Quantum Groups and Braid Group Statistics in Conformal Current Algebra Models

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Abstract

A quantum universal enveloping algebra U_q and the braid group on n strands $\mathcal{B}_n$ mutually commute when acting on the n-fold tensor product of a U_q-module. Their combined action is applied to low dimensional systems – the only ones that admit a nontrivial monodromy and hence a braid group (rather than a permutation group) statistics.

The lectures introduce the notions of braid group and Hopf algebra and apply them to examples of 2-dimensional (rational) conformal field theory. The case of the $su(n)$ current algebra model, for which the deformation parameter q is an even root of unity, is considered in some detail. We survey, in particular, the canonical approach to the classical Wess-Zumino-Novikov-Witten (WZNW) model and its quantization. The solution to the Schwarz problem for the $su(2)$ Knizhnik-Zamolodchikov equation is also reviewed.

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Contents

Preface

These lectures are meant as an introduction to quantum groups (with emphasis on quantum universal enveloping algebras (QUEA)), braid groups and their application to 2-dimensional conformal field theory. With a view of an audience of mixed background they purport to introduce the basic concepts encountered on the way: Hopf algebras, permutation and braid groups, the conformal group in two and higher dimensions, axiomatic quantum field theory – in various degrees of detail. Thus the first four sections and Appendices A, B form a minicourse on braid groups and Hopf algebras viewing them in the context of (a deformed) Schur-Weyl duality. Section 4 and Appendix C also contain some less standard material: the general form of n-point $U_q(s\ell_2)$ invariants based on joint work with P. Furlan and Ya. S. Stanev of the 1990's. The Drinfeld double and the universal R-matrix are treated rather schematically (in Section 5); mastering this subject would require more work and further reading. This is even more true for the sketch of Wightman axioms (Appendix D) and of the axioms for a chiral vertex algebra (Appendix E) – subjects of monographs outlined here on a couple of pages.

The next three sections (6, 7, 8) provide another introductory course (for more advanced students) – on conformal field theory (CFT). We begin with the axioms of quantum field theory (supplemented with the requirement of conformal invariance) in D space-time dimension in order to stress the special features of the case $D = 2$ to which the rest of the lectures is devoted. Sections 7 and 8 deal with the $u(1)$ conformal current algebra – the simplest $2D$ CFT – and its local extensions, introducing on the way fractional charge fields with anyonic statistics. The axiomatic survey of the $su(2)$ current algebra model (Section 9) starts with an explicit form of the *Knizhnik-Zamolodchikov* (KZ) equation adapted to this special case and only indicates its generalization at the end of the section. This initial specialization is exploited to write down, following a joint work with Yassen Stanev, a regular basis of solutions of the $su(2)$ KZ equation, corresponding to the basis of $U_q(s\ell_2)$ invariants of Section 4 and Appendix D. Sections 10-12 review the classical and quantum Wess-Zumino-Novikov-Witten canonical approach to the same model. We introduce, following [78], the canonical 3-form that describes both the dynamics and the symplectic structure of the model, gradually proceeding to more advanced material. In treating the chiral zero modes (Sections 11, 12) we use (and extend) the U_q oscillator algebra of Pusz and Woronowicz which can be viewed as a deformation of Schwinger's model for $su(2)$.

The two topics, braid groups and QUEA, on one hand, and $2D$ CFT, on the other, are combined in Sections 13, 14 into the study of monodromy representations of the braid group $\mathcal{B}_4$. As an application we survey in Section 13 the solution of the Schwarz problem for the Knizhnik-Zamolodchikov equation (worked out by Stanev and I.T.). Section 14 introduces and studies the restricted and the Lusztig QUEA for q an even root of unity and reviews recent work of Paolo Furlan and the authors on the subject. Section 15, the last one, contains an overview and provides references to adjacent topics (including Chern-Simons theory) that have been left out.

We thank our long term collaborators mentioned above who contributed to the understanding of the subject matter of these notes.

