# Integrable Systems in Noncommutative Spaces

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# Integrable Systems in Noncommutative Spaces

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**Abstract**

We discuss extension of soliton theories and integrable systems into non-commutative spaces. In the framework of NC integrable hierarchy, we give infinite conserved quantities and exact soliton solutions for many NC integrable equations, which are represented in terms of Strachan's products and quasi-determinants, respectively. We also present a relation to an NC Anti-Self-Dual Yang-Mills equation, and make comments on how "integrability" should be considered in noncommutative spaces.

# 1 Introduction

Non-Commutative (NC) extension of field theories is not just a generalization of them but a fruitful study direction in both physics and mathematics. First of all, we introduce motivation and goal of it.

## 1.1 Motivation to extend to NC spaces

Noncommutative spaces are characterized by the noncommutativity of the spatial coordinates:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{1.1}$$

where the anti-symmetric tensor $\theta^{\mu\nu}$ is called the *NC parameter*. In this paper, the NC parameter is a real constant and closely related to existence of a background flux.

We summarize some properties of field theories on NC spaces.

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- Resolution of singularities

  Eq. (1.1) looks like the canonical commutation relation $[q, p] = i\hbar$ in quantum mechanics and would lead to "space-space uncertainty relation." Hence the singularity which exists on commutative spaces could resolve on noncommutative spaces. This is one of the distinguished features of noncommutative theories and gives rise to various new physical objects such as $U(1)$ instantons [43].

- Equivalence between NC gauge theory and commutative gauge theory in background magnetic fields

  In the context of effective theory of D-branes, NC gauge theories are found to be equivalent to ordinary gauge theories in the presence of background magnetic fields and have been studied intensively for the last several years (For reviews, see e.g. [8, 32, 53].) NC solitons especially play important roles in the study of D-brane dynamics, such as the confirmation of Sen's conjecture on tachyon condensation. (For reviews, see e.g. [17, 29, 51].) We note that $U(1)$ part of the gauge group is necessary and plays important roles as in $U(1)$ instantons.

- Easy Treatment

  Solitons special to noncommutative spaces are sometimes so simple that we can calculate various physical quantities, such as the energy, the fluctuation around the soliton configuration and so on. Because of resolution of singularities, singular configurations becomes smooth and become suitable for the usual calculation. Furthermore, we can take large noncommutativity limit where the situations become simple. The successful application to D-brane dynamics are actually due to this point.

## 1.2 Towards NC Integrable Systems

NC extension of integrable equations such as the KdV equation is also one of the hot topics. These equations imply no gauge field and NC extension of them perhaps might have no physical picture or no good property on integrabilities. To make matters worse, NC extension of $(1+1)$-dimensional equations introduces infinite number of time derivatives, which makes it hard to discuss or define the integrability. Those equations had been examined one by one. Now it was time to discuss the geometrical and physical origin of the special properties and integrabilities, in more general framework.

We proposed the following study programs as future directions:

- NC twistor theory together with NC Ward's conjecture

Twistor theory is the most essential framework in the study of integrability of ASDYM eqs. (See, e.g. [39, 59].) NC extension of twistor theories are already discussed by several authors, e.g. [3, 28, 30, 31, 54]. This would give a geometrical foundation of integrabilities of ASDYM eqs.

NC Ward's conjecture is very important to give physical pictures to lower-dimensional integrable equations and to make it possible to apply analysis of NC solitons to that of the corresponding D-branes. Origin of the integrable-like properties would be also revealed from the viewpoints of NC twistor theory and preserved supersymmetry in the D-brane systems.

- NC Sato's theory

  Sato's theory is known to be one of the most beautiful theories of solitons and reveals essential aspects of the integrability, such as, the construction of exact multi-soliton solutions, the structure of the solution space, the existence of infinite conserved quantities, and the hidden symmetry of them, on commutative spaces. So it is reasonable to extend Sato's theory onto NC spaces in order to clarify various integrable-like aspects directly.

