\sim \eta_i^2$ quadratic in those fields $\eta_i$ in the Lagrangian and the corresponding $\gamma_1^i \equiv \gamma_1^{\eta_i^2}$ combine with $\gamma_1^r$ to give the $\beta$-function in our sign conventions as

$$\beta^r = x[\gamma_1^r + \sum_i \gamma_1^i/2] \ .$$

Here, the monomial $r$, for $r$ a vertex amplitude, comes along with a probability $x$ for the tree-level scattering process described by $r$ to happen, and this probability — necessarily smaller than one —, furnishes a natural expansion parameter for the series $\gamma_1$. We consider only theories which have a single vertex amplitude in this paper, and hence have a unique expansion parameter.

Following now [18], [19], and [26] we can reduce the Dyson–Schwinger equations to a system of differential equations for the anomalous dimensions $\gamma_1^r$.

We outline the argument as follows. As we are interested only in the high-energy sector of the theory, we reduce one-particle irreducible Green functions to depend on a single scale $L = \log q^2/\mu^2$. The coupling constant will be denoted $x$. Then we expand the Green functions in $L$ as above.

From the renormalization group equation we obtain [18, 26]

$$\gamma_k^r(x) = -\frac{1}{k}\left(\pm\gamma_1^r(x) + \sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x)x\partial_x\right)\gamma_{k-1}^r(x)\ . \tag{1}$$

where again the sign is positive for a vertex and negative for a propagator and where the $s_j$ are defined, in accordance with the fields coupling at a vertex as above, by

$$\beta(x) = x\sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x)$$

where $\beta(x)$ is said $\beta$-function of the theory, and $\mathcal{R}$ is the set of all amplitudes needing renormalization. In the single equation case (1) reads

$$\gamma_k = \pm\frac{1}{k}\gamma_1(x)(1 - sx\partial_x)\gamma_{k-1}(x)\ .$$

As in [19] the Dyson-Schwinger equations can be rewritten in terms of derivatives of the Mellin transforms for the primitives. By Mellin transforms we simply mean the analytically regularized Feynman integrals for the primitives. Again by combinatorics on the Hopf algebra we can reduce to Mellin transforms in a single variable $\rho$, that is to a single insertion place. Finally by shifting unwanted powers of $\rho$ at a given loop order to lower powers of $\rho$ at a higher loop order, as in [19], [26], we can relate the coefficients of $L$ and $L^2$ which, in view of (1) gives us the system

$$\gamma_1^r(x) = P_r(x) \mp \gamma_1^r(x)^2 + \left(\sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x)\right) x\partial_x\gamma_1^r(x) \tag{2}$$

as $r$ runs over $\mathcal{R}$, the residues of the theory. $P_r$ is a modified version of the values of the primitives. The modification comes from two places. First the reduction to a single insertion place is purely combinatorial and leads essentially to the need to consider in the contribution to $P_r$ primitives which are not merely single graphs, but sums of graph. Second by exchanging powers of $\rho$ for powers of the coupling constant $x$ we also modify $P_r$. This reduction is not yet as well understood, but none-the-less it is simply a rearrangement of the analytic information contained in the original primitives [26].

## 1.2 QED as a special case

Most of our analysis will be confined to the case with only one equation and $s > 0$.

$$\gamma_1(x) = P(x) - \gamma_1(x)^2 + s\gamma_1(x)x\partial_x\gamma_1(x) \tag{3}$$

This case is general enough to cover gauge theories for the following reasons. First of all, we note that in QED thanks to the Ward identity, the $\beta$-functions is computable from the anomalous dimension of the photon $\gamma_1^{\frac{1}{4}F^2}$ alone, and similarly for non-abelian gauge theories in a background field gauge [1]. Furthermore, using a Baker–Johnson–Willey gauge [12], QED is a finite theory were it not for the (gauge-invariant) photon propagator. So it is indeed a single equation case.

In particular, it is the case $s = 1$, reflecting the fact that the lowest order term in the $\beta$-function comes from a graph which itself has no internal photon propagation. The variable $s$ measures the power of the Green function appearing in the invariant charge, or the power of the nonlinear part of the recursive appearance of the Green function in the Dyson-Schwinger equation. More specifically the power of the recursive appearance of the Green function at loop order $k$ is $1 - sk$. So with $s = 1$ and $k = 1$ we the power is $0$ representing the fact that in QED the one loop photon graph doesn't have an internal photon edge and the correct counting continues to hold at higher loop orders. For the Yukawa theory example of [7] the Green function appears recursively with power $-1$ at $k = 1$ leading to $s = 2$.

In the general non-abelian case the co-ideal and Hochschild cohomology structure of the Hopf algebra underlying the expansion of a non-abelian gauge theory in the coupling [15, 23, 24] allow for similar simplifications in particular in the background field method. An analysis of this method will be given in future work.

Notice that the $\beta$-function is showing up as the coefficient of $(\gamma_1^r)'(x)$ in (2) above, namely

$$\beta(x) = x \sum_{j \in \mathcal{R}} |s_j| \gamma_1^j(x)$$

in the system case and

$$\beta(x) = xs\gamma_1(x)$$

in the single equation case with $s > 0$. Consequently this differential equation is well suited to improving our understanding of the $\beta$-function. Furthermore, solving for the derivative $\gamma_1^{r\prime}(x)$ shows the appearance of the $\beta$-function in the denominator. A zero of the $\beta$-function is hence a degenerate singular case, even if it corresponds to a simple scaling behavior of quantum fields reflected by an abelian renormalization group flow (or, equivalently, a co-commutative expansion in the Hopf algebra of perturbation theory) [17].

In particular in the single equation case, we see immediately from (3) that any zeroes of $\beta(x)$ must occur either where $P(x) = 0$ — so we have no quantum corrections driving the equations of motion at some particular value of the coupling — or where $\gamma_1'(x)$ is infinite. The second of these possibilities is not physically reasonable for a finite value of $x$ and is indeed only realized by solutions for $\gamma_1^r$ which are multi-valued.

For QED taken out to four loops and correcting the primitives for our setup and using values from [10] we have

$$P(x) = \frac{x}{3} + \frac{x^2}{4} + (-0.0312 + 0.06037)x^3 + (-0.6755 + 0.05074)x^4 \; .$$

$P(x)$ is decreasing for $x > 0.653\ldots$, which will not be permissible below, and it has a zero at $x = 0.992\ldots$. We expect the zero to be spurious as it would immediately lead to a zero of the $\beta$-function, and believe both problems are due only to taking the 4 loop approximation beyond where it is valid. On the other hand the estimate

$$c_k \sim_{k>1} (-1)^k k! k^3 \; ,$$

where $P(x) = \sum_k c_k x^k$, of [11] suggests via resummation that $P(x)$ is bounded for large $x$, which tantalizingly leads to the possibility of the absence of a Landau pole in an analysis of the $\beta$-function beyond perturbation theory. The behavior $P(x)$ in the QED case, clearly deserves more attention from this viewpoint.

The system case is not quite so simple. Assume $\beta(x) = 0$. If we rule out infinite $(\gamma_1^r)'(x)$, then we can only conclude that for each $r \in \mathcal{R}$

$$\gamma_1^r(x) + \gamma_1^r(x)^2 - P_r(x) = 0 \,.$$

## 1.3 Exposition of the main results

In the remainder of this article, we will restrict ourselves to the single equation case (3) with $s > 0$, which we can rewrite as

$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}x} = f(\gamma_1(x), x)\,, \quad \text{where} \quad f(\gamma_1, x) = \frac{\gamma_1 + \gamma_1^2 - P(x)}{sx\gamma_1}\,. \tag{4}$$

The main assumptions we make on the primitive skeleton function are:

H1: $P$ is a twice differentiable function on $\mathbf{R}^+$, with $P(0) = 0$ and $P(x) > 0$ if $x > 0$.

