Discrete Minimal Surface Algebras

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Abstract. We consider Discrete Minimal Surface Algebras (DMSA) as non-commutative analogues of minimal surfaces in higher dimensional spheres. These algebras appear naturally in the context of Membrane Theory, where sequences of their representations are used as a regularization of the theory. After showing that the defining relations of the algebra are consistent, and that one can compute a basis of the universal enveloping algebra, we give several explicit examples of DMSAs in terms of subsets of $\mathfrak{sl}_n$ (any semi-simple Lie algebra providing a trivial example by itself). A special class of DMSAs are Yang-Mills algebras.

The representation graph is introduced to study representations of DMSAs of dimension $d \leq 4$, and properties of representations are related to properties of graphs. The representation graph of a tensor product is (generically) the Cartesian product of the corresponding graphs. We provide explicit examples of irreducible representations and, for coinciding eigenvalues, classify all the unitary representations of the corresponding algebras.

1. Introduction

Noncommutative analogues of manifolds have been studied in many different contexts. One way of constructing such objects is to relate the Poisson structure of a manifold M to the commutator structure of a sequence of matrix algebras $\mathcal{A}_\hbar$ (parametrized by $\hbar > 0$), where the dimension of the matrices increases as $\hbar \to 0$. Namely, for some set of values of the parameter $\hbar$, one defines a map $T^\hbar : C^\infty(M) \to \mathcal{A}_\hbar$ such that

$$\lim_{\hbar\to 0} \left|\left| T^\hbar\big(\{f,g\}\big) - \frac{1}{i\hbar}\big[T^\hbar(f), T^\hbar(g)\big] \right|\right| = 0 \tag{1.1}$$

for all $f, g \in C^\infty(M)$.

For surfaces, the map $T^\hbar$ has been constructed in several different ways. One rather concrete approach is to consider the surface Σ as embedded in an ambient manifold M with embedding coordinates $x_1(\sigma_1,\sigma_2),\dots,x_d(\sigma_1,\sigma_2)$. If the Poisson brackets satisfy

$$\{x_i, x_j\} = p_{ij}(x_1,\dots,x_d)$$

where $p_{ij}(x_1,\dots,x_d)$ are polynomials, then one defines an algebra of non-commuting variables $X_1,\dots,X_d$ such that the following relations hold

$$\big[X_i, X_j\big] = i\hbar\Psi(p_{ij})(X_1,\dots,X_d)$$

where Ψ is an ordering map, mapping commutative polynomials to non-commutative ones (such that when composed with the projection back to commutative polynomials one gets the identity map). Thus, for any pair of polynomials p, q it holds that

$$\Psi(\{p,q\}) - \frac{1}{i\hbar}[\Psi(p),\Psi(q)] = O(\hbar),$$

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and by considering representations of the above defined algebra one obtains a sequence of matrix algebras and maps $T^\hbar$ such that relation (1.1) holds for all polynomial functions (see [Arn08a] for details). It is natural to demand that a notion of noncommutative analogues of manifolds should have some features that can be traced back to the geometry of the original manifold. For surfaces, the genus is the obvious invariant, and one can show that the above procedure gives rise to algebras whose representation theory encodes geometric data [ABH$^+$09].

For a surface Σ embedded in $\mathbb{R}^d$, the Laplace-Beltrami operator on Σ acting on the embedding coordinates $x_1(\sigma_1,\sigma_2),\ldots,x_d(\sigma_1,\sigma_2)$ can be written as

$$\Delta(x_i)=\sum_{j=1}^{d}\{\{x_i,x_j\},x_j\}$$

where $\{f,h\}(\sigma_1,\sigma_2)=\frac{1}{\sqrt{g}}(\partial_1 f\partial_2 h-\partial_1 h\partial_2 f)$ and g is the determinant of the induced metric on Σ. With this notation, minimal surfaces in S^{d-1} can be found by constructing embedding coordinates such that

$$\sum_{j=1}^{d}\{\{x_i,x_j\},x_j\}=-2x_i$$

subject to the constraint $\sum_{j=1}^{d}x_i^2=1$. In the above spirit of replacing Poisson brackets by commutators, corresponding noncommutative minimal surfaces are defined by the relations

$$\hbar^2\sum_{j=1}^{d}\big[[X_i,X_j],X_j\big]=2X_i. \tag{1.2}$$

Another example where equations like (1.2) arise is in the context of a physical theory of “Membranes” [Hop82]. The equations of motion for a membrane moving in $d+1$ dimensional Minkowski space (with a particular choice of coordinates) can be written as

$$\partial_t^2 x_i=\sum_{j=1}^{d}\{\{x_i,x_j\},x_j\}\qquad\text{and}\qquad\sum_{j=1}^{d}\{\partial_t x_j,x_j\}=0,$$

A regularized theory is given by d time-dependent hermitian matrices X_i satisfying the equations

$$\partial_t^2 X_i=-\hbar^2\sum_{j=1}^{d}\big[[X_i,X_j],X_j\big]\qquad\text{and}\qquad\sum_{j=1}^{d}[\partial_t X_i,X_j]=0. \tag{1.3}$$

In the first equation, we can separate time from the matrices by making the Ansatz

$$X_i(t)=\frac{a}{\hbar}\big(e^{A(at+b)}\big)_{ij}M_j, \tag{1.4}$$

(see also [Hop97, AHT04]) where A is a $d\times d$ antisymmetric matrix such that $A^2=\operatorname{diag}(-\mu_1,\ldots,-\mu_d)$ with $\mu_{2i-1}=\mu_{2i}$ for $i=1,2,\ldots,\lfloor d/2\rfloor$. Then X_i (as defined in (1.4)) solves the first equation in (1.3) provided

$$\sum_{j=1}^{d}\big[[M_i,M_j],M_j\big]=\mu_i M_i. \tag{1.5}$$

Motivated by the above examples, we set out to study algebras generated by relations (1.5) (for *arbitrary* μ_i's). In particular, we shall study their representation theory for $d\le 4$.

In Section 2 we introduce Discrete Minimal Surface Algebras, and after showing that a basis of the universal enveloping algebra can be computed via the Diamond lemma, some properties are investigated and several examples are given. Section 3 deals with hermitian representations of the first non-trivial algebras, for which the representation graph is introduced, and properties of representations are related to properties of graphs. Finally, we provide explicit examples of irreducible representations and their corresponding graphs and, in the case when $\mathrm{Spec}(\mathcal{A}) = \{\mu\}$, we classify all unitary representations.

2. Discrete Minimal Surface Algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. For any finite subset $\mathcal{X} = \{x_1, x_2, \dots, x_d\} \subseteq \mathfrak{g}$ we define a linear map $\Delta_{\mathcal{X}} : \mathfrak{g} \to \mathfrak{g}$ by

$$\Delta_{\mathcal{X}}(a) = \sum_{j=1}^{d} \big[[a, x_j], x_j\big],$$

and let $\langle \mathcal{X} \rangle$ denote the vectorspace spanned by the elements in $\mathcal{X}$.

Definition 2.1 (DMSA)**.** Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{X} = \{x_1, x_2, \dots, x_d\}$ be a set of linearly independent elements of $\mathfrak{g}$. We call $\mathcal{A} = (\mathfrak{g}, \mathcal{X})$ a *Discrete Minimal Surface Algebra* (DMSA) (of dimension d) if there exists complex numbers $\mu_1, \dots, \mu_d$ such that $\Delta_{\mathcal{X}}(x_i) = \mu_i x_i$ for $i = 1, \dots, d$. The set $\{\mu_1, \mu_2, \dots, \mu_d\}$ is called *the spectrum of* $\mathcal{A}$ and is denoted by $\mathrm{Spec}(\mathcal{A})$.

Definition 2.2. Two DMSAs $\mathcal{A} = (\mathfrak{g}, \mathcal{X})$ and $\mathcal{B} = (\mathfrak{g}', \mathcal{Y})$ are *isomorphic* if there exists a Lie Algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ such that $\phi|_{\langle \mathcal{X} \rangle}$ is a vectorspace isomorphism of $\langle \mathcal{X} \rangle$ onto $\langle \mathcal{Y} \rangle$.