It is a common observation (see, e.g., the Introduction to [61]) that public attention to fundamental physics is declining. In the light of this global phenomenon it was particularly rewarding to the first named author to witness the keen interest of the young (and not so young) audience at the Universidade Federal do Espírito Santo in Vitória, Brazil, during the course of these lectures.

I.T. thanks Clistenis Constantinidis, Olivier Piguet and Galen Sotkov for their hospitality in Vitoria where these lectures were presented. The hospitality and support of l'Institut des Hautes Études Scientifiques, Bures-sur-Yvette, where most of these notes were written is also gratefully acknowledged. We thank, in particular, Cécile Cheikhchoukh for her expert and expeditious typing which allowed to produce both an early version and the present one without delay. I.T also thanks the Theory Unit of the Department of Physics of CERN for its hospitality and support during the final weeks of completing and improving the manuscript. L.H. thanks INFN, Sezione di Trieste and SISSA, Trieste for their long lasting hospitality and their support of his work on the subject. This work is supported in part by the Research Training Network of the European Commission under contract MRTN-CT-2004-00514 and by the Bulgarian National Council for Scientific Research under contracts Ph-1406 and DO-02-257.

1 Introduction

The concept of a group seems to be tailor made to match the notion of symmetry. It is economical and general: it just assumes that a composition of maps (or transformations) g_1 and g_2 is again a map, g_1g_2 (of a set into itself), that the product is associative, $(g_1g_2)g_3 = g_1(g_2g_3)$, and that for each transformation g there is an inverse, g^{-1} such that $gg^{-1} = g^{-1}g = 1$ (1 standing for the identity map which has the property $g \cdot 1 = 1 \cdot g = g$). For transformations depending on continuous parameters (like translations and rotations) one has the powerful concept of a Lie[1] group which allows to reduce in most cases the study of a symmetry to a local problem of Lie algebra.

Why then should we look for a more general concept like Hopf[2] algebra or "quantum group" (or even for some further extension thereof)?

A historical account answering this question from a mathematical point of view can be found in the lectures of Pierre Cartier [33] (see also the complementary more recent publication [7]). We shall single out one aspect of his answer which also has a physical interpretation. Another view of the history of quantum groups is provided by Ludwig Faddeev [56] starting with integrable systems, in particular, spin chains.

A first principle of quantum theory is the *principle of superposition.* It tells us that *quantum states* give rise to a vector space and symmetry groups act by (linear) representations on this space. If we assume, as usual, the standard probabilistic interpretation of state vectors, then we have to deal with a Hilbert state-space and with *unitary representations* of the symmetry group. (More precisely, as pure states correspond to *unit rays* in a complex Hilbert space, one is faced with *projective* or *ray representations* of a symmetry group G which can only be lifted to ordinary vector representations of a suitable *central extension* of G. If G is a semi-simple Lie group then we have to deal with just an appropriate covering of G. Thus half-integer-spin ray representations of the 3-dimensional rotation group $SO(3)$ are lifted to vector representations of its double covering group $SU(2)$.) Furthermore, the state space of a pair of non-interacting systems is the *tensor product* of the spaces of individual systems. This leads us to considering the ring of representations closed under tensor products and direct sums.

Consider now a system of n identical non-relativistic particles of coordinates x_i and internal quantum number s_i ($i = 1, \ldots, n$). Assume further, for the sake of definiteness, that each s_i takes k values and that the internal symmetry group is $U(k)$. The state of such a system is described by a (fixed time, in general, multicomponent) wave function $\psi(x_1, s_1; \ldots; x_n, s_n)$ ($\in \mathcal{H}_1^{\otimes n}$ where $\mathcal{H}_1$ is the 1-particle space). It possesses two types of symmetry which commute with each other: (i) the internal symmetry, described by the n-fold tensor product of fundamental representations of $U(k)$ acting on the variables s_i; (ii) symmetry under permutation of the pairs of arguments (x_i, s_i), reflecting the indistinguishability of identical particles. For 1-component wave function we have the Fermi[3]-Bose[4] alternative: ψ is either invariant or changes sign under transposition of two (pairs of) arguments. Here and in the sequel we consider *pure states* that transform under irreducible representations (IR) of the permutation group. A "superparticle" that is a superposition of a boson and a fermion is not a pure state. This terminology originates with the notion of a *superselection rule* of [154]. In general (for a multicomponent ψ), it may

[1]Marius Sophus Lie (1812-1899) Norvegian mathematician.