In this article, we report recent developments of NC extension of soliton theories and integrable systems. We prove the existence of infinite conserved quantities and exact multi-soliton solutions in the framework of NC integrable hierarchy. We also give an example of reduction of NC Anti-Self-Dual Yang-Mills equation into NC KdV eq.

## 1.3 NC Field Equations in the sense of Moyal deformations

NC field theories are given by the replacement of ordinary products in the commutative field theories with the *star-products* and realized as deformed theories from the commutative ones. The star-product is defined for ordinary fields on flat spaces, explicitly by

$$\begin{aligned} f \star g(x) &:= \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^{(x')}\partial_\nu^{(x'')}\right) f(x')g(x'')\Big|_{x'=x''=x} \\ &= f(x)g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x) + \mathcal{O}(\theta^2), \end{aligned} \tag{1.2}$$

where $\partial_i^{(x')} := \partial/\partial x'^i$ and so on. This explicit representation is known as the *Moyal product* [41].

The star-product has associativity: $f\star(g\star h) = (f\star g)\star h$, and reduces to the ordinary product in the commutative limit: $\theta^{\mu\nu} \to 0$. The modification of the product makes the ordinary spatial coordinate "noncommutative" which means : $[x^\mu, x^\nu]_\star := x^\mu\star x^\nu - x^\nu\star x^\mu = i\theta^{\mu\nu}$.

We note that the fields themselves take c-number values as usual and the differentiation and the integration for them are well-defined as usual. A nontrivial point is that NC field equations contain infinite number of derivatives in general. Hence the integrability of the equations are not so trivial as commutative cases, especially for space-time noncommutativity.

In this article, we mainly studies NC KP and KdV equations:

- NC KP equation in $(2+1)$-dimension ($[x,y]_\star = i\theta$ or $[t,x]_\star = i\theta$)

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^3 u}{\partial x^3} + \frac{3}{4}\left(\frac{\partial u}{\partial x}\star u + u \star \frac{\partial u}{\partial x}\right) + \frac{3}{4}\partial_x^{-1}\frac{\partial^2 u}{\partial y^2} - \frac{3}{4}\left[u, \partial_x^{-1}\frac{\partial u}{\partial y}\right]_\star, \tag{1.3}$$

  where $t$ and $x, y$ are time and spatial coordinates, respectively, and $\partial_x^{-1} f(x) = \int^x dx' f(x')$.

- NC KdV equation in $(1+1)$-dimension ($[t,x]_\star = i\theta$)

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^3 u}{\partial x^3} + \frac{3}{4}\left(\frac{\partial u}{\partial x}\star u + u \star \frac{\partial u}{\partial x}\right). \tag{1.4}$$

The ordering of non-linear terms is crucial to preserve some special integrable properties and determined in the Lax formalism as we will see later. For NC KP and KdV eqs., the non-linear term $2u \cdot \partial_x u$ becomes symmetric: $\partial_x u \star u + u \star \partial_x u$.

# 2 NC Integrable Systems

In this section, we discuss some integrable aspects of NC integrable equations focusing on NC KdV eq.

## 2.1 NC integrable Hierarchies

Firstly, we derive various NC integrable equations in terms of pseudo-differential operators which include negative powers of differential operators.

An $n$-th order pseudo-differential operator $A$ is represented as follows

$$A = a_n \partial_x^n + a_{n-1}\partial_x^{n-1} + \cdots + a_0 + a_{-1}\partial_x^{-1} + a_{-2}\partial_x^{-2} + \cdots, \tag{2.1}$$

where $a_i$ is a function of $x$ associated with noncommutative associative products (here, the Moyal products). When the coefficient of the highest order $a_N$ equals to 1, we call it *monic*. Here we introduce useful symbols:

$$\begin{aligned} A_{\geq r} &:= \partial_x^n + a_{n-1}\partial_x^{n-1} + \cdots + a_r\partial_x^r, && (2.2)\\ A_{\leq r} &:= A - A_{\geq r+1} = a_r\partial_x^r + a_{r-1}\partial_x^{r-1} + \cdots. && (2.3)\\ \text{res}_r A &:= a_r. && (2.4)\end{aligned}$$

The symbol $\text{res}_{-1}A$ is especially called the *residue* of $A$.