Note that $\mathrm{Spec}(\mathcal{A})$ is not an invariant within an isomorphism class. Let $\mathfrak{g}$ be a Lie algebra with structure constants f_{ij}^k (relative to the basis $x_1, \dots, x_n$) and let $\mathfrak{g}'$ be a Lie algebra with structure constants cf_{ij}^k (relative to the basis $y_1, \dots, y_n$), for some non-zero complex number c. Then the two DMSAs $\mathcal{A} = (\mathfrak{g}, \mathcal{X} = \{x_1, \dots, x_d\})$ and $\mathcal{B} = (\mathfrak{g}', \mathcal{Y} = \{y_1, \dots, y_d\})$ will be isomorphic (through $\phi(x_i) = y_i/c$), and if $\mathrm{Spec}(\mathcal{A}) = \{\mu_1, \dots, \mu_d\}$ then $\mathrm{Spec}(\mathcal{B}) = \{c^2\mu_1, \dots, c^2\mu_d\}$.

Definition 2.3. Let $\mathcal{A} = (\mathfrak{g}, \mathcal{X})$ be a DMSA with $\mathrm{Spec}(\mathcal{A}) = \{\mu_1, \dots, \mu_d\}$ and let $\mathbb{C}\langle \mathcal{X} \rangle$ denote the free associative algebra over $\mathbb{C}$, generated by the set $\mathcal{X} = \{x_1, x_2, \dots, x_d\}$. The universal enveloping algebra of $\mathcal{A}$ is defined as the quotient $\mathfrak{U}_d(\mathcal{A}) = \mathbb{C}\langle \mathcal{X} \rangle / \langle \Delta_{\mathcal{X}}(x_i) - \mu_i x_i \rangle$, where $[a, b] = ab - ba$.

Proposition 2.4. *A basis of* $\mathfrak{U}_d(\mathcal{A})$ *is provided by the set of words on* $\{x_1, \dots, x_d\}$ *that do not contain any of the following subwords*

$$x_d^2 x_1,\ x_d^2 x_2,\ \dots,\ x_d^2 x_{d-1},\ x_d x_{d-1}^2.$$

Proof. In the following proof we will use the notation and terminology of [Ber78], to which we refer for details. The defining relations of the algebra

$$\sum_{j=1}^{d} \big[[x_i, x_j], x_j\big] = \mu_i x_i$$

for $i = 1, \dots, d$ will be put into the following reduction system S

$$\sigma_i = (W_i, f_i) = \Big(x_d^2 x_i, 2x_d x_i x_d - x_i x_d^2 + \lambda_i x_i - \sum_{j=1}^{d-1} \big[[x_i, x_j], x_j\big]\Big) \quad 1 \le i < d$$

$$\sigma_d = (W_d, f_d) = \Big(x_d x_{d-1}^2, 2x_{d-1} x_d x_{d-1} - x_{d-1}^2 x_d + \lambda_d x_d - \sum_{j=1}^{d-2} \big[[x_i, x_j], x_j\big]\Big).$$

Next, we need to define a semi-group partial ordering on words, that is compatible with S, i.e. every word in f_i should be less than W_i. Let w_1, w_2 be two words on $x_1, \dots, x_d$; if w_1 has smaller length than w_2 then we set $w_1 < w_2$. If w_1 and w_2 has the same length, we set $w_1 < w_2$ if w_1 preceeds w_2 lexicographically, where the lexicographical ordering is induced by $x_1 < x_2 < \cdots < x_d$. These definitions imply that if $w_1 < w_2$ then $aw_1b < aw_2b$ for any words a, b (which defines a semi-group partial ordering). It is also easy to check that this ordering is compatible with S. Does the ordering satisfy the descending chain condition? Let $w_1 \geq w_2 \geq w_3 \geq \cdots$ be an infinite sequence of decreasing words. Clearly, since the length of w_i is a positive integer, it must eventually become constant. Thus, for all $i > N$ the length of w_i is the same. But this implies that the series eventually become constant because there is only a finite number of words preceeding a given word lexicographically. Hence, the ordering satisfies the descending chain condition.

Now, we are ready to apply the Diamond Lemma. If we can show that all ambiguities in S are resolvable, then a basis for the algebra is provided by the irreducible words. In this case there is only one ambiguity in the reduction system S. Namely, there are two ways of reducing the word $x_d^2 x_{d-1}^2$: Either we write it as $x_d(x_d x_{d-1}^2)$ and apply σ_d or we write it as $(x_d^2 x_{d-1})x_{d-1}$ and apply σ_{d-1}. Let us prove that $A \equiv x_d f_d - f_{d-1}x_{d-1} = 0$, i.e. the ambiguity is resolvable.

$$
\begin{aligned}
A =& - x_{d-1}\overbrace{x_d^2 x_{d-1}}^{\to f_{d-1}} + \overbrace{x_d x_{d-1}^2}^{\to f_d} x_d + \lambda_{d-1}x_{d-1}^2 - \lambda_d x_d^2 \\
&- \sum_{j=1}^{d-1}\big[[x_{d-1}, x_j], x_j\big]x_{d-1} + x_d\sum_{j=1}^{d-2}\big[[x_d, x_j], x_j\big] \\
=& \sum_{j=1}^{d-2}\Big[x_{d-1}, \big[[x_{d-1}, x_j], x_j\big]\Big] + \sum_{j=1}^{d-2}\Big[x_d, \big[[x_d, x_j], x_j\big]\Big]
\end{aligned}
$$

Now, let us rewrite the second commutator above as follows:

$$
\begin{aligned}
\Big[x_d, \big[[x_d, x_j], x_j\big]\Big] =& (\overbrace{x_d^2 x_j}^{\to f_j})x_j - 2x_d x_j x_d x_j - x_j^2 x_d^2 + 2x_j x_d x_j x_d \\
=& 2x_j x_d x_j x_d - x_j^2 x_d^2 + \lambda_j x_j^2 - x_j\overbrace{x_d^2 x_j}^{\to f_j} - \sum_{k=1}^{d-1}\big[[x_j, x_k], x_k\big]x_j \\
=& \sum_{k=1}^{d-1}\Big[x_j, \big[[x_j, x_k], x_k\big]\Big].
\end{aligned}
$$

By introducing the sum, we obtain

$$
\begin{aligned}
\sum_{j=1}^{d-2}\Big[x_d, \big[[x_d, x_j], x_j\big]\Big] &= -\sum_{j=1}^{d-2}\Big[x_{d-1}, \big[[x_{d-1}, x_j], x_j\big]\Big] + \sum_{j,k=1}^{d-2}\Big[x_j, \big[[x_j, x_k], x_k\big]\Big] \\
&= -\sum_{j=1}^{d-2}\Big[x_{d-1}, \big[[x_{d-1}, x_j], x_j\big]\Big],
\end{aligned}
$$

which implies that $A = 0$. From the Diamond Lemma, we can now conclude that the set of all irreducible words, with respect to the reduction system S, provide a basis of the algebra. □

Let us start by noting that any semi-simple Lie algebra $\mathfrak{g}$ is itself a DMSA. Namely, if we let $\mathcal{X} = \{x_1, x_2, \dots, x_d\}$ be a basis of $\mathfrak{g}$ such that $K(x_i, x_j) = \delta_{ij}$ (where

K denotes the Killing form), then the structure constants will be totally antisymmetric which implies that

$$\Delta_{\mathcal{X}}(x_i) = \sum_{j=1}^{d} \big[[x_i, x_j], x_j\big] = \sum_{j,k,l=1}^{d} f_{ij}^k f_{kj}^l x_l = \sum_{l=1}^{d} K(x_i, x_l) x_l = x_i.$$

Thus, in such a basis $(\mathfrak{g}, \mathfrak{g})$ is a DMSA for any semi-simple Lie algebra $\mathfrak{g}$. On the other hand, if $\mathfrak{g}$ is nilpotent, it follows that the map $\Delta_{\mathcal{X}}$ is nilpotent. Thus, for a DMSA related to a nilpotent Lie algebra, it must hold that $\mathrm{Spec}(\mathcal{A}) = \{0\}$. Note that the class of DMSA for which $\mathrm{Spec}\,\mathcal{A} = \{0\}$ has also been studied under the name of *(Lie) Yang-Mills Algebras* [CDV02], and their representation theory has recently been studied in [HS08].

