Two-Dimensional Topological Strings Revisited

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Two-Dimensional Topological Strings Revisited

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Abstract. The topological string of the type A with a two-dimensional target space is studied, an explicit formula for the string partition function is found and the target space field theory reproducing this partition function is proposed. This field theory has an infinite set of additional deformations overlooked by the standard definition of the topological string. It can be in turn coupled to gravity, thereby realizing the "worldsheets for worldsheets" idea. We also exhibit the wave function nature of the string partition function and suggest a new relation to quantum integrable systems.

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1. Introduction

Topological strings are a continuous source of inspiration for gauge and string theorists. They can be studied on their own, for the purely mathematical reasons. Sometimes the amplitudes of the topological string can be viewed as the subset of the "physical" superstrings. The topological strings produce exact all-string-loop results [1], from which one hopes to gain some intuition about the quantum theory of gravity, perhaps even at the non-perturbative level. For example, the topological strings give a realization of the quantum space foam picture [2]. The topological strings of the A and B [3] type play a crucial rôle in describing the compactifications [4] of the type II string theories on Calabi–Yau threefolds [1], which gives rise to the $\mathcal{N}=2$ theories in four dimensions. The partition function $\mathcal{Z}(t)$ of a topological string, of A or B type, depends on a set of couplings t, which correspond to the cohomology of the target space of string theory, valued in some sheaf. For example, for the B model on a Calabi–Yau manifold X of complex dimension d, the coupling constants t belong to

$$H_B(X) = \bigoplus_{p,q=0}^{d} H^p(X, \Lambda^q \mathcal{T}_X) \approx H^{d-*,*}(X),$$

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while for A model the couplings are valued in

$$H_A(X) = \mathbf{i}lus^d_{p,q=0} H^p(X, \Lambda^q \mathcal{T}^*_X) \approx H^{*,*}(X),$$

In addition, every operator $\mathcal{O}$, describing these couplings, comes with the so-called gravitational descendants $\sigma_k(\mathcal{O}), k = 1, 2, \ldots$. Thus the full set of couplings of the topological string is an infinite dimensional space

$$L_+ H_{A,B}(X)$$

of the positive loops $\mathbf{t}(z)$ valued in $H_{A,B}(X)$ where we are using a formal variable z to label the gravitational descendants:

$$\sigma_k(\mathcal{O}) \leftrightarrow \mathcal{O} \otimes z^k, \qquad k \geq 0$$

In the case $d = 3$ the gravitational descendants decouple for $k > 0$, except for the dilaton $\sigma_1(1)$, which corresponds to the string coupling constant $\hbar$. The (disconnected) partition function of the topological string on a threefold

$$\mathcal{Z}_X(t; \hbar) = \exp \sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g(t),$$

where $t \in H_{A,B}(X)$, is a generating function of the genus g topological string diagrams. For the B model these diagrams can be identified with the Feynman diagrams of a certain quantum field theory on X, the so-called Kodaira–Spencer field theory [1]. For the A model the analogous theory, the so-called theory of Kähler gravity [5] is expected to be non-local and is constructed only in the large volume limit where the non-local effects are exponentially suppressed. We shall construct the Kähler gravity theory for the two-dimensional X and will find that it is a local theory of an infinite number of fields.

Duality CY versus $\mathbf{R}^4$: *topological string–supersymmetric gauge theory.* The duality between the topological strings on local Calabi–Yau manifolds and the chiral sector in the four-dimensional $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetric gauge theories keeps attracting attention. The simplest example of that duality is the geometrical engineering of [6]. One starts with an ADE singularity, i.e. a quotient $\mathbf{C}^2 / \Gamma_G$, fibered over a $\mathbf{CP}^1$ so that the total space is a (singular) Calabi–Yau manifold. By resolving the singularities one obtains a smooth non-compact Calabi–Yau manifold X_G. If one views the IIA string on $X_G \times \mathbf{R}^{1,3}$ as a large volume limit of a compactification on a Calabi–Yau manifold with the locus of ADE singularities over an isolated rational curve, then the effective four-dimensional theory will decouple from gravity. Moreover one can be model the effective theory on the four-dimensional $\mathcal{N} = 2$ theory with the MacKay dual ADE gauge group G, where the resolution of singularities of X_G corresponds to fixing a particular vacuum expectation value of the adjoint scalar. Then the prepotential of the low-energy effective theory is given by the genus zero prepotential of the type A topological string on X_G (more precisely, it is the prepotential of the five-dimensional gauge theory compactified on a circle which arises in this way [7,8], in order to see the four-dimensional prepotential one has to go to a certain scaling limit in the CY moduli space [6]).

Duality Σ *versus* $\mathbf{R}^4$: *topological string–supersymmetric gauge theory.* Another remarkable duality between the chiral sector of the four-dimensional $\mathcal{N}=2$ theories and the topological strings on the two-dimensional manifolds was discovered in [9] and further studied in [10]. It is based on the comparison of the instanton calculus in the four-dimensional gauge theory [11] and the Gromov–Witten/Hurwitz correspondence of [12]. The physics of that correspondence involves the theory on a fivebrane wrapped on a Riemann surface. One can actually stretch the validity of the duality beyond the realm of the physical superstrings. The conjecture that the powerful S-duality holds at the level of the topological strings [13] leads to the topological string version of M-theory, or Z-theory [14,15].

The duality of [9] (see also a paper on the mathematically related subject [16] and recent works on the duality with $\mathcal{N}=1$ four-dimensional theories [17,18]) identified the disconnected partition function of the topological string on $\mathbf{CP}^1$ in the background with the arbitrary topological descendants of the Kähler form $\sigma_k(\omega)$ turned on. The couplings t_k^{ω} (up to a k-dependent factor) are identified with the couplings of the operators

$$\int d^4\vartheta \;\; \mathrm{tr}\, \Phi^{k+2}$$

in the $\mathcal{N}=2$ gauge theory:

$$\sum_{k=0}^{\infty} t_k^{\omega} \int_C \sigma_k(\omega) \leftrightarrow \sum_{k=0}^{\infty} \frac{t_k^{\omega}}{(k+2)!} \int_{\mathbf{R}^{4|4}} d^4x d^4\vartheta \;\; \mathrm{tr}\, \Phi^{k+2} \tag{1.1}$$

where in the left hand side we write the worldsheet couplings. In this paper we shall deepen the duality discovered in Ref. [9]. In particular, by replacing $\mathbf{CP}^1$ with a genus h Riemann surface Σ one gets a relation between the $\mathcal{N}=2$ theory with h adjoint hypermultiplets and the Gromov–Witten theory of Σ, which is natural in view of the work [19].

Duality Σ *versus* Σ *topological string–two-dimensional gauge theory.* About 15 years ago Gross has suggested [20] to attack the problem of finding the description of a large N gauge theory in terms of some string theory via the analysis of the two-dimensional gauge theories. By carefully analyzing the large N' t Hooft limit of the two-dimensional pure Yang–Mills theory on a Riemann surface Gross and Taylor [21] have identified many features of the corresponding string theory, while [22,23] have proposed a new kind of topological string theory. An important aspect of the construction of [22,23] was the realization of the fact that the topological Yang–Mills theory (which is the perturbative limit of the physical Yang–Mills

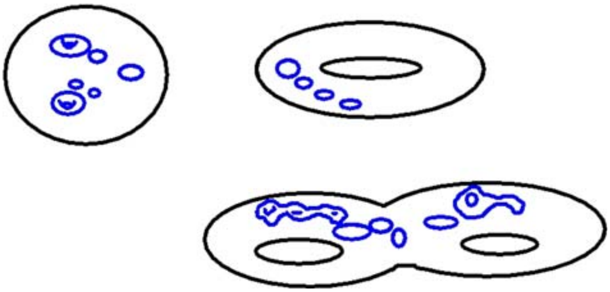

Figure 1. Quantum gravity in target space versus worldsheet.

theory) can be described by the Hurwitz theory. The latter counts ramified coverings of a Riemann surface Σ. In this paper we shall find a different version of the string field theory, the one corresponding to the type A topological strings on a Riemann surface Σ. It will turn out to be a kind of an infinite N gauge theory, but most likely not the ordinary 't Hooft large N limit of the gauge group with the finite dimensional gauge group like $SU(N)$ or $SO(N)$.

Worldsheets for worldsheets. In [24] Green has proposed to study the two-dimensional string backgrounds as the worldsheet theories of yet another string theories. With the advent of the string dualities a few interesting examples of this construction were found. For example, M-theory fivebrane wrapped on **K3** becomes a heterotic string on $\mathbf{T}^3$. This is not exactly a realization of the [24] idea as we are using the localized soliton to generate the string. One could try to study the **CY4** or **Spin(7)** compactifications of the Type II string [25]. but this is difficult due to the lack of the detailed knowledge of the moduli spaces of these manifolds. In this paper we shall approach this problem in the context of the topological string (Figure 1).

Very large phase space of the topological string. In the conventional formulation of the A model to every cohomology class $\mathbf{e}_\alpha \in H^*(X)$ of the target space X one assigns an infinite sequence of observables $\sigma_k(\mathbf{e}_\alpha), k = 0, 1, 2, \ldots$. The corresponding couplings t_k^α parameterize the so-called *large phase space*. For $k = 0$ one gets the *small phase space*. Viewed from the worldsheet, the observable $\sigma_k(\mathbf{e}_\alpha)$ is the kth descendant of $\mathbf{e}_\alpha$. However, if we think of these observables in terms of the target space we should say that $\sigma_k(\mathbf{e}_\alpha)$ is the $\dim(X) - \deg(\mathbf{e}_\alpha)$-descendant of some local BRST invariant observable $\mathcal{O}_k$:

$$\sigma_k(\mathbf{e}_\alpha) \sim \int_X \mathbf{e}_\alpha \wedge \mathcal{O}_k^{(\dim(X) - \deg(\mathbf{e}_\alpha))} \tag{1.2}$$

The gravitational descendants of the top cohomology class of X therefore correspond to the zero-observables $\mathcal{O}_k^{(0)}$ of the target space theory, and as such they are the simplest to study. This is why we shall use as the starting point the so-called *stationary sector* of the theory [12], where only the couplings of these observables are turned on. The observables which are the hardest ones to study are the descendants of the puncture, i.e. the descendants of the unit operator. They correspond to the $\dim(X)$-observables constructed out of $\mathcal{O}_k$. In the standard paradigm of the topological field theory they correspond to the deformations of the space–time Lagrangian $\mathcal{L}$.

When the topological theory is the twisted version of the supersymmetric field theory, these deformations correspond to the F-terms (for the B model) or to the twisted F-terms (for the A model) of the supersymmetric theory. In two dimensions they are the (twisted) superpotential deformations, in four dimensions they are the prepotential deformations.

However, the operators $\mathcal{O}_k$ do not exhaust the full set of the (twisted) F-terms. Indeed, the products of the local observables $\mathcal{O}_k$ and $\mathcal{O}_l$ cannot be expressed, in general, as the linear combination of $\mathcal{O}_k$'s. Thus the target space theory (twisted) superpotential can be deformed by the terms like $\mathbf{O}_\kappa = \mathcal{O}_{k_1}\mathcal{O}_{k_2}\cdots\mathcal{O}_{k_p}$, for any partition $\kappa = (k_1 \geq k_2 \geq \cdots \geq k_p)$. In analogy with the gauge theory which we shall make more precise, the observables $\mathcal{O}_k$ correspond to the *single trace operators*, while the products $\mathbf{O}_\kappa$, for $\ell(\kappa) > 1$, correspond to the *multi-trace operators*:

$$\mathcal{L} \longrightarrow \mathcal{L} + \sum_\kappa T_\kappa \mathbf{O}_\kappa \tag{1.3}$$

The couplings T_κ parametrize a formal neighbourhood of a point $T = 0$ in the moduli space $\mathcal{M}$ of the target space theories (which is the moduli space of the F-terms of some supersymmetric field theory). The tangent space to $\mathcal{M}$ at $T = 0$ is the Fock space

$$T_0\mathcal{M} = \mathbf{H} = \oplus_{k=0}^{\infty} S^k L_+ \mathbf{C} \tag{1.4}$$

of a single bosonic oscillator: $\mathcal{O}_k \leftrightarrow \alpha_{-k}$. It has two bases labelled by the partitions, the bosonic and the fermionic one. The bosonic basis corresponds to the operators $\mathbf{O}_\kappa$. The fermionic basis ($\mathbf{\Psi}_\lambda$) is the basis which diagonalizes the chiral ring:

$$\mathbf{O}_\kappa \cdot \mathbf{\Psi}_\lambda = \mathcal{E}_\kappa(\lambda)\mathbf{\Psi}_\lambda \tag{1.5}$$

At the origin $\mathbf{t} = 0$ of the deformation space the eigenvalues $\mathcal{E}_\kappa(\lambda)$ are the characters $\chi_\lambda(\kappa)$ of the representation λ of the symmetric group $\mathcal{S}_{|\lambda|}$ evaluated on the conjugacy class κ (cf. [12]). As we move away from the origin, these characters are replaced by some interesting formal power seria in T_κ.

The target space theory can be also coupled to the topological gravity. Then the full space of deformations of the target space theory will be parameterized by the couplings $\mathbf{T}_\kappa^{\alpha,n}$, where α labels the cohomology of X, n labels the gravitational descendants in the sense of the topological gravity on X (in the problem studied in this paper, X is a two-dimensional manifold and n is a non-negative integer), and κ is a partition labelling the multi-trace operators:

$$\mathcal{S} \longrightarrow \mathcal{S} + \sum_{\alpha,\kappa} \sum_{n=0}^{\infty} T_{\kappa}^{\alpha,n} \int_X \sigma_n(\mathbf{O}_\kappa) \wedge \mathbf{e}_\alpha \tag{1.6}$$

We shall call the space of all these couplings the *Very large phase space*. We shall write an expression for the partition function of the topological string on the *Very large phase space* in genus zero (i.e. for target space being a two-sphere). The problem of finding the special coordinates on the *Very large phase space*, which is in a sense equivalent to the problem of constructing the full quantum gravity dressed string theory partition function, is beyond the scope of the present paper. The formulation of the problem for the general target space X looks more important then the possible solution of the problem we can anticipate from the gauge theory analogy for $X = \Sigma$, a Riemann surface.

The partition function $\mathcal{Z}_X$. In this paper we study the case where X is a Riemann surface of genus h. The partition function $\mathcal{Z}_X$ of the A-model on a Riemann surface X is a function of an infinite set of couplings, $\mathbf{t} = \left(t_k^\alpha\right)$ where $\alpha = 0, \ldots, \dim H^*(X) - 1 = 2h+1$, $\mathbf{e}_0 = \mathbf{1}$, and $k \in \mathbf{Z}_{\geq 0}$. We introduce some additive basis $\mathbf{e}_\alpha$ of the cohomology of X, $\mathbf{e}_\alpha \in H^*(X, \mathbf{C})$. We have:

$$\mathcal{Z}_X(\mathbf{t}; \hbar, q) = \exp \sum_{g,n;\beta=0}^{\infty} \frac{\hbar^{2g-2} q^\beta}{n!} \sum_{\vec{k},\vec{\alpha}} \prod_{i=1}^{n} t_{k_i}^{\alpha_i} \int_{\overline{\mathcal{M}_{g,n}(X,\beta)}} \bigwedge_{i=1}^{n} \mathrm{ev}_i^*(\mathbf{e}_{\alpha_i}) \wedge \psi_i^{k_i} \tag{1.7}$$

where we used the standard notations [26] for the moduli space $\overline{\mathcal{M}_{g,n}(X,\beta)}$ of degree β genus g stable maps to X with n punctures, the evaluation maps:

$$\mathrm{ev}_i : \overline{\mathcal{M}_{g,n}(X,\beta)} \longrightarrow X \tag{1.8}$$

defined as:

$$\mathrm{ev}_i(C, x_1, \ldots, x_n; \phi) = \phi(x_i) \tag{1.9}$$

where $(C, x_1, \ldots, x_n; \phi)$ is the stable map with the n punctures $x_1, \ldots, x_n$. Finally, in (1.7) we use the first Chern class $\psi_i = c_1(T_{x_i}^* C)$ of the cotangent line at the i'th marked point. Following [27] it is convenient to think of the partition function $\mathcal{Z}_X$ as of the functional on the space of positive loops valued in $H^*(X)$. Thus, let us introduce the $H^*(X)$-valued function:

$$\mathbf{t}(z) = \sum_{n=0}^{\infty} \mathbf{t}_n \frac{z^n}{n!}, \quad \mathbf{t}_n = \sum_{\alpha=0}^{2h+1} t_n^\alpha \, \mathbf{e}_\alpha \in H^*(X) \tag{1.10}$$

of a formal variable z. In addition we introduce several other functions, related to $\mathbf{t}(z)$, the expansion coefficients of its antiderivative

$$\partial^{-1}(z - \mathbf{t}(z)) = \frac{z^2}{2} - \sum_{\alpha=0}^{2h-1} t^\alpha(z) e_\alpha \tag{1.11}$$

and its Legendre transform:[1]

$$\mathbf{F}_{\mathbf{t}}(x) = x\,\mathbf{z}(x) - \frac{1}{2}\mathbf{z}^2(x) + \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \mathbf{t}_k\, \mathbf{z}^{k+1}(x), \tag{1.12}$$

where $\mathbf{z}(x) \in H^*(X)$ solves:

$$x = \mathbf{z}(x) - \mathbf{t}(\mathbf{z}(x)) \tag{1.13}$$

and is given by the following formal power series in $\mathbf{t}_k$'s:

$$\mathbf{z}(x) \equiv x + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\mathbf{t}^n(x)\right]^{(n-1)} = x + \mathbf{t}(x) + \mathbf{t}(x)\cdot \mathbf{t}'(x) + \cdots \tag{1.14}$$

Note that

$$\mathbf{z}(-t_0^{\mathbf{1}}) = 0 \tag{1.15}$$

and

$$\frac{\partial \mathbf{F}_{\mathbf{t}}(x)}{\partial x} = \mathbf{z}[x], \qquad \frac{\partial \mathbf{F}_{\mathbf{t}}(x)}{\partial \mathbf{t}_k} = \frac{\mathbf{z}[x]^{k+1}}{(k+1)!} \tag{1.16}$$

Even though $x \in \mathbf{C} = H^0(X)$, both $\mathbf{z}(x), \mathbf{F}_{\mathbf{t}}(x) \in H^*(X)$ are the inhomogeneous cohomology classes, and can be expanded,

$$\begin{aligned} \mathbf{z}(x) &= z[x]\cdot \mathbf{1} + \cdots \\ \mathbf{F}_{\mathbf{t}}(x) &= F_{\mathbf{t}}(x)\cdot \mathbf{1} + \cdots \end{aligned} \tag{1.17}$$

where $\cdots$ stands for the higher degree classes in $H^*(X)$.