H2: $P$ is everywhere increasing.

To motivate the detailed study of (4) we will conduct in the remainder of this paper, we consider briefly the simple examples $P(x) = x$ and $s = 1$ or $s = 2$. For a qualitative overview, see Figure 1.

If $s = 1$, we can solve (4) by specifying $\gamma_1(x)$ at $x = 1$, finding

$$\gamma_1(x) = x + x\,W\Big((\gamma_1(1) - 1)\exp\big(\gamma_1(1) - \tfrac{1}{x}\big)\Big)\,, \tag{5}$$

where $W$ is the Lambert $W$ function. A few of such solutions, along with the direction field associated with (4) are displayed in figure 2. Note first that $\frac{\gamma_1(x)}{x} \to 1$ as $x \to 0$, irrespective of $\gamma_1(1)$. On the other hand, A careful study of (5) shows that there is a preferred value $\gamma_1^\star(1) = 1 + W(-\mathrm{e}^{-2}) = 0.8414\ldots$ that separates initial conditions at $x = 1$ in two disjoint intervals, $\mathrm{I}(1) = (0, \gamma_1^\star(1))$ and $\mathrm{G}(1) = [\gamma_1^\star(1), \infty)$. All solutions with $\gamma_1(1) \in \mathrm{G}(1)$ are global, i.e. they exist for all $x \geq 0$, while all solutions with $\gamma_1(1) \in \mathrm{I}(1)$ satisfy $\gamma_1(x^\star) = 0$ for some $x^\star > 1$ depending on $\gamma_1(1)$ and cannot be continued beyond $x = x^\star$. The solution $\gamma_1^\star(x)$ of (4) with $\gamma_1(1) = \gamma_1^\star$ is thus the smallest global solution and is called the separatrix for that reason. Furthermore, $\gamma_1^\star(x) \sim \sqrt{2x} - \frac{2}{3} + \mathcal{O}(x^{-1/2})$ as $x \to \infty$. See also figure 2.

If $s = 2$, on the other hand, one can only obtain an implicit formula for the solutions, as was originally done in [7]:

$$\sqrt{x} = \sqrt{x_0}\,\mathrm{e}^{\Gamma_1(x_0)^2 - \Gamma_1(x)^2} - \sqrt{2}\mathrm{e}^{-\Gamma_1(x)^2}\int_{\Gamma_1(x)}^{\Gamma_1(x_0)} \mathrm{e}^{z^2}\mathrm{d}z \quad \text{where} \quad \Gamma_1(x) = \frac{1 + \gamma_1(x)}{\sqrt{2x}}\,. \tag{6}$$

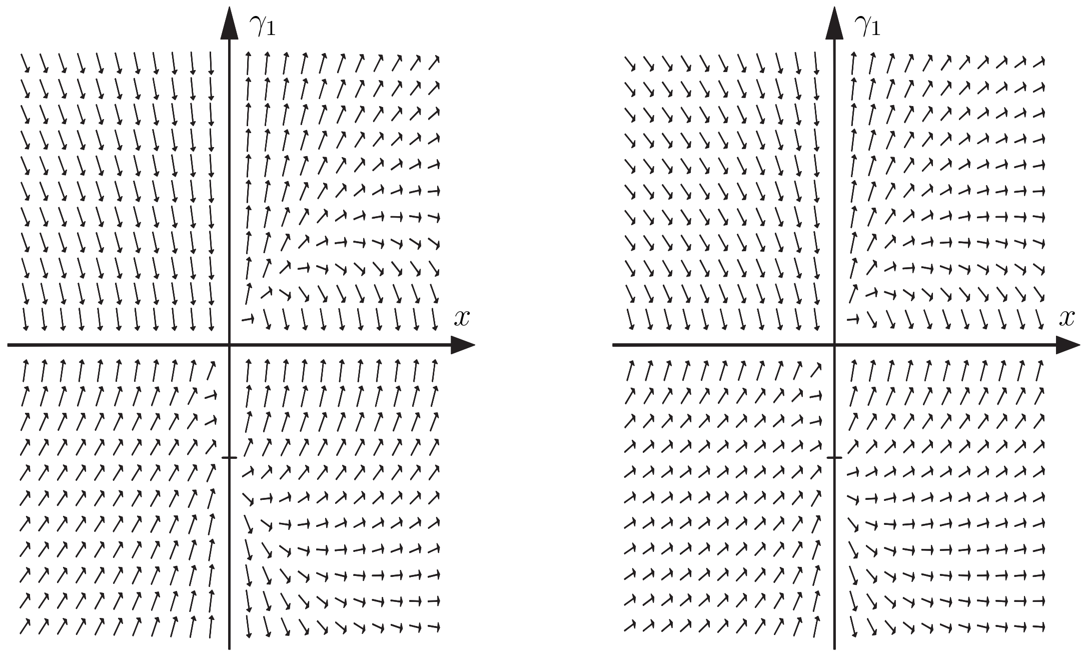


Figure 1: Direction fields for $P(x) = x$, $s = 1$ on left panel, $s = 2$ on right panel. Note how the two pictures look similar; yet, as will be shown, the $s = 1$ case admits global positive solutions, and $s = 2$ does not.

Note that $\Gamma_1(x) \leq \Gamma_1(x_0)$ for all $x \geq x_0$ (as long as $\gamma_1(x) \geq 0$, or as long as $\gamma_1(x)$ exists), since

$$\frac{\mathrm{d}\Gamma_1(x)}{\mathrm{d}x} = -\frac{1}{2^{3/2}\sqrt{x}\gamma_1(x)} \leq 0 \ .$$

In particular, (6) gives an upper bound on the maximal interval of existence of solutions:

$$x \leq x_0 \, \mathrm{e}^{2\Gamma_1(x_0)^2} < \infty \ .$$

We thus see that if $P(x) = x$ and $s = 2$, there are *no* global solutions of (4).

Going beyond $P(x) = x$, we first define the following (possibly infinite) quantities

$$\mathcal{D}_s(P) = \int_{x_0}^{\infty} \frac{P(z)}{z^{1+2/s}} \mathrm{d}z \quad \text{and} \quad \mathcal{L}(P) = \int_{x_0}^{\infty} \frac{2\mathrm{d}z}{z(\sqrt{1+4P(z)}-1)} \ . \tag{7}$$

We can now state our rigorous results, which will be proven later by a strategy largely inspired by [22]:

- Under the hypothesis H1 alone, there are no global solutions if $\mathcal{D}_s(P) = \infty$, while there exist some global solutions if $\mathcal{D}_s(P) < \infty$. Note that in the $P(x) = x$ case, we recover the previous analysis, since $\mathcal{D}_s(P(x) = x) < \infty$ if and only if $s < 2$.