[2]Heinz Hopf (1894-1971) introduces the concept of Hopf algebra (in a topological context) in 1941 – see references to the original papers in [33].

[3]The Italian (later American) physicist Enrico Fermi (1901-1954) did his work on the Fermi-Dirac statistics while in Florence (1925-26). He received the Nobel Prize in Physics in 1938 for his work on induced radioactivity.

[4]Satyendra Nath Bose (1894-1974) is an Indian Bengali mathematical physicist. His work of 1922 on the Bose statistics was first rejected and then only accepted for publication after the author sent his manuscript to Einstein who presented it together with his own paper on the same subject to Zeitschrift für Physik in 1924.

transform under a more general irreducible representation (IR) of the permutation group $\mathcal{S}_n$. (This relates to the Schur-Weyl duality reviewed in Appendix A.)

All this is fine if the *configuration space* $X_n = Y_n/\mathcal{S}_n$, where Y_n is the space of points $(x_1, \ldots, x_n)$ such that $x_i \neq x_j$ for $i \neq j$, is simply connected and hence carries single valued analytic functions. This is the case for space dimensions larger than two. If the x_i are points in a 2-dimensional plane however then the configuration space X_n is no longer simply connected. We shall see that the natural generalization of $\mathcal{S}_n$ in this case is the braid group $\mathcal{B}_n$ on n strands that will be introduced in Section 2. What is important for us here is the realization that when the permutation group acting on the tensor product of, say $U(k)$, representations is substituted by the braid group then the condition that "the symmetry commutes with the statistics" implies that tensor product of representations should be deformed to a *coproduct* (in general, not co-commutative) meaning that the concept of a symmetry group should be substituted by the more general notion of a Hopf algebra or quantum group.

Conformal field theory (CFT) gives rise to an important class of *quantum field theory* (QFT) models providing, in the 2-dimensional ($2D$) case, a framework for studying multivalued correlation functions that exhibit braid group statistics. We introduce both the axiomatic and the canonical approach to conformal current algebra models (in Sections 6–9 and Appendices D, E, and in Sections 10–14, respectively). We also give a glimpse of the *minimal conformal models* that triggered (with the work of Belavin, Polyakov and Zamolodchikov, [17]) the interest to $2D$ CFT. First, in Section 7.2, we consider the *critical Ising*[5] *model*, the simplest among the minimal models, introducing on this example the notion of a *null vector* and apply it to deduce the *fusion rules* for the primary fields of the model. Next, in Section 9.3, we sketch the coset space construction [83] of minimal models. Section 10 acquaints the reader with the first order *covariant Hamiltonian formalism*, which seems to have no standard textbook treatment and is being reviewed in the introductory sections of research articles such as [78] and [98]. A central role in it is played by a $(D+1)$-form ω (for a theory in D space-time dimensions) which may exist even when there is no single valued Lagrangian D-form. For the $D = 2$ *Wess-Zumino-Novikov-Witten* (WZNW) model ω includes a term proportional to the Wess-Zumino form θ (10.12) (whose vertical projection is the canonical 3-form on the group manifold).

A special feature of our treatment of the monodromy representation of the braid group is the introduction (in Section 4, Appendix C, and Section 13) of a distinguished basis of conformal blocks and an associated basis of quantum group invariants in which the braid group generators are represented by triangular matrices that also make sense in non-unitary indecomposable representations of the braid group. (This sums up a development of the 1990's in [76, 143, 142, 92], triggered by an idea of Yassen Stanev.)