The action of a differential operator $\partial_x^n$ on a multiplicity operator $f$ is formally defined as the following generalized Leibniz rule:

$$\partial_x^n \cdot f := \sum_{i\geq 0} \begin{pmatrix} n \\ i \end{pmatrix} (\partial_x^i f)\partial^{n-i}, \tag{2.5}$$

where the binomial coefficient is given by

$$\begin{pmatrix} n \\ i \end{pmatrix} := \frac{n(n-1)\cdots(n-i+1)}{i(i-1)\cdots 1}. \tag{2.6}$$

We note that the definition of the binomial coefficient (2.6) is applicable to the case for negative $n$, which just define the action of negative power of differential operators.

The composition of pseudo-differential operators is also well-defined and the total set of pseudo-differential operators forms an operator algebra. For a monic pseudo-differential operator $A$, there exist the unique inverse $A^{-1}$ and the unique $m$-th root $A^{1/m}$ which commute with $A$. (These proofs are all the same as commutative ones.) For more on pseudo-differential operators and integrable hierarchies, see e.g. [33, 2, 4, 1].

In order to define the NC KP hierarchy, let us introduce a Lax operator:

$$L = \partial_x + u_2\partial_x^{-1} + u_3\partial_x^{-2} + u_4\partial_x^{-3} + \cdots, \quad u_k = u_k(x; x_1, x_2, x_3, \ldots). \tag{2.7}$$

The noncommutativity is introduced into the coordinates $(x_1, x_2, \ldots)$. The differential operator $B_m$ is given by

$$B_m := (\underbrace{L \star \cdots \star L}_{m \text{ times}})_{\geq 0}. \tag{2.8}$$

The NC KP hierarchy is defined as

$$\partial_m L = [B_m, L]_\star, \quad m = 1, 2, \ldots, \tag{2.9}$$

where the action of $\partial_m := \partial/\partial x_m$ on the pseudo-differential operator $L$ should be interpreted to be coefficient-wise, that is, $\partial_m L := [\partial_m, L]_\star$ or $\partial_m \partial_x^k = 0$. The KP hierarchy gives rise to a set of infinite differential equations with respect to infinite kind of fields from the coefficients in Eq. (2.9) for a fixed $m$. Hence it contains huge amount of differential (evolution) equations for all $m$. The LHS of Eq. (2.9) becomes $\partial_m u_k$ which shows a kind of flow in the $x_m$ direction.

If we put the constraint $(L^l)_{\leq -1} = 0$ or equivalently $L^l = B_l$ on the NC KP hierarchy (2.9), we get a reduced NC KP hierarchy which is called the *l-reduction* of the NC KP hierarchy, or the *NC lKdV hierarchy*, or the $l$-th *NC Gelfand-Dickey hierarchy*. We can easily show

$$\frac{\partial u_k}{\partial x^{Nl}} = 0, \tag{2.10}$$

for all $N, k$ because $\partial L^l/\partial x^{Nl} = [B_{Nl}, L^l]_\star = [(L^l)^N, L^l]_\star = 0$, which implies Eq. (2.10). In particular, the 2-reduction of the NC KP hierarchy is just the NC KdV hierarchy.

Let us see explicit examples.

- NC KP hierarchy

  The coefficients of each powers of (pseudo-)differential operators in the NC KP hierarchy (2.9) yield a series of infinite NC "evolution equations." For example,

  - for $m = 1$

    $$\partial_x^{1-k}) \quad \partial_1 u_k = \partial_x u_k, \quad k = 2, 3, \ldots, \tag{2.11}$$

    which implies $x^1 \equiv x$.