We present our result in two forms:

The mathematical formula represents $\mathcal{Z}_X(\mathbf{t})$ as a sum over partitions.[2] Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, let $f_\lambda(x)$ denote its profile function (Figure 2):

$$\begin{aligned} f_\lambda(x) = |x| + \sum_{i=1}^{\infty} & |x - \hbar(\lambda_i - i + 1)| - |x - \hbar(\lambda_i - i)| + \\ & + |x - \hbar(-i)| - |x - \hbar(-i+1)| \end{aligned} \tag{1.18}$$

Define $\mathcal{S}_\lambda(\mathbf{t}) \in H^*(X)$ as:

$$\begin{aligned} \mathcal{S}_\lambda(\mathbf{t}) = \frac{1}{2\hbar} \int_{\mathbf{R}} & \mathrm{d}x f_\lambda''(x) \mathbf{U}_{\mathbf{t}}(x) + \\ & + \frac{e(X)}{8} \int\int_{\mathbf{R}^2} \mathrm{d}x_1 \mathrm{d}x_2 f_\lambda''(x_1) f_\lambda''(x_2) \mathbf{G}_{\mathbf{t}}(x_1, x_2) \end{aligned} \tag{1.19}$$

[1] In order to avoid the confusion we shall give the explicit formulae for the expansion of $t^{\mathbf{1}}(z), s^I(z), t^{\omega}(z)$ later on.

[2] The basic notions of the theory of partitions are recalled in the Appendix A.

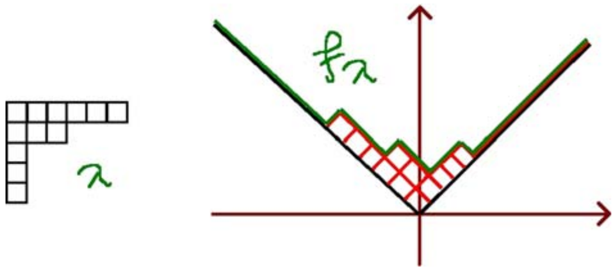

Figure 2. A partition $\lambda = (6, 3, 1, 1, 1)$, its Young diagram(s), and profile in *green*.

where $e(X) = c_1(TX)$ is the Euler class of X,

$$\chi(X) = \int_X e(X) = 2 - 2h,$$

and the functions $\mathbf{U_t}$ and $\mathbf{G_t}$ are the particular solutions to the finite difference equations:

$$\mathbf{U_t}\left(x + \frac{1}{2}\hbar\right) - \mathbf{U_t}\left(x - \frac{1}{2}\hbar\right) = \mathbf{F_t}(x) + e(X) \cdot \left((x + \mathbf{t}_0)\log\left(\frac{x + \mathbf{t}_0}{\mathbf{z}(x)}\right) - \right.$$
$$\left. - \sum_{l=2}^{\infty}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{l}\right)\frac{\mathbf{t}_l}{l!}\mathbf{z}^l(x)\right), \qquad (1.20)$$

(the right hand side is a formal power series in x)

$$\mathbf{G_t}\left(x_1 + \frac{1}{2}\hbar, x_2 + \frac{1}{2}\hbar\right) - \mathbf{G_t}\left(x_1 - \frac{1}{2}\hbar, x_2 + \frac{1}{2}\hbar\right) -$$
$$-\mathbf{G_t}\left(x_1 + \frac{1}{2}\hbar,\ x_2 - \frac{1}{2}\hbar\right) + \mathbf{G_t}\left(x_1 - \frac{1}{2}\hbar,\ x_2 - \frac{1}{2}\hbar\right) =$$
$$= \log\left(\frac{\mathbf{z}(x_1) - \mathbf{z}(x_2)}{\mu}\right), \qquad (1.21)$$

which we specify in the main body of the paper. The scale μ is related to $\hbar$. The mathematical formula is:

$$\boxed{\mathcal{Z}_X(\mathbf{t}; \hbar, q) = \sum_{\lambda}(-q)^{|\lambda|}\exp\left(\frac{1}{\hbar}\int_X \mathcal{S}_\lambda(\mathbf{t})\right)} \qquad (1.22)$$

The physical formula. The A model version identifies $\mathcal{Z}_X(\mathbf{t})$ with the partition function of a two-dimensional gauge theory on X. The gauge theory in question is a twisted $\mathcal{N} = 2$ super–Yang–Mills theory with the gauge group $\mathbf{G}$, to be specified momentarily, perturbed by all single-trace operators, commuting with the scalar supercharge Q. More precisely, $\mathcal{Z}_X$ is equal to the generating function of the correlators of all 2, 1, and 0-observables (we remind the relevant notions in the main body of the paper), constructed out of the single-trace operators

$$\mathcal{O}_k = \mathrm{Coeff}_{u^k}\, \mathrm{Tr}_{\mathcal{H}}\, e^{u\phi}, \tag{1.23}$$

that is:

$$\boxed{\begin{aligned} \mathcal{Z}_X(\mathbf{t};\hbar,q) &= \left\langle \exp - \int_X \left[\sum_{k=0}^{\infty} \sum_{\alpha=1}^{2h+2} \hbar^{k-1+\frac{\deg e_\alpha}{2}}\, \widehat{t}_k^{\alpha}\, e_\alpha \wedge \mathcal{O}_{k+1}^{(2-\deg e_\alpha)} \right] \right\rangle \\ \widehat{t}_0^{2h+2} &= t_0^{2h+2} - \log(q) + \chi(X)\log(\hbar) \\ \widehat{t}_1^{1} &= t_1^1 - 1 \end{aligned}} \tag{1.24}$$

and the other times $\widehat{t}_k^{\alpha} = t_k^{\alpha}$. The gauge group $\mathbf{G}$ consists of certain unitary transformations of a Hilbert space $\mathcal{H}$. Its definition will be discussed in Section 4.

The physical formula. The B model version represents $\mathcal{Z}_X(\mathbf{t})$ as a partition function of a Landau–Ginzburg theory with the worldsheet X. The $\mathcal{N}=2$ supersymmetric Landau–Ginzburg theory without topological gravity is determined by the following data: a target space, which is a complex manifold $\mathcal{U}$, a holomorphic function $\mathcal{W}$, and a top degree holomorphic form Ω on $\mathcal{U}$. The target space $\mathcal{U}$ is an infinite-dimensional disconnected space. Its connected components $\mathcal{U}_\lambda$ are labelled by partitions λ. Each component is isomorphic to $\mathbf{C}^\infty$, the space of finite sequences of complex numbers. The superpotential $\mathcal{W}_\lambda : \mathcal{U}_\lambda \to \mathbf{C}$ is given by the regularized infinite sum

$$\mathcal{W}_\lambda = \sum_{i=1}^{\infty} \left[\left(\lambda_i - i + \frac{1}{2} \right) z_i - \frac{1}{2} z_i^2 + \sum_{k=0}^{\infty} \frac{t_k}{(k+1)!} z_i^{k+1} \right] \tag{1.25}$$

The top degree form is given by the formal product:

$$\Omega = \bigwedge_{i=1}^{\infty} \mathrm{d}\varepsilon_i \tag{1.26}$$

where

$$1 + \sum_{i=1}^{\infty} \varepsilon_i t^i = \prod_{i=1}^{\infty} (1 + t z_i).$$

Of course the infinite-dimensionality of various ingredients involved means that this is not the conventional B model. However the theory provides a regularization of the infinite products and sums above.

Very large phase space extension. In the worldsheet formulation the Very large phase space observables are non-local. In the topological string language, the observable $\mathcal{O}_{0,0,\dots,0}(x_1,\dots,x_l)\,(\mathbf{e}_\alpha)$ corresponds to the condition that the points $x_1,\dots,x_l$ of the worldsheet are mapped to the same point $f\in X$ sitting in a cycle representing $[\alpha]\in H_*(X)$, the Poincare dual of $\mathbf{e}_\alpha$. As a cohomology class of the moduli space of stable maps:

$$\mathcal{O}_{0,0,\dots,0}(x_1,\dots,x_l)\,(\mathbf{e}_\alpha) = \text{Poincare dual}\left[(\mathrm{ev}_1\times\mathrm{ev}_2\times\cdots\times\mathrm{ev}_l)^{-1}\,[\alpha]^{\times l}\right]. \tag{1.27}$$

Analogously, one defines the descendant observables:

$$\mathcal{O}_{k_1,k_2,\dots,k_l}(x_1,\dots,x_l)(\mathbf{e}_\alpha) = \psi_1^{k_1}\wedge\ldots\wedge\psi_l^{k_l}\wedge\mathcal{O}_{0,\dots,0}(x_1,\dots,x_l)(\mathbf{e}_\alpha) \tag{1.28}$$

Note that the non-local string theories describing multi-trace deformations of gauge theories were recently studied in the context of the AdS/CFT correspondence [28].

On the very large phase space the function $\mathcal{W}_\lambda$ is perturbed by the generic symmetric function of z_i's:

$$\begin{aligned}\mathcal{W}_\lambda &= \sum_{i=1}^{\infty}\left(\lambda_i - i + \tfrac{1}{2}\right) z_i - \mathcal{O}_2 + \sum_{\kappa} T_\kappa \Phi_\kappa\\ \Phi_\kappa &= \mathcal{O}_{k_1}\mathcal{O}_{k_2}\cdots\mathcal{O}_{k_l}\\ \kappa &= (k_1\ge k_2\ge\cdots\ge k_l)\end{aligned} \tag{1.29}$$

where

$$\mathcal{O}_k = \mathrm{Coeff}_{u^k}\sum_{i=1}^{\infty} e^{u z_i}, \tag{1.30}$$

while the holomorphic top form is given by (1.26). The three point function on a sphere is given by the regularized version of the Grothendieck residue:

$$\mathbf{C}_{\alpha\beta\gamma} = \sum_{\lambda}\sum_{z_\lambda : d\mathcal{W}_\lambda(z_\lambda)=0}\frac{\Phi_\alpha(z_\lambda)\Phi_\beta(z_\lambda)\Phi_\gamma(z_\lambda)}{\mathrm{Hess}_\Omega^{\mathcal{W}_\lambda}(z_\lambda)} \tag{1.31}$$

where α,β,γ are partitions, and $\mathrm{Hess}_\Omega^{\mathcal{W}_\lambda}$ is the regularized determinant, defined as a ratio of the determinant of the derivative map $\mathrm{Det}(d\mathcal{W}_\lambda):\det\mathcal{T}\longrightarrow\det\mathcal{T}^*$ and Ω^2 viewed as an element of $\det^{\otimes 2}\mathcal{T}^*$, where $\mathcal{T}$ is the tangent space $T_{p_\lambda}\mathcal{U}$ at the critical point p_λ of $\mathcal{W}_\lambda$.

The couplings T_κ in (1.29) are most likely not the flat (or special) coordinates $\mathbf{T}_\kappa$ on the Very large phase space. The flat coordinates $\mathbf{T}_\kappa$ are obtained from T_κ by a formal diffeomorphism. The defining feature of these coordinates is the gradient form of the three-point functions:

$$\mathbf{C}_{\alpha\beta\gamma} = \frac{\partial^3\mathbf{F}}{\partial\mathbf{T}_\alpha\partial\mathbf{T}_\beta\partial\mathbf{T}_\gamma} \tag{1.32}$$

for some prepotential function $\mathbf{F}(T)$. The target space theory observables Φ_κ also have their own gravitational descendants $\sigma_k(\Phi_\kappa)$.

2. Two-Dimensional Target *X*

Let $X=\Sigma$ be a Riemann surface of genus h. The cohomology of X is generated by $\mathbf{1}\in H^0(X)$, $\varpi_I\in H^1(X)$, $I=1,\dots,2h$, and $\omega\in H^2(X)$. The product of the odd classes is governed by the intersection form $\eta_{IJ}=-\eta_{JI}$,

$$\varpi_I\cap\varpi_J=\eta_{IJ}\cdot\omega \tag{2.1}$$

We normalize:

$$\int_X \omega=1,\quad \int_X \varpi_I\wedge\varpi_J=\eta_{IJ} \tag{2.2}$$

Let $e(X)\in H^2(X)$ denote the Euler class of the tangent bundle of X,

$$\int_X e(X)=2-2h$$

The moduli space $\overline{\mathcal{M}_{g,n}(X;\beta)}$ of holomorphic maps $f:C\to X$ of genus $g(C)=g$, and degree $f_*[C]=\beta\in H_2(X)$ has the (virtual) dimension:

$$\dim\overline{\mathcal{M}_{g,n}(X;\beta)}=\beta(2-2h)+2g-2+n=b+n \tag{2.3}$$

where $b=2(\beta(1-h)+g-1)$ is the number of branch points of a typical f. The moduli space $\overline{\mathcal{M}_{g,n}(X;\beta)}$ is a compactification of the configuration space of b points on X, and n points on C.

2.1. THE MATHEMATICAL DEFINITION OF THE CORRELATORS

There exists a mathematical definition of the observables of the topological string. Unfortunately we did not succeed in fully deriving this definition (which we recall immediately) from the field theory. Perhaps this difficulty is related to the difficulty of finding the proper definition of the three-dimensional quantum gravity using the gauge theory techniques [31].

Consider the moduli space (stack) of stable maps $\overline{\mathcal{M}_{g,n}(X,\beta)}$ of degree $\beta\in H_2(X)$, genus g, with n marked points. This is the space of triples: $(C;x_1,x_2,\dots,x_n;\phi)$, where C is a compact connected complex curve with at most double point singularities (i.e. the singularities which locally look like a coordinate cross $xy=0$ in $\mathbf{C}^2$), $x_i\in C$ are distinct marked points on C which do not coincide with the double points, $\phi:C\to X$ is a holomorphic map, such that $[\phi(C)]=\beta$. There is a (possibly disconnected) smooth curve $\widetilde{C}=\amalg_\alpha\widetilde{C}_\alpha$ which is obtained from C be separating the double points. The genus g of C is computed as from the topological Euler characteristics $2-2g=\chi_{\text{top}}(C)$. It is easy to express it in terms of the genera

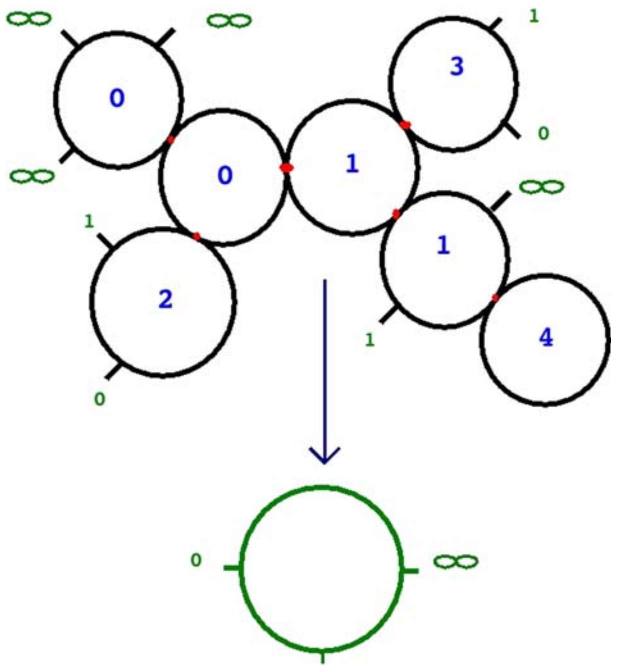

Figure 3. A typical genus zero $\beta = 11$ stable map to $X = \mathbf{CP}^1$. The preimages of the points $0, 1, \infty \in X$ and the degrees β_α of the maps of the smooth components C_α are shown explicitly. The double points are *red*.

of the components $\widetilde{C}_\alpha$, the numbers ν_α of the preimages of the double points on each $\widetilde{C}_\alpha$ and the number ν of the double points on C:

$$2 - 2g = \nu + \sum_\alpha (2 - 2g_\alpha - \nu_\alpha)$$

There are additional conditions on the allowed triples, which guarantee the absence of continuous infinitesimal automorphisms. Namely, every component C_α of genus zero must either have at least three marked or double points, or the restriction of the holomorphic map ϕ on C_α be non-trivial, $\beta_\alpha = \phi[C_\alpha] \neq 0$. Every component C_α of genus one, where ϕ is constant, must have at least one marked or double point (Figure 3).