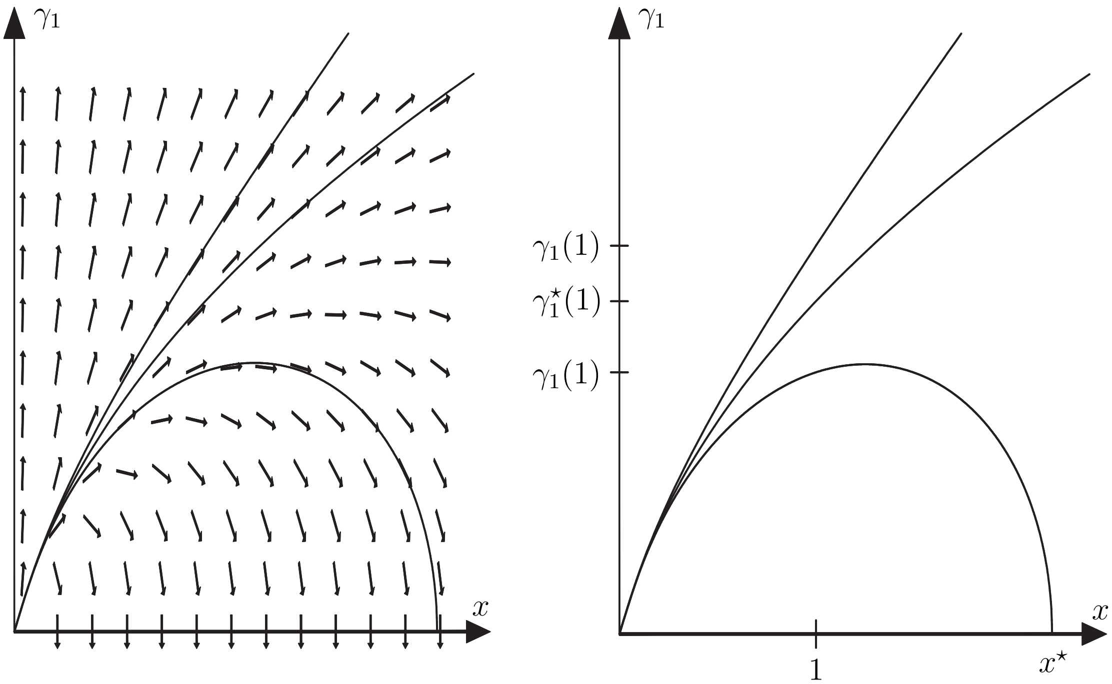


Figure 2: $P(x) = x$, $s = 1$ illustrating that all solutions of (4) tend to $0$ as $x \to 0$, but that some choices of $\gamma_1(1)$ lead to solutions that extend as $x \to \infty$, while others lead to $\gamma_1(x^\star) = 0$ at some finite $x^\star$ and cease to exist beyond that point.

- Under the additional hypothesis H2, there is a (non-trivial) minimal solution $\gamma_1^\star(x)$ which exists for all $x > 0$ and separates global solutions (above $\gamma_1^\star(x)$) from solutions that exist only for finite $x$ (below $\gamma_1^\star(x)$).

The separatrix $\gamma_1^\star(x)$, in the case when it exists, is thus the minimal physical solution, and it matches perturbation theory near the origin. Further, with appropriate conditions on $P(x)$, its behavior in terms of the running coupling is extremely special as we will discuss below.

Consequently we conjecture that it is the solution chosen by nature. Note that this solution does not give us a preferred value for $x$: if we vary $x$ in accordance with the renormalization group equation for the running coupling,

$$\frac{\mathrm{d}x(x_0, L)}{\mathrm{d}L} = \beta(x(x_0, L)) \ ,$$

we just move along our distinguished curve, but there is no preferred value of $x$ from the existence of a distinguished solution.

Following the proof of these results we interpret them in terms of the running coupling, by using the renormalization group equation, which in this case reduces to

$$\frac{\mathrm{d}x}{\mathrm{d}L} = \beta(x) = sx\gamma_1(x) \ .$$

If $\mathcal{D}_s(P) < \infty$, then we will show that if also $\mathcal{L}(P) < \infty$ then all global solutions give Landau poles, whereas the separatrix is the only global solution that does *not* lead to a Landau pole if $\mathcal{L}(P) = \infty$, or, in particular, if $\lim_{x\to\infty} P(x) < \infty$.

The remainder of this paper is organized as follows: in Section 2, we will consider the existence/absence of global solutions of (4), prove the existence of the separatrix in the appropriate case, and state the asymptotic properties of the global solutions. Then, in Section 3, we will interpret the results of Section 2 in terms of the running coupling. The paper concludes with Section 4, in which we give the details left over in the proofs of Section 2 and 3.

# 2 Main results

Building on our analysis of the $P(x) = x$, $s = 1$ case, and since, $f(\gamma_1, x)$ is singular at both $x = 0$ and $\gamma_1 = 0$, we first avoid those singularities by considering instead of (4) the initial value problem

$$\frac{\mathrm{d}\gamma_1(x)}{\mathrm{d}x} = \frac{\gamma_1(x) + \gamma_1(x)^2 - P(x)}{sx\gamma_1(x)}\ , \quad \gamma_1(x_0) = \gamma_0 > 0\ , \tag{8}$$

for some $x_0 > 0$. Since $f(\gamma_1, x)$ is regular away from $x = 0$ and $\gamma_1 = 0$, solutions of (8) exist *locally* around $x = x_0$. Furthermore these solutions are unique and are continuous w.r.t. the initial condition $\gamma_0$. These three statements (local existence, uniqueness and continuity) can be rigorously proved with standard techniques, using that if $\gamma_1$ and $\gamma_2$ are solutions of (8), they satisfy the integral equations

$$\gamma_i(x) = \left(\frac{x}{x_0}\right)^{1/s} (1 + \gamma_i(x_0)) - 1 - x^{1/s} \int_{x_0}^{x} \frac{P(z)}{sz^{1+1/s}\gamma_i(z)}\mathrm{d}z\ , \tag{9}$$

$$\gamma_1(x) - \gamma_2(x) = \Big(\gamma_1(x_0) - \gamma_2(x_0)\Big)\ \exp\bigg(\int_{x_0}^{x} \frac{1}{sz} + \frac{P(z)}{sz\gamma_1(z)\gamma_2(z)}\mathrm{d}z\bigg)\ , \tag{10}$$

as long as they exist (and are strictly positive).

We now prove that global solutions of (8) exist if and only if

$$\int_{x_0}^{\infty} \frac{P(z)}{z^{1+2/s}}\mathrm{d}z < \infty\ , \tag{11}$$

for some finite $x_0 > 0$. Note that in the particular case $P(x) = x$, (11) reduces to $s < 2$, which agrees with our previous analysis. This is the content of the following theorem:

**Theorem 2.1** *Let $s > 0$ and $P$ be a $\mathcal{C}^2$ everywhere positive function. There exist positive global solutions of (8) if and only if $P$ satisfies the integrability condition (11) for some $x_0 > 0$.*

Before proving Theorem 2.1, we note that condition (11) places a strong restriction on the asymptotic behavior of $P(x)$ as $x \to \infty$. For example in the case of QED, $s = 1$, $P(x)$ can grow at most like $o(x^2)$

as $x \to \infty$ for global positive solution of (8) to exist. On the other hand, if $\lim_{x\to\infty} P(x) < \infty$, (11) is satisfied *for all* $s > 0$.