  - for $m = 2$

    $$\begin{array}{lrcl} \partial_x^{-1}) & \partial_2 u_2 &=& u_2'' + 2u_3', \\ \partial_x^{-2}) & \partial_2 u_3 &=& u_3'' + 2u_4' + 2u_2 \star u_2' + 2[u_2, u_3]_\star, \\ \partial_x^{-3}) & \partial_2 u_4 &=& u_4'' + 2u_5' + 4u_3 \star u_2' - 2u_2 \star u_2'' + 2[u_2, u_4]_\star, \\ \partial_x^{-4}) & \partial_2 u_5 &=& \cdots, \end{array} \tag{2.12}$$

    which implies that infinite kind of fields $u_3, u_4, u_5, \ldots$ are represented in terms of one kind of field $2u_2 \equiv u$ as is seen in Eq. (2.12).

  - for $m = 3$

    $$\partial_x^{-1}) \quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4' + 3u_2' \star u_2 + 3u_2 \star u_2',$$

$$
\begin{aligned}
\partial_x^{-2}) \quad \partial_3 u_3 &= u_3''' + 3u_4'' + 3u_5' + 6u_2 \star u_3' + 3u_2' \star u_3 + 3u_3 \star u_2' + 3[u_2, u_4]_\star, \\
\partial_x^{-3}) \quad \partial_3 u_4 &= u_4''' + 3u_5'' + 3u_6' + 3u_2' \star u_4 + 3u_2 \star u_4' + 6u_4 \star u_2' \\
&\quad -3u_2 \star u_3'' - 3u_3 \star u_2'' + 6u_3 \star u_3' + 3[u_2, u_5]_\star + 3[u_3, u_4]_\star, \\
\partial_x^{-4}) \quad \partial_3 u_5 &= \cdots .
\end{aligned}
\tag{2.13}
$$

These just imply the $(2+1)$-dimensional NC KP equation [46, 33] with $2u_2 \equiv u, x^2 \equiv y, x^3 \equiv t$ and $\partial_x^{-1} f(x) = \int^x dx' f(x')$:

$$
\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^3 u}{\partial x^3} + \frac{3}{4}\frac{\partial (u \star u)}{\partial x} + \frac{3}{4}\partial_x^{-1}\frac{\partial^2 u}{\partial y^2} - \frac{3}{4}\left[u, \partial_x^{-1}\frac{\partial u}{\partial y}\right]_\star . \tag{2.14}
$$

And higher-order flow gives an infinite set of higher-order KP equations. The order of nonlinear terms are determined in this way.

- NC KdV Hierarchy (2-reduction of the NC KP hierarchy)

  Taking the constraint $L^2 = B_2 =: \partial_x^2 + u$ for the NC KP hierarchy, we get the NC KdV hierarchy. This time, the following NC hierarchy

$$
\frac{\partial u}{\partial x^m} = \left[B_m, L^2\right]_\star , \tag{2.15}
$$

  include neither positive nor negative power of (pseudo-)differential operators for the same reason as commutative case and gives rise to the $m$-th KdV equation for each $m$. For example,

  - for $m = 3$, identifying the time coordinate as $x^3 \equiv t$:

$$
\dot{u} = \frac{1}{4}u''' + \frac{3}{4}\left(u' \star u + u \star u'\right), \tag{2.16}
$$

    which is just the $(1+1)$-dimensional NC KdV equation.

  - for $m = 5$ identifying the time coordinate $x^5 \equiv t$:

$$
\dot{u} = \frac{1}{16}u''''' + \frac{5}{16}(u \star u''' + u''' \star u) + \frac{5}{8}(u' \star u' + u \star u \star u)', \tag{2.17}
$$

    which is the $(1+1)$-dimensional 5-th NC KdV equation.

  We note that the time coordinate is defined for each flow equation. This point is important for discussion on conserved quantities of NC integrable equations.

In this way, we can generate infinite set of the $l$-reduced NC KP hierarchies. More explicit examples are seen in e.g. [18]. (See also [44, 56].) The present discussion is also applicable to other NC hierarchies, such as, the NC Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy [7], the NC Toda field hierarchy [48] the NC toroidal KdV hierarchy [19] and so on.