The linear operator $\Delta_{\mathcal{X}}$ is invariant under orthogonal transformations of the elements in $\mathcal{X}$ in the following sense:

Lemma 2.5. *Let $\mathcal{X} = \{x_1, x_2, \dots, x_d\}$ and $\mathcal{X}' = \{x_1', \dots, x_d'\}$ be subsets of a Lie algebra $\mathfrak{g}$ such that $x_i' = \sum_{j=1}^{d} R_{ij} x_j$ for some orthogonal $d \times d$-matrix R. Then $\Delta_{\mathcal{X}}(a) = \Delta_{\mathcal{X}'}(a)$ for all $a \in \mathfrak{g}$.*

Proof. The proof is given by the following calculation:

$$\begin{aligned}\Delta_{\mathcal{X}'}(a) &= \sum_{j,k,l=1}^{d} R_{jk} R_{jl} \big[[a, x_k], x_l\big] = \sum_{k,l=1}^{d} \big(R^T R\big)_{kl} \big[[a, x_k], x_l\big] \\ &= \sum_{k,l=1}^{d} \delta_{kl} \big[[a, x_k], x_l\big] = \sum_{j=1}^{d} \big[[a, x_j], x_j\big] = \Delta_{\mathcal{X}}(a).\end{aligned}$$

□

Remark 2.6. Note that it is not necessarily true that $(\mathfrak{g}, \mathcal{X}')$ is a DMSA when $(\mathfrak{g}, \mathcal{X})$ is a DMSA. However, if we let $\mathrm{Spec}\big((\mathfrak{g}, \mathcal{X})\big) = \{\mu_1, \dots, \mu_k\}$ and $m_1, \dots, m_k$ be the multiplicities of each eigenvalue, then any block-diagonal orthogonal matrix $R = \mathrm{diag}(R_1, \dots, R_k)$ (where R_i has dimension m_i) will generate a DMSA. In other words, we can always choose to make an orthogonal transformation among those x_i that belong to the same eigenvalue.

The preceding lemma enables us to make the following observation. Let $\mathcal{X}$ be a set of linearly independent elements in a n-dimensional Lie algebra $\mathfrak{g}$, and let $\langle \mathcal{X} \rangle$ denote the linear span of the elements in $\mathcal{X}$. Furthermore, assume that $\langle \mathcal{X} \rangle$ is closed under the action of $\Delta_{\mathcal{X}}$, i.e. $\Delta_{\mathcal{X}}(x) \in \langle \mathcal{X} \rangle$ for all $x \in \langle \mathcal{X} \rangle$. Relative to a basis where $x_1, \dots, x_d$ are chosen to be the first d basis elements, the $n \times n$ matrix of $\Delta_{\mathcal{X}}$ has the block form

$$\Delta_{\mathcal{X}} = \begin{pmatrix} X_0 & A \\ 0 & B \end{pmatrix}$$

where X_0 is a $d \times d$ matrix. If X_0 is diagonalizable by an orthogonal matrix R then the elements $x_i' = R_{ij} x_j$ will be eigenvectors of $\Delta_{\mathcal{X}'}$, since the action of $\Delta_{\mathcal{X}}$ is invariant under orthogonal transformations in $\mathcal{X}$. Thus, $(\mathfrak{g}, \mathcal{X}')$ is a DMSA. In particular, if we choose an orthonormal basis of $\mathfrak{g}$, then the matrices ad_{x_i} are antisymmetric, which implies that $\Delta_{\mathcal{X}}$ is symmetric. In this case, X_0 will be diagonalizable by an orthogonal matrix.

We will now concentrate on subsets $\mathcal{X}$ of the Lie Algebra $\mathfrak{sl}_n$, such that $\mathcal{A} = (\mathfrak{sl}_n, \mathcal{X})$ is a DMSA. To preform calculations the following set of conventions will be used: $\alpha_1, \dots, \alpha_{n-1}$ denotes the simple roots and for every positive root α, we choose elements $e_\alpha, e_{-\alpha}, h_\alpha$ such that

$$[h, e_\alpha] = \alpha(h) e_\alpha$$
$$[e_\alpha, e_{-\alpha}] = h_\alpha,$$

and h_α is the element of the Cartan subalgebra $\mathfrak{h}$ such that $\alpha(h) = K(h_\alpha, h)$ for all $h \in \mathfrak{h}$. For any pair of roots α, β we define the constants $N(\alpha,\beta)$ by

$$[e_\alpha, e_\beta] = N(\alpha,\beta) e_{\alpha+\beta},$$

and when $\alpha+\beta$ is not a root, we set $N(\alpha,\beta)=0$. In $\mathfrak{sl}_n$ all roots have the same length, and we denote $(\alpha,\alpha) \equiv K(h_\alpha, h_\alpha) = \alpha(h_\alpha) = l^2$. With these conventions, the constants $N(\alpha,\beta)$ satisfies the relations

$$N(\alpha,\beta) = N(\beta,\gamma) = N(\gamma,\alpha) \quad \text{if } \alpha+\beta+\gamma = 0$$
$$N(\alpha,\beta)N(-\alpha,-\beta) = -\frac{l^2}{2} q(p+1)$$

where p, q are positive integers such that $\beta - p\alpha, \dots, \beta, \dots, \beta + q\alpha$ are roots. Furthermore, in $\mathfrak{sl}_n$ it holds that if $\beta+\alpha$ is a root then $\beta-\alpha$ is not a root and $\beta \pm 2\alpha$ is never a root. Therefore, if $N(\alpha,\beta)$ is non-zero, we have that

$$N(\alpha,\beta)N(-\alpha,-\beta) = -\frac{1}{2} l^2.$$

Although the following result does not depend on it, we will for definiteness choose each $N(\alpha,\beta)$ such that $N(-\alpha,-\beta) = -N(\alpha,\beta)$.

Lemma 2.7. *For every positive root α in $\mathfrak{sl}_n$, we set*

$$e_\alpha^+ = ic\big(e_\alpha + e_{-\alpha}\big) \qquad \text{and} \qquad e_\alpha^- = c\big(e_\alpha - e_{-\alpha}\big),$$

for an arbitrary $c \in \mathbb{R}$. Then the following holds

(1) $\big[[e_\alpha^+, e_\beta^+], e_\beta^+\big] = \big[[e_\alpha^+, e_\beta^-], e_\beta^-\big] = -\frac{1}{2}c^2 l^2 e_\alpha^+$ *(when $\alpha \pm \beta$ is a root)*
(2) $\big[[e_\alpha^-, e_\beta^+], e_\beta^+\big] = \big[[e_\alpha^-, e_\beta^-], e_\beta^-\big] = -\frac{1}{2}c^2 l^2 e_\alpha^-$ *(when $\alpha \pm \beta$ is a root)*
(3) $\big[[e_\alpha^\pm, e_\alpha^\mp], e_\alpha^\mp\big] = -2c^2 l^2 e_\alpha^\pm$
(4) $\big[[e_\alpha^\pm, h_\beta], h_\beta\big] = (\alpha,\beta)^2 e_\alpha^\pm$
(5) $\big[[h_\alpha, e_\beta^\pm], e_\beta^\pm\big] = \mp 2c^2(\alpha,\beta) h_\beta$

From this lemma, it is easy to construct a couple of examples.

Example 2.8. Let $\mathcal{X} = \{e_{\beta_1}^\pm, \dots, e_{\beta_d}^\pm\}$ (where the signs are chosen independently) for any positive roots β_i. In this case, $\big[[x_i, x_j], x_j\big]$ is proportional to x_i for all $x_i, x_j \in \mathcal{X}$.