Now, the moduli space of stable maps has n natural line bundles L_i, $i = 1, \ldots, n$. The fiber of L_i over $(C; x_1, \ldots, x_n; \phi)$ is simply $T^*_{x_i}C$, the holomorphic cotangent bundle at the i'th marked point. In addition, there are n evaluation maps: $\mathrm{ev}_i : \overline{\mathcal{M}_{g,n}(X, \beta)} \to X$,

$$\mathrm{ev}_i(C; x_1, \ldots, x_n; \phi) = \phi(x_i) \tag{2.4}$$

Let $\psi_i = c_1(L_i)$ denote the first Chern class of the line bundle L_i. It is a degree two cohomology class of $\overline{\mathcal{M}_{g,n}(X, \beta)}$. One defines the observables of the topological string:

$$\mathbf{e}_\alpha \in H^*(X) \longrightarrow \sigma_k(\mathbf{e}_\alpha)(x_i) = \mathrm{ev}_i^*(\mathbf{e}_\alpha)\wedge\psi_i^k \tag{2.5}$$

The mathematical definition of the topological string partition function at genus g is given by (1.7).

2.2. THE STATIONARY SECTOR

The stationary sector of the topological string on X is defined as the restriction on the set of couplings $\mathbf{t}^\alpha$. Namely, we turn on the gravitational descendants only for the top degree form, $t_k^\alpha \neq 0$ for $k>0$ only for α, s.t. $\int_X \mathbf{e}_\alpha \neq 0$. In the case of $X=\Sigma$ this means that the only non-vanishing times are $t_k^\omega \equiv t_k^{2h+1}$. The logarithm of the partition function gives the free energy of the topological string, which has a genus expansion:

$$\mathcal{Z}_X(\mathbf{t}^\omega;\hbar,q) = \exp\sum_{g=0}^{\infty}\hbar^{2g-2}\mathcal{F}_g^X(\mathbf{t}^\omega;q) \tag{2.6}$$

where the genus g free energy (the contribution of the connected genus g worldsheets) $\mathcal{F}_g^X(\mathbf{t}^\omega;q)$, in turn, has the worldsheet instanton expansion:

$$\mathcal{F}_g^X(\mathbf{t}^\omega;q) = \sum_{\beta\in\mathbf{Z}_{\geq 0}} q^\beta \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\vec{k}} t_{k_1}^\omega t_{k_2}^\omega\cdots t_{k_n}^\omega \times$$
$$\times\int_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \mathrm{ev}_1^*\omega\wedge\psi_1^{k_1}\wedge\cdots\wedge\mathrm{ev}_n^*\omega\wedge\psi_n^{k_n} \tag{2.7}$$

In the case $X=\Sigma$ the topological string computes the Hurwitz numbers [12], i.e. the number of coverings of Σ with given branching conditions.

This number can be computed [12] by the simple harmonic analysis on the permutation group – the structure group of the covers.

The two-sphere. We first state the result for $\Sigma = \mathbf{CP}^1$. Let

$$t^\omega(x) = \sum_{k=0}^{\infty} t_k^\omega \frac{x^{k+1}}{(k+1)!}, \tag{2.8}$$

then the partition function

$$\mathcal{Z}_{\mathbf{CP}^1}(t^\omega;\hbar,q) \;=\; \sum_\lambda (-q)^{|\lambda|}\mathcal{Z}_\lambda^{\mathbf{CP}^1}(t^\omega;\hbar) \tag{2.9}$$

is the sum over the partitions λ, which label the representations of permutation group. The partition λ contributes to (2.9) as follows:

$$\mathcal{Z}_\lambda^{\mathbf{CP}^1}(\mathbf{t}^\omega;\hbar) = \left(\frac{\dim\lambda}{\hbar^{|\lambda|}|\lambda|!}\right)^2 \exp\sum_{k=0}^{\infty} t_k^\omega\hbar^k c_{k+1}(\lambda) \tag{2.10}$$

where $c_k(\lambda), k>0$ are the Casimirs of the partition λ, defined in the Appendix.

Higher genus. For the genus h Riemann surface the result is but a small modification of (2.9):

$$\mathcal{Z}_{\Sigma}(\mathbf{t}^{\omega};\hbar,q)=\sum_{\lambda}(-q)^{|\lambda|}\mathcal{Z}_{\lambda}^{\Sigma}(\mathbf{t}^{\omega};\hbar)$$
$$\mathcal{Z}_{\lambda}^{\Sigma}(\mathbf{t}^{\omega};\hbar)=\left(\frac{\dim\lambda}{\hbar^{|\lambda|}|\lambda|!}\right)^{2-2h}\exp\sum_{k=0}^{\infty}t_{k}^{\omega}\hbar^{k}c_{k+1}(\lambda) \tag{2.11}$$

Unlike the genus zero partition function, which converges for all values of q, t_0^{ω} and is a formal power series in t_k^{ω} for $k>0$, the higher genus partition function is a formal power series in $qe^{t_0^{\omega}}$ for $h>1$ and has a finite radius of convergence in $qe^{t_0^{\omega}}$ for $h=0$.

2.3. THE NONSTATIONARY EXTENSION ON $X=\mathbf{CP}^1$

We now turn on the descendants $\sigma_k(\mathbf{1})$ of the unit operator, with the corresponding couplings $(t_k^1)_{k=0}^{\infty}$. We continue with $X=\mathbf{CP}^1$. Recall the Virasoro constraints obeyed by $\mathcal{Z}_X$, derived in [32]:

$$L_k\,\mathcal{Z}_X=0,\quad k\geq -1, \tag{2.12}$$

where L_k are given by (A.1-3). Equations (2.12) can be viewed as the infinite-dimensional version of the heat evolution equation. Now, since the initial condition (2.10) is a superposition of the plane waves[3] in the t^{ω}-space, the $t^{\mathbf{1}}$-dependent solution to the Virasoro constraints is also a linear combination of the plane waves:

$$\mathcal{Z}_{\mathbf{CP}^1}(t^{\mathbf{1}},t^{\omega};\hbar,q)=\sum_{\lambda}(-q)^{|\lambda|}\,\mathcal{Z}_{\lambda}^{\mathbf{CP}^1}(t^{\mathbf{1}},t^{\omega};\hbar)$$
$$\mathcal{Z}_{\lambda}^{\mathbf{CP}^1}(t^{\mathbf{1}},t^{\omega};\hbar)=\exp\left(2\,\mathbf{r}(\lambda;\hbar,t^{\mathbf{1}})+\frac{1}{\hbar}\sum_{k=0}^{\infty}t_k^{\omega}c_{k+1}(\lambda;\hbar,t^{\mathbf{1}})\right) \tag{2.13}$$

where now:

$$c_{k+1}(\lambda;\hbar,t^{\mathbf{1}}) = \mathrm{Coeff}_{u^{k+1}}\ \mathcal{C}_{\lambda}(u;\hbar,t^{\mathbf{1}}) \tag{2.14}$$

are the u-expansion coefficients of the regular part of the $t^{\mathbf{1}}$-evolved Chern character

$$\mathcal{C}_{\lambda}(u;\hbar,t^{\mathbf{1}})=\sum_{i=1}^{\infty}e^{u\,z\left[\hbar(\lambda_i-i+\frac{1}{2})\right]} \tag{2.15}$$

[3] An analogue of e^{ikx}.

where the formal function $z[x]$ is defined by (1.13) and (1.17). Finally the **r**-function is expressed in terms of the profile $f_\lambda(x)$ of the partition λ:

$$\mathbf{r}(\lambda;\hbar,t^{\mathbf{1}})=\frac{1}{8}\int\int \mathrm{d}x_1\mathrm{d}x_2\, f_\lambda''(x_1)f_\lambda''(x_2)G_t(x_1,x_2)+\frac{1}{2\hbar}\int \mathrm{d}x\, f_\lambda''(x)\Delta_t(x) \tag{2.16}$$

where the kernel G_t and the function Δ_t are given by the explicit formula

$$\begin{aligned} G_t(x_1,x_2)&=\mathcal{I}_{\widetilde{x}_1}\mathcal{I}_{\widetilde{x}_2}\left[\log\left(\frac{z[\widetilde{x}_1]-z[\widetilde{x}_2]}{\widetilde{x}_1-\widetilde{x}_2}\right)\right](x_1,x_2)+\gamma_\hbar(x_1-x_2)\\ \Delta_t(x)&=\mathcal{I}_x\left[(x+t_0^{\mathbf{1}})\log\left(\frac{x+t_0^{\mathbf{1}}}{z[x]}\right)-\sum_{l=2}^{\infty}\left\{\sum_{m=2}^{l}\frac{1}{m}\right\}\frac{1}{l!}t_l^{\mathbf{1}}\, z[x]^l\right]\end{aligned} \tag{2.17}$$

which uses certain integral operator $\mathcal{I}_x$ and a special function $\gamma_\hbar(x)$. Both are described in the Appendix, together with the derivation of the formula (2.16). The t^ω-dependent part of (2.13) can also be expressed using the profile $f_\lambda(x)$ as follows:

$$\frac{1}{2\hbar}\int_{\mathbf{R}} \mathrm{d}x\, f_\lambda''(x)\mathcal{I}[t^\omega(z[x])] \tag{2.18}$$

The formula (2.13) is equivalent to (1.22) for $X=\mathbf{CP}^1$. To show that, use (1.16) to relate the integral $\int_X \mathbf{U_t}$ to $\mathcal{I}[\mathbf{F_t}]$ and (2.16).

2.4. THE GENERAL TWO-DIMENSIONAL TARGET SPACE

Now let us consider the general genus h two-dimensional Riemann surface Σ as our target space, $X=\Sigma$. The topological string partition function depends on another infinite sequence of coupling constants: $\left(s_k^I\right)_{k=0,1,2,\dots}^{I=1,\dots,2h}$. They correspond to the odd-dimensional cohomology classes $\varpi_I\in H^1(\Sigma)$ of Σ and their gravitational descendants.

We fix the s_k^I-dependence by solving the generalizations of the Virasoro constraints [32]. The result is:

$$\begin{aligned}\mathcal{Z}_\Sigma(\mathbf{t};\hbar,q)=\sum_\lambda(-q)^{|\lambda|}e^{(2-2h)\mathbf{r}(\lambda;\hbar,\mathbf{t^1})}\times\exp\Bigg(&\frac{1}{\hbar}\sum_{k=0}^{\infty}t_k^\omega c_{k+1}(\lambda;\hbar,\mathbf{t^1})+\\ &+\frac{1}{2}\eta_{IJ}\sum_{k,l\geq 0}\binom{k+l}{l}\sum_{I,J=1}^{2h}s_k^Is_l^J\,\mathbf{g}_{k+l}(\lambda;\hbar,\mathbf{t^1})\Bigg)\end{aligned} \tag{2.19}$$

where η_{IJ} is the intersection pairing (2.1) and $\mathbf{g}_k(\lambda;\hbar,\mathbf{t^1})$ are the new invariants of λ_i's[4]defined as:

$$\mathbf{g}_k(\lambda;\hbar,\mathbf{t^1}) = \mathrm{Coeff}_{u^k}\,\Sigma_\lambda(u;\hbar,\mathbf{t^1}) \tag{2.21}$$

where (we introduce a nilpotent variable ϑ, $\vartheta^2=0$):

$$\mathcal{C}_\lambda(u;\hbar,\mathbf{t^1}) + \vartheta\,\Sigma_\lambda(u;\hbar,\mathbf{t^1}) = \sum_{i=1}^{\infty} e^{u\,z\left[\hbar\left(\lambda_i - i + \frac{1}{2}\right)+\vartheta\right]} \tag{2.22}$$

The u-expansion coefficients of this function are well-defined as the formal power series in $\mathbf{t^1}$. Indeed, for the empty partition $\lambda=0$ one has:

$$\begin{aligned}\Sigma_0(u;\mathbf{t^1}) &= \frac{1}{e^{\frac{\hbar\partial_x}{2}} - e^{-\frac{\hbar\partial_x}{2}}}\Bigg|_{x=0} e^{u\mathbf{z}(x)}\mathbf{z}'(x) + \text{negative in } u \text{ terms}\\ \Sigma_0(u;\mathbf{t^1})^+ &= \frac{1}{\hbar u}\sum_{g=0}^{\infty}\frac{\mathbf{b}_{2g}}{(2g)!}\hbar^{2g}\partial_x^{2g}\Bigg|_{x=0}(\exp u\,z(x)-1) = \mathcal{I}\left[e^{uz[x]}z'(x)\right]\end{aligned} \tag{2.23}$$

where the numbers $\mathbf{b}_{2g}$ are given in the Appendix. For $\lambda\neq 0$ the difference $\Sigma_\lambda - \Sigma_0$ is a finite sum of exponents in u.

From (2.23) we derive:

$$\mathbf{g}_k(0;\mathbf{t^1}) = \mathcal{I}\left[z^k(x)z'(x)\right] \tag{2.24}$$

3. Two-Dimensional Gauge Theories

In this section we prepare the grounds for identification of the field theory on Σ whose partition function reproduces the disconnected topological string partition function. This field theory is a twisted version of a two-dimensional $\mathcal{N}=2$ supersymmetric Yang–Mills theory. We shall first remind the relevant notions of the two-dimensional Yang–Mills theory with the gauge group a compact Lie group G, then discuss its $\mathcal{N}=2$ supersymmetric version, then discuss their relation, and then discuss the modifications needed to reproduce the string partition function.

3.1. TWO-DIMENSIONAL PURE YANG–MILLS THEORY

Group theory notations

Let G denote a compact Lie group, $\mathbf{g}=\mathrm{Lie}G$ its Lie algebra, $T\subset G$ its maximal torus, and $\mathbf{t}\subset\mathbf{g}$ its Lie algebra, i.e. the Cartan subalgebra of $\mathbf{g}$. Let $r=\dim\mathbf{t}$ denote

[4]They are the regularized sums

$$\mathbf{g}_k(\lambda;\mathbf{t^1}) \sim \frac{1}{k!}\sum_{i=1}^{\infty}\frac{y_i^k}{1-\mathbf{t^1}'(y_i)}, \tag{2.20}$$

the rank of G. We have Cartan decomposition: $\mathbf{g}=\mathbf{t}\oplus\mathbf{n}_+\oplus\mathbf{n}_-$. Let $\mathcal{W}$ denote the Weyl group of G, Δ_+ the set of positive roots, $\Delta=\Delta_+\cup\Delta_-$ the set of all roots. Each root $\alpha\in\Delta_+$ corresponds to an element $e_\alpha\in\mathbf{n}_+$, also sometimes called a positive root and to an element $e_{-\alpha}\in\mathbf{n}_-$, called the negative root. The root e_α being an eigenvector for the adjoint action of $\mathbf{t}$ on $\mathbf{g}$ also defines an element of $\mathbf{t}^*$. Let ρ denote half the sum of the positive roots:

$$\rho=\frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha \tag{3.1}$$

which we view as an element of $\mathbf{t}^*$. Finally, $\Lambda_{\mathrm{w}}\subset\mathbf{t}^*$ denotes the weight lattice. It contains the root lattice Λ_{r}, which is integrally generated by $e_\alpha\in\mathbf{t}^*,\alpha\in\Delta$. The quotient $\Lambda_{\mathrm{w}}/\Lambda_{\mathrm{r}}$ is isomorphic to the center $Z(G)$ of G.

We shall be using the notation ϕ for vectors in $\mathbf{g}$, φ for vectors in $\mathbf{t}$. An $Ad(G)$-invariant function $F(\phi)$ on $\mathbf{g}$ is uniquely determined by its restriction $f(\varphi)$ on $\mathbf{t}$, where it defines a $\mathcal{W}$-invariant function. As such, it can be also expressed in terms of the Chevalley generators (elementary symmetric polynomials in the case of $G=SU(r+1)$), $\sigma_1,\ldots,\sigma_r$:

$$\mathbf{C}[\mathbf{t}]^{\mathcal{W}}=\mathbf{C}[\sigma_1,\ldots,\sigma_r] \tag{3.2}$$

We shall sometimes use the same notation for the $\mathcal{W}$-invariant function f on $\mathbf{t}$ and for the function on $\mathbf{t}/\mathcal{W}$:

$$f(\varphi)\sim f(\sigma_1,\ldots,\sigma_r) \tag{3.3}$$

An important rôle is played by the identity:

$$d\sigma_1\wedge\cdots\wedge d\sigma_r=\prod_{\alpha\in\Delta_+}\langle\alpha,\varphi\rangle\ d\varphi_1\wedge\cdots\wedge d\varphi_r \tag{3.4}$$

Two-dimensional gauge theory

Let $\mathcal{P}$ a principal G-bundle over Σ. Let $\mathrm{ad}(\mathcal{P})=\mathbf{g}\times_{Ad}\mathcal{P}$ be the adjoint vector bundle over Σ (with fibers isomorphic to $\mathbf{g}$). The principal G-bundles $\mathcal{P}$ over Σ are classified topologically by the elements $c_{\mathcal{P}}\in H^2(\Sigma,\pi_1(G))$. If $\widetilde{G}$ is a simply connected cover of G, then $c_{\mathcal{P}}$ can be represented by an element $z_{\mathcal{P}}$ of the center $Z(\widetilde{G})$ (one trivializes the bundle $\mathcal{P}$ over the complement $\Sigma\backslash p$ to a point $p\in\Sigma$, and then the obstruction to trivialize $\mathcal{P}$ is a small loop in G, which lifts to a path in $\widetilde{G}$. The endpoints of that path are related by the element $z_{\mathcal{P}}\in\widetilde{G}$). Given a dominant weight $\lambda\in\Lambda_{\mathrm{w}}^+$ consider the corresponding highest weight $\widetilde{G}$-module V_λ. The character of $z_{\mathcal{P}}$ in V_λ will be denoted as

$$\chi_\lambda(\mathcal{P})=\mathrm{tr}_{V_\lambda}(z_{\mathcal{P}}) \tag{3.5}$$

The fields of the two-dimensional gauge theory are: the gauge field A, a fermionic one-form ψ, valued in the adjoint bundle, and a scalar ϕ, also in the adjoint representation. Mathematically, A is a connection in some principal G-bundle $\mathcal{P}$ over a two-dimensional Riemann surface Σ, $\psi \in \Gamma(\Pi \mathrm{ad}(\mathcal{P}) \otimes \Omega^1(\Sigma))$, $\phi \in \Gamma(\mathrm{ad}(\mathcal{P}) \otimes \Omega^0(\Sigma))$. The field ϕ is the momentum conjugate to A, sometimes called the electric field.