## 2.2 Conservation Laws

Here we prove the existence of infinite conservation laws for the wide class of NC soliton equations. The existence of infinite number of conserved quantities would lead to infinite-dimensional hidden symmetry from Noether's theorem.

First we would like to comment on conservation laws of NC field equations [26]. The discussion is basically the same as commutative case because both the differentiation and the integration are the same as commutative ones in the Moyal representation.

Let us suppose the conservation law

$$\partial_t \sigma(t, x^i) = \partial_i J^i(t, x^i), \tag{2.18}$$

where $\sigma(t, x^i)$ and $J^i(t, x^i)$ are called the *conserved density* and the *associated flux*, respectively. The conserved quantity is given by spatial integral of the conserved density:

$$Q(t) = \int_{\text{space}} d^D x \sigma(t, x^i), \tag{2.19}$$

where the integral $\int_{\text{space}} dx^D$ is taken for spatial coordinates and the surface term of the integrand $J_i(t, x^i)$ is supposed to vanish.

Here let us return back to NC hierarchy. In order to discuss the conservation laws, we have to specify for what equations the conservation laws are. The specified equations possess space and time coordinates in the infinite coordinates $x_1, x_2, x_3, \cdots$. Identifying $t \equiv x^m$, we can get infinite conserved densities for the NC hierarchies as follows ($n = 1, 2, \ldots$) [18]:

$$\sigma_n = \text{res}_{-1} L^n + \theta^{im} \sum_{k=0}^{m-1} \sum_{l=0}^{k} \begin{pmatrix} k \\ l \end{pmatrix} \partial_x^{k-l} \text{res}_{-(l+1)} L^n \diamond \partial_i \text{res}_k L^m, \tag{2.20}$$

where the suffices $i$ must run in the space-time directions only. The symbol "$\diamond$" is called the *Strachan product* [52] and defined by

$$f(x) \diamond g(x) := \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} \left( \frac{1}{2} \theta^{\mu\nu} \partial_\mu^{(x')} \partial_\nu^{(x'')} \right)^{2s} f(x') g(x'') \Big|_{x'=x''=x}. \tag{2.21}$$

This is a commutative and non-associative product.

We can easily see that deformation terms appear in the second term of Eq. (2.20) in the case of space-time noncommutativity. On the other hand, in the case of space-space noncommutativity, the conserved density is given by the residue of $L^n$ as commutative case.

For examples, explicit representation of the NC KP equation is as follows:

- space-space noncommutativity $[x, y]_\star = i\theta$:

$$\sigma_n = \mathrm{res}_{-1} L^n, \tag{2.22}$$

which is essentially the same as commutative one. In this case, the equation is the first order differential equation w.r.t. time and notion of time evolution and Hamiltonian structure are well-defined as in commutative situation. In particular, the trace of a pseudo-differential operator $A$ (2.1) should be defined as $\mathrm{tr}\, A := \int dx dx^i \mathrm{res}_{-1} A$, where the integration $\int dx^i$ must be done over all spatial directions.

- space-time noncommutativity $[t, x]_\star = i\theta$:

$$\sigma_n = \mathrm{res}_{-1} L^n - 3\theta \left( (\mathrm{res}_{-1} L^n) \diamond u_3' + (\mathrm{res}_{-2} L^n) \diamond u_2' \right). \tag{2.23}$$

This time, the deformation part is non-trivial. However the meaning of the existence of infinite conserved quantities is hard to discuss because the equation contains infinite time derivatives and it is hard to discuss time evolution, Hamiltonian structure, Poisson brackets and so on. One possible direction is to find the corresponding commutative description via the Seiberg-Witten map [50].

## 2.3 Exact Soliton Solutions

Here we show the existence of exact multi-soliton solutions of NC integrable hierarchy by giving the explicit formula in terms of quasideterminants.