Example 2.9. Let $\mathcal{X} = \{h_{\beta_1}, \dots, h_{\beta_k}, e_{\gamma_1}^+, e_{\gamma_1}^-, \dots, e_{\gamma_l}^+, e_{\gamma_l}^-\}$. Now, $\big[[h_{\beta_i}, e_{\gamma_j}^+], e_{\gamma_j}^+\big]$ might not be proportional to h_{β_i}. However, since both $e_{\gamma_j}^+, e_{\gamma_j}^- \in \mathcal{X}$ this term will cancel against $\big[[h_{\beta_i}, e_{\gamma_j}^-], e_{\gamma_j}^-\big]$. Thus, $\Delta_{\mathcal{X}}(h_{\beta_i}) = 0$ for $i = 1, \dots, k$.

Example 2.10. Let $\mathcal{X} = \{h_{\beta_1}, \dots, h_{\beta_k}, e_{\beta_1}^\pm, \dots, e_{\beta_k}^\pm\}$ (where the signs are chosen to be the same). In this case $\Delta_{\mathcal{X}}(h_{\beta_i})$ will not be proportional to h_{β_i}. However, the matrix $\Delta_{\mathcal{X}}$ will be symmetric, which implies that there exists an orthogonal $k \times k$ matrix R such that $(\mathfrak{sl}_n, \{x_1, \dots, x_k, e_{\beta_1}^\pm, \dots, e_{\beta_k}^\pm\})$ is a DMSA if $x_i = R_{ij} h_{\beta_j}$.

When the dimension $d = 2m$ (i.e. even) and every eigenvalue in $\mathrm{Spec}(\mathcal{A})$ has an even multiplicity (which is relevant for one of the applications mentioned in the introduction) there is a convenient complexified basis provided by

$$t_i = x_{2i-1} + i x_{2i}$$
$$s_i = x_{2i-1} - i x_{2i}$$

for $i = 1, \ldots, m$. The defining relations of a DMSA can then be written as

$$2\mu_i t_i = \sum_{j=1}^{m} \Big(\big[[t_i, s_j], t_j\big] + \big[[t_i, t_j], s_j\big] \Big)$$

$$2\mu_i s_i = \sum_{j=1}^{m} \Big(\big[[s_i, t_j], s_j\big] + \big[[s_i, s_j], t_j\big] \Big).$$

The lowest dimensional non-trivial DMSA has dimension 2. In this case the algebra is generated by the relations

$$\big[[x, y], y\big] = \lambda x$$

$$\big[[y, x], x\big] = \mu y,$$

and by defining $z = -i[x, y]$ we see that $\{x, y, z\}$ spans a 3-dimensional Lie algebra. By rescaling the elements we obtain the following result:

- $\lambda \neq 0, \mu \neq 0$: $\mathcal{A}$ is isomorphic to $\mathfrak{sl}_2$,
- $\lambda = \mu = 0$: $\mathcal{A}$ is isomorphic to the Heisenberg algebra,
- $\lambda \neq 0, \mu = 0$ or $\lambda = 0, \mu \neq 0$: $\mathcal{A}$ is isomorphic to the Lie Algebra VII_1 in the Bianchi Classification [Bia98]. This algebra is defined by the relations: $[u, v] = -w$, $[v, w] = 0$ and $[w, u] = -v$.

3. Hermitian representations of $\mathfrak{U}_4(\mathcal{A})$

In general, any representation of the Lie algebra $\mathfrak{g}$ gives rise to a representation of the DMSA $(\mathfrak{g}, \mathcal{X})$. In the following, we shall however concentrate on finding *hermitian* representations. Hermitian representations ϕ of $\mathfrak{U}_d(\mathcal{A})$ are given by $\phi(x_i) = X_i$ where $X_1, \ldots, X_d$ are hermitian matrices satisfying

$$\sum_{j=1}^{d} \big[[X_i, X_j], X_j\big] = \mu_i X_i \quad \text{for } i = 1, \ldots, d. \tag{3.1}$$

We note that, unless all μ_i are real, no hermitian (or anti-hermitian) representations can exist (except for the trivial one: $\phi(x_i) = 0$ for all i). Hence, from now on we will assume the spectrum to be real and all representations to be hermitian.

Since the matrix algebra generated by $\{X_1, \ldots, X_d\}$ is invariant under hermitian conjugation, the following result is immediate.

Proposition 3.1. *Any hermitian representation of* $\mathfrak{U}_d(\mathcal{A})$ *is completely reducible.*

As DMSAs of dimension 2 are isomorphic to Lie algebras, we will start by considering the case when $d = 4$. We expect that these algebras have a rich structure of representations even for the case when $\mu_1 = \ldots = \mu_4$. Namley, since the equations defining a 4-dimensional DMSA can be thought of as discrete analogues of minimal surface equations in S^3 (see the introduction), and minimal surfaces of any genus exist in S^3 [Law70], we believe that there will be representations corresponding to many (if not all) of these surfaces.

Since the defining relations of $\mathfrak{U}_d(\mathcal{A})$ are expressed entirely in terms of commutators, the tensor product of Lie algebra representations, i.e.

$$\big(\phi \otimes \phi'\big)(x) = \phi(x) \otimes \mathbb{1} + \mathbb{1} \otimes \phi'(x),$$

also defines a tensor product for representations of DMSAs. In contrast to Lie algebras, the tensor product of two irreducible representations might again be irreducible (as we shall explicitly see for $\mathfrak{U}_4(\mathcal{A})$). Thus, when studying the representation theory of $\mathfrak{U}_d(\mathcal{A})$ it becomes natural to look, not only for irreducible representations, but also for *prime representations*, i.e. irreducible representations that can not be written as a tensor product of two other representations.

We shall associate a directed graph embedded in $\mathbb{C}$ to each hermitian representation ϕ of $\mathfrak{U}_4(\mathcal{A})$, such that the vertices of the graph are placed at the characteristic roots of $\phi(x_1+ix_2)$. The edges of the graph are determined by the matrix $\phi(x_3+ix_4)$ as desribed below. We note that this construction can be carried out for hermitian representations of any algebra on at most four generators. In the following, we will use the notation $t_1 = x_1 + ix_2$, $t_2 = x_3 + ix_4$, $\phi(t_1) = \Lambda$ and $\phi(t_2) = T$. Let us start by recalling the directed graph of a matrix.

Definition 3.2. Let T be a $n \times n$ matrix and let $G = (V, E)$ be a directed graph on n vertices with vertex set $V = \{1, \ldots, n\}$ and edge set $E \subseteq V \times V$. We say that G is the *directed graph of* T, and write $G = G_T$, if it holds that

$$T_{ij} \neq 0 \quad \Longleftrightarrow \quad (i,j) \in E.$$

for $i, j = 1, \ldots, n$.

The idea is now to associate a graph to every representation, such that each vertex is assigned an eigenvalue of Λ and the graph itself being the directed graph of T. Needless to say, the graph of T depends on the basis chosen and therefore we will introduce a particular choice of basis in which all graphs will be refered to.

Definition 3.3. Let Λ and T be linear operators on $V = \mathbb{C}^n$, and let $\mathcal{B}$ be the $*$-algebra generated by $\Lambda, \Lambda^\dagger, T, T^\dagger$. Furthermore, let $V = V_1 \oplus \cdots \oplus V_m$ be a decomposition of V into irreducible subspaces with respect to $\mathcal{B}$. For each i, let $v_1^{(i)}, \ldots, v_{n_i}^{(i)}$ denote a Jordan basis for $\Lambda\big|_{V_i}$. Then $v_1^{(1)}, \ldots, v_{n_1}^{(1)}, \ldots, v_1^{(m)}, \ldots, v_{n_m}^{(m)}$ is a basis for V and is called *a Jordan basis of* Λ *with respect to* T.

Definition 3.4. Let ϕ be a n-dimensional representation of $\mathfrak{U}_4(\mathcal{A})$ and let $v_1, \ldots, v_n$ denote a Jordan basis of $\Lambda = \phi(t_1)$ with respect to $T = \phi(t_2)$. Define the matrix α by defining its matrix elements through

$$Tv_i = \sum_{j=1}^{n} \alpha_{ji} v_j.$$

Futhermore, let $G_\alpha = (\{1, 2, \ldots, n\}, E)$ denote the directed graph of α and let $\lambda : V \to \mathbb{C}$ be defined by $\lambda(i) = \lambda_i$, where λ_i is the eigenvalue corresponding to v_i. We set $G_\phi = (G_\alpha, \lambda)$ and call G_ϕ *a representation graph of* ϕ.