The action of the theory is defined with the help of the $H^*(\Sigma)$-valued $Ad(G)$-invariant function[5] $\mathbf{W}$ on $\mathbf{g}$:

$$S = \int_\Sigma \mathbf{W}(\phi + \psi + F_A) \tag{3.6}$$

where $F_A \in \Gamma(\mathrm{ad}(\mathcal{P}) \otimes \Omega^2(\Sigma))$ is the curvature of the connection A. Being G-invariant, the function $\mathbf{W}$ is uniquely determined by its restriction onto the Cartan subalgebra $\mathbf{t}$, where it defines a $\mathcal{W}$-invariant function $\boldsymbol{\Omega}$.

We can expand:

$$\begin{aligned} \mathbf{W} &= W + S^I \varpi_I + T^\omega \omega \\ \boldsymbol{\Omega} &= w + s^I \varpi_I + t^\omega \omega \end{aligned} \tag{3.7}$$

The functions W, S^I, T^ω are the Ad-invariant functions on $\mathbf{g} \cdot W$ and T^ω are bosonic functions, while S^I are fermionic. These functions are uniquely determined by their restrictions w, s^I, t^ω on $\mathbf{t}$. These functions w, s^I, t^ω are $\mathcal{W}$-invariant. Again, w, t^ω are bosonic while s^I are fermionic.

The space of fields (ϕ, ψ, A) has a canonical measure:

$$\frac{1}{\mathrm{Vol}(\mathcal{G})} DAD\psi D\phi \tag{3.8}$$

where $\mathcal{G}$ denotes the group of gauge transformations of $\mathcal{P}$. We identify ϕ as an element of the Lie algebra of $\mathcal{G}$, and, as this group has a Haar measure, we can define $D\phi/\mathrm{Vol}(\mathcal{G})$ to be simply the restriction onto the Lie algebra, identified with the tangent space at the identity in the group, of the normalized Haar measure on the group. The remaining measure $DAD\psi$ is the canonical Berezin measure on the split superspace.

The gauge theory partition function is:

$$\mathcal{Z}(\mathbf{W}; \Sigma, \mathcal{P}) = \frac{1}{\mathrm{Vol}(\mathcal{G})} \int DAD\psi D\phi \, \exp \int_\Sigma \mathbf{W}(\phi + \psi + F_A) \tag{3.9}$$

Of course, (3.9) does not exhaust all the possible correlation functions of the physical two-dimensional Yang–Mills theory. We cannot study the expectation values of the Wilson loops, for example. However, by differentiating (3.9) with respect

[5]Note that $\mathbf{W}$ can be viewed as an element of $H^*_G(\Sigma)$, where G acts on Σ trivially. Moreover, one can consider a generalization where the center $Z(G)$ of G acts on Σ. It would be interesting to relate this possibility to the constructions of [33].

to the parameters of **W** one gets the generating function of the correlators of the Q-invariant observables, where Q acts as follows:

$$QA = \psi, \quad Q\psi = D_A\phi, \quad Q\phi = 0 \tag{3.10}$$

3.2. SUPERSYMMETRIC YANG–MILLS THEORY

The fields of the two-dimensional $\mathcal{N}=2$ supersymmetric Yang–Mills theory are: a complex adjoint scalar σ, its conjugate $\overline{\sigma}$, the gauge field A, and the gluinos, i.e. the adjoint-valued fermions $\lambda_+, \lambda_-, \overline{\lambda}_+, \overline{\lambda}_-$, where $\pm$ denote the worldsheet chirality:

$$\lambda_\pm, \overline{\lambda}_\pm \in \Gamma\left(K_\Sigma^{\pm 1/2} \otimes \mathrm{ad}\,(\mathcal{P})\right)$$

3.2.1. *Lagrangian and global symmetries*

The Lagrangian of the pure $\mathcal{N}=2$ super-Yang–Mills theory is ($\mathrm{vol}_g = \star 1$):

$$L = \frac{1}{2g^2}\int_\Sigma \mathrm{tr}\,\Big(F_A \wedge \star F_A + D_A\sigma \wedge \star D_A\overline{\sigma} + [\sigma,\overline{\sigma}]^2\,\mathrm{vol}_g + \\ + \overline{\lambda}_+\overline{\partial}_A\lambda_+ + \overline{\lambda}_-\partial_A\lambda_- + \overline{\lambda}_+[\sigma,\overline{\lambda}_-] + \lambda_+[\overline{\sigma},\lambda_-]\Big) \tag{3.11}$$

The theory has four supercharges $\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm$, which obey the $\mathcal{N}=2$ superalgerba:

$$\begin{aligned} \{\mathcal{Q}_+, \overline{\mathcal{Q}}_+\} &= P_+, \quad \{\mathcal{Q}_-, \overline{\mathcal{Q}}_-\} = P_-, \\ \{\mathcal{Q}_+, \mathcal{Q}_-\} &= 0, \quad \{\overline{\mathcal{Q}}_+, \overline{\mathcal{Q}}_-\} = 0, \\ \{\mathcal{Q}_+, \overline{\mathcal{Q}}_-\} &= 0, \quad \{\overline{\mathcal{Q}}_+, \mathcal{Q}_-\} = 0 \end{aligned} \tag{3.12}$$

where $P_\pm = H \pm P$ are the left and right lightcone translations respectively. The theory has a global $U(1)_A R$-symmetry, under which λ's have charge $+1$, $\overline{\lambda}$'s have charge -1, the supercharges $\mathcal{Q}, \overline{\mathcal{Q}}$ have the opposite charges, the scalar σ has a charge $+2$, and $\overline{\sigma}$ has a charge -2. The theory has another global $U(1)_V R$-symmetry, under which $(\lambda_+, \overline{\lambda}_-)$ have charge $+1$, and $(\lambda_-, \overline{\lambda}_+)$ have charge -1, while the rest of the fields is neutral. The supercharges $\mathcal{Q}_+, \overline{\mathcal{Q}}_-$ have charge -1, while $\mathcal{Q}_-, \overline{\mathcal{Q}}_+$ have charge $+1$.

3.2.2. *Topological twist*

The twisted super-Yang–Mills theory has the same set of fields except for their Lorentz transformation properties are modified. The Euclidean rotation group of the twisted theory is the diagonal $U(1)$ subgroup in the product $U(1)_{\mathrm{Lorentz}} \times U(1)_V$ of the symmetry groups of the original physical super-Yang–Mills theory. Since only the fermions of the original physical theory are charged under $U(1)_V$, only the fermions change their Lorentz spin upon twist.

The fields (λ_+, λ_-) become the $(1,0)$ and $(0,1)$ components of the adjoint-valued one-form $\psi = \psi_z dz + \psi_{\bar{z}} d\bar{z}$, while $\overline{\lambda}_\pm$ become the scalars, which we shall denote as $\chi \pm \eta$. Out of the four supercharges, two, $\mathcal{Q}_+$ and $\mathcal{Q}_-$, become scalars and two become a one-form. Let us denote the twisted version of $\mathcal{Q}_+$ as $Q^{1,0}$ while the twisted version of $\mathcal{Q}_-$ as $Q^{0,1}$. In fact, only the sum $Q = Q^{1,0} + Q^{0,1}$ is a scalar. The difference, $\widetilde{Q} = Q^{1,0} - Q^{0,1}$ is a pseudo-scalar, it changes sign under the world-sheet parity. The topological supercharge Q acts as follows:

$$
\begin{aligned}
&QA = \psi, \quad Q\psi = D_A\sigma, \quad Q\sigma = 0\\
&Q\chi = H, \quad QH = [\chi, \sigma]\\
&Q\overline{\sigma} = \eta, \quad Q\eta = [\overline{\sigma}, \sigma]
\end{aligned}
\tag{3.13}
$$

where we introduced the auxiliary field H, a scalar in adjoint representation. On-shell, $H = \star F_A$. The other charge $\widetilde{Q}$ acts as follows:

$$
\begin{aligned}
&\widetilde{Q}A = \star\psi, \quad \widetilde{Q}\psi = \star D_A\sigma, \quad \widetilde{Q}\sigma = 0\\
&\widetilde{Q}\chi = \star[\sigma, \overline{\sigma}], \quad \widetilde{Q}\overline{\sigma} = \star\chi\\
&\widetilde{Q}\eta = \star H, \quad \widetilde{Q}H = \star[\eta, \sigma]
\end{aligned}
\tag{3.14}
$$

The action of the twisted theory:

$$
\begin{aligned}
S &= \frac{1}{2g^2}\left\{Q, \left[\widetilde{Q}, \int \mathrm{tr}\,(\overline{\sigma}\star F_A + \chi\eta)\right]\right\} =\\
&= \frac{1}{2g^2} Q \int \mathrm{tr}\left(\chi\left(F_A - \frac{1}{2}\star H\right) + \psi\star D_A\overline{\sigma} + \eta\star[\sigma, \overline{\sigma}]\right)
\end{aligned}
\tag{3.15}
$$

upon elimination of the auxiliary field H reads, in components:

$$
\begin{aligned}
&\frac{1}{2g^2}\int \mathrm{tr}\,(F_A \wedge \star F_A + D_A\sigma \wedge \star D_A\overline{\sigma} + [\sigma, \overline{\sigma}]^2\,\mathrm{vol}_g) +\\
&\quad + \mathrm{tr}\,(\chi \wedge D_A\psi - \psi \wedge \star D_A\eta + \chi\star[\chi, \sigma] + \eta\star[\sigma, \eta] + \psi \wedge \star[\psi, \overline{\sigma}])
\end{aligned}
\tag{3.16}
$$

Explicitly, the current of the Q operator, $Q = \oint J$ has the form:

$$
J = \mathrm{tr}\,(H\psi + (\eta D_A\sigma + \chi\star D_A\sigma))
$$

The twisted theory has the observables $\mathcal{O}$, which are annihilated by the Q operator. As all the metric dependence in (3.16) comes from the Q-exact terms, the correlation functions of these observables are metric independent, i.e. they are topological. The main observation of [34] was that these correlation functions can be computed, up to a certain subtlety, by a two-dimensional gauge theory (3.9) where essentially ϕ is identified with σ, while the quartet of fields $\chi, H, \overline{\sigma}, \eta$ is eliminated.

3.2.3. *Physical versus topological observables*

The observables of the topological theory are the Q-invariant gauge invariant functionals of σ, ψ, A. Indeed, the quartet $\chi, H, \overline{\sigma}, \eta$ does not contribute to the cohomology, as one can explicitly see by adding a term:

$$i\bar{t}Q\int \mathrm{tr}\,\overline{\sigma}\chi \tag{3.17}$$

to the action, sending $\bar{t}$ to infinity, thereby making $\chi, H, \overline{\sigma}, \eta$ infinitely massive. Thus we are left with the observables which are: local observables, which are the gauge invariant functions of σ:

$$\mathcal{O}_P^{(0)} = P(\sigma) \tag{3.18}$$

and their cohomological descendants:

$$\begin{aligned} \int_C \mathcal{O}_P^{(1)} &= \int_C \frac{\partial P}{\partial \sigma^a}\psi^a, \quad C \in H_1(\Sigma) \\ \int_\Sigma \mathcal{O}_P^{(2)} &= \int_\Sigma \frac{1}{2}\frac{\partial^2 P}{\partial\sigma^a\partial\sigma^b}\psi^a\wedge\psi^b + \frac{\partial P}{\partial\sigma^a}F^a \end{aligned} \tag{3.19}$$

Here $a, b = 1, \ldots, \dim\mathbf{g}$ are the Lie algebra indices. The main property of the sequence $\mathcal{O}_P^{(0)}, \mathcal{O}_P^{(1)}, \mathcal{O}_P^{(2)}$ of local operators, constructed out of a single gauge invariant P is that they obey the descent equation:

$$\mathrm{d}\mathcal{O}_P^{(i)} = \{Q, \mathcal{O}_P^{(i+1)}\} \tag{3.20}$$

which implies that

$$\int_\Sigma t^{(2)}\mathcal{O}_{P_2}^{(0)} + s^{(1)}\wedge\mathcal{O}_{P_1}^{(1)} + t^{(0)}\mathcal{O}_{P_0}^{(2)} \tag{3.21}$$

is a Q-invariant deformation of the Lagrangian of the theory (3.11) for the c-number valued closed differential forms $t^{(0,2)} \in H^{\mathrm{even}}(\Sigma), s^{(1)} \in H^{\mathrm{odd}}(\Sigma)$:

$$\mathbf{W} \longrightarrow \mathbf{W} + t^{(2)}P_2 + s^{(1)}P_1 + t^{(0)}P_0 \tag{3.22}$$

In the language of the two-dimensional $\mathcal{N}=2$ theories, the 2-observables $\mathcal{O}_W^{(2)}$ correspond to the *twisted superpotential* terms, the Q-exact term (3.17) comes from the conjugate *twisted superpotential* term $\overline{W} = \frac{i}{2}\bar{t}\,\mathrm{tr}\,\overline{\sigma}^2$.

Now, by identifying the field σ of the supersymmetric Yang–Mills theory with the field ϕ of the ordinary Yang–Mills theory one embeds the latter as a subsector of the former. Moreover, the correlation functions of the Q-invariant observables of the physical Yang–Mills theory are identified with the correlation functions of the Q-invariant observables of the $\mathcal{N}=2$ twisted theory, modulo small subtlety which we discuss later.

3.2.4. *Gauge theory partition function*

The result which can be derived following the strategy of [34], is:

$$\boxed{\begin{aligned} &\mathcal{Z}(\mathbf{W};\Sigma,\mathcal{P}) = \frac{1}{|\mathcal{W}|}\sum_{\lambda\in\Lambda_{\mathrm{w}}}\chi_\lambda(\mathcal{P})\exp\int_\Sigma \mathbf{\Omega}^{\mathrm{eff}}(\mathbf{\Phi}_\lambda) + \langle\mathbf{\Phi}_\lambda,\lambda+\rho\rangle \\ &\frac{\partial\mathbf{\Omega}}{\partial\varphi}(\mathbf{\Phi}_\lambda) = \lambda+\rho, \quad \mathbf{\Phi}_\lambda\in H^*(\Sigma;\mathbf{t}) \\ &\mathbf{\Omega}^{\mathrm{eff}} = \mathbf{\Omega} - \frac{1}{2}e(\Sigma)\log\mathrm{Det}\left\|\frac{\partial\varphi}{\partial\sigma}\frac{\partial^2\mathbf{\Omega}}{\partial\varphi\partial\varphi}\frac{\partial\varphi}{\partial\sigma}\right\| \end{aligned}} \tag{3.23}$$

An equivalent form of the answer (3.23):

$$\boxed{\begin{aligned} &\mathcal{Z}(\mathbf{W};\Sigma,\mathcal{P}) = \sum_{\lambda\in\Lambda_{\mathrm{w}}^+}\chi_\lambda(\mathcal{P})\left[\left(\frac{\mathrm{Det}_{\mathbf{t}}\frac{\partial^2 w}{\partial\varphi\partial\varphi}}{\mathrm{Det}_{\mathbf{g}/\mathbf{t}}\,\mathrm{ad}(\varphi)}\right)^{h-1}\exp\widetilde{t}(\varphi)\right]_{\varphi=\varphi_\lambda} \\ &\frac{\partial w}{\partial\varphi}(\varphi_\lambda) = (\lambda+\rho), \quad \varphi_\lambda\in\mathbf{t} \\ &\widetilde{t}(\varphi) = t^{\omega}(\varphi) - \frac{1}{2}\eta_{IJ}\frac{\partial s^I}{\partial\varphi_i}\frac{\partial s^I}{\partial\varphi_i}\left[\left(\frac{\partial^2 w}{\partial\varphi\partial\varphi}\right)^{-1}\right]^{ij} \end{aligned}} \tag{3.24}$$

We now remind three possible derivations of (3.23) and (3.24).