*Proof.* Consider first that (11) holds and let $x_0 > 0$. Choose then

$$\gamma_1(x_0) = x_0^{1/s} \left( \frac{2}{s} \int_{x_0}^{\infty} \frac{P(z)}{z^{1+2/s}} \mathrm{d}z + \epsilon^2 \right)^{1/2} \tag{12}$$

for some $\epsilon > 0$. Assume *ab absurdum* that the corresponding solution $\gamma_1(x)$ has a maximal finite interval of existence $[x_0, x_1]$ for some $x_1 > x_0$. It follows that either $\gamma_1(x_1) = \infty$ or $\gamma_1(x_1) = 0$. The first case cannot happen since from (9), we find

$$\gamma_1(x) \le \left(\frac{x}{x_0}\right)^{1/s} (1 + \gamma_1(x_0)) \tag{13}$$

for all $x \in [x_0, x_1]$, hence $\gamma_1(x_1) = 0$. This also leads to a contradiction, since rewriting (8) as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(\gamma_1(x)^2) = \frac{\gamma_1(x)^2}{sx} + \frac{\gamma_1(x)}{sx} - \frac{P(x)}{sx} \ge \frac{\gamma_1(x)^2}{sx} - \frac{P(x)}{sx} \tag{14}$$

(using $\gamma_1(x) \ge 0$ for $x \in [x_0, x_1]$), integrating that inequality on $[x_0, x_1]$ and using (12) gives

$$\gamma_1(x_1)^2 \ge x_1^{2/s} \left( \frac{\gamma_1(x_0)^2}{x_0^{2/s}} - \frac{2}{s} \int_{x_0}^{x_1} \frac{P(z)}{z^{1+2/s}} \mathrm{d}z \right) = \left(\epsilon\, x_1^{1/s}\right)^2 > 0\ . \tag{15}$$

This contradicts our *ab absurdum* assumption that $\gamma_1(x_1) = 0$, and so $x_1 = \infty$.

To prove the converse, assume *ab absurdum* that there exist a global positive solution of (8) for any $\gamma_1(x_0) > 0$ if

$$\lim_{x\to\infty} \int_{x_0}^{x} \frac{P(z)}{z^{1+2/s}} \mathrm{d}z = \infty\ .$$

Since the solution is global, (13) holds for all $x \ge x_0$, and inserting (13) into (9) gives

$$\gamma_1(x) \le x^{1/s} \left( \frac{1 + \gamma_1(x_0)}{x_0^{1/s}} - \frac{x_0^{1/s}}{s(1 + \gamma_1(x_0))} \int_{x_0}^{x} \frac{P(z)}{z^{1+2/s}} \mathrm{d}z \right) - 1\ , \tag{16}$$

a contradiction, since (16) becomes negative as $x \to \infty$. ■

In the case where global positive solutions do exist, we now prove that the notion of *smallest* global positive solution (the separatrix) is well defined, at least if $P$ is strictly increasing:

**Theorem 2.2** *Let $x_0 > 0$, $s > 0$ and assume that $P$ is a $\mathcal{C}^2$ everywhere positive increasing function that satisfies (11). Then there exist a unique value $\gamma_1^\star(x_0)$ such that the solution of (8) exists globally* *if and only if* $\gamma_1(x_0) \geq \gamma_1^\star(x_0)$. *Furthermore, for every global solution* $\gamma_1(x)$*, there exists a constant* $C > 0$ *such that for all* $x \geq 0$*,*

$$\gamma_c(x) < \gamma_1^\star(x) \leq \gamma_1(x) \leq \gamma_c(x) + Cx^{\frac{1}{s}} + \begin{cases} 0 & \textit{if} \;\; x \geq x_0 \\ B_s(x,x_0) & \textit{if} \;\; x \leq x_0 \end{cases} \tag{17}$$

*where* $\gamma_1^\star(x)$ *is the solution of (8) that corresponds to the initial condition* $\gamma_1^\star(x_0)$ *and*

$$\gamma_c(x) = \frac{\sqrt{1+4P(x)}-1}{2}\,, \tag{18}$$

$$B_s(x,x_0) = x^{1/s}\int_x^{x_0} \frac{\mathrm{d}z}{z^{1/s}} = \begin{cases} \mathcal{O}(x) & \textit{as } x \to 0 \;\; \textit{if} \;\; s<1 \\ \mathcal{O}(x\,|\ln(x)|) & \textit{as } x \to 0 \;\; \textit{if} \;\; s=1 \\ \mathcal{O}(x^{1/s}) & \textit{as } x \to 0 \;\; \textit{if} \;\; s>1 \end{cases}. \tag{19}$$

*In particular,* $\lim_{x\to 0}\gamma_1(x) = 0$ *for every positive global solution.*

*Proof.* The technical details of the proof will be given in Section 4 below. We first note that solutions can have at most one global maximum, and no local minima, and that the global maximum can only occur on the nullcline $\gamma_c(x)$ as defined in (18). Namely, if $x^\star$ is an extremum, then

$$\gamma_1(x^\star) = \gamma_c(x^\star)\,, \quad \gamma_1'(x^\star) = 0 \quad \text{and} \quad \gamma_1''(x^\star) = -\frac{P'(x^\star)}{sx^\star\gamma_c(x^\star)} < 0\,. \tag{20}$$

These relations have the following consequences. First, solutions of (8) with $\gamma_1(x_0) \geq \gamma_c(x_0)$ cannot have a global maximum at some $x_1 < x_0$, nor a local minimum at such a point, and hence must decay monotonically to $0$ as $x \to 0$, while satisfying $\gamma_1(x) > \gamma_c(x)$ for all $x \in [0, x_0]$. Second, solutions of (8) with $\gamma_1(x_0) < \gamma_c(x_0)$ will have a global maximum at some $x_1 < x_0$, and will then decay monotonically to $0$ as $x \to 0$, while satisfying $\gamma_1(x) > \gamma_c(x)$ for all $x \in [0, x_1]$ by the above argument. In particular, all solutions of (8) can be continued as $x \to 0$ and more refined arguments (see Lemma 4.3) show that they satisfy (17) for all $x \in [0, x_0]$.

The relations (20) also show that a solution that satisfies $\gamma_1(x_0) < \gamma_c(x_0)$ must decrease monotonically for all $x \geq x_0$, and more refined arguments (see Lemma 4.1) will show that those solutions indeed satisfy $\gamma_1(x_1) = 0$ for some finite $x_1 > x_0$ and thus cannot be continued as $x \to \infty$.

Furthermore, since $\gamma_c$ is itself monotonically increasing, solutions that start with $\gamma_1(x_0) = \gamma_c(x_0) + \epsilon$ with $\epsilon \ll 1$ necessarily cross the nullcline at some $x > x_0$, and thus also cannot be continued indefinitely as $x \to \infty$, see Lemma 4.2 below. On the other hand, when (11) holds, Theorem 2.1 shows that there are large enough initial conditions whose corresponding solutions can be continued as $x \to \infty$, and thus never cross the nullcline.

By the above arguments, continuity of solutions with respect to initial conditions and equation (10),

$$\mathrm{I}(x_0) = \{\gamma_1(x_0) > \gamma_c(x_0) \mid \exists x > x_0 \;\; \text{with} \;\;\; \gamma_1(x) < \gamma_c(x)\}$$

is a single open, bounded, non-empty interval. Define now $\gamma_1^\star(x_0)$ as the supremum of $\mathrm{I}(x_0)$. From (10), no solution starting below $\gamma_1^\star(x_0)$ can exist globally, and all solutions starting above must stay above the solution corresponding to $\gamma_1^\star(x_0)$, and (13) then implies (17) as $x \geq x_0$. ■

We conclude this section with the following corollary about the growth of global solutions as $x \to \infty$.