A bibliographical note.

The relation between the possible particle statistics and the topology (in particular, the fundamental group) of configuration space was first pointed out by mathematicians (in particular, by Arnold [8] in 1968 – see Section 2), then by physicists, ending up with Leinaas and Myrheim [109] (for a more detailed historical account – see [19] and [147]). For a thought provoking recent review of spin and quantum statistics – see [61].

Soon after Drinfeld's quantum group paper [49] (see also [50]) was reported at Berkeley, its relation to CFT was recognized – see, e.g., [6, 69, 125, 90]. The relation to integrable models is also highlighted in [125].

[5] Ernst Ising (1900-1998) is known for a single paper, his doctoral thesis on the 1-dimensional spin chain, suggested by his adviser Wilhelm Lenz (1888-1957) in Hamburg.

There are, by now, a number of texts on quantum groups – see, e.g., [34, 62, 103, 111, 115, 116]. Concerning braid groups – see [20, 21, 104], as well as the historical survey [114]. For the axiomatic approach to QFT we refer to [144, 97, 24] and [87] (see also [131]). [42] is regarded as a standard text on $2D$ CFT; the review [75] makes, on the other hand, connection with axiomatic QFT and thus prepares the ground for the mathematical theory of chiral vertex algebras [25, 60, 82, 47, 100, 59, 120, 122, 123, 13].

The reader will recognize in most of the footnotes (and in some of the references) our bias towards historical remarks.

Appendix A. Young diagrams, Young tableaux and Schur-Weyl duality

A *Young diagram*[6] of n boxes is a graphical expression of a *partition* of the natural number n into a sum of decreasing integers $n_1 \geq n_2 \geq \ldots n_k (\Sigma\, n_i = n)$. It consists of a finite number of boxes arranged in rows of decreasing length. All Young diagrams of three boxes are displayed on Figure A1

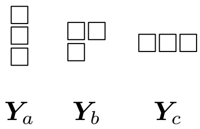


Figure A1: Young diagrams of three boxes

Young tableaux are Young diagrams in which each box carries a number. *Standard Young tableaux* of n boxes carry the numbers $1, \ldots, n$ of increasing order along rows and columns. There are four standard Young tableaux of three boxes displayed on Figure A2

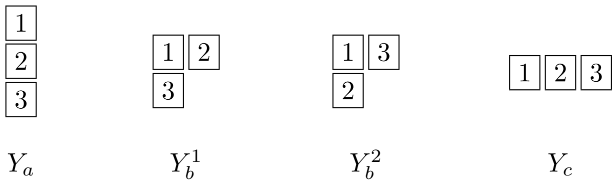


Figure A2: Standard Young tableaux of three boxes

Young diagrams Y of n boxes label the *irreducible representations* (IR) of the *symmetric group* $\mathcal{S}_n$ of permutations of n objects. The standard Young tableaux Y corresponding to a given diagram $\boldsymbol{Y}$ form a *basis* in the representation space of the IR $\boldsymbol{Y}$.

In general, the dimension of the representation corresponding to a Young diagram can be computed without writing down explicitely all Young tableaux corresponding to a given diagram $\boldsymbol{Y}$. To this end we shall introduce the *hook length* $h(x)$ of a box x of $\boldsymbol{Y}$. It is equal to the sum of the number of boxes to the right of x in the same row plus the number of boxes in the same column below x plus 1 (for x itself). In Figure A3 we give examples of hook lengths for two different diagrams

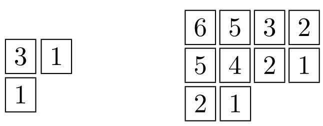


Figure A3: The number in each box gives its hook length

Then the dimension of $\boldsymbol{Y}$ is given by

$$d(\boldsymbol{Y}) := \dim \boldsymbol{Y} = n!/ \prod_{x \in Y} h(x)\,. \tag{A.1}$$

[6]The English mathematician Alfred Young (1873-1940) introduced these diagrams in 1900 while in Cambridge. For a systematic survey of Young tableaux and their applications – see [66].