Nonabelian derivation. The idea of this derivation [34] is to make a field redefinition under the path integral to reduce the problem to the case $W=\frac{1}{2}t(\phi,\phi)$ and then use the known results on the two-dimensional pure Yang–Mills theory.

$$Z = \sum_{\lambda\in P_+}\left(\left(\frac{\sqrt{t}}{2\pi}\right)^{\dim G}\frac{\mathrm{Vol}(G)}{\dim R_\lambda}\right)^{2h-2}\exp\sum_k t^{-k}\epsilon_k c_k(R_\lambda), \tag{3.25}$$

the sum being over the dominant weights of G, i.e. the set of all irreducible representations (there are some subtleties when $\pi_1(G)\neq 0$, see [34]).

Abelianization. The idea of this derivation [35] is to project the nonabelian fields onto the abelian ones, integrate along the fibers of the projection, and then analyze the effective abelian theory.

Localization. The idea of this derivation is to first look at the path integral with fixed σ, and then use the localization onto the fixed locus of the Q-symmetry, i.e. onto the space of gauge fields, preserved by the gauge transformation generated by σ, see [36]. For generic σ this set is empty. Indeed, the equation $d_A\sigma = 0$ implies at the generic point on Σ that the gauge field A belongs to the maximal torus of G determined by σ, and that all the gauge-invariant functions of σ are constant:

$$\mathrm{d}\mathcal{O}_P^{(0)} = \frac{\partial P}{\partial \sigma^a} d_A \sigma^a = 0$$

Thus the path integral over all fields but σ is a distribution in the functional space of σ's, supported at the locus of the constant (up to a gauge transformation) fields [37]. The determinants one sees in (3.23) and (3.24) are the standard Bott–Duistermaat–Heckmann type products of weights of the action of σ on the normal bundle to the Q-fixed locus.

The case of $G = U(N)$. In this case the formulae (3.23) and (3.24) can be made rather explicit. The gauge invariant functions W, S^I, T^ω are the $\mathcal{S}_N$-invariant functions $w(\varphi_1, \ldots, \varphi_N)$, $s^I(\varphi_1, \ldots, \varphi_N)$, $t^\omega(\varphi_1, \ldots, \varphi_N)$ of the N eigenvalues $i\,(\varphi_1, \ldots, \varphi_N)$ of the $N \times N$ anti-Hermitian matrix ϕ. The principal G-bundles $\mathcal{P}$ over Σ are classified by a first Chern class $c_1(\mathcal{P})$. Instead of computing the partition function (3.9) for fixed $\mathcal{P}$ it is convenient to sum over all topological classes of $\mathcal{P}$, with the Upsilon angle term

$$\mathrm{e}^{\frac{\vartheta}{2\pi}\int_\Sigma \mathrm{tr}\, F_A}$$

to distinguish them. The Upsilon term is actually a particular deformation of $\mathbf{W}$:

$$\mathbf{W} \longrightarrow \mathbf{W} + \frac{\vartheta}{2\pi}\,\mathrm{tr}\,\phi \tag{3.26}$$

so that we shall not write it out explicitly. The partition function (3.9) with these modifications understood, can be written, more explicitly, as:

$$\begin{aligned}
\mathcal{Z}(\mathbf{W};\Sigma) = {} & \frac{1}{\mathrm{Vol}(\mathcal{G})}\int DA_{ij}\,D\psi_{ij}\,D\varphi_i \prod_{i\neq j}\mathrm{Det}_{\Gamma(\Omega^0(\Sigma)\otimes\mathcal{E}_{ij})}\left(\varphi_i - \varphi_j\right)\times \\
& \times\exp\int_\Sigma \sum_{i=1}^N \frac{\partial w}{\partial\varphi_i}\mathrm{d}A_{ii} + \frac{1}{2}\sum_{i,j=1}^N\int_\Sigma \frac{\partial^2 w}{\partial\varphi_i\partial\varphi_j}\psi_{ii}\wedge\psi_{jj} + \\
& + \frac{1}{2}\sum_{i,j=1}^N\int_\Sigma\left(\frac{\partial w}{\partial\varphi_i} - \frac{\partial w}{\partial\varphi_j}\right)\left(\frac{\psi_{ij}\wedge\psi_{ji}}{\varphi_i-\varphi_j} + A_{ij}\wedge A_{ji}\right) + \\
& + \sum_{I=1}^{2h}\sum_{i=1}^N\int_\Sigma \frac{\partial s^I}{\partial\varphi_i}\psi_{ii}\wedge\varpi_I + \int_\Sigma t^\omega(\varphi)\,\omega
\end{aligned} \tag{3.27}$$

where we fixed the gauge $\phi = i\,\mathrm{diag}\,(\varphi_1, \ldots, \varphi_N)$, and used

$$\frac{\partial W}{\partial\phi_{ij}} = \delta_{ij}\frac{\partial w}{\partial\varphi_i}, \quad \frac{\partial^2 W}{\partial\phi_{ij}\partial\phi_{kl}} = \delta_{ij}\delta_{kl}\frac{\partial^2 w}{\partial\varphi_i\partial\varphi_k} + \delta_{jk}\delta_{il}\frac{1-\delta_{ik}}{\varphi_i-\varphi_k}\left(\frac{\partial w}{\partial\varphi_i} - \frac{\partial w}{\partial\varphi_k}\right) \tag{3.28}$$

which can be derived using the well-known formulae from the quantum mechanical perturbation theory:

$$\delta\varphi_i = \delta\phi_{ii} + \sum_{j\neq i} \frac{\delta\phi_{ij}\delta\phi_{ji}}{\varphi_i - \varphi_j} + \cdots \tag{3.29}$$

In (3.27) we have the Faddeev–Popov determinant

$$\prod_{i\neq j} \mathrm{Det}_{\Gamma(\Omega^0(\Sigma)\otimes\mathcal{E}_{ij})}(\varphi_i - \varphi_j) \tag{3.30}$$

associated with the gauge fixing. The structure group $G = U(N)$ of the gauge bundle reduces to the maximal torus $\mathbf{T} = U(1)^N$ commuting with ϕ. The diagonal components A_{ii} of the gauge field are the corresponding abelian gauge fields. The fluxes

$$n_i = \frac{1}{2\pi i}\int_\Sigma \mathrm{d}A_{ii} \tag{3.31}$$

are to be summed over. The line bundle $\mathcal{E}_{ij}$ we see in (3.30) and (3.27) is associated with the root $e_i - e_j$. It has the first Chern class $n_i - n_j$. Upon integrating out A_{ij} and ψ_{ij} for $i \neq j$ one gets a determinant in denominator, similar to (3.30), except that the operator of multiplication by $\varphi_i - \varphi_j$ acts on the one-forms twisted by $\mathcal{E}_{ij}$. The integral over A_{ii} in the given topological class fixes φ to be a constant on Σ[6] The integral over the bosonic fluctuations of φ_i, and A_{ii} is almost cancelled by the integral over the fermionic fluctuations of ψ_{ii}. The mismatch is due to the difference of the number of the zero modes: there are h zero modes[7] of $\psi_{ii}^{0,1}$ and there is one zero mode of φ_i. As fr the determinants of the $\mathcal{E}_{ij}$ fluctuations, the difference of the number of the corresponding zero modes is $\dim H^0(\Sigma)\otimes\mathcal{E}_{ij} - \dim H^{0,1}(\Sigma)\otimes\mathcal{E}_{ij} = 1 - h + n_i - n_j$. Assembling all the factors, taking into account the contribution of the sources s^I, t^ω, and ignoring the $(\mathrm{Vol}U(N))^{2h-2}$ factors we arrive at

[6] In comparing with [34] it is useful to remember that the Hessian of W on $\mathrm{Lie}\,U(N)$ can be expressed through the derivatives of $w(\varphi_1, \ldots, \varphi_N)$:

$$\mathrm{Det}_{a,b=1}^{N^2}\left\| \frac{\partial^2 W}{\partial\phi^a\partial\phi^b}\right\| = \det_{i,j=1}^{N}\left\|\frac{\partial^2 w}{\partial\varphi_i\partial\varphi_j}\right\| \times \prod_{i\neq j=1}^{N}\left(\frac{\frac{\partial w}{\partial\varphi_i} - \frac{\partial w}{\partial\varphi_j}}{\varphi_i - \varphi_j}\right). \tag{3.32}$$

[7] We use (0, 1)-components to account for the difference between the Pfaffians and the determinants.

[cf. (3.24)]:

$$
\begin{aligned}
\mathcal{Z}(\mathbf{W};\Sigma) &= \frac{1}{N!}\sum_{\vec{n}}\int \mathrm{d}\varphi_1\cdots\mathrm{d}\varphi_N \left(\mathrm{Det}_{i,j}\left\|\frac{\partial^2 w}{\partial\varphi_i\partial\varphi_j}\right\|\right)^h \prod_{i\neq j}\left(\varphi_i-\varphi_j\right)^{1-h+n_i-n_j}\times \\
&\quad\times \mathrm{e}^{2\pi i\sum_i \frac{\partial w}{\partial\varphi_i}n_i + t^{\omega}(\varphi) - \frac{1}{2}\eta_{IJ}\left(\partial^2 w\right)^{-1,ij}\frac{\partial s^I}{\partial\varphi_i}\frac{\partial s^J}{\partial\varphi_j}} = \\
&= \sum_{\lambda}\left(\mathrm{Det}_{i,j}\left\|\frac{\partial^2 w}{\partial\varphi_i\partial\varphi_j}\right\|\prod_{i\neq j}\left(\varphi_i-\varphi_j\right)^{-1}\right)^{h-1}\mathrm{e}^{\widetilde{t}(\varphi)}\Bigg|_{\varphi=\varphi^{\lambda}}
\end{aligned}
\tag{3.33}
$$

$$
\varphi^{\lambda}:\ \frac{\partial w}{\partial\varphi_i} = \lambda_i - i + \frac{1}{2}, \qquad \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \lambda_N)
$$

$$
\widetilde{t}(\varphi) = t^{\omega}(\varphi) - \frac{1}{2}\eta_{IJ}\left(\partial^2 w\right)^{-1,ij}\frac{\partial s^I}{\partial\varphi_i}\frac{\partial s^J}{\partial\varphi_j}
$$

Note that we can write:

$$
\varphi_i^{\lambda} = z\left[\lambda_i - i + \frac{1}{2}\right] \tag{3.34}
$$

where $z[x]$ is the function, which is the formal inverse to

$$
y(x) = \sum_{k=0}^{N-1}\frac{\partial\mathbf{w}}{\partial\mathcal{O}_{k+1}}\frac{1}{k!}x^k \tag{3.35}
$$

with

$$
\mathcal{O}_k = \frac{1}{k!}\,\mathrm{tr}\,\phi^k \tag{3.36}
$$

where we represented the symmetric function $w(\varphi_1,\ldots,\varphi_N)$ as the function of $\mathcal{O}_1,\ldots,\mathcal{O}_N$:

$$
w(\varphi_1,\ldots,\varphi_N) = \mathbf{w}(\mathcal{O}_1,\ldots,\mathcal{O}_N)
$$

The subtlety. The correlation functions of the supersymmetric Yang–Mills theory are perturbative in the couplings $\alpha^{(i)}$. Indeed, these correlation functions are given by the integrals over the finite-dimensional moduli space $\mathcal{M}_G$ of flat connections, where the operators $\mathcal{O}_P^{(i)}$ are identified with the Donaldson classes, obtained by the slant products of the characteristic classes of the universal bundle over $\mathcal{M}_G\times\Sigma$ with the homology classes of Σ.

The correlation functions of the "physical" Yang–Mills theory are more complicated – in addition to the integrals over $\mathcal{M}_G$, which are the correlation functions of the $\mathcal{N}=2$ theory, they also get some contributions from the "higher critical points" of the Yang–Mills function. These contributions are non-perturbative in the couplings ϵ_k in (3.25) and can be separated from the result of the topological theory by analyzing the Poisson resummation of (3.25) as in [34].

4. The Synthesis

We are now approaching our main point. The partition function (1.22) has exactly the form (3.23), (3.24), (3.33) where the rôle of the gauge invariant functions **W** is played by the power-sum symmetric functions:

$$
\begin{aligned}
w(\varphi) &= -\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\tilde{t}^{\mathbf{1}}_{k}\sum_{i}\varphi_i^{k+1}\\
s^I(\varphi) &= \sum_{k=0}^{\infty}\frac{1}{(k+1)!}s^I_k\sum_{i}\varphi_i^{k+1}\\
t^{\omega}(\varphi) &= \sum_{k=0}^{\infty}\frac{1}{(k+1)!}t^{\omega}_k\sum_{i}\varphi_i^{k+1}
\end{aligned}
\tag{4.1}
$$

where the gauge invariant operator $\frac{1}{k!}\operatorname{tr}\phi^k$ in the $U(N)$ gauge theory is replaced by the $\mathbf{t^1}$-evolved Chern character $c_k(\lambda;\hbar,\mathbf{t^1}) = \mathrm{Coeff}_{u^k}\operatorname{tr}\mathrm{e}^{u\phi}$ where

$$\phi = z[\hbar\Xi]$$

and the infinite matrix Ξ is in the formal neighborhood of the operator L_0 acting in the first quantized Hilbert space of the single free fermion on a circle:

$$\Xi \sim \mathrm{diag}\left(\lambda_i - i + \frac{1}{2}\right)_{i=1}^{\infty} \tag{4.2}$$

Indeed, the evolved Plancherel measure $\exp\mathbf{r}(\lambda;\hbar,\mathbf{t^1})$ can be interpreted as the regularization of the infinite N version of:

$$
\begin{aligned}
\mathbf{r}(\lambda;\hbar,\mathbf{t^1}) &\sim -\frac{1}{2}\log\left(\mathrm{Det}\left\Vert\frac{\partial^2 w}{\partial\varphi_i\partial\varphi_j}\right\Vert\prod_{i\neq j}\frac{1}{\varphi_i-\varphi_j}\right) =\\
&= -\frac{1}{2}\sum_i \log\left(1-\mathbf{t^1}'(\varphi_i)\right) + \sum_{i<j}\log\left(\varphi_i-\varphi_j\right)\sim\\
&\sim -\frac{1}{2}\left(\frac{d}{ds}\bigg|_{s=0}\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{\mathrm{d}u}{u}u^s\sum_i \mathrm{e}^{iuz[\lambda_i-i+\frac{1}{2}+i\varepsilon]}\sum_j \mathrm{e}^{-iuz[\lambda_j-j+\frac{1}{2}-i\varepsilon]}\right)_{\varepsilon^0}
\end{aligned}
\tag{4.3}
$$

while the $\mathbf{g}_k(\lambda;\hbar,t^{\mathbf{1}})$ symmetric functions (2.21) are the regularization of the sums

$$\mathbf{g}_k(\lambda;\hbar,\mathbf{t^1}) \sim \frac{1}{k!}\sum_i\frac{\varphi_i^k}{w''(\varphi_i)}$$

which we see in

$$\frac{1}{2}\eta_{IJ}\left(\partial^2 w\right)^{-1,ij}\frac{\partial s^I}{\partial\varphi_i}\frac{\partial s^J}{\partial\varphi_j} = \frac{1}{2}\eta_{IJ}\sum_{k,l=0}^{\infty}\frac{1}{k!l!}s^I_k s^J_l\sum_i\frac{\varphi_i^{k+l}}{w''(\varphi_i)}$$

Thus, we can identify the target space theory of the topological string with a generalization of the two-dimensional gauge theory, a kind of $U(\infty)$ theory. The precise definition of the gauge group **G** should involve some sort of a formal functional analysis, which we are not aware of. It is amusing to note, however, that the **G**-invariant functions on the Lie algebra of **G** are given by the regularized traces related to Dixmier traces. We are thus suggesting our string field theory may serve as a definition of the quantum theory with the spectral action, as in [38].

The topological string amplitudes at fixed β and g are the finite polynomials in the couplings t_k^{ω}, due to the dimensional counting. Given the remark at the end of the previous section we are forced to claim that it is the $\mathcal{N}=2$ supersymmetric theory (in its twisted form) which should be identified with the string field theory, rather then the ordinary Yang–Mills theory (in this sense we are not giving the string theory of the large N two-dimensional QCD, as studied in [20]). However, should a non-perturbative definition of the topological string theory come along, one might wish to reconsider this identification. In particular, it may happen that the "instantons", i.e. the non-trivial Yang–Mills connections of the Hilbert space gauge group **G** are precisely the non-perturbative corrections to the topological string amplitudes.

Multi-trace deformations

The partition function (1.22) does not map to the full gauge theory partition function. What is lacking are the multi-trace deformations

$$\mathbf{W} \longrightarrow \mathbf{W} + \sum_{\kappa=(k_1\geq k_2\geq\cdots\geq k_p)} \widetilde{T}_\kappa \frac{\mathrm{tr}\,\phi^{k_1}\,\mathrm{tr}\,\phi^{k_2}\cdots\,\mathrm{tr}\,\phi^{k_p}}{k_1!k_2!\cdots k_p!}.$$

They correspond to the non-local (on the worldsheet) enumerative invariants of the topological string, defined in (1.27) and (1.28).

Equivariant gauge theory in two dimensions

When the topologically twisted $\mathcal{N}=2$ supersymmetric gauge theory is studied on a worldsheet with isometries, the theory can be deformed by modifying its scalar supercharge by making it into the equivariant differential.