Let us introduce the following functions,

$$f_s(\vec{x}) = e_\star^{\xi(\vec{x};\alpha_s)} + a_s e_\star^{\xi(\vec{x};\beta_s)}, \quad \xi(\vec{x}; \alpha) = x_1\alpha + x_2\alpha^2 + x_3\alpha^3 + \cdots, \tag{2.24}$$

and $\alpha_s$, $\beta_s$ and $a_s$ are constants. Star exponential functions are defined by

$$e_\star^{f(x)} := 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \underbrace{f(x) \star \cdots \star f(x)}_{n \text{ times}}. \tag{2.25}$$

An $N$-soliton solution of the NC KP hierarchy (2.9) is given by a quasideterminant of the Wronski matrix [9]:

$$L = \Phi_N \star \partial_x \Phi_N^{-1}, \tag{2.26}$$

where

$$\Phi_N \star f \quad = \quad |W(f_1, \ldots, f_N, f)|_{N+1,N+1},$$

$$= \begin{vmatrix} f_1 & f_2 & \cdots & f_N & f \\ f_1' & f_2' & \cdots & f_N' & f' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_1^{(N-1)} & f_2^{(N-1)} & \cdots & f_N^{(N-1)} & f^{(N-1)} \\ f_1^{(N)} & f_2^{(N)} & \cdots & f_N^{(N)} & \boxed{f^{(N)}} \end{vmatrix}. \tag{2.27}$$

Definition of quasideterminants is seen in Appendix A. The Wronski matrix $W(f_1, f_2, \cdots, f_m)$ is as usual:

$$W(f_1, f_2, \cdots, f_m) := \begin{pmatrix} f_1 & f_2 & \cdots & f_m \\ f_1' & f_2' & \cdots & f_m' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(m-1)} & f_2^{(m-1)} & \cdots & f_m^{(m-1)} \end{pmatrix}, \tag{2.28}$$

where $f_1, f_2, \cdots, f_m$ are functions of $x$ and $f' := \partial f/\partial x,\ f'' := \partial^2 f/\partial x^2,\ f^{(m)} := \partial^m f/\partial x^m$ and so on.

In the commutative limit, $\Phi_N \star f$ is reduced to

$$\Phi_N \star f \longrightarrow \frac{\det W(f_1, f_2, \ldots, f_N, f)}{\det W(f_1, f_2, \ldots, f_N)}, \tag{2.29}$$

which just coincides with commutative one [4]. In this respect, quasi-determinants are fit to this framework of the Wronskian solutions.

From Eq. (2.26), we have a more explicit form as [9]:

$$u_2 = \partial_x \left( \sum_{s=1}^{N} W_s' \star W_s^{-1} \right), \quad W_s := |W(f_1, \ldots, f_s)|_{ss}. \tag{2.30}$$

The $l$-reduction condition $(L^l)_{\leq -1} = 0$ or $L^l = B_l$ is realized at the level of the soliton solutions by taking $\alpha_s^l = \beta_s^l$ or equivalently $\alpha_s = \epsilon\beta_s$ for $s = 1, \cdots, N$, where $\epsilon$ is the $l$-th root of unity. For the KdV eq., $\alpha_s = -\beta_s$.

Physical interpretation of the configurations is non-trivial because even when $f(x)$ and $g(x)$ are real, $f(x) \star g(x)$ is not in general. However, the $N$-soliton solutions can be real in the following situations.

- One-soliton solutions

  First, let us comment on one-soliton solutions [**?**, 26]. Defining $z := x+vt$, $\bar{z} := x-vt$, we easily see

$$f(z) \star g(z) = f(z)g(z) \tag{2.31}$$

because the star-product (1.2) is rewritten in terms of $(z,\bar{z})$ as

$$f(z,\bar{z})\star g(z,\bar{z}) = e^{iv\theta(\partial_{\bar{z}'}\partial_{z''}-\partial_{z'}\partial_{\bar{z}''})} f(z',\bar{z}')g(z'',\bar{z}'')\Big|_{\substack{z'=z''=z\\ \bar{z}'=\bar{z}''=\bar{z}.}} \tag{2.32}$$

Hence NC one soliton-solutions are essentially the same as commutative ones and hence can be real in all region of the space-time.