Two representation graphs $G_\phi = (\{1, \ldots, n\}, E, \lambda)$ and $G_{\phi'} = (\{1, \ldots, n\}, E', \lambda')$ are *isomorphic* if there exists a permutation $\sigma \in S_n$ such that $(i,j) \in E \Leftrightarrow \big(\sigma(i), \sigma(j)\big) \in E'$ and $\lambda = \lambda' \circ \sigma$. In particular, $(\{1, \ldots, n\}, E)$ and $(\{1, \ldots, n\}, E')$ are isomorphic as directed graphs.

Note that two representation graphs corresponding to the same representation need not be isomorphic. This could be resolved by further fixing the basis in which the directed graph of T is calculated. However, let us postpone this choice and study the properties of representation graphs that follow from the above definition.

The first property that one might wish for, is a correspondence between disconnected components of the representation graph and the irreducible components of the representation. That a connected graph corresponds to an irreducible representation follows immediately from the definition of the Jordan basis with respect to T.

Proposition 3.5. *Let G_ϕ be a representation graph of ϕ. If G_ϕ is connected then ϕ is irreducible.*

Proof. Let G_ϕ be a connected representation graph of ϕ. If ϕ is reducible, then G_ϕ consists of at least two components, since the matrix α is block diagonal with at least two blocks, by the construction of the Jordan basis. Hence, ϕ must be irreducible. □

For convenience, let us introduce some terminology indicating when the matrices of a representation have certain properties.

Definition 3.6. Let ϕ be a representation of $\mathfrak{U}_4(\mathcal{A})$. If $\Lambda = \phi(t_1)$ is diagonalizable then ϕ is called diagonalizable. If all eigenvalues of Λ are distinct, then ϕ is called nondegenerate. If Λ is normal then ϕ is called semi-normal. If both Λ and $T = \phi(t_2)$ are normal then ϕ is called normal. If Λ and T are unitary, then ϕ is called unitary.

For semi-normal representations, the result in Proposition 3.5 can be strengthened to an if and only if statement.

Proposition 3.7. *Let ϕ be a semi-normal representation of $\mathfrak{U}_4(\mathcal{A})$ and let G_ϕ be a representation graph of ϕ. Then ϕ is irreducible if and only if G_ϕ is connected.*

Proof. Assuming that G_ϕ is connected, the first implication follows from Proposition 3.5. Now, assume that ϕ is irreducible. When Λ is normal, the Jordan basis is given by a set of orthogonal vectors (the eigenvectors of Λ). Therefore, any matrix P, bringing Λ to the (diagonal) Jordan normal form $P^{-1}\Lambda P$, can be written as $P = UD$ where U is a unitary matrix and D is an invertible diagonal matrix (reflecting the choice of length of the eigenvectors). If we define the matrix $\tilde{\alpha}$ through

$$T^\dagger v_i = \sum_{j=1}^{n} \tilde{\alpha}_{ji} v_j.$$

then α and $\tilde{\alpha}$ are related by $\tilde{\alpha}^\dagger = D^{-2}\alpha D^2$. Since conjugation by an invertible diagonal matrix does not change the structure of the corresponding directed graph (i.e. $(D^{-2}\alpha D^2)_{ij} \neq 0 \Leftrightarrow \alpha_{ij} \neq 0$) $G_{\tilde{\alpha}}$ is obtained from G_α by reversing all arrows. In particular, $G_{\tilde{\alpha}}$ is connected if and only if G_α is connected. From this it follows that if G_ϕ is disconnected then $G_{\tilde{\alpha}}$ is disconnected which implies that T and $T^\dagger$ has a common invariant subspace generated by a collection of the basis elements. Since both Λ and $\Lambda^\dagger$ acts diagonally on this basis, the subspace will be invariant for all operators. This contradicts the fact that ϕ is irreducible. Hence, G_ϕ must be connected. □

When the eigenvalues of $\Lambda = \phi(t_1)$ are distinct, one can easily show that all representation graphs of ϕ are isomorphic.

Proposition 3.8. *Let ϕ be a nondegenerate representation. Then any Jordan basis for Λ is a Jordan basis for Λ with respect to T up to a permutation of the basis vectors. Moreover, all representation graphs of ϕ are isomorphic.*

Proof. When all eigenvalues of Λ are distinct, the only freedom in choosing a basis in which Λ is diagonal lies in the length of the eigenvectors and the ordering of the basis vectors. Hence, given two Jordan bases for Λ it is always possible to apply a permutation to obtain one basis from the other, up to a rescaling of the vectors. Furthermore, a rescaling of the basis vectors does not change the block diagonal form of a matrix. Hence, any Jordan basis of Λ is a Jordan basis of Λ with respect to T up to a permutation. In particular, this implies that any two representation graphs are related by a permutation of the vertices. □

Proposition 3.8 has the consequence that if one constructs a representation ϕ, in which Λ is diagonal and has distinct eigenvalues, then the directed graph of T is the unique representation graph of ϕ.

Let us now study the representation graph of the tensor product. For directed graphs, forming the tensor product

$$\hat{T} = T \otimes \mathbb{1} + \mathbb{1} \otimes T'$$

amounts to taking the *Cartesian product* of G_T and $G_{T'}$ [Sab60, HT66]. The Cartesian product of two graphs $G = (V, E)$ and $H = (U, F)$ is defined as the graph $G' = (V \times U, E')$ such that

$$\begin{aligned}&\big((v_1, u_1), (v_2, u_2)\big) \in E' \quad \Longleftrightarrow \\ &\big\{v_1 = v_2 \text{ and } (u_1, u_2) \in F\big\} \text{ or } \big\{u_1 = u_2 \text{ and } (v_1, v_2) \in E\big\}.\end{aligned}$$

Now, one might ask if the Cartesian product of two representation graphs is a representation graph of the tensor product? This is not always true, but we have the following result.

Proposition 3.9. *Let ϕ and ϕ' be representations such that $\phi \otimes \phi'$ is a nondegenerate representation. Then ϕ and ϕ' are nondegenerate and $G_{\phi\otimes\phi'}$ is the Cartesian product of G_ϕ and $G_{\phi'}$.*

Proof. Let $\Lambda = \phi(t_1)$ and $\Lambda' = \phi'(t_1)$ and let P and Q be matrices whose column vectors are Jordan bases of Λ and Λ' with respect to $T = \phi(t_2)$ and $T' = \phi'(t_2)$. By assumption, the eigenvalues of $\hat{\Lambda} = \Lambda \otimes \mathbb{1} + \mathbb{1} \otimes \Lambda'$ are distinct, which implies that the eigenvalues of the matrix

$$M = \big(P \otimes Q\big)^{-1}\Big[\Lambda \otimes \mathbb{1} + \mathbb{1} \otimes \Lambda'\Big]\big(P \otimes Q\big) = \big(P^{-1}\Lambda P\big) \otimes \mathbb{1} + \mathbb{1} \otimes \big(Q^{-1}\Lambda' Q\big)$$

are distinct. Since $P^{-1}\Lambda P$ and $Q^{-1}\Lambda' Q$ are upper triangular (and hence has their eigenvalues on the diagonal) the matrix M will also be upper triangular. The diagonal elements of M (its eigenvalues) will be all possible sums of eigenvalues from Λ and Λ'. Since the eigenvalues of M are distinct the eigenvalues of Λ and Λ' must be distinct. This proves the first part of the statement.