Exercise A.1. Find the dimension $d(\boldsymbol{Y})$ of the IRs of $\mathcal{S}_5$ and verify the formula $\sum_{\boldsymbol{Y}} d^2(\boldsymbol{Y}) = 5!$.

To see how one reconstructs the action of the elements of $\mathcal{S}_n$ on a basis of Young tableaux, we will first say something more about the structure of the symmetric group.

$\mathcal{S}_n$ can be defined as a (finite) group of $(n-1)$ *generators* $s_1, \dots, s_{n-1}$ (where $s_i = P_{i\,i+1}$ plays the role of *transposition* (permutation) of the "objects" i and $i+1$), satisfying three sets of relations (the first of which tells us that the s_i are *reflections*):

$$s_i^2 = 1\,,\ i = 1, \dots, n-1\,;\ s_i\, s_j = s_j\, s_i \quad \text{for} \quad |i-j| > 1\,;$$

$$P_{i,i+2} = s_i\, s_{i+1}\, s_i = s_{i+1}\, s_i\, s_{i+1}\,,\ i = 1, \dots, m-2 \tag{A.2}$$

(P_{ij} playing the role of transposition of the objects (i,j) and satisfying $P_{ij}^2 = 1$; verify that indeed $P_{i\,i+2}^2 = 1$, as a consequence of (A.2)). Iterating the last relation (A.2) we can express any P_{ij}, $i \neq j$ as a word in the generators (of length $2\,|i-j|-1$. If the indices i, j belong to a single column of the Young tableau Y then, by definition, $P_{ij}\, Y = -Y$. The permutation of two columns of equal lengths in a tableau Y leaves, by definition, Y invariant. The product of basic reflections determines the conjugacy class of the *Coxeter element*[7] of order n; in particular

$$c_{1n} := s_1 \dots s_{n-1} = c_{n1}^{-1} \quad (c_{n1} = s_{n-1} \dots s_1)\,, \quad c_{1n}^n = \mathbb{I}\,. \tag{A.3}$$

Exercise A.2.
(a) Prove that the transposition P_{1n} has the form $P_{1n} = s_1 \dots s_{n-2}\ s_{n-1}\, s_{n-2} \dots s_1$; verify the relation $s_i\, P_{1n} = P_{1n}\, s_i$ for $i = 2, \dots, n-1$.

(b) Prove (A.3). (*Hint* : use induction in n proving $(s_1 \dots s_n)^n = (s_1 \dots s_{n-1})^{n-1}\ P_{1n+1}$.)

Let us now describe, as a next exercise, the IRs of $\mathcal{S}_3$. The IRs $\boldsymbol{Y}_a$ and $\boldsymbol{Y}_c$ being 1-dimensional are easy to describe: $\boldsymbol{Y}_c$ is the trivial representation while $\boldsymbol{Y}_a$ is the *alternating* one: $s_1\, Y_a = s_2\, Y_a = -Y_A$. To construct the 2-dimensional representation $\boldsymbol{Y}_b$ we first note that the generators s_i are represented by 2×2 matrices of eigenvalues ± 1 (hence, $\det s_1 = \det s_2 = -1$, $\operatorname{tr} s_1 = \operatorname{tr} s_2 = 0$).

Exercise A.3. Using the relations $s_2\, Y_b^1 = Y_b^2$, $s_2\, Y_b^2 = Y_b^1$, $s_1\, Y_b^2 = -Y_b^2$ find $s_1\, Y_b^1$ and the matrix P_{13}.

(*Answer* : $s_1\, Y_b^1 = Y_b^1 - Y_b^2\,;\ P_{13} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$.)