- Asymptotic region of $N$-soliton solutions

  In order to analyze the asymptotic behavior of $N$-soliton solutions, we usually take a new coordinate comoving with the $I$-th soliton. Then we can see that in the asymptotic region, the configuration just coincides with the commutative one. Hence, asymptotic behavior of the multi-soliton solutions is all the same as commutative one. As the results, the $N$-soliton solutions possess $N$ localized energy densities. In the general scattering process without resonances, they never decay and preserve their shapes and velocities of the localized solitary waves. The phase shifts also occur by the same degree as commutative ones. These observations are crucially due to special properties of quasideterminants. Detailed discussion is seen in [21].

## 2.4 Reduction of NC ASDYM Eq.

Here we briefly discuss reductions of NC ASDYM equation into lower-dimensional NC integrable equations such as the NC KdV equation. let us summarize the strategy for reductions of NC ASDYM equation into lower-dimensions. Reductions are classified by a choice of gauge group, a choice of symmetry, such as, translational symmetry, a choice of gauge fixing, and a choice of constants of integrations in the process of reductions. Gauge groups are in general $GL(N)$. We have to take $U(1)$ part into account in NC case. A choice of symmetry reduces NC ASDYM equations to simple forms. We note that noncommutativity must be eliminated in the reduced directions because of compatibility with the symmetry. Hence within the reduced directions, discussion about the symmetry is the same as commutative one. A choice of gauge fixing is the most important ingredient in this paper which is shown explicitly at each subsection. The residual gauge symmetry sometimes shows equivalence of a few reductions. Constants of integrations in the process of reductions sometimes lead to parameter families of NC reduced equations, however, in this paper, we set all integral constants zero for simplicity.

NC ASDYM equations can be represented in complex representation as follows (Notation is the same as the book of Mason-Woodhouse [39]):

$$F_{wz} = \partial_w A_z - \partial_z A_w + [A_w, A_z]_\star = 0,$$

$$
\begin{aligned}
F_{\tilde{w}\tilde{z}} &= \partial_{\tilde{w}} A_{\tilde{z}} - \partial_{\tilde{z}} A_{\tilde{w}} + [A_{\tilde{w}}, A_{\tilde{z}}]_\star = 0, \\
F_{z\tilde{z}} - F_{w\tilde{w}} &= \partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z + \partial_{\tilde{w}} A_w - \partial_w A_{\tilde{w}} + [A_z, A_{\tilde{z}}]_\star - [A_w, A_{\tilde{w}}]_\star = 0, \quad (2.33)
\end{aligned}
$$

where $z, w, \tilde{z}, \tilde{w}$ are linear combinations of the coordinates of the 4-dimensional spaces $(x^0, x^1, x^2, x^3)$, and $A_z, A_w, A_{\tilde{z}}, A_{\tilde{w}}$ denote the gauge fields in the Yang-Mills theory. This is actually equivalent to the condition of anti-self-duality of the gauge fields :$F_{\mu\nu} = -*F_{\mu\nu}$ where the symbol $*$ is the Hodge dual.

Here, we present non-trivial reductions of NC ASDYM equation with $G = GL(2)$ to the NC KdV equation.

First, let us take a dimensional reduction by null translations:

$$X = \partial_w - \partial_{\tilde{w}}, \ Y = \partial_{\tilde{z}}. \quad (2.34)$$

and identify space-time coordinates as $t \equiv z, \ x = w + \tilde{w}$, and put the following non-trivial reduction conditions on the gauge fields

$$A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ u/2 & 0 \end{pmatrix}, \ A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ A_w = \begin{pmatrix} 0 & -1 \\ u & 0 \end{pmatrix}, \ A_z = \frac{1}{4} \begin{pmatrix} u' & -2u \\ u'' + 2u \star u & -u' \end{pmatrix},$$

then we can see Eq. (2.33) reduces to the NC KdV equation:

$$\dot{u} = \frac{1}{4} u''' + \frac{3}{4} \left( u' \star u + u \star u' \right). \quad (2.35)$$

In this non-trivial way, the NC KdV equation is actually derived. Many other NC integrable equations are proved to be derived from NC ASDYM equation by reduction, which is summarized in [20, 21].