Since Λ and Λ' has distinct eigenvalues, the matrices $P^{-1}\Lambda P$ and $Q^{-1}\Lambda' Q$ are in fact diagonal, which implies that the matrix M is diagonal, so $P \otimes Q$ clearly provides us with a Jordan basis for $\hat{\Lambda}$. Since $\phi \otimes \phi'$ is nondegenerate, Proposition 3.8 tells us that the (unique) representation graph is given by the directed graph of

$$\big(P \otimes Q\big)^{-1}\Big[T \otimes \mathbb{1} + \mathbb{1} \otimes T'\Big]\big(P \otimes Q\big) = \big(P^{-1}TP\big) \otimes \mathbb{1} + \mathbb{1} \otimes \big(Q^{-1}T' Q\big)$$

Now, since $P^{-1}TP$ and $Q^{-1}T'Q$ defines G_ϕ and $G_{\phi'}$, we conclude that $G_{\phi\otimes\phi'}$ is given by the Cartesian product of G_ϕ and $G_{\phi'}$. $\square$

Let us now proceed to construct representations of $\mathfrak{U}_4(\mathcal{A})$. As noted earlier, even the case when $\mu_1 = \ldots = \mu_4$ is expected to have a rich representation theory. Therefore, we will start by concentrating on the case for which $\mu_1 = \mu_2 = \mu$ and $\mu_3 = \mu_4 = \rho$ (which is also relevant for applications, as mentioned in the introduction). In this case, representations are found by solving the matrix equations

$$2\mu\Lambda = \big[[\Lambda, T], T^\dagger\big] + \big[[\Lambda, T^\dagger], T\big] + \big[[\Lambda, \Lambda^\dagger], \Lambda\big] \tag{3.2}$$

$$2\rho T = \big[[T, \Lambda], \Lambda^\dagger\big] + \big[[T, \Lambda^\dagger], \Lambda\big] + \big[[T, T^\dagger], T\big]. \tag{3.3}$$

The action of the group $O(2) \times O(2)$ can be explicitly realized by letting $\Lambda \to e^{i\theta}\Lambda$ and $T \to e^{i\theta'}T$, which gives a new representation for any $\theta, \theta' \in \mathbb{R}$; this representation will be deoted by $\phi_{\theta\theta'}$ and is in general not equivalent to ϕ since the eigenvalues of e.g. Λ will be different. This enables us to construct new irreducible representations from a given one via the tensor product. Namely, let ϕ be a nondegenerate irreducible representation; then one can always choose θ, θ' such that $\phi \otimes \phi_{\theta\theta'}$ is a nondegenerate representation. By Proposition 3.9 the representation graph of $\phi \otimes \phi_{\theta\theta'}$ will be the Cartesian product of two connected graphs (the representation graphs of ϕ and $\phi_{\theta\theta'}$), which implies that it is connected [HT66]. Hence, it follows from Proposition 3.5 that $\phi \otimes \phi_{\theta\theta'}$ is irreducible.

3.1. **The Fuzzy sphere.** As any semi-simple Lie algebra is itself a DMSA, it follows that hermitian representations of $\mathfrak{su}(2)$ should induce hermitian representations of $\mathfrak{U}_4(\mathcal{A})$. Indeed, choosing hermitian $n\times n$ matrices S_1, S_2, S_3, with non-zero elements

$$\begin{aligned}
\big(S_1\big)_{k,k+1} &= \frac{1}{2}\sqrt{k(n-k)} = \big(S_1\big)_{k+1,k} \qquad k=1,\dots,n-1\\
\big(S_2\big)_{k,k+1} &= -\frac{i}{2}\sqrt{k(n-k)} = -\big(S_2\big)_{k+1,k} \qquad k=1,\dots,n-1\\
\big(S_3\big)_{k,k} &= \frac{1}{2}(n+1-2k) \qquad k=1,\dots,n,
\end{aligned}$$

satisfying $[S_i,S_j]=i\epsilon_{ijk}S_k$, yields a representation ϕ by defining

$$\Lambda = \phi(t_1) = e^{i\theta}S_3. \tag{3.4}$$

$$T = \phi(t_2) = S_1 + iS_2 \tag{3.5}$$

for any $\theta\in\mathbb{R}$ (no θ' appears in $\phi(t_2)$ since it can always be removed by conjugating with a diagonal unitary matrix). One easily calculates that this is a representation of $\mathfrak{U}_4(\mathcal{A})$ with $\mathrm{Spec}(\mathcal{A})=\{2\}$. Note that in the special case when $\theta=0$, in which case Λ is hermitian, this provides a representation of $\mathfrak{U}_3(\mathcal{A})$.

This is a nondegenerate semi-normal representation, and the representation graph takes the form as in Figure 1.

FIGURE 1. The representation graph of the Fuzzy Sphere.

Furthermore, this representation is irreducible by Proposition 3.5, and its representation graph is prime with respect to the Cartesian product since any Cartesian product graph with n vertices has at least n edges (whereas the above graph has $n-1$ edges). For increasing n, the algebras that are generated by these matrices (with an appropriate normalization) are recognized as a sequence converging to the Poisson algebra of functions on S^2 [Hop82].

Let us for this case demonstrate the tensor product and construct the corresponding Cartesian product of the representation graphs. For simplicitly, we let ϕ_2 and ϕ_3 be a two- respectively three-dimensional representation of the type described above, and set

$$\begin{aligned}
\phi(t_1) &= \phi_3(t_1)\otimes \mathbb{1}_2 + \mathbb{1}_3\otimes\phi_2(t_1)\\
\phi(t_2) &= \phi_3(t_2)\otimes \mathbb{1}_2 + \mathbb{1}_3\otimes\phi_2(t_2).
\end{aligned}$$

If we denote the arbitrary phases by θ_2 and θ_3 the representation graph takes the form as in Figure 2.

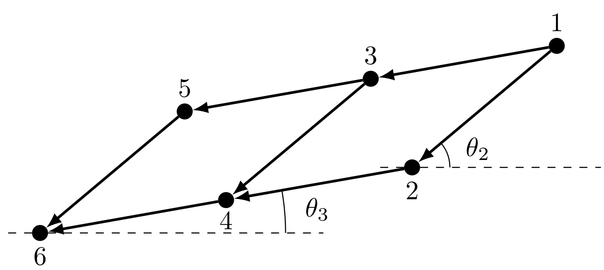

FIGURE 2. The representation graph of a tensor product of two Fuzzy sphere representations.

Since this representation is non-degenerate it will be irreducible by Proposition 3.5, and in general we obtain inequivalent representations for different choices of θ_2 and θ_3.

We note that the matrices S_1, S_2, S_3 gives rise to another representation by setting

$$\Lambda = zS_3 \tag{3.6}$$

$$T = w(S_1 + aS_2) \tag{3.7}$$

for arbitrary $z, w \in \mathbb{C}$ and $a \in \mathbb{R}$. This gives a representation of $\mathfrak{U}_4(\mathcal{A})$ with $\mathrm{Spec}(\mathcal{A}) = \{|z|^2, |w|^2(1+a^2)\}$. The representation graph can be seen in Figure 3.

FIGURE 3. The representation graph of a normal representation constructed from $\mathfrak{su}(2)$.

We conclude that this is an irreducible nondegenerate normal representation, which is not equivalent to the Fuzzy sphere, since the corresponding graphs are not isomorphic. Moreover, its representation graph is prime with respect to the Cartesian product.

3.2. **The Fuzzy torus.** The fuzzy torus algebra (cp. [FFZ89, Hop89]) is generated by the matrices g and h, with non-zero elements

$$(h)_{k,k+1} = 1, \quad (h)_{n,1} = 1 \qquad\qquad k = 1, \dots, n-1$$
$$(g)_{kk} = q^{k-1} \qquad\qquad k = 1, \dots, n,$$

fulfilling the relation $hg = q \cdot gh$ with $q^n = 1$. It is known that they generate matrix sequences that converge to functions on T^2. A representation ϕ of $\mathfrak{U}_4(\mathcal{A})$, with $\mathrm{Spec}(\mathcal{A}) = \{|1-q|^2/2\}$, is obtained by setting

$$\phi(T) = e^{i\theta'}h$$
$$\phi(\Lambda) = e^{i\theta}g,$$

for any $\theta, \theta' \in \mathbb{R}$. This is an irreducible nondegenerate unitary representation, with a representation graph as in Figure 4.