Physically this is equivalent to studying the theory in the Ω-background [11]. The supercharge Q is modified to Q_V, which acts as follows:

$$\begin{aligned} Q_V A &= \psi, \quad Q_V\psi = D_A\sigma + \iota_V F_A \\ Q_V\chi &= H, \quad Q_V H = [\chi,\sigma] + \iota_V D_A\chi \\ Q_V\overline{\sigma} &= \eta, \quad Q_V\eta = [\overline{\sigma},\sigma] + \iota_V D_A\overline{\sigma} \end{aligned} \tag{4.4}$$

The square Q_V^2 is a gauge transformation generated by $\sigma + \iota_V A$ and a translation $\mathcal{L}_V$ along the vector field V (recall that $\mathcal{L}_V A = \iota_V F_A + D_A(\iota_V A)$). The gauge theory action in the Ω-background is given by ($\overline{V}$ is another isometry, which commutes with V, $[V,\overline{V}]=0$):

$$S=\frac{1}{2g^2}Q_V\int \mathrm{tr}\left(\chi\left(F_A-\tfrac{1}{2}\star H\right)+\psi\star\left(D_A\overline{\sigma}-\iota_{\overline{V}}F_A\right)+\right.$$
$$\left.+\left(\eta-\iota_{\overline{V}}\psi\right)\star\left([\sigma,\overline{\sigma}]-\iota_V D_A\overline{\sigma}+\iota_{\overline{V}}D_A\sigma+\iota_V\iota_{\overline{V}}F_A\right)\right) \quad (4.5)$$

Given the two-dimensional nature of our problem we can only work on the two-dimensional sphere $\mathbf{S}^2$, and take $V=\epsilon\partial_\varphi$, $\overline{V}=\overline{\epsilon}\partial_\varphi$. The dependence on $\overline{\epsilon}$ is Q_V-exact. We take the limit $\overline{\epsilon}\to\infty$, and throw away the terms $Q_V(\chi F_A+(\eta-\iota_{\overline{V}}\psi)\star[\sigma,\overline{\sigma}])$. The path integral would localize onto the V-invariant gauge field configurations, whose field strength is concentrated near one of the two fixed points on X. For the gauge group $U(N)$ the corresponding partition function is given by:

$$\mathcal{Z}^{U(N)}_{\mathbf{CP}^1}=\sum_{\vec{n}\in\mathbf{Z}^N}\int \mathrm{d}\phi_1\cdots\mathrm{d}\phi_N\prod_{i<j}\left(\phi_{ij}+\frac{\varepsilon}{2}n_{ij}\right)\left(\phi_{ij}-\frac{\varepsilon}{2}n_{ij}\right)\times$$
$$\times\exp\frac{1}{\varepsilon}\sum_{k=0}^{\infty}\sum_{i=1}^{N}x_k\mathrm{ch}_k(\mathcal{E}_0)-x_k^*\mathrm{ch}_k(\mathcal{E}_\infty) \quad (4.6)$$

where $\phi_{ij}=\phi_i-\phi_j$, $n_{ij}=n_i-n_j$ and

$$\mathrm{ch}(\mathcal{E}_0)=\sum_{i=1}^{N}\mathrm{e}^{(\phi_i+\frac{\varepsilon}{2}n_i)},\qquad \mathrm{ch}(\mathcal{E}_\infty)=\sum_{i=1}^{N}\mathrm{e}^{(\phi_i-\frac{\varepsilon}{2}n_i)}$$

are the localized Chern classes of the gauge bundle, n_i are the Chern classes of the line bundles L_i. In the limit $\varepsilon\to 0$, with $x_k=\frac{1}{\varepsilon}t_k^{\mathbf{1}}$, $x_k^*=-\frac{1}{\varepsilon}t_k^{\mathbf{1}}+t_k^{\omega}$, (4.6) goes over to: (3.33).

Given the general pattern of the string theory/gauge theory correspondence, we should test our proposed target space theory on the new ground. Namely, the equivariant Gromov–Witten theory of $\mathbf{CP}^1$ was studied in [39] and the partition function was shown to be given by a particular τ-function of a 2-Toda hierarchy. Can one reproduce this result using the equivariant version of the super-Yang–Mills theory? This question is addressed in [29].

5. Quantum Gravity in the Target Space

So far we discussed the standard type A topological string on X. The target space turns out to be a topological field theory. The reason is rather simple. Indeed, the type A amplitudes are independent of the complex structure of X and depend only on its Kähler class. In two dimensions the only diffeomorphism invariants of a symplectic manifold X are the topology of X and the area, i.e. the Kähler class.

The topological field theory in two dimensions comes with its moduli space M of deformations. At a given point T of the moduli space M of the topological field theories this is a commutative associative algebra $\mathbf{A}_T$, and a functional

$\langle\cdot\rangle : \mathbf{A}_T \to \mathbf{C}$. In fact, the tangent space $T_T M$ is isomorphic to $\mathbf{A}_T$, thereby making M the so-called Frobenius manifold [40]. In our case the algebra is the algebra of (formal) functions on the space of partitions $\{\lambda\}$. It has a basis $\mathcal{O}_k$, $k = 0, 1, 2, \ldots$, where the function $\mathcal{O}_k$ evaluates to (we set $\hbar = 1$):

$$\mathcal{O}_k(T, \lambda) = \mathrm{Coeff}_{u^k} \sum_{i=1}^{\infty} \mathrm{e}^{u\, z_T\left[\lambda_i - i + \frac{1}{2}\right]} \tag{5.1}$$

with the formal function $z_T[x]$ which depends on $T \in M$ in the way we describe below, and the functional:

$$\langle \mathcal{O} \rangle_T = \sum_{\lambda} (-q)^{|\lambda|} \mathrm{e}^{2\mathbf{r}(\lambda; t(T,\lambda))} \mathcal{O}(T, \lambda) \tag{5.2}$$

Note that in the standard definition of the topological field theory the factor $(-q)^{|\lambda|}$ in (5.2) corresponds to the insertions of 0-observables, and is not usually considered. The infinite-dimensional nature of the space of deformations of our theory makes the q-factors necessary.

The space of deformations has the coordinates, the couplings T_κ. On the large phase space where only the single trace deformations $t_k^{\mathbf{1}} \mathcal{O}_{k+1}$ are turned one, the formula (5.2) holds, with the function $z_T[x]$ our old friend (1.14). On the very large space, where all the multi-trace deformations are included, the formula (5.2) holds when written in terms of the effective couplings $t_k(T, \lambda)$ which are defined analogously to (3.35): first define

$$t_k(T, \lambda) = \left. \frac{\partial \mathcal{W}_T}{\partial \mathcal{O}_{k+1}} \right|_{\lambda} \tag{5.3}$$

where

$$\mathcal{W}_T = \mathcal{O}_2 - \sum_{\kappa} T_\kappa \mathcal{O}_{k_1} \cdots \mathcal{O}_{k_{\ell(\kappa)}}, \tag{5.4}$$

then form

$$t_{T,\lambda}(z) = \sum_{k=0}^{\infty} t_k^{\mathbf{1}}(T, \lambda) \frac{z^{k+1}}{(k+1)!}$$

define $z_T[x]$ to be the formal inverse to $x - t_{T,\lambda}(x)$, and then impose the consistency equations (cf. [41]):

$$\left. \mathcal{O}_k \right|_{\lambda} = \mathrm{Coeff}_{u^{k+1}} \sum_{i=1}^{\infty} \mathrm{e}^{z_T[\lambda_i - i + \frac{1}{2}]} \tag{5.5}$$

The two-dimensional topological field theory can be coupled to the topological gravity. This is the target space gravity.

We can describe its observables directly in target space. We can also discuss its worldsheet definition. The latter is potentially interesting for more realistic quantum gravity theories.

The target space definition is the following. Consider the moduli space $\mathcal{M}_X$ of complex structures on X. The topological string amplitudes are independent of the choice of the complex structure on X. However, one can generalize them, so that they would define the closed differential forms on $\mathcal{M}_X$. Moreover, we can consider non-compact Riemann surfaces X, i.e. curves with punctures. The worldsheet realization of this idea is analogous to the construction in [42] for the three-dimensional targets X. Green has proposed in [24] to study the two-dimensional string backgrounds as the theories of worldsheets for yet another string theories. Our construction is the first almost practical realization of this idea.

6. Conclusions

In this paper we have calculated explicitly the full partition function

$$\mathcal{Z}(\mathbf{t}; q, \hbar) = \exp \sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g(\mathbf{t}; q) \tag{6.1}$$

of the topological string of type A with the general two-dimensional target space Σ. The couplings $\mathbf{t} = (\mathbf{t}_k)_{k=0}^{\infty}$ take values in $H^*(\Sigma)$, q is a formal parameter counting the worldsheet instantons, and $\hbar$ is a topological string coupling constant.

The first surprise is that the disconnected partition function $\mathcal{Z}$ has an additive form. It is given by the sum

$$\mathcal{Z} = \sum_{\lambda} (-q)^{|\lambda|} \mathcal{Z}_{\lambda} \tag{6.2}$$

over all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0)$. The contribution $\mathcal{Z}_{\lambda}$ is a generalized "plane wave" on the space of the couplings. It is natural, therefore, to view it as a kind of a wavefunction, obtained by the quantization of the loop space of the cohomology of the target space, in agreement with [27]. That the disconnected partition function of the type A topological string on the three-dimensional Calabi–Yau manifold is a wavefunction of some seven-dimensional theory has been suspected for a long time [43–46]. Our topological string has the non-critical target space, which explains the need for the use of the gravitational descendants. What about the Chern–Simons nature of the target space theory? Our theory looks like a two-dimensional gauge theory with the infinite-dimensional gauge group $\mathbf{G}$. It is tempting to try to rewrite this theory as a higher-dimensional gauge theory.

One possible approach is inspired by the Donaldson–Thomas/Gromov–Witten correspondence [47]. Consider a three-dimensional non-compact Calabi–Yau manifold, the total space $\mathcal{X}_{\Sigma}$ of the vector bundle $K_{\Sigma}^{1/2} \otimes \mathbf{C}^2$ over Σ. The topological string on $\mathcal{X}_{\Sigma}$ can be defined by means of the $\mathbf{T} = \mathbf{C}^{\times} \times \mathbf{C}^{\times}$ equivariant Gromov–Witten theory, where the torus acts on the fiber. If one chooses a one-dimensional torus $\mathbf{C}^{\times} \subset \mathbf{T}$ which preserves the holomorphic top degree form on $\mathcal{X}_{\Sigma}$, then the corresponding Gromov–Witten invariants will be those of Σ [9]. On the other hand, the correspondence [2,47] would map those to the correlation functions in the six-dimensional "$U(1)$ gauge theory". Thus the infinite-dimensional gauge group **G** in two dimensions may well come from the $U(1)$ gauge group (or rather its noncommutative version) in six dimensions.

Another surprise is the need to extend the definition of the topological string. The gauge theory which we identified with the target space theory has many more gauge invariant deformations then we see in (1.22).

It is also interesting to compare the form (1.22) of the string partition function with that of more general $\mathcal{N}=2$ supersymmetric gauge theories in two dimensions. It has been recently suggested [48] that the equations $\mathcal{W}'(z_i)=0$, with $\mathcal{W}$ depending on λ_i's as in (1.25) can be interpreted as Bethe equations for some quantum integrable system. In that language the couplings $t_k^{\mathbf{1}}$ can be viewed as the effective couplings, which result in integrating out some massive matter fields. If some of the matter fields are kept in the Lagrangian, one may hope to get the gauge theory describing topological string on more interesting target spaces. For example, it is tempting to conjecture that the **G** version of the theory studied in [49–51] would describe the (equivariant) topological string on $T^*\Sigma$.

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Appendix A. Toolbox: Partitions, Measures, Casimirs, Useful Functions

Recall that *partition* is a finite sequence of non-negative integeres:

$$\lambda = \left(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell_\lambda} > 0\right). \tag{A.1}$$

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[9]IHES seminars in June 2007, the conference on Algebraic Geometry and String theory at RIMS, Kyoto University in January 2008, the London Mathematical Society lectures at the Imperial College in April 2008.

A.1. PARTITION MEASUREMENTS

The number ℓ_λ of non-negative integers in λ is called the length of the partition λ, while the sum $\lambda_1+\lambda_2+\cdots+\lambda_{\ell_\lambda}=|\lambda|$ is called the size of the partition λ. We shall identify λ with an infinite sequence $\lambda_1\geq\lambda_2\geq\cdots$ by setting $\lambda_{\ell_\lambda+1}=\lambda_{\ell_\lambda+2}=\cdots=0$.

The Casimirs $c_k(\lambda)$ are defined as follows. Introduce the generating function $c_\lambda(t)$, called the Chern character of the partition λ:

$$c_\lambda(t)\equiv\sum_{i=1}^{\infty}\mathrm{e}^{t\left(\lambda_i-i+\frac{1}{2}\right)} \tag{A.2}$$

For the empty λ:

$$c_0(t)=\frac{1}{2\sinh\left(\frac{t}{2}\right)} \tag{A.3}$$

The small t expansion of $c_\lambda(t)$ gives the Casimirs $c_k(\lambda)$:

$$c_\lambda(t)=\frac{1}{t}+\sum_{k=1}^{\infty}c_k(\lambda)\,t^k \tag{A.4}$$

Explicitly,

$$c_k(\lambda)=\frac{\mathbf{b}_{k+1}}{(k+1)}+\sum_{i=1}^{\ell_\lambda}\sum_{l=0}^{k-1}\binom{k}{l}\left(-i+\frac{1}{2}\right)^l\lambda_i^{k-l} \tag{A.5}$$

For example:

$$\begin{aligned}c_1(\lambda)&=|\lambda|-\frac{1}{24}\\ c_2(\lambda)&=\frac{1}{2}\sum_i\lambda_i\,(\lambda_i-2i+1)\end{aligned} \tag{A.6}$$

The numbers $\mathbf{b}_k$ in (A.5) are essentially the Bernoulli numbers, $\mathbf{b}_{2g+1}=0$, $\mathbf{b}_{2g}=\left(2^{1-2g}-1\right)B_{2g}$:

$$\frac{\mathbf{b}_k}{k!}=\ \mathrm{Coeff}_{t^{k-1}}\frac{1}{2\sinh\left(\frac{t}{2}\right)}\ , \tag{A.7}$$

For example, $\mathbf{b}_0=1$, $\mathbf{b}_2=-\frac{1}{12}$, $\mathbf{b}_4=\frac{7}{240}$, $\mathbf{b}_6=-\frac{31}{1344}$, $\mathbf{b}_8=\frac{127}{3840}$.

A.2. PLANCHEREL MEASURE

The factor

$$\mu^2(\lambda;\hbar)=\left(\frac{\dim\lambda}{\hbar^{|\lambda|}\,|\lambda|!}\right)^2 \tag{A.8}$$

in (2.10) is the so-called *Plancherel measure* on the space of representations of the symmetric group $S_{|\lambda|}$. It can be written as a product over the boxes of the Young diagram of the partition λ of the hook-lengths

$$\mu^2(\lambda;\hbar) = \prod_{\square\in\lambda} \frac{1}{(\hbar h_\square)^2}$$

The Plancherel measure can also be viewed as a regularized version of the infinite-dimensional version of the Vandermonde determinant of the matrix with the eigenvalues $y_i = \lambda_i - i + \frac{1}{2},\ i = 1,2,\ldots$:

$$\frac{\dim\lambda}{|\lambda|!} = \prod_{1\le i<j}^{\infty} \frac{\lambda_i - \lambda_j - i + j}{j-i} \tag{A.9}$$

The latter product is understood as follows. First, split the product over j as the product over $1 \le j \le \ell_\lambda$ and then as the product over $j > \ell_\lambda$:

$$\frac{\dim\lambda}{|\lambda|!} = \prod_{i=1}^{\ell_\lambda}\left[\prod_{i<j\le\ell_\lambda} \frac{\lambda_i - \lambda_j - i + j}{j-i} \prod_{j=\ell_\lambda+1}^{\infty} \frac{\lambda_i - i + j}{j-i}\right] \tag{A.10}$$

The first product is finite. The second product looks infinite, but in fact almost all terms cancel, leaving only a finite number of terms in the denominator:

$$\prod_{j=\ell_\lambda+1}^{\infty}\left(\frac{\lambda_i - i + j}{j-i}\right) = \prod_{j=1}^{\lambda_i} \frac{1}{j - i + \ell_\lambda} = \frac{(\ell_\lambda - i)!}{(\lambda_i - i + \ell_\lambda)!}$$

With the help of the Chern character c_λ we can rewrite the Plancherel measure as follows:

$$\frac{\dim\lambda}{\hbar^{|\lambda|}\,|\lambda|!} = \exp - \left.\frac{d}{ds}\right|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} du\ \{c_\lambda(\hbar u)\, c_\lambda(-\hbar u) - c_0(\hbar u)\, c_0(-\hbar u)\}_+ \tag{A.11}$$

where the operation $\{\ldots\}_+$ is defined as:

$$\left\{e^{-ux}\right\}_+ = e^{-ux}, \quad \mathrm{Re}\,(x) > 0 \tag{A.12}$$

and zero otherwise. By writing $c_\lambda(\hbar u) = c_0(\hbar u) + \delta c_\lambda(\hbar u)$, where

$$\delta c_\lambda(\hbar u) = \sum_{i=1}^{\ell_\lambda}\left(e^{\hbar u \lambda_i} - 1\right) e^{\hbar u\left(-i+\frac{1}{2}\right)}$$

is a finite sum of exponents, we show that

$$c_\lambda(\hbar u)\, c_\lambda(-\hbar u) - c_0(\hbar u)\, c_0(-\hbar u) = -\frac{\delta c_\lambda(\hbar u) - \delta c_\lambda(-\hbar u)}{2\sinh\left(\frac{\hbar u}{2}\right)} + \delta c_\lambda(\hbar u)\delta c_\lambda(-\hbar u)$$

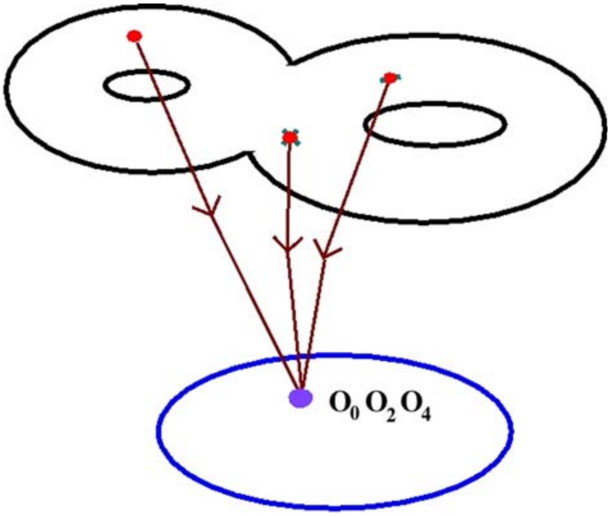

Figure 4. Multi-trace operators on the worldsheet.

is also a finite sum of exponents.[10] Thus the integrand in (A.11) is well-defined.