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# A Brief Introduction to Quasi-determinants

In the appendix, we make a brief introduction of quasi-determinants introduced by Gelfand and Retakh [13] and present a few properties of them which play important roles in the following sections. The detailed discussion is seen in e.g. [12].

Quasi-determinants are not just a generalization of usual commutative determinants but rather related to inverse matrices. From now on, we suppose existence of all the inverses.

Let $A = (a_{ij})$ be a $N \times N$ matrix and $B = (b_{ij})$ be the inverse matrix of $A$, that is, $A \star B = B \star A = 1$. Here all products of matrix elements are supposed to be star-products, though the present discussion hold for more general situation where the matrix elements belong to a noncommutative ring.

Quasi-determinants of $A$ are defined formally as the inverse of the elements of $B = A^{-1}$:

$$|A|_{ij} := b_{ji}^{-1}. \tag{A.1}$$

In the commutative limit, this is reduced to

$$|A|_{ij} \stackrel{\theta \to 0}{\longrightarrow} (-1)^{i+j} \frac{\det A}{\det A^{ij}}, \tag{A.2}$$

where $A^{ij}$ is the matrix obtained from $A$ deleting the $i$-th row and the $j$-th column.

We can write down more explicit form of quasi-determinants. In order to see it, let us recall the following formula for a block-decomposed square matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - B \star D^{-1} \star C)^{-1} & -A^{-1} \star B \star (D - C \star A^{-1} \star B)^{-1} \\ -(D - C \star A^{-1} \star B)^{-1} \star C \star A^{-1} & (D - C \star A^{-1} \star B)^{-1} \end{pmatrix},$$

where $A$ and $D$ are square matrices. We note that any matrix can be decomposed as a $2 \times 2$ matrix by block decomposition where one of the diagonal parts is $1 \times 1$. Then the above formula can be applied to the decomposed $2 \times 2$ matrix and an element of the inverse matrix is obtained. Hence quasi-determinants can be also given iteratively by:

$$\begin{aligned} |A|_{ij} &= a_{ij} - \sum_{i'(\neq i), j'(\neq j)} a_{ii'} \star ((A^{ij})^{-1})_{i'j'} \star a_{j'j} \\ &= a_{ij} - \sum_{i'(\neq i), j'(\neq j)} a_{ii'} \star (|A^{ij}|_{j'i'})^{-1} \star a_{j'j}. \end{aligned} \tag{A.3}$$

It is sometimes convenient to represent the quasi-determinant as follows:

$$|A|_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & & \boxed{a_{ij}} & & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}. \tag{A.4}$$

Examples of quasi-determinants are, for a $1 \times 1$ matrix $A = a$

$$|A| = a,$$

and for a $2 \times 2$ matrix $A = (a_{ij})$

$$|A|_{11} = \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12} \star a_{22}^{-1} \star a_{21}, \quad |A|_{12} = \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{11} \star a_{21}^{-1} \star a_{22},$$

$$|A|_{21} = \begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix} = a_{21} - a_{22} \star a_{12}^{-1} \star a_{11}, \quad |A|_{22} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} = a_{22} - a_{21} \star a_{11}^{-1} \star a_{12},$$

and for a $3 \times 3$ matrix $A = (a_{ij})$

$$\begin{aligned} |A|_{11} &= \begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} - (a_{12}, a_{13}) \star \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}^{-1} \star \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix} \\ &= a_{11} - a_{12} \star \begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1} \star a_{21} - a_{12} \star \begin{vmatrix} a_{22} & a_{23} \\ \boxed{a_{32}} & a_{33} \end{vmatrix}^{-1} \star a_{31} \\ &\quad - a_{13} \star \begin{vmatrix} a_{22} & a_{23} \\ \boxed{a_{32}} & a_{33} \end{vmatrix}^{-1} \star a_{21} - a_{13} \star \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} \star a_{31}, \end{aligned}$$

and so on.

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