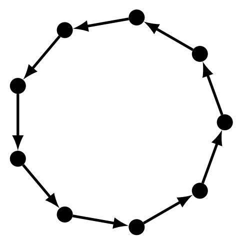

FIGURE 4. The representation graph of the Fuzzy torus.

Furthermore, this graph is prime with respect to the Cartesian product. Let us now show that this is essentially the only irreducible unitary representation when $\mathrm{Spec}(\mathcal{A}) = \{\mu\}$.

3.3. **Unitary representations.** When Λ and T are unitary and $\operatorname{Spec}(\mathcal{A}) = \{\mu\}$, the equations can be written as

$$\lambda\Lambda = T^\dagger \Lambda T + T\Lambda T^\dagger \tag{3.8}$$

$$\lambda T = \Lambda^\dagger T\Lambda + \Lambda T\Lambda^\dagger, \tag{3.9}$$

where we have introduced $\lambda = 2 - \mu$. By multiplying the first equation from the right by T and the second equation from the left by Λ we note that $[\Lambda T, T\Lambda] = 0$. Hence, by a unitary transformation, we can always choose a basis such that $D = \Lambda T$ and $\tilde{D} = T\Lambda$ are diagonal. Let us denote the eigenvalues by

$$\begin{aligned} D &= \operatorname{diag}(d_1, \dots, d_n) = \operatorname{diag}\left(e^{i\varphi_1}, \dots, e^{i\varphi_n}\right) \\ \tilde{D} &= \operatorname{diag}(\tilde{d}_1, \dots, \tilde{d}_n) = \operatorname{diag}\left(e^{i\tilde{\varphi}_1}, \dots, e^{i\tilde{\varphi}_n}\right). \end{aligned}$$

The equations (3.8) and (3.9) (together with $D = \Lambda T$ and $\tilde{D} = T\Lambda$) can equivalently be written as

$$\lambda\Lambda = \Lambda\tilde{D}^\dagger D + \tilde{D}\Lambda\tilde{D}^\dagger \tag{3.10}$$

$$\lambda\Lambda\tilde{D}^\dagger = \tilde{D}^\dagger\Lambda + \Lambda D^\dagger \tag{3.11}$$

$$\Lambda\tilde{D} = D\Lambda \tag{3.12}$$

$$T = \tilde{D}\Lambda^\dagger. \tag{3.13}$$

Thus, given unitary $\Lambda, D, \tilde{D}$ satisfying (3.10)–(3.12), we obtain a solution to the original equations by defining $T = \tilde{D}\Lambda^\dagger$. Written out in components, the three first equations become

$$\Lambda_{ij}\left[\lambda - \bar{\bar{\tilde{d}}}_j d_j - \tilde{d}_i \bar{\bar{\tilde{d}}}_j\right] = 0 \tag{3.14}$$

$$\bar{\Lambda}_{ij}\left[\lambda\tilde{d}_j - \tilde{d}_i - d_j\right] = 0 \tag{3.15}$$

$$\Lambda_{ij}\left[\tilde{d}_j - d_i\right] = 0. \tag{3.16}$$

If $\Lambda_{ij} \neq 0$ then we obtain the following relations

$$\begin{pmatrix} d_j \\ \tilde{d}_j \end{pmatrix} = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d_i \\ \tilde{d}_i \end{pmatrix} \equiv s \begin{pmatrix} d_i \\ \tilde{d}_i \end{pmatrix} \tag{3.17}$$

since (3.15) and (3.16) together imply (3.14). Now, consider the directed graph $G_\Lambda = (V, E)$ of Λ, where we have assigned the vector $\vec{x}_i = (d_i, \tilde{d}_i)$ to each vertex $i \in V$. We can restrict ourselves to connected graphs, since if G_Λ is disconnected then the representation will trivially be reducible. The above considerations tell us that whenever there is an edge $(i,j) \in E$, it must hold that $\vec{x}_j = s(\vec{x}_i)$. In particular, since D and $\tilde{D}$ are unitary matrices, the map s must take $\vec{x} \in S^1 \times S^1$ to another vector in $S^1 \times S^1$. If $\vec{x} = (e^{i\varphi}, e^{i\tilde{\varphi}})$ then this is true only if $\lambda = 0$ or

$$\lambda = 2\cos(\varphi - \tilde{\varphi}). \tag{3.18}$$

This observation leads to the following result.

Proposition 3.10. *Let $\mathcal{A}$ be a DMSA with $\operatorname{Spec}(\mathcal{A}) = \{\mu\}$. If $\mu < 0$ or $\mu > 4$ then there exists no unitary representation of $\mathfrak{U}_4(\mathcal{A})$.*

Proof. Assume that Λ and T provides a unitary representation and that a basis has been chosen in which D and $\tilde{D}$ are diagonal. Since Λ is a unitary matrix at least one of its matrix element has to be non-zero, say $\Lambda_{ij} \neq 0$. Then equation (3.18) must hold, which is impossible if $\lambda < -2$ or $\lambda > 2$. □

From the above result it follows that whenever a unitary representation exists, then there exists a $\beta \in [0, \pi/2]$ such that $\lambda = 2\cos 2\beta$. We will now proceed in analogy with the proofs in [ABH$^+$09, Arn08b] to which we refer for details.

Let us start by studying the case when $\lambda \neq 0$, and let $\vec{x} = (e^{i\varphi}, e^{i\tilde{\varphi}})$ be a vector such that $\varphi - \tilde{\varphi} = \pm 2\beta$. Then it is easy to calculate that $s(\vec{x}) = (e^{i(\varphi \pm 2\beta)}, e^{i\varphi})$. Thus, the maps s preserves the condition (3.18) provided that we start with a vector fulfilling the condition.

This implies that if G_Λ has a loop (i.e. a directed cycle) on k vertices, then we must have that $s^k(\vec{x}_i) = \vec{x}_i$ for some $i \in V$. Moreover, it is a trivial fact that every directed graph of a unitary matrix must have a loop. Hence, β must be such that $e^{i2k\beta} = 1$ for some integer $k > 0$. Since the map s is invertible, given any $\vec{x}_i$ in the graph uniquely determines $\vec{x}_j$ for all other vertices in the graph. Hence, we can partition the vertices into subsets $V_1, \dots, V_k$ such that $\vec{x}_i = \vec{x}_j$ if and only if $i, j \in V_l$ for some $l \in \{1, \dots, k\}$. It follows that all edges of G_Λ are of the form (i, j) with $i \in V_l$ and $j \in V_{l+1}$ (where we identify $k + 1 \equiv 1$). Thus, we can permute the vertices to bring the matrix Λ to the following form

$$\Lambda = \begin{pmatrix} 0 & \Lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \Lambda_{k-1} \\ \Lambda_k & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with Λ_i being unitary matrices for $i = 1, \dots, k$. Moreover, there exists a unitary matrix such that $U^\dagger \Lambda U$ is of the above form but each Λ_i is diagonal. This means that the directed graph of $U^\dagger \Lambda U$ is a direct sum of k loops (this also holds for T since $T = \tilde{D}\Lambda^\dagger$), and each of these loops correspond to an irreducible representation. However, to caculate the representation graph, we must go to the basis in which Λ is diagonal. The matrix corresponding to a single loop on n vertices has the n roots of unity as eigenvalues. Therefore, in the basis in which Λ is diagonal, the directed graph of T will be the representation graph. It is easy to see that the matrices D and $\tilde{D}$ will act as shift operators on the eigenvectors of Λ, which implies that $T = \tilde{D}\Lambda^\dagger$ will also act as a shift operator in this basis. Thus, the directed graph of T will be a single loop. We conclude that this representation is precisely the Fuzzy torus representation presented above.

Let us turn to the case when $\lambda = 0$, i.e. $\mu = 2$. In this case, there is no restriction on $\varphi - \tilde{\varphi}$, but instead one notes that $s^4(\vec{x}) = \vec{x}$ for any $\vec{x} \in \mathbb{C}^2$. Thus, the vertices of G_Λ can be split into four disjoint subsets, and we conclude that all irreducible representations are 4-dimensional. However, since $\varphi - \tilde{\varphi}$ does not have to be related to β, the action of T on the eigenbasis of Λ will not simply be a shift. Therefore, the representation graph will be one of the two in Figure 5.