A.3. PROFILE REPRESENTATION

To advance further the representation (A.11) is not very convenient. As is routinely done in the theory of random partitions, one assigns to the partition λ a piece-wise linear function $f_\lambda(x)$, which is the Young diagram rotated by 135^0 and completed by two semi-infinite lines

Explicitly [see Figure 4; (1.18)]:

$$f_\lambda(x) = |x| + g_\lambda\left(x + \frac{1}{2}\hbar\right) - g_\lambda\left(x - \frac{1}{2}\hbar\right)$$
$$g_\lambda(x) = \sum_{i=1}^{\ell_\lambda} \left|x - \hbar\left(\lambda_i - i + \frac{1}{2}\right)\right| - \left|x - \hbar\left(-i + \frac{1}{2}\right)\right| \tag{A.13}$$

where ℓ_λ is the length of the partition λ, i.e. the number of non-zero terms $\lambda_i > 0$. The function $f_\lambda(x)$ equals $|x|$ for sufficiently large $|x|$. Using $f_\lambda(x)$ we rewrite the

[10]Recall the formula for the $SU(2)$ character: for $d \in \mathbf{Z}$,

$$\frac{\sinh\left(\frac{\hbar u d}{2}\right)}{\sinh\left(\frac{\hbar u}{2}\right)} = \sum_{m=\frac{1-d}{2}}^{\frac{d-1}{2}} \mathrm{e}^{m\hbar u}.$$

Chern character $c_\lambda(\hbar u)$ as follows:

$$c_\lambda(\hbar u) = \frac{1}{4\sinh\left(\frac{\hbar u}{2}\right)} \int_{\mathbf{R}} \mathrm{d}x\, f''_\lambda(x)\mathrm{e}^{ux} \ , \tag{A.14}$$

or, using $g_\lambda(x)$:

$$c_\lambda(\hbar u) = c_0(\hbar u) + \frac{1}{2}\int_{\mathbf{R}} \mathrm{d}x\, g''_\lambda(x)\mathrm{e}^{ux} \tag{A.15}$$

Substituting (A.14) into (A.11) and changing the order of integration we get the following "two-body interaction" representation of the Plancherel measure:[11]

$$\mu^2(\lambda) \equiv \mathrm{e}^{2\mathbf{r}(\lambda;0)} = \hbar^{\frac{1}{24}}\mathrm{e}^{\frac{\zeta'(-1)}{2}} \exp \frac{1}{8}\int\int \mathrm{d}\xi\mathrm{d}\eta\, f''_\lambda(\xi) f''_\lambda(\eta)\gamma_\hbar(\xi-\eta) \tag{A.16}$$

where we use

A.4. THE $\gamma_\hbar(x)$-FUNCTION

The $\gamma_\hbar$-function [52] accompanies many computations in topological string theory. It is defined, for $\mathrm{Re}(x+\hbar) > 0$, $\mathrm{Re}\hbar > 0$, by the explicit integral:

$$\gamma_\hbar(x) = \zeta'(-1) + \left.\frac{d}{ds}\right|_{s=0} \frac{\hbar^s}{\Gamma(s)} \int_0^\infty \frac{\mathrm{d}u}{u}\, u^s \frac{\mathrm{e}^{-ux}}{4\sinh^2\left(\frac{\hbar u}{2}\right)} \tag{A.17}$$

The function (A.17) obeys:

$$\gamma_\hbar(x+\hbar) + \gamma_\hbar(x-\hbar) - 2\gamma_\hbar(x) = \log\left(\frac{\mu}{x}\right) \tag{A.18}$$

and has the following asymptotic expansion as $x/\hbar \to \infty$:

$$\gamma_\hbar(x) = -\frac{x^2}{2\hbar^2}\left(\log\left(\frac{x}{\hbar}\right) - \frac{3}{2}\right) + \frac{1}{12}\log\left(\frac{x}{\hbar}\right) + \sum_{g=2}^\infty \frac{B_{2g}}{2g(2g-2)}\left(\frac{\hbar}{x}\right)^{2g-2} \tag{A.19}$$

The strange-looking constant $\zeta'(-1) \approx -0.165421$[12] in front of the integral in (A.17) is introduced so as to have:

$$\gamma_\hbar(0) = -\frac{1}{12}\log\hbar \tag{A.20}$$

[11] Use the relation:

$$\left.\frac{d}{ds}\right|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty u^s \frac{du}{u}\, c_0(\hbar u)\, c_0(-\hbar u) = -\zeta'(-1) - \frac{1}{12}\log\hbar$$

[12] Which can also be written as $\frac{1-\gamma-\log(2\pi)}{12} + \frac{\zeta'(2)}{2\pi^2}$.

We shall also need the “positive-frequency” part of $\gamma_\hbar(x)$:

$$\gamma_\hbar(x)_+ = \gamma_\hbar(-x) = \\ = -\frac{d}{ds}\left[\int_0^\infty \frac{u^{s-1}du}{\Gamma(s)} \frac{e^{ux}}{\left(e^{\frac{\hbar u}{2}} - e^{-\frac{\hbar u}{2}}\right)^2}\right]_{s=0}, \quad \text{for } \mathrm{Re}(x) < 0, \quad \mathrm{Re}(\hbar) \neq 0 \tag{A.21}$$

and zero otherwise. The right hand side of (A.21) defines a function $\gamma_\hbar(x)$ on the whole complex plane of the parameters $(x, \hbar)$ by analytic continuation. It solves the following finite difference equation:

$$\gamma_\hbar(x+\hbar) + \gamma_\hbar(x-\hbar) - 2\gamma_\hbar(x) = \log(-x) \tag{A.22}$$

which allows to extend the definition (A.21) to $\mathrm{Re}(x) \geq 0$. In addition, it is an even function of $\hbar$:

$$\gamma_{-\hbar}(x) = \gamma_\hbar(x) \tag{A.23}$$

The limit $x \to 0$ can be evaluated directly,[13] since the integral in (A.21) converges for $\mathrm{Re}(x \pm \hbar) < 0$.

Appendix B. Virasoro Constraints

It is shown in [32] that the following operators:

$$L_{-1} = -\frac{\partial}{\partial t_0^{\mathbf{1}}} + \frac{1}{\hbar^2} t_0^{\mathbf{1}} t_0^{\omega} + \\ + \sum_{l=0}^{\infty} \left(t_{l+1}^{\mathbf{1}} \frac{\partial}{\partial t_l^{\mathbf{1}}} + t_{l+1}^{\omega} \frac{\partial}{\partial t_l^{\omega}} \right) \tag{B.1}$$

$$L_0 = -\frac{\partial}{\partial t_1^{\mathbf{1}}} + \frac{1}{\hbar^2} t_0^{\mathbf{1}} t_0^{\mathbf{1}} - \\ -2\frac{\partial}{\partial t_0^{\omega}} + 2\sum_{l=0}^{\infty} t_{l+1}^{\mathbf{1}} \frac{\partial}{\partial t_l^{\omega}} + \\ + \sum_{l=0}^{\infty} \left(l t_l^{\mathbf{1}} \frac{\partial}{\partial t_l^{\mathbf{1}}} + (l+1) t_l^{\omega} \frac{\partial}{\partial t_l^{\omega}} \right) \tag{B.2}$$

[13] In particular,

$$-\frac{d}{ds}\left[\int_0^\infty \frac{u^{s-1}du\, e^{-\hbar u}}{\Gamma(s)\left(1-e^{-\hbar u}\right)^2}\right]_{s=0} = -\frac{d}{ds}\left[\hbar^{-s}\zeta(s-1)\right]_{s=0} = -\frac{1}{24}\log\left(\hbar^2\right) - \zeta'(-1) \tag{A.24}$$

and for $k > 0$:

$$\begin{aligned} L_k = \ & -(k+1)! \frac{\partial}{\partial t^{\mathbf{1}}_{k+1}} - \\ & -2(1)_{k+1} h_{k+1} \frac{\partial}{\partial t^{\omega}_{k}} + \\ & + \sum_{l=0}^{\infty} \left((l)_{k+1} t^{\mathbf{1}}_l \frac{\partial}{\partial t^{\mathbf{1}}_{k+l}} + (l+1)_{k+1} t^{\omega}_l \frac{\partial}{\partial t^{\omega}_{k+l}} \right) + \\ & +2 \sum_{l=0}^{\infty} (l)_{k+1} (h_{k+l} - h_{l-1}) t^{\mathbf{1}}_l \frac{\partial}{\partial t^{\omega}_{k+l-1}} + \\ & + \hbar^2 \sum_{l=0}^{k-2} (l+1)!(k-l-1)! \ \frac{\partial^2}{\partial t^{\omega}_l \partial t^{\omega}_{k-l-2}} \end{aligned} \tag{B.3}$$

where

$$(a)_b = \frac{(a+b-1)!}{(a-1)!},$$

annihilate the full string partition function of the topological string on $X = \mathbf{CP}^1$:

$$L_k \mathcal{Z} = 0, \quad k \geq -1 \tag{B.4}$$

Let $\mathcal{V}$ denote the subalgebra of the Virasoro algebra, generated by (B.1)–(B.3).

B.1. GEOMETRIZATION OF THE VIRASORO ACTION

Let $\tilde{t}^{\mathbf{1}}_k = t^{\mathbf{1}}_k - \delta_{k,1}$ (the dilaton shift). The constraints (B.4) become more transparent if we express them in terms of the functions $y(z), t^{\omega}(z)$:

$$\begin{aligned} y(z) &= z - \mathbf{t}^{\mathbf{1}}(z) = -\sum_{k=0}^{\infty} \tilde{t}^{\mathbf{1}}_k \frac{z^k}{k!} \\ t^{\omega}(z) &= \sum_{k=0}^{\infty} t^{\omega}_k \frac{z^{k+1}}{(k+1)!} \end{aligned} \tag{B.5}$$

Now the transformation $z \mapsto z + \epsilon_k z^{k+1}$ generates the transformation on the functions (y, t^{ω}), which is equivalent to the first and the third lines in (B.3). In fact, let us introduce a useful notation:

$$\begin{aligned} \nabla_k = \ & -(k+1)! \frac{\partial}{\partial t^{\mathbf{1}}_{k+1}} + \\ & + \sum_{l=0}^{\infty} \left((l)_{k+1} t^{\mathbf{1}}_l \frac{\partial}{\partial t^{\mathbf{1}}_{k+l}} + (l+1)_{k+1} t^{\omega}_l \frac{\partial}{\partial t^{\omega}_{k+l}} \right) \quad k \geq -1 \end{aligned} \tag{B.6}$$

The presence of other terms in (B.3) means that the partition function $\mathcal{Z}$ is not just a $\mathcal{V}$-invariant functional of $y(z)$ and $t^{\omega}(z)$, but rather a section of a line bundle. Let us introduce an auxiliary function $z[x]$, the formal inverse of $y(z)$:

$$y(z[x])=x, \quad z[y(x)]=x \tag{B.7}$$

The Lagrange inversion theorem gives the formula for $z[x]$:

$$z[x]=x+\sum_{n=0}^{\infty}\partial_x^n\left(\frac{\mathbf{t}^{\mathbf{1}}(x)^{n+1}}{(n+1)!}\right) \tag{B.8}$$

We have the following identities, for fixed x, and $k\geq -1$:

$$\nabla_k y(x)=x^{k+1}\partial_x y(x), \quad \nabla_k z[x]=-z[x]^{k+1} \tag{B.9}$$

Also, for $k>-1$, we have

$$\nabla_k t^{\omega}(y)=y^{k+1}\partial_y t^{\omega}(y) \tag{B.10}$$

which imply, again for fixed x:

$$\nabla_k t^{\omega}(z[x])=0 \ , \ \ k>-1 \tag{B.11}$$

For $k=-1$ and fixed x, y:

$$\nabla_{-1}t^{\omega}(y)=\partial_y t^{\omega}(y)-t_0^{\omega}, \quad \nabla_{-1}t^{\omega}(z[x])=-t_0^{\omega} \tag{B.12}$$

B.2. USEFUL FUNCTIONS AND OPERATORS

The function:

$$\Upsilon(x)=\sum_{l=0}^{\infty}\frac{\widetilde{t}_l^{\mathbf{1}}z[x]^l}{l!}(\log z[x]-h_l) \tag{B.13}$$

where

$$h_l=\sum_{m=1}^{l}\frac{1}{m}, \quad h_0=0 \tag{B.14}$$

obeys, for fixed x:

$$\nabla_k\Upsilon(x)=-\sum_{l=0}^{\infty}\widetilde{t}_l^{\mathbf{1}}\frac{z[x]^{k+l}}{l!}\sum_{m=0}^{k}\frac{l}{m+l} \tag{B.15}$$

In particular:

$$\begin{aligned}&\nabla_{-1}\Upsilon(x)=-t_0^{\mathbf{1}}z[x]^{-1}, \quad \nabla_0\Upsilon(x)=-\sum_{l=0}^{\infty}\widetilde{t}_l^{\mathbf{1}}\frac{z[x]^l}{l!}=x\\&\nabla_k\Upsilon(x)=-\sum_{l=0}^{\infty}\widetilde{t}_l^{\mathbf{1}}\frac{h_{k+l}-h_{l-1}}{(l-1)!}z[x]^{k+l}, \quad k>0\end{aligned} \tag{B.16}$$

with the convention

$$\frac{(h_{k+l}-h_{l-1})}{(l-1)!}=1, \qquad l=0, \quad k>-1 \tag{B.17}$$

We shall also use a related function:

$$\psi(x)=\Upsilon(x)+(x+t_0^{\mathbf{1}})\left(\log(x+t_0^{\mathbf{1}})-1\right)-t_0^{\mathbf{1}}\log z[x] \tag{B.18}$$

which obeys:

$$\begin{aligned}\nabla_{-1}\psi(x) &= \widetilde{t}_1^{\mathbf{1}}\log\left(\frac{x+t_0^{\mathbf{1}}}{z[x]}\right)\\ \nabla_0\psi(x) &= x+t_0^{\mathbf{1}}\\ \nabla_k\psi(x) &= \nabla_k\Upsilon(x)+t_0^{\mathbf{1}}z^k(x)\\ &= -\sum_{l=1}^{\infty}\widetilde{t}_l^{\mathbf{1}}\frac{h_{k+l}-h_{l-1}}{(l-1)!}z[x]^{k+l}, \quad k>0\end{aligned} \tag{B.19}$$

and

$$\psi(-t_0^{\mathbf{1}})=0 \tag{B.20}$$

Finally, $\psi(x)$ has a power series expansion in x near $x=0$ whose coefficients are power series[14] in $(t_k^{\mathbf{1}})$.

[14]Indeed, we can rewrite $\psi(x)$ as:

$$\psi(x)=(x+t_0^{\mathbf{1}})\left(\log\left(\frac{x+t_0^{\mathbf{1}}}{z[x]}\right)-1\right)-\sum_{l=1}^{\infty}\widetilde{t}_l^{\mathbf{1}}\frac{h_l}{l!}z[x]^l \tag{B.21}$$

First, we expand ψ near $x=-t_0^{\mathbf{1}}$ in powers of $(x+t_0^{\mathbf{1}})$:

$$\psi(x)=(x+t_0^{\mathbf{1}})\left(\log(1-t_1^{\mathbf{1}})+\sum_{k=1}^{\infty}\psi_k(\mathbf{t})\left(\frac{x+t_0^{\mathbf{1}}}{1-t_1^{\mathbf{1}}}\right)^k\right) \tag{B.22}$$

It is easy to see that the coefficient $\psi_k(\mathbf{t})$ is a homogeneous degree k polynomial in $\widetilde{t}_k=\frac{t_k^{\mathbf{1}}}{1-t_1^{\mathbf{1}}}$, $2\leq k\leq p+1$, where we assign a degree $k-1$ to $\widetilde{t}_k$. The first few coefficients are:

We use in our computations the following linear operator $\mathcal{I}$:

$$\mathcal{I}[f](x)=\frac{1}{\hbar}\int_{-t_0^{\mathbf{1}}}^{x} f(\xi)\mathrm{d}\xi+\mathcal{B}[f](x) \tag{B.24}$$

where

$$\mathcal{B}[f](x)=\frac{i}{\hbar}\int_0^\infty \frac{f(x+i\xi)-f(x-i\xi)}{1+\exp\left(\frac{2\pi\xi}{\hbar}\right)}\mathrm{d}\xi=\sum_{g=1}^{\infty}\frac{\mathbf{b}_{2g}}{(2g)!}\hbar^{2g-1}f^{(2g-1)}(x) \tag{B.25}$$

The operator (B.24) is well-defined when acting on the formal power seria $f(x)$ whose coefficients are in turn the formal power seria in $t_k^{\mathbf{1}}$ with a certain bound on the growth, which is obeyed in our construction.