The IRs of the group $U(k)$ of complex unitary $k \times k$ matrices are again labeled by Young diagrams – of any number of boxes but of no more than k rows. The corresponding basis vectors can be represented by *semi-standard Young tableaux* in which the allowed numbers are $(1, \dots, k)$ that should increase monotoneously along rows and strictly along columns. Thus the representation corresponding to a single column of k boxes is 1-dimensional (given by the determinant). The IR associated with a single box is k-dimensional (with basis $\boxed{i}$, $i = 1, \dots, k$)

[7]Harold Scott MacDonald Coxeter (1907-2003) born in London but worked for 60 years at the University of Toronto; he studied the product of generators in 1951 – see [38].

and so is the representation of a single column of $(k-1)$-boxes. In order to compute the dimensions of more general IRs of $U(k)$ we shall introduce some auxiliary notions.

Let us label the rows and columns of a Young diagram by consecutive integers i and j starting with 0. Each box x of the resulting Young chess-board Y is thus labeled by a pair (i,j) of non-negative integers. We define, following Macdonald, [112], the *content* $c(x)$ of a box x by

$$c(x) \equiv c(i,j) = j - i\,. \tag{A.4}$$

An illustration is provided on Figure A4:

0	1	2	3
-1	0	1	
-2	-1		
-3			

Figure A4: The number in each box x is its content $c(x)$.
For a Young diagram symmetric with respect to the diagonal
the tableau of $c(x)$ values is skewsymmetric.

Statement A.1. *The dimension of the IR of* $U(k)$ *corresponding to the Young diagram* $\boldsymbol{Y}\,(=\boldsymbol{Y}_{U(k)})$ *is given by*

$$d(\boldsymbol{Y}) = \prod_{x\in\boldsymbol{Y}} \frac{k+c(x)}{h(x)}\,. \tag{A.5}$$

We shall not reproduce the proof of this statement which can be extracted from [112]. We leave it to the reader to verify that Eq. (A.5) indeed gives the number of semi-standard Young tableaux corresponding to a given Young diagram of $U(k)$ (at least for examples of diagrams with a small number of boxes).

Exercise A.4. Verify that Eq. (A.5) reproduces the number of semi-standard tableaux of three boxes (Figure A1):

$$d_a := d(\boldsymbol{Y}_a) = \binom{k}{3}\;,\; d_b = 2\binom{k+1}{3}\;,\; d_c = \binom{k+2}{3} \qquad (\boldsymbol{Y} = \boldsymbol{Y}_{U(k)})\,.$$

Eqs. (A.1) and (A.5) are examples of *combinatorial formulae* for the dimensions of IRs of $\mathcal{S}_n$ and $U(k)$, respectively. The better known analytic expression for $d(\boldsymbol{Y}_{U(k)})$ in terms of the $U(k)$ weights will be reproduced in Appendix F below.

The Schur[8]-Weyl[9] theory concerns the decomposition of the n-fold tensor product of the defining (k-dimensional) representation $\square$ of $U(k)$ into IRs of $U(k)\times\mathcal{S}_n$. (The permutations $s\in\mathcal{S}_n$ of different copies of the $U(k)$ module $\mathbb{C}^k$ commute with the $U(k)$ action.) We have

[8] Issai Schur (January 10, 1875, Mogilov, Belarus, Russian empire – January 10, 1941, Tel Aviv, Palestine) studied and worked in Berlin; regarded himself as German and declined invitations to leave Germany for the US and Britain in 1934; dismissed from his chair in 1935 eventually emmigrated to Palestine in 1939. He is known for Schur's lemma and Schur's polynomials among many others.

[9] Hermann Weyl (1885, Elmshorn, near Hamburg – 1955, Zürich) worked in Göttingen, Zürich and Princeton. The duality in question appears in Weyl's 1928 book Gruppentheorie und Quantenmechanik. Concerning the Schur-Weyl duality – see, e.g., [158].