**Corollary 2.3** *Let $s > 0$ and assume that $P$ satisfies (11). Then every global solution of (8) with $\gamma_1(x_0) > \gamma_1^\star(x_0)$ satisfies $C_1\, x^{\frac{1}{s}} \leq \gamma_1(x) \leq C_2\, x^{\frac{1}{s}}$ as $x \to \infty$ for some $0 < C_1 < C_2$, while the separatrix itself satisfies*

$$\gamma_c(x) < \gamma_1^\star(x) \leq \min\Big(\lim_{x\to\infty} \gamma_c(x)\ ,\ C\, x^{\frac{1}{s}}\Big)$$

*for some $C > 0$. In particular, if $\lim\limits_{x\to\infty} P(x) < \infty$, the separatrix is the only global bounded solution of (8).*

*Proof.* Let $\gamma_1(x_0) > \gamma_1^\star(x_0)$, and consider the corresponding solution of (8). The upper bound $\gamma_1(x) \leq C_2\, x^{\frac{1}{s}}$ follows immediately from (9). For the lower bound, we note that from (10), we have

$$\gamma_1(x) \geq \gamma_1^\star(x) + \big(\gamma_1(x_0) - \gamma_1^\star(x_0)\big)\ \exp\Big(\int_{x_0}^{x} \frac{\mathrm{d}z}{sz}\Big) \geq C_1\, x^{\frac{1}{s}}\ ,$$

for some $C_1 > 0$ since $\gamma_1(x_0) > \gamma_1^\star(x_0)$.

As for the separatrix itself, first note that the lower bound is already contained in Theorem 2.2. If $\lim_{x\to\infty} P(x) = \infty$, the upper bound $\gamma_1^\star(x) \leq C\, x^{\frac{1}{s}}$ follows again from (9). If $\lim_{x\to\infty} P(x) < \infty$, we first set $\gamma_\infty = \lim_{x\to\infty} \gamma_c(x) < \infty$. Consider then $\gamma_1(x_0) = \gamma_\infty$. The corresponding solution $\gamma_1(x)$ of (8) must initially increase above $\gamma_\infty$ for $x$ sufficiently close to $x_0$ since

$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}x}\Big|_{x=x_0} = \frac{\gamma_\infty + \gamma_\infty^2 - P(x_0)}{s x_0 \gamma_\infty} = \frac{\lim_{x\to\infty} P(x) - P(x_0)}{s x_0 \gamma_\infty} > 0\ .$$

Once the solution is above $\gamma_\infty$, it cannot have a local maximum at an $x > x_0$ and hence can be continued as $x \to \infty$. If $\lim_{x\to\infty} P(x) < \infty$, we thus have a one parameter family of global solutions, indexed by $x_0$, the point at which $\gamma_1(x_0) = \gamma_\infty$. Since the separatrix $\gamma_1^\star$ is the smallest global solution, we get $\gamma_1^\star(x) \leq \gamma_\infty$ for all $x > 0$, which concludes the proof. ■

# 3 The running coupling

We now interpret the above analysis in view of the running of the coupling constant. With appropriate conventions this introduces the second differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}L} = \beta(x(L))\ . \tag{21}$$

In the single equation case, combining (21) with (8), we obtain the following system

$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}L} = \gamma_1 + \gamma_1^2 - P \ , \quad \frac{\mathrm{d}x}{\mathrm{d}L} = s\ x\ \gamma_1 \ , \tag{22}$$

which we supplement with initial conditions at $L = 0$:

$$x(L=0) = x_0 \quad \text{and} \quad \gamma_1(L=0) = \gamma_1(x_0) \ .$$

Before considering the fate of non-global solutions of (8), we first explain how (almost all) global solutions of (8) are Landau poles.

**Theorem 3.1** *Assume that $P$ is a $\mathcal{C}^2$, positive, everywhere increasing function that satisfies (11). The separatrix $\gamma_1^\star$ is a Landau pole if and only if*

$$\mathcal{L}(P) = \int_{x_0}^{\infty} \frac{\mathrm{d}z}{z\ \gamma_c(z)} = \int_{x_0}^{\infty} \frac{2\mathrm{d}z}{z(\sqrt{1+4P(z)}-1)} < \infty \ .$$

*All other global solutions of (8) are Landau poles, irrespective of the value of $\mathcal{L}(P)$.*

*Proof.* We first note that global solutions of (8) give solutions of (22) via the reparametrization

$$L = \int_{x_0}^{x(L)} \frac{\mathrm{d}z}{s\ z\ \gamma_1(z)} \ .$$

In particular, a global solution of (22) with $x(L=0) = x_0$ and $\gamma_1(L=0) = \gamma_1(x_0)$ reaches $x = \infty$ at

$$L^\star = \int_{x_0}^{\infty} \frac{\mathrm{d}z}{s\ z\ \gamma_1(z)} \ . \tag{23}$$

From Corollary (2.3), we know that any global solution of (8) that is not the separatrix grows at least like $x^{\frac{1}{s}}$ as $x \to \infty$. In particular, the integral in the r.h.s. of (23) converges to some finite $L^\star$, and $\gamma_1(L)$ diverges as $L \to L^\star$, signaling that this solution is a Landau pole. By Corollary 2.3, the separatrix is also a Landau pole if $P(x)$ grows fast enough as $x \to \infty$ so that $\mathcal{L}(P) < \infty$.

If $\lim_{x\to\infty} P(x) < \infty$, then Corollary (2.3) shows that $\gamma_1^\star \leq \lim_{x\to\infty} \gamma_c(x) < \infty$, which makes the integral in the r.h.s. of (23) divergent. In particular, the separatrix is the only global solution of (8) that *is not* a Landau pole when written in terms of the running coupling $L$. In section 4 below, we will show that this actually holds not only if $\lim_{x\to\infty} P(x) < \infty$ but also for all $P$ that grow sufficiently slowly as $x \to \infty$ so that $\mathcal{L}(P) = \infty$. ■

Consider now $\gamma_1(x)$ a solution of (8) that only exist on maximal finite interval $x \in [0, x^\star]$. By the results of section 2, we necessarily have $\gamma_1(x^\star) = 0$. As is apparent from (22), the introduction of the running coupling removes the singularity of (8) at $\gamma_1 = 0$. There is thus a 1-1 correspondence between solutions of (8) that exist only on finite intervals with the family of solutions of (22) with $x(L=0) = x^\star$ and $\gamma_1(L=0) = 0$. More precisely, we have the

**Theorem 3.2** *For each $x^\star > 0$, there is a unique solution of (22) that satisfies $x(L=0) = x^\star$ and $\gamma_1(L=0) = 0$. This solution is an heteroclinic orbit of the system (22) connecting the two equilibrium points $(x, \gamma_1) = (0, 0)$ at $L = -\infty$ to $(x, \gamma_1) = (0, -1)$ at $L = \infty$.*

Note that this theorem implies that solutions of (8) that exist only on finite intervals are actually double-valued as functions of $x$, but exists for all $L \in \mathbf{R}$. Note that in particular, such solutions come back to $x = 0$ as a dipole-ghost [21]: we gained a full integer in scaling weight for the photon, see left panel of figure 3.