Let us list a useful formula:

$$\mathcal{I}[x]=\frac{x^2}{2\hbar}-\frac{(t_0^{\mathbf{1}})^2}{2\hbar}-\frac{\hbar}{24} \tag{B.26}$$

The crucial property of the $\mathcal{B}$-operator is that it commutes with the Virasoro generators ∇_k. The operator $\mathcal{I}$ fails to commute with ∇_k in the following manner:

$$\mathcal{I}[\nabla_k f(x)]=\nabla_k\mathcal{I}[f(x)]+\widetilde{t}_1^{\mathbf{1}}\delta_{k,-1}\ f(-t_0^{\mathbf{1}}) \tag{B.27}$$

Finally,

$$\mathcal{I}[f]\left(x+\frac{1}{2}\hbar\right)-\mathcal{I}[f]\left(x-\frac{1}{2}\hbar\right)=f(x) \tag{B.28}$$

Footnote 14 continued

$$\begin{aligned}\psi_1(\mathbf{t}) &= -\frac{3}{4}\widetilde{t}_2\\ \psi_2(\mathbf{t}) &= -\frac{5}{8}\widetilde{t}_2^2-\frac{11}{36}\widetilde{t}_3\\ \psi_3(\mathbf{t}) &= -\frac{35}{48}\widetilde{t}_2^3-\frac{5}{8}\widetilde{t}_2\widetilde{t}_3-\frac{25}{288}\widetilde{t}_4\\ &\ldots\end{aligned} \tag{B.23}$$

Now expand (B.22) in x. The coefficients are the power series in $t_k^{\mathbf{1}}$.

As an example of application of the operators $\mathcal{I}$ and $\mathcal{B}$, we can express the logarithm of the Γ-function in terms of the ordinary logarithm,[15] for $\hbar = 1$:

$$\log \frac{\Gamma\left(x+\frac{1}{2}\right)}{\sqrt{2\pi}} = x\,(\log(x)-1) + \mathcal{B}[\log(x)] = \mathcal{I}[\log](x), \tag{B.30}$$

the last equality holding for $t_0^{\mathbf{1}} = 0$.

Appendix C. Plane Wave Ansatz

Let us substitute the ansatz

$$\mathcal{Z} = \sum_{\lambda} \mathcal{Z}_{\lambda} \tag{C.1}$$

where

$$\mathcal{Z}_{\lambda} = \exp\left[2\mathbf{r}(\lambda;\hbar,\mathbf{t^1}) + \frac{1}{\hbar}\sum_{k=0}^{\infty} t_k^{\omega} c_{k+1}(\lambda;\hbar,\mathbf{t}^1)\right] \tag{C.2}$$

and impose the Virasoro constraints:

$$L_k \mathcal{Z} = 0, \quad k \geq -1 \tag{C.3}$$

We get a system of equations (omitting the arguments λ,$\hbar$ and $\mathbf{t^1}$):

$$\begin{aligned} \nabla_k c_{p+1} &= -\frac{(p+k+1)!}{p!} c_{p+k+1}, \quad (k,p) \neq (-1,0) \\ \nabla_{-1} c_1 &= -\frac{1}{\hbar} t_0^{\mathbf{1}} \end{aligned} \tag{C.4}$$

and

$$\begin{aligned} \nabla_{-1}\mathbf{r} &= 0 \\ \nabla_0 \mathbf{r} &= -\frac{1}{2\hbar^2} t_0^{\mathbf{1}} t_0^{\mathbf{1}} - \frac{1}{\hbar}\sum_{l=1}^{\infty} \tilde{t}_l^{\mathbf{1}} c_l \\ \nabla_k \mathbf{r} &= -\frac{1}{2}\sum_{l=1}^{k-1} l!(k-l)!\, c_l c_{k-l} - \\ &\quad -\frac{1}{\hbar}\sum_{l=0}^{\infty} \frac{(k+l)!}{(l-1)!}(h_{k+l} - h_{l-1})\tilde{t}_l^{\mathbf{1}} c_{k+l}, \quad k>0 \end{aligned} \tag{C.5}$$

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[15]Use the following properties of Γ-function:

$$\begin{aligned} \log\ \Gamma\left(x+\frac{1}{2}\right) &= \log\left(\frac{\Gamma(2x)}{\Gamma(x)}\right) - (2x - \tfrac{1}{2})\log(2) + \log\sqrt{2\pi} \\ \log\ \Gamma(x) &= \left(x - \frac{1}{2}\right)\log(x) - x + \log\sqrt{2\pi} + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} x^{1-2n}, \end{aligned} \tag{B.29}$$

C.1. EVOLUTION OF THE CHERN CHARACTER

It is convenient to organize the c_p's into a generating function:

$$\mathcal{C}_\lambda(u;\hbar,\mathbf{t^1})^+ = \sum_{k=0}^{\infty} c_{k+1}(\lambda;\hbar,\mathbf{t^1})u^{k+1} \tag{C.6}$$

The Virasoro constraints (C.4) translate to the equations:

$$\nabla_k \mathcal{C}_\lambda(u;\hbar,\mathbf{t^1})^+ = -u\partial_u^{k+1}\mathcal{C}_\lambda(u;\hbar,\mathbf{t^1})^+ - \frac{1}{\hbar}t_0^{\mathbf{1}} u\delta_{k,-1} \tag{C.7}$$

The equation (C.7) implies that the Laplace transform of $\mathcal{C}_\lambda(u;\mathbf{t^1})^+$ is obtained from that of $\mathcal{C}_\lambda(u;0)^+$ by a diffeomorphism, which sends the function y to $w(y)$. The correction term $\propto \delta_{k,-1}$ indicates some kind of anomaly, which we shall make explicit immediately.

First of all, let us rewrite the t_k^ω-dependent part of (C.1) for the stationary sector, i.e. for $w(y)=y$, in terms of v:

$$\begin{aligned} R_\lambda\left(0,\mathbf{t}^\omega\right) &= \frac{1}{\hbar}\sum_{k=0}^{\infty} t_k^\omega c_{k+1}(\lambda;\hbar,0) = \\ &= \mathcal{B}[v(x)] + \frac{1}{\hbar}\sum_{i=1}^{\ell_\lambda}\left[v\left(\hbar\left(\lambda_i - i + \frac{1}{2}\right)\right) - v\left(\hbar\left(-i+\frac{1}{2}\right)\right)\right] \end{aligned} \tag{C.8}$$

Now, with $\mathbf{t^1}$ turned on we replace (C.8) by

$$\begin{aligned} R_\lambda\left(\mathbf{t^1},\mathbf{t}^\omega\right) &= \frac{1}{\hbar}\sum_{k=0}^{\infty} t_k^\omega c_{k+1}(\lambda;\hbar,\mathbf{t^1}) \\ &= \mathcal{I}[v(z(x))] + \frac{1}{\hbar}\sum_{i=1}^{\ell_\lambda}[v(y_i) - v(z_i)] \\ y_i &= z\left(\hbar\left(\lambda_i - i + \frac{1}{2}\right)\right), \quad z_i = z\left(\hbar\left(-i+\frac{1}{2}\right)\right) \end{aligned} \tag{C.9}$$

The first term in the right hand side of (C.9) is due to the "anomalous" term $t_0^{\mathbf{1}}t_0^\omega$ in the expression for L_{-1}. One might be tempted to interpret (C.9) as a spectral decomposition of an operator acting in a superspace, with both a discrete and a continuous spectra. For the purposes of the conventional spectral theory it would be wrong, since the "density" of the imaginary eigenvalues is imaginary. In a sense, the "continuous" part of (C.8) is a regularized contribution of the infinite branch of the discrete spectrum:

$$\sum_{i=1}^{\infty} v(z_i) \sim \mathcal{I}[v(z[x]] \tag{C.10}$$

A quick way to understand (C.9) is to study the exponential generating function:

$$\mathcal{C}_\lambda(u;0)=\sum_{i=1}^{\infty}\exp\left(u\hbar\left(\lambda_i-i+\frac{1}{2}\right)\right) \tag{C.11}$$

The notation (C.13) is not accidental. Indeed, the generating function $\mathcal{C}_\lambda(u;0)^+$ differs from $\mathcal{C}_\lambda(u;0)$ only by the term $\frac{1}{\hbar u}$:

$$\mathcal{C}_\lambda(u;0)=\sum_{i=1}^{\ell_\lambda}\left(\mathrm{e}^{u\hbar\lambda_i}-1\right)\ \mathrm{e}^{u\hbar(-i+\frac{1}{2})}+\sum_{i=1}^{\infty}\mathrm{e}^{u\hbar(-i+\frac{1}{2})}=\mathcal{C}_\lambda(u;0)^+ +\frac{1}{\hbar u} \tag{C.12}$$

Looking at (C.12) we see that the Laplace transform is a distribution supported at the discrete set $\Lambda=\{\ y_i^0=\hbar(\lambda_i-i+\frac{1}{2})\}_{i=1}^{\infty}$. The (inverse) diffeomorphism $z=w^{-1}$ maps this set onto the set: $z(\Lambda)=\{\ y_i=z(\hbar(\lambda_i-i+\frac{1}{2}))\}_{i=1}^{\infty}$. The inverse Laplace transform of the delta-distribution supported at that set gives:

$$\mathcal{C}_\lambda(u;\mathbf{t}^{\mathbf{1}})=\sum_{i=1}^{\infty}\mathrm{e}^{uy_i}=\mathcal{C}_0(u;\mathbf{t}^{\mathbf{1}})+\sum_{i=1}^{\ell_\lambda}\left[\mathrm{e}^{uy_i}-\mathrm{e}^{uz_i}\right] \tag{C.13}$$

The expression (C.13) is rather tame, its small u expansion can be evaluated with the help of the Euler–Maclauren trick:[16]

$$\mathcal{C}_0(u;\mathbf{t}^{\mathbf{1}})=\frac{1}{\hbar}\sum_{k=0}^{\infty}(-1)^{k-1}\frac{\delta_{k,1}-t_k^{\mathbf{1}}}{u^k}\ +\sum_{k=1}^{\infty}c_k(0;\mathbf{t}^{\mathbf{1}})u^k \tag{C.14}$$

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$$\mathcal{C}_0(u;\mathbf{t}^{\mathbf{1}})=\sum_{i=1}^{\infty}\mathrm{e}^{uz_i}=\left.\frac{e^{-\frac{\hbar\partial_x}{2}}}{1-\mathrm{e}^{-\hbar\partial_x}}\right|_{x=0}\mathrm{e}^{uz[x]}$$

assuming $\mathrm{Re}u>0$ and $t_k^{\mathbf{1}}$ infinitesimal

$$=\frac{1}{\hbar}\int_{-\infty}^{0}\mathrm{e}^{uz[x]}\mathrm{d}x+\mathcal{B}\left[\mathrm{e}^{uz}\right](0)=\frac{1}{\hbar}\int_{-\infty}^{-t_0^{\mathbf{1}}}\mathrm{e}^{uz[x]}\mathrm{d}x+\mathcal{I}\left[\mathrm{e}^{uz}\right](0)$$

The singular part of the u expansion of $\mathcal{C}_0(u;\mathbf{t}^{\mathbf{1}})$ near $u=0$ comes from the integral:

$$\int_{-\infty}^{-t_0^{\mathbf{1}}}\mathrm{e}^{uz[x]}\mathrm{d}x=\int_{-\infty}^{0}\mathrm{e}^{uy}w''(y)\mathrm{d}y$$

We summarize:

$$\boxed{\begin{aligned} c_k(\lambda;\hbar,\mathbf{t}^{\mathbf{1}}) &= c_k(0;\hbar,\mathbf{t}^{\mathbf{1}}) + \frac{1}{\hbar}\sum_{i=1}^{\ell_\lambda} z^k\left[\hbar\left(\lambda_i - i + \frac{1}{2}\right)\right] - z^k\left[\hbar\left(-i+\frac{1}{2}\right)\right] \\ c_k(0;\hbar,\mathbf{t}^{\mathbf{1}}) &= \mathcal{I}\left[z^k(x)\right](0) \end{aligned}} \tag{C.15}$$

C.2. PROFILE REPRESENTATION

Using the profile $f_\lambda(x)$ of the partition λ we can rewrite (C.9) as:

$$R_\lambda\left(\mathbf{t}^{\mathbf{1}},\mathbf{t}^{\omega}\right) = \frac{1}{2\hbar}\int \mathrm{d}x\ f_\lambda''(x)\nu(x)\ , \tag{C.16}$$

with $\nu(x)$ being a formal power series in x,

$$\nu(x) = \mathcal{I}[\upsilon\circ z](x) = \frac{1}{\hbar}\int_{-t_0^{\mathbf{1}}}^{x} \upsilon(z(x'))\mathrm{d}x' + \mathcal{B}[\upsilon(z[x])], \tag{C.17}$$

solving:

$$\nu\left(x+\frac{1}{2}\hbar\right) - \nu\left(x-\frac{1}{2}\hbar\right) = \upsilon(z[x]) \tag{C.18}$$

as a consequence of (B.28). The function $\nu(x)$ satisfies:

$$\nabla_k \nu(x) = -\frac{1}{\hbar}\delta_{k,-1}t_0^{\omega}t_0^{\mathbf{1}} \tag{C.19}$$

We summarize:

$$\boxed{\rho_p(\lambda;\mathbf{t}^{\mathbf{1}}) = \frac{1}{2}\int \mathrm{d}x\ f_\lambda''(x)\mathcal{I}[z^{p+1}](x)} \tag{C.20}$$

C.3. EVOLUTION OF THE PANCHEREL MEASURE

The Plancherel measure evolves along the $t_k^{\mathbf{1}}$-directions to the generalization of the Vandermonde determinant. Now that we know the evolution of the Chern character (C.20) the equations (C.5) become simple:

$$\nabla_0 \mathbf{r}(\lambda;\mathbf{t^1}) = -\frac{1}{2\hbar^2} t_0^{\mathbf{1}} t_0^{\mathbf{1}} - \frac{1}{2\hbar}\sum_{l=1}^{\infty}\frac{\tilde{t}_l^{\mathbf{1}}}{l!}\int \mathrm{d}x\, f_\lambda''(x)\mathcal{I}[z^l](x) =$$
$$= -\frac{1}{2\hbar^2} t_0^{\mathbf{1}} t_0^{\mathbf{1}} + \frac{1}{2\hbar}\int \mathrm{d}x\, f_\lambda''(x)\,\left(\frac{1}{2\hbar}(x+t_0^{\mathbf{1}})^2 - \frac{\hbar}{24}\right) = |\lambda| - \frac{1}{24}$$
$$\nabla_k \mathbf{r}(\lambda;\mathbf{t^1}) = -\frac{1}{2\hbar}\int \mathrm{d}x\, f_\lambda''(x)\sum_{l=0}^{\infty}\frac{\left(h_{k+l}-h_{l-1}\right)}{(l-1)!}\tilde{t}_l^{\mathbf{1}}\,\mathcal{I}\left[z^{k+l}\right](x) - \tag{C.21}$$
$$-\frac{1}{8}\int\int \mathrm{d}x_1 \mathrm{d}x_2\, f_\lambda''(x_1) f_\lambda''(x_2)\sum_{l=0}^{k-2}\mathcal{I}[z^{l+1}](x_1)\mathcal{I}[z^{k-l-1}](x_2)\;,$$
$$k>0$$

The geometric series $z_1^{l+1} z_2^{k-l-1}$ sums up to

$$\frac{z_1^{k+1}-z_2^{k+1}}{z_1-z_2} - z_1^k - z_2^k = -\nabla_k\left[\log\left(\frac{z_1-z_2}{z_1 z_2}\right)\right]$$

This observation suggests the following solution to (C.21):

$$\mathbf{r}(\lambda;\mathbf{t^1}) = \frac{1}{8}\int\int \mathrm{d}x_1\mathrm{d}x_2\, f_\lambda''(x_1) f_\lambda''(x_2)\mathbf{G}(x_1,x_2;\mathbf{t^1}) + \frac{1}{2\hbar}\int \mathrm{d}x\, f_\lambda''(x)\mathcal{I}[\Psi](x) \tag{C.22}$$

where the kernel $\mathbf{G}$ is given by:

$$\mathbf{G}(x_1,x_2;\mathbf{t^1}) = \mathcal{I}_{\tilde{1}}\mathcal{I}_{\tilde{2}}\left[\log\left(\frac{z(\tilde{x}_1)-z(\tilde{x}_2)}{\tilde{x}_1-\tilde{x}_2}\right)\right](x_1,x_2) + \gamma_\hbar(x_1-x_2) \tag{C.23}$$

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