When $\lambda = 2$ ($\mu = 0$) then $\varphi = \tilde{\varphi}$ and we see that any vector of the form $(e^{i\varphi}, e^{i\varphi})$ is a fixpoint of s. Hence, all irreducible representations are 1-dimensional. This agrees with the result in [HS08] which states that, when $\operatorname{Spec}(\mathcal{A}) = \{0\}$, all irreducible finite dimensional representations of $\mathfrak{U}_d(\mathcal{A})$ are 1-dimensional.

Proposition 3.11. *Assume that $\mu \neq 2$ and let $\mathcal{A}$ be a DMSA with* $\operatorname{Spec}(\mathcal{A}) = \{\mu\}$. *If ϕ is an n-dimensional irreducible unitary representation of $\mathfrak{U}_4(\mathcal{A})$ then ϕ is equivalent to a representation ϕ' with $\phi'(t_1) = e^{i\theta} g$ and $\phi'(t_2) = e^{i\theta'} h$ for some $\theta, \theta' \in \mathbb{R}$. Moreover, there exists a $\beta \in \mathbb{R}$ such that $\mu = 4\sin^2(\beta)$ and $e^{i2n\beta} = 1$.*

Proposition 3.12. *Let $\mathcal{A}$ be a DMSA with* $\operatorname{Spec}(\mathcal{A}) = \{2\}$ *and let ϕ be an irreducible unitary representation of $\mathfrak{U}_4(\mathcal{A})$. Then ϕ is 4-dimensional and the representation graph of ϕ is one of the two in Figure 5.*

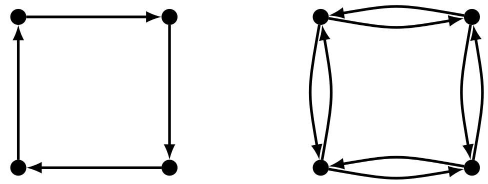

FIGURE 5. The two different types of representation graphs for irreducible unitary representations when $\mu_1 = \cdots = \mu_4 = 2$.

The graph to the right in Figure 5 is the Cartesian product of two representation graphs corresponding to 2-dimensional representations defined by (3.6) and (3.7). However, one can check that there are 4-dimensional unitary representations that can not be written as a tensor product of two such representations.

3.4. **Representations induced by $\mathfrak{sl}_3$.** We will now present a DMSA $\mathcal{A}$ constructed from $\mathfrak{sl}_3$, whose representations give rise to normal representations of $\mathfrak{U}_4(\mathcal{A})$. The vertices of the representation graph will be the weight diagram of the $\mathfrak{sl}_3$-representation.

Let α and β be the simple roots of $\mathfrak{sl}_3$. By setting

$$t_1 = e^{i\theta}\Big(h_\alpha + e^{i\pi/3}h_\beta\Big) \qquad s_1 = e^{-i\theta}\Big(h_\alpha + e^{-i\pi/3}h_\beta\Big)$$
$$t_2 = e^{i\theta'}\Big(e_\alpha + e^{i\varphi_1}e_\beta + e^{i\varphi_2}e_{-\alpha-\beta}\Big)$$
$$s_2 = e^{-i\theta'}\Big(e_{-\alpha} + e^{-i\varphi_1}e_{-\beta} + e^{-i\varphi_2}e_{\alpha+\beta}\Big)$$

we obtain a DMSA with $[t_1, s_1] = [t_2, s_2] = 0$ and

$$\big[[t_1, t_2], s_2\big] = \frac{3}{2}l^2 t_1 \qquad \big[[s_1, t_2], s_2\big] = \frac{3}{2}l^2 s_1$$
$$\big[[t_2, t_1], s_1\big] = \frac{3}{4}l^4 t_2 \qquad \big[[s_2, t_1], s_1\big] = \frac{3}{4}l^4 s_2.$$

In the current convention, a compact real form of $\mathfrak{sl}_3$ is provided by

$$ih_\alpha, ih_\beta, e^+_\alpha, e^-_\alpha, e^+_\beta, e^-_\beta, e^+_{\alpha+\beta}, e^-_{\alpha+\beta}$$

as defined in Lemma 2.7. Hence, any representation is equivalent to one where these elements are represented by *anti-hermitian* matrices, which implies that $\phi(e_\gamma)^\dagger = \phi(e_{-\gamma})$ and $\phi(h_\gamma)^\dagger = \phi(h_\gamma)$ for $\gamma = \alpha, \beta, \alpha+\beta$. It follows that $\phi(t_1)^\dagger = \phi(s_1)$ and $\phi(t_2)^\dagger = \phi(s_2)$. The weight diagram of a representation is usually presented as vectors with repsect to an orthonormal basis of the Cartan subalgebra. In $\mathfrak{sl}_3$ we can construct an orthonormal basis by setting

$$h_1 = \frac{1}{l}h_\alpha \qquad h_2 = \frac{1}{l\sqrt{3}}(h_\alpha + 2h_\beta),$$

and from this we calculate that

$$t_1 = \frac{l\sqrt{3}}{2}e^{i(\theta-\pi/6)}(h_1 + ih_2).$$

Hence, the eigenvalues of $\phi(t_1)$ will be the weights of the (scaled and rotated) weight diagram of the representation ϕ. As an example, let us study the representations of the kind $\{n, 0\}$, i.e. representations of highest weight nw_1, where w_1, w_2 are the fundamental weights. These representations have dimension $(n+1)(n+2)/2$ and all weights have multiplicity one. Therefore, in a representation ϕ, where the elements of the Cartan subalgebra are diagonal, the representation graph is given by the directed graph of $T = \phi(t_2)$. Since t_2 is a linear combination of e_α, e_β and $e_{-\alpha-\beta}$

we can construct the representation graph by drawing arrows in the direction of these roots in the weight diagram, see Figure 6.

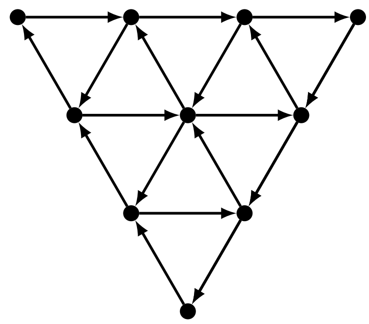

FIGURE 6. The representation graph corresponding to the $\{3, 0\}$ representation of $\mathfrak{sl}_3$.

4. Summary

Motivated by several examples, in which double commutator matrix equations arise, we have considered the relations

$$\sum_{j=1}^{d} \big[[x_i, x_j], x_j\big] = \mu_i x_i \tag{4.1}$$

in a general (Lie) algebraic setting. Some examples can easily be constructed from subsets of semi-simple Lie algebras. Via the Diamond lemma we can show that it is consistent to impose relations (4.1) in a free associative algebra, and a basis for the corresponding universal enveloping algebra was computed.

In contrast to the case when $\mu_i = 0$ for $i = 1, \dots, d$ (in which case all irreducible finite-dimensional representations are one dimensional [HS08]), the representation theory for aribtrary μ_i's has a rich structure. We have considered the case when $d \leq 4$ in detail and introduced the *representation graph*, which encodes the structure of a finite-dimensional representation in terms of a directed graph. The connectivity of the graph provides information on the irreducible components of the representation, and the tensor product can (generically) be described by the Cartesian product of graphs. All unitary representations when $\mathrm{Spec}(\mathcal{A}) = \{\mu\}$ were then classified, and it was shown that essentially all such representations are equivalent to the Fuzzy Torus algebra. Several other examples were provided to demonstrate that the representation theory is non-trivial. A particular feature, that in general distinguishes the representation theory from that of Lie algebras, is that the tensor product of two irreducible representations can again be irreducible.

While we think relations (4.1) are interesting in themselves – as a class of algebras containing all semi-simple Lie algebras – let us stress a particular application. Namely, we expect matrix sequences corresponding to surfaces of genus $g \geq 2$ to exist, even for $d \leq 4$ and $\mu_1 = \ldots = \mu_4$.

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