*Proof.* Fix $x^\star > 0$, and consider the solution of (22) that satisfies $x(L=0) = x^\star$ and $\gamma_1(L=0) = 0$. Note first that the vector field associated with (22) is perpendicular to the $x$-axis, and crosses the $x = x^\star$ vertical line from left to right above the $x$-axis, see also the right panel of figure 3. As a consequence, and by local existence of solutions of (22), there exists a *finite* $L^- < 0$ such that $\gamma_1(L^-) > 0$ and $0 < x(L^-) < x^\star$. By the results of section (2), the solution of (8) with $x_0 = x(L^-) > 0$ and $\gamma_1(x_0) = \gamma_1(x(L^-)) > 0$ can be extended up to $x = 0$ and satisfies $\gamma_1(x) \sim \gamma_c(x)$ as $x \to 0$. In particular, the solution of (22) satisfying $x(L=0) = x^\star$ and $\gamma_1(L=0) = 0$ tends to $(x, \gamma_1) = (0, 0)$ as $L \to -\infty$ since

$$L = L^- - \int_{x(L)}^{x(L^-)} \frac{\mathrm{d}z}{s\, z\, \gamma_1(z)} \to -\infty \quad \text{as} \quad x(L) \to 0\ .$$

We now prove that $(x, \gamma_1) \to (0, -1)$ as $L \to \infty$. Again, since the vector field associated with (22) is perpendicular to the $x$-axis, and crosses the $x = x^\star$ vertical line from right to left below the $x$-axis, there exists a finite $L^+ > 0$ such that $-1 < \gamma_1(L^+) < 0$ and $0 < x(L^+) < x^\star$ (the value $\gamma_1(L^+)$ is the dashed line on the right panel of figure 3). Note then that the vector field points inside the rectangle $R = [0, x^\star] \times [\gamma_1(L^+), -1 - \gamma_c(x^\star)]$, except on the $\gamma_1$ axis where it is tangent and points towards $(x, \gamma_1) = (0, -1)$, see also the right panel of figure 3. It thus follows that

$$0 \le x(L) \le x^\star \quad \text{and} \quad -1 - \gamma_c(x^\star) \le \gamma_1(L) \le \gamma_1(L^+) \quad \forall L \ge L^+\ .$$

In particular,

$$-c_1\, x \equiv -sx(1 + \gamma_c(x^\star)) \le \frac{\mathrm{d}x}{\mathrm{d}L} \le sx\gamma_1(x(L^+)) \equiv -c_2\, x\ ,$$

for some $c_1, c_2 > 0$, and thus

$$x(L^+)\mathrm{e}^{-c_1(L-L^+)} \le x(L) \le x(L^+)\mathrm{e}^{-c_2(L-L^+)}\ ,$$

which shows that $x(L) \to 0$ as $L \to \infty$. Since the vector field points towards $(x, \gamma_1) = (0, -1)$ on the $\gamma_1$ axis, it also follows that $\gamma_1(L) \to -1$ as $L \to \infty$, and concludes the proof. ■

# 4 Technical proofs

This section contains the technical details needed for a complete proof of Theorem 2.2 above. Throughout this section, we assume $P$ is a $\mathcal{C}^2$, positive, strictly increasing function of $x$, which satisfies (11).

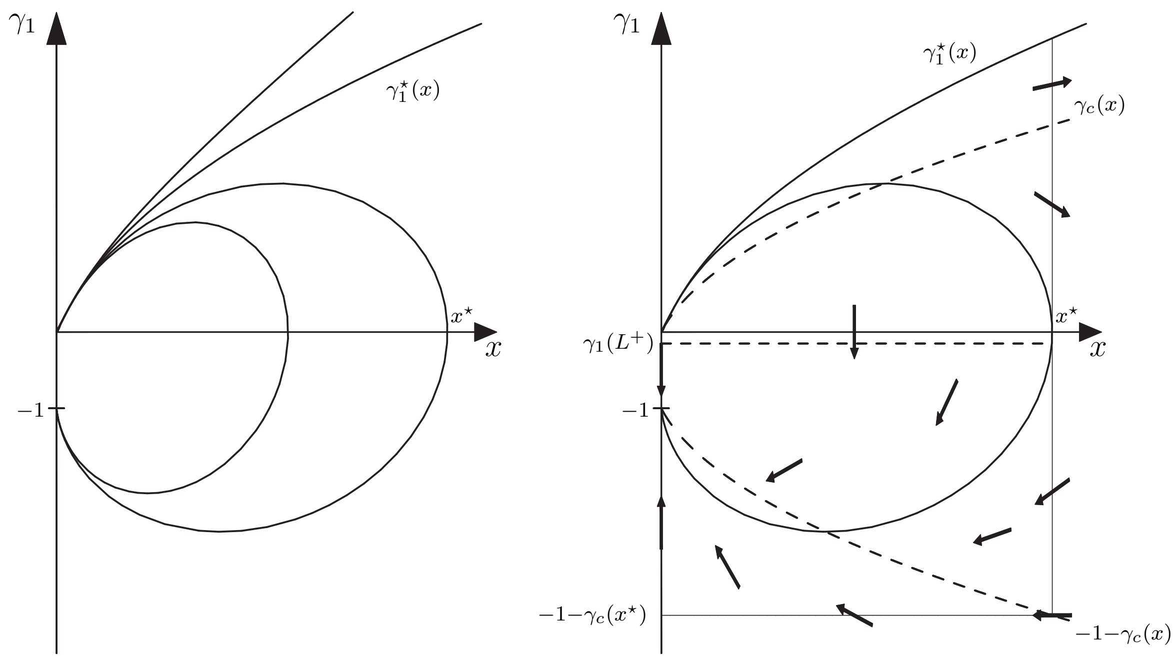


Figure 3: $P(x) = x$, $s = 1$ illustrating that, as a function of $L$, non-global solutions of (8) turn around and head to $-1$ as $L \to \infty$.

Our first step is to show that solutions that start below the nullcline $\gamma_c(x_0)$ cannot be continued as $x \to \infty$. Note that this does not follow directly from (20), since $\gamma_1(x)$ could a priori decrease indefinitely as $x \to \infty$ without ever reaching $\gamma_1 = 0$.

**Lemma 4.1** *Let $\gamma_1(x_0) < \gamma_c(x_0)$ then the solution of (8) satisfies $\gamma_1(x_1) = 0$ for some finite $x_1 > x_0$.*

*Proof.* Let $\gamma_1(x_0) \equiv \gamma_c(x_0) - \epsilon$ for some $0 < \epsilon < \gamma_c(x_0)$. We first note that $\gamma_1(x) \leq \gamma_1(x_0)$ for all $x \geq x_0$ such that the solution exists, otherwise there would be a local minimum at some $x^\star \in [x_0, x]$, which is precluded by (20). Since $P(x)$ is increasing, we find

$$\begin{aligned} \frac{\mathrm{d}\gamma_1(x)}{\mathrm{d}x} &\leq \frac{\gamma_c(x_0) - \epsilon + (\gamma_c(x_0) - \epsilon)^2 - P(x_0)}{sx(\gamma_c(x_0) - \epsilon)} \\ &\leq -\frac{\epsilon(1 + 2\gamma_c(x_0) - \epsilon)}{sx(\gamma_c(x_0) - \epsilon)} \equiv -\frac{R(x_0, \epsilon)}{x}, \end{aligned} \tag{24}$$

for some $R(x_0, \epsilon) > 0$. Integrating (24) on $[x_0, x]$ gives

$$\gamma_1(x) \leq \gamma_1(x_0) - R(x_0, \epsilon) \int_{x_0}^{x} \frac{\mathrm{d}z}{z} = \gamma_c(x_0) - \epsilon - R(x_0, \epsilon) \ln\left(\frac{x}{x_0}\right) ,$$

which shows that $\gamma_1(x_1) = 0$ for some $x_1 \leq x_0 \exp\left(\frac{\gamma_c(x_0) - \epsilon}{R(x_0, \epsilon)}\right) < \infty$ as claimed. ■

Our next step is to show that solutions that start close enough, but above the nullcline at $x_0$ cross the nullcline at some $x > x_0$, and thus cannot be continued as $x \to \infty$ by Lemma 4.1.

**Lemma 4.2** *Assume* $\gamma_1(x_0) = \gamma_c(x_0) + \delta^2$*. There exist* $\delta > 0$ *sufficiently small such that if* $\gamma_1(x)$ *solves (8), then* $\gamma_1(x_0 + \delta) < \gamma_c(x_0 + \delta)$*.*

*Proof.* First note that by (9), $\gamma_1(x) \leq (\frac{x}{x_0})^{1/s}(1 + \gamma_1(x_0)) - 1$. Under the assumption that $\gamma_1(x_0) = \gamma_c(x_0) + \delta^2$, we thus find

$$\sup_{x \in [x_0, x_0+\delta]} \gamma_1(x) \leq \gamma_c(x_0) + C\delta$$

for some constant $C = C(x_0, s) > 0$. We now use the following estimate on $f(\gamma_1(x), x)$

$$\begin{aligned}\sup_{x \in [x_0, x_0+\delta]} f(\gamma_1(x), x) &\leq \frac{\gamma_c(x_0) + C\delta + (\gamma_c(x_0) + C\delta)^2 - P(x_0)}{s x_0 (\gamma_c(x_0) + C\delta)} \\ &\leq \frac{C\delta}{s x_0} \left( 2 + \frac{1}{\gamma_c(x_0) + C\delta} \right) \leq M\delta\end{aligned}$$

for some constant $M = M(x_0, s) > 0$. We thus find, upon integration of (8) that

$$\gamma_1(x_0 + \delta) \leq \gamma_c(x_0) + (1 + M)\delta^2. \tag{25}$$

Now by Taylor's theorem, there exists a constant $N(x_0)$ such that

$$\gamma_c(x_0 + \delta) \geq \gamma_c(x_0) + \gamma_c'(x_0)\delta + N(x_0)\delta^2 \ . \tag{26}$$

Since $\gamma_c'(x_0) > 0$, we can choose $\delta$ sufficiently small so that

$$(1 + M - N(x_0))\delta^2 < \gamma_c'(x_0)\delta \ ,$$

which implies that $\gamma_1(x_0 + \delta) < \gamma_c(x_0 + \delta)$ and completes the proof. ■

Our next step is to show that every local solution of (8) can be continued as $x \to 0$. We will also show that all solutions behave asymptotically like $\gamma_c(x)$ as $x \to 0$.

**Lemma 4.3** *Let* $\gamma_1(x)$ *be a (local) solution of (8) with* $\gamma_1(x_0) > 0$*. Then that solution can be continued for all* $x \in [0, x_0]$*. Furthermore, there exist* $0 < x_1 \leq x_0$ *(with* $x_1 = x_0 \Leftrightarrow \gamma_1(x_0) \geq \gamma_c(x_0)$*) and a constant* $C > 0$ *such that* $\gamma_c(x) < \gamma_1(x) \leq \gamma_c(x) + Cx^{\frac{1}{s}} + CB_s(x, x_1)$ *for all* $x \in [0, x_1]$*, where* $B_s(x, x_1)$ *is defined in (19).*

*Proof.* We first note that $\gamma_1(x_0) > 0$ guarantees that the solution exists locally around $x_0$. We now prove that it satisfies

$$\min(\gamma_c(x), \gamma_1(x_0)) \leq \gamma_1(x) \leq \max(\gamma_c(x_0), \gamma_1(x_0)) \quad \forall x \in [0, x_0] \ , \tag{27}$$

and hence can be continued up to $x = 0$. Recall from (20) that a solution can have at most one global maximum, and no local minimum. We now consider two cases, $\gamma_1(x_0) \geq \gamma_c(x_0)$ and $\gamma_1(x_0) < \gamma_c(x_0)$.

In the first case, we claim that

$$\gamma_1(x_0) \geq \gamma_c(x_0) \quad \Rightarrow \quad \gamma_c(x) \leq \gamma_1(x) \leq \gamma_1(x_0) \quad \forall x \in [0, x_0] \; . \tag{28}$$

Namely, if $\gamma_1(x_0) \geq \gamma_c(x_0)$, then $\gamma_1(x)$ must decrease as $x$ decreases, at least for all $x$ sufficiently close to $x_0$ with $x < x_0$. This follows because either $\gamma_1'(x_0) > 0$ if $\gamma_1(x_0) > \gamma_c(x_0)$ or $\gamma_1'(x_0) = 0$ and $\gamma_1''(x_0) < 0$ if $\gamma_1(x_0) = \gamma_c(x_0)$). We thus get that $\gamma_1(x) < \gamma_1(x_0)$ holds for all $0 \leq x < x_0$ since the solution cannot have a local minimum. Also, the solution cannot have a maximum at $x_1 < x_0$ either, since at that maximum, $\gamma_1(x_1) < \gamma_1(x_0)$, which would require a local minimum at some intermediate value $x^\star \in (x_1, x_0)$, and hence $\gamma_1(x) > \gamma_c(x)$ for all $x < x_0$.

In the case $\gamma_1(x_0) < \gamma_c(x_0)$, we claim that there exist $0 < x_1 < x_0$ such that $\gamma_1(x_1) = \gamma_c(x_1)$, or, in other words, the solution crosses the nullcline at some $x_1 < x_0$. Namely, the solution must increase initially as $x$ decreases (since $\gamma_1'(x_0) < 0$). Since the solution cannot have a local minimum, $\gamma_1(x) \geq \gamma_1(x_0)$ as long as it is below the nullcline, and hence it must cross the nullcline (and have a global maximum) at some $x_1 \in (\gamma_c^{-1}(\gamma_1(x_0)), x_0)$. In particular, the global maximum $\gamma_1(x_1)$ satisfies $\gamma_1(x_1) \leq \gamma_c(x_0)$ since $\gamma_c$ is strictly increasing. We thus find

$$\gamma_1(x_0) \leq \gamma_1(x) \leq \gamma_c(x_0) \quad \forall x \in [x_1, x_0] \; .$$

Since $\gamma_1(x_1) = \gamma_c(x_1)$, we apply (28) with $x \in [0, x_1]$ and get (using also $\gamma_c(x_1) \leq \gamma_c(x_0)$) that

$$\gamma_c(x) \leq \gamma_1(x) \leq \gamma_1(x_1) \leq \gamma_c(x_0) \quad \forall x \in [0, x_1] \; .$$

This completes the proof of (27).

Note now that in all cases, there exists $x_1 \leq x_0$ such that $\gamma_1(x) \geq \gamma_c(x)$ for all $x \in [0, x_1]$. In particular, $\frac{P(z)}{\gamma_1(z)} \leq \frac{P(z)}{\gamma_c(z)} = 1 + \gamma_c(z)$ for all $z \in [0, x_1]$. Let now $x \in [0, x_1]$. From (9), we find

$$\gamma_1(x) \leq \left(\frac{x}{x_1}\right)^{1/s} (1 + \gamma_1(x_1)) - 1 + x_1^{1/s} \int_x^{x_1} \frac{1 + \gamma_c(z)}{s z^{1+1/s}} \mathrm{d}z \; , \tag{29}$$

which, after integrating by parts, gives

$$\gamma_1(x) \leq \left(\frac{x}{x_1}\right)^{1/s} (\gamma_1(x_1) - \gamma_c(x_1)) + \gamma_c(x) + x^{1/s} \int_x^{x_1} \frac{\gamma_c'(z)}{z^{1/s}} \mathrm{d}z \; . \tag{30}$$

Since $\gamma_c'(z) \leq C$ for all $z \in [0, x_1]$, we get $\gamma_1(x) \leq \gamma_c(x) + C x^{\frac{1}{s}} + C B_s(x, x_1)$ for all $x \in [0, x_1]$, which completes the proof. ∎

We now have all the tools to prove Theorem 2.2, which we restate now in the form

**Theorem 4.4** *Assume that (11) holds. The set*

$$\mathrm{I}(x_0) = \{\gamma_1(x_0) > \gamma_c(x_0) \mid \exists x > x_0 \quad \textit{with} \quad \gamma_1(x) < \gamma_c(x)\} \; ,$$

*is a single, open, non-empty and bounded interval. Moreover the solution $\gamma_1^\star(x)$ of (8) with*

$$\gamma_1^\star(x_0) = \sup(\mathrm{I}(x_0))$$

*is the smallest solution that exists for all $x \in [0, \infty)$, and its graph defines the separatrix, in the sense that any global solution $\gamma_1(x)$ of (8) satisfies*

$$\gamma_c(x) < \gamma_1^\star(x) \leq \gamma_1(x) \leq \gamma_c(x) + Cx^{1/s} + C \begin{cases} B_s(x) & \textit{if} \quad x \leq x_0 \\ 0 & \textit{if} \quad x > x_0 \end{cases} \tag{31}$$

*for all $x \in [0, \infty)$.*

*Proof.* We first note that by Lemma 4.2, $\mathrm{I}(x_0) \neq \emptyset$, and by Lemma 4.1 solutions that start in $\mathrm{I}(x_0)$ cannot be continued as $x \to \infty$. Since global solutions exist by Theorem 2.1 for all $\gamma_1(x_0)$ large enough, $\mathrm{I}(x_0)$ is bounded above. Also, $\mathrm{I}(x_0)$ is open by continuity of solutions with respect to initial conditions. Consider now $\gamma_1(x)$ and $\gamma_2(x)$, to be two solutions of (8), for which $\gamma_c(x_0) < \gamma_2(x_0) < \gamma_1(x_0)$ and $\gamma_1(x_0) \in \mathrm{I}(x_0)$. We now claim that $\gamma_2(x_0)$ must also be in $\mathrm{I}(x_0)$. Namely, since $\gamma_1(x_0) \in \mathrm{I}(x_0)$, there must be an $x_1 > x_0$ such that $\gamma_1(x_1) < \gamma_c(x_1)$. By (10), we have $\gamma_2(x) < \gamma_1(x)$ as long as both solutions exist. In particular, either $\gamma_2(x)$ exists on $[x_0, x_1]$, and (10) shows that $\gamma_2(x_1) < \gamma_1(x_1) < \gamma_c(x_1)$ and thus $\gamma_2(x_0) \in \mathrm{I}(x_0)$, or $\gamma_2(x)$ cannot be continued up to $x = x_1$ and since it cannot diverge to infinity by (13), we must have $\gamma_2(x_2) = 0$ for some $x_2 < x_1$, which also implies that $\gamma_2(x_0) \in \mathrm{I}(x_0)$. This shows that $\mathrm{I}(x_0)$ is a single open interval. We thus define

$$\gamma_1^\star(x_0) = \sup(\mathrm{I}(x_0)) \ .$$

Evidently, $\gamma_1^\star(x_0) \notin \mathrm{I}(x_0)$, and so the corresponding solution $\gamma_1^\star(x)$ of (8) satisfies $\gamma_c(x) < \gamma_1^\star(x) \leq Cx^{1/s}$ for all $x \geq x_0$ (the lower bound follows by definition of $\mathrm{I}(x_0)$, the upper bound by (13)), while Lemma 4.3 shows that $\gamma_1^\star(x)$ exists for all $x \in [0, x_0]$ and satisfies $\gamma_c(x) < \gamma_1^\star(x) \leq \gamma_c(x) + Cx^{1/s} + CB_s(x, x_0)$ for all $x \in [0, x_0]$. Using (10) and (13) again shows that the solution corresponding to any $\gamma_1(x_0) \geq \gamma_1^\star(x_0)$ can also be continued as $x \to \infty$ and satisfies (31). ■

We conclude this section with a last result concerning the growth of the separatrix in the case where $P$ is a slowly increasing function.

**Lemma 4.5** *Assume that $P$ satisfies*

$$\int_{x_0}^{\infty} \frac{\mathrm{d}z}{z\, \gamma_c(z)} = \int_{x_0}^{\infty} \frac{2\mathrm{d}z}{z\sqrt{1 + 4P(z)} - 1} = \infty \ , \tag{32}$$

*for some $x_0 > 0$. Then there exists a constant $C > 0$ such that the separatrix $\gamma_1^\star$ satisfies $\gamma_c(x) < \gamma_1^\star(x) \leq \gamma_c(x) + C\ln(x)$ as $x \to \infty$. In particular,*

$$\int_{x_0}^{\infty} \frac{\mathrm{d}z}{z\, \gamma_1^\star(z)} = \infty \ .$$

*Proof.* We first derive from (32) the following bounds on the asymptotics of $P(x)$ and $P'(x)$ as $x \to \infty$:

$$P(x) < C_1 \ln(x)^4 \quad \text{and} \quad \frac{P'(x)x}{\sqrt{1+4P(x)}} \leq C_2 \ln(x) \,.$$

The first one is obvious, and assuming $\frac{P'(x)x}{\sqrt{1+4P(x)}} > C_2 \ln(x)$ gives $P(x) \geq C_1(1+\ln(x))^4$ which contradicts (32). In particular, (11) is satisfied for all $s > 0$, and we are guaranteed by Theorem 4.4 that the separatrix $\gamma_1^\star$ exists and satisfies $\gamma_1^\star(x) > \gamma_c(x)$ for all $x > 0$.

Let now $\overline{\gamma_1}(x) = \gamma_c(x) + \overline{C}\ln(x)$ for some $\overline{C} > 0$, and consider the one parameter family of solutions of (8) obtained by fixing $\gamma_1(x_0) = \overline{\gamma_1}(x_0)$ for different $x_0 > 1$. At least, those solutions that start with $x_0$ sufficiently large can be extended as $x \to \infty$, since

$$\frac{\mathrm{d}}{\mathrm{d}x}\overline{\gamma_1}(x_0) < \frac{\mathrm{d}}{\mathrm{d}x}\gamma_1(x_0) = \frac{\overline{\gamma_1}(x_0) + \overline{\gamma_1}(x_0)^2 - P(x_0)}{s x_0 \overline{\gamma_1}(x_0)} \,,$$

because

$$\frac{\mathrm{d}}{\mathrm{d}x}\overline{\gamma_1}(x_0) = \frac{\overline{C} + C_2\ln(x_0)}{x_0} \quad \text{as} \quad x_0 \to \infty$$

$$\frac{\overline{\gamma_1}(x_0) + \overline{\gamma_1}(x_0)^2 - P(x_0)}{s x_0 \overline{\gamma_1}(x_0)} = \frac{2\overline{C}\ln(x_0)}{s\, x_0} + \mathcal{O}(x_0^{-1}) \,.$$

This shows that a solution that starts on the curve $\overline{\gamma_1}(x) = \gamma_c(x) + \overline{C}\ln(x)$ at $x = x_0$ cannot cross it at any $x$ with $x > x_0$, and thus is a global solution. In particular, the separatrix must be smaller than any of these solutions, and we get $\gamma_1^\star(x) \leq \gamma_c(x) + \overline{C}\ln(x)$. From this last estimate, we get immediately that

$$\int_{x_0}^{\infty} \frac{\mathrm{d}z}{z\gamma_1^\star(z)} \geq \frac{1}{2}\min\Bigl(\int_{x_0}^{\infty}\frac{\mathrm{d}z}{z\gamma_c(z)} \,,\; \int_{x_0}^{\infty}\frac{\mathrm{d}z}{z\overline{C}\ln(z)}\Bigr) = \infty \,,$$

which completes the proof. ∎

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