Proposition A.1. *Let $\boldsymbol{Y}$ run through the n-box Young diagrams with no more than k rows; then*

$$\square_{U(k)}^{\otimes n} = \bigoplus_{\boldsymbol{Y}} \boldsymbol{Y}_{U(k)} \otimes \boldsymbol{Y}_{\mathcal{S}_n} \, . \tag{A.6}$$

In other words the representation $\square_{U(k)}^{\otimes n}$ splits into a sum of tensor products of IRs of $U(k) \otimes \mathcal{S}_n$, the two IRs in each term corresponding to the *same Young diagram.*

Exercise A.5. Using the result of Exercise A.4 verify that $k^3 = d_a + 2\, d_b + d_c$. Do the same exercise for $\square_{U(k)}^{\otimes 4}$ finding first the dimensions of all IRs of $\mathcal{S}_4$ and the dimensions of the IRs of $U(k)$ labeled by Young diagrams of four boxes.

The *special unitary group* $SU(k)$ is the subgroup of $U(k)$ consisting of (unitary, $k \times k$) matrices with unit determinant. Its IRs are labeled by Young diagrams with no more that $k-1$ rows (same as $U(k-1)$). The IRs of $U(k)$ remain irreducible when restricted to $SU(k)$, the only difference coming from the representation of the determinant. Accordingly, all $U(k)$ representations, corresponding to Young diagrams that differ only in the (number of) columns of length k, restrict to a single $SU(k)$ representation – the one labeled by the Young diagram obtained after all such columns have been removed.

Exercise A.6. Show that the combinatorial formula (A.5) for the dimension of a representation gives the same result for Young diagrams which only differ in their content of columns of length k.

The topology of $SU(k)$ and $U(k)$ is, on the other hand, quite different. While $SU(k)$ is *simply connected*, i.e. has a trivial fundamental group, the group $U(k)$ has an infinite sheeted *universal cover* (which is by definition simply connected) isomorphic to $\mathbb{R} \times SU(k)$.

Remark A.1. The groups $SU(2)$ and $SU(4)$ are two-fold coverings of the special orthogonal groups $SO(3)$ and $SO(6)$, respectively, that coincide with their universal covering groups. The universal covering of $SO(4)$ is the direct product $SU(2) \times SU(2)$ (cf. Appendix F). The group $SU(2)$ is topologically isomorphic to the 3-sphere $\mathbb{S}^3$. $U(2)$, on the other hand, is isomorphic to the compactified Minkowski space $\bar{M}$ for $D = 4$ (see Eq. (6.8) of Section 6 for the definition of $\bar{M}$ in D dimensions):

$$U(2) \simeq \bar{M} = \mathbb{S}^3 \times \mathbb{S}^1 / \pm 1 \, . \tag{A.7}$$

2 Braid groups and Hecke algebras

In order to describe the fundamental group π_1 of the configuration space, we first introduce the Artin[10] *braid group* $\mathcal{B}_n$ *on* n *strands*. It is an infinite discrete group which can be defined in analogy with the symmetric group $\mathcal{S}_n$ (cf. Appendix A) as a group of $n-1$ generators $b_1, \ldots, b_{n-1}$ (and their inverses) obeying two sets of defining *braid relations* :

$$b_i\, b_{i+1}\, b_i = b_{i+1}\, b_i\, b_{i+1}\,, \ i = 1, \ldots, n-2\,; \quad b_i\, b_j = b_j\, b_i \quad \text{for} \quad |i-j| > 1\,. \tag{2.1}$$

Their intuitive meaning of $b_i\, b_i^{-1} = \mathbb{1}$ is illustrated on Figure 2.1; that of the first equation (2.1) is pictured on Figure 2.2:

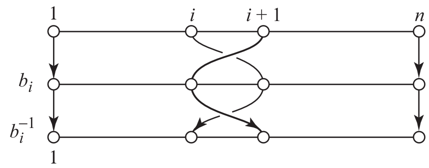


Figure 2.1: $b_i\, b_i^{-1} = \mathbb{1}$

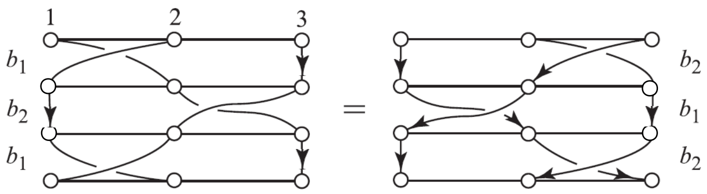


Figure 2.2: $b_1\, b_2\, b_1 = b_2\, b_1\, b_2$

If we ignore the path of a braid transformation and only follow its end point we obtain a permutation. This defines a homomorphism of $\mathcal{B}_n$ onto $\mathcal{S}_n$ whose kernel is, by definition, the *pure braid* or *monodromy* subgroup $\mathcal{M}_n$. This means that the following sequence of group homomorphisms is exact:

$$1 \to \mathcal{M}_n \to \mathcal{B}_n \to \mathcal{S}_n \to 1\,. \tag{2.2}$$

We are now ready to formulate a result that goes back to Hurwitz which has been then repeatedly rediscovered (see [114] for a historical survey by an active participant in this work).

Let $(z_1, \ldots, z_n)$ be n different points in the complex plane $\mathbb{C}$ and let z_0 be a fixed point in $\mathbb{C}$, different from $(z_1, \ldots, z_n)$. Let $Y_n = \mathbb{C} \backslash (z_1, \ldots, z_n)$ be the *n-punctured plane* and $X_n = Y_n / \mathcal{S}_n$ – the configuration space.

[10]Emil Artin (1898-1962), Theorie der Zöpfe (Hamburg, 1925). According to Wilhelm Magnus [114] braid groups were implicit in Adolf Hurwitz's (1859-1919) work on monodromy (1891).

Theorem 2.1. [8] *The fundamental group of the configuration space* $\pi_1(X_n, z_0)$ *coincides with the braid group* $\mathcal{B}_{n+1}$*. The fundamental group of the* n*-punctured plane coincides with its monodromy subgroup,* $\mathcal{M}_{n+1}$*:*

$$\pi_1(X_n, z_0) \simeq \mathcal{B}_{n+1}\,, \quad \pi_1(Y_n, z_0) \simeq \mathcal{M}_{n+1}\,. \tag{2.3}$$

We just note that a path connecting two punctures, say z_i and z_{i+1}, is viewed as a closed path in X_n.

Introduce the analogues of the Coxeter elements (A.3):

$$B_{1n} = b_1 \dots b_{n-1}\,, \quad B_{n1} = b_{n-1} \dots b_1\,. \tag{2.4}$$

The powers of B_{1n} give rise to automorphisms that intertwine the $\mathcal{B}_n$ generators b_i among themselves:

$$B_{1n}^i\, b_1\, B_{1n}^{-i} = b_{i+1}\,, \quad i = 1, \dots, n-2\,. \tag{2.5}$$

We shall cite the following result on the structure of $\mathcal{B}_n$ (see [114] for a concise review and references to the original papers).

Proposition 2.2. *The centre* $Z_n = Z(\mathcal{B}_n)$ *of* $\mathcal{B}_n$ *consists of all powers of the element (of infinite order)*

$$\theta = B_{1n}^n \quad (= B_{n1}^n)\,. \tag{2.6}$$

The two elements θ *and*

$$\Omega = B_{1n}\, B_{n1} \quad \text{subject to the relation} \quad \Omega^n = \theta^2 \tag{2.7}$$

generate a normal subgroup $\mathcal{N}_n$ *of* $\mathcal{B}_n$*.*

The fundamental group $\pi_1(\mathbb{S}^2, n)$ of the 2-sphere with n punctures can be presented as the quotient F_n^* of the free group F_n on n generators, $x_1, \dots, x_n$, by the single relation

$$x_1 \dots x_n = 1 \tag{2.8}$$

(expressing the fact that a loop encircling all n points on the sphere is contractible). The braid group $\mathcal{B}_n$ acts by automorphisms on F_n and on its quotient F_n^* ([114]).

Proposition 2.3. *The automorphisms*

$$\beta_\nu(x_\nu) = x_{\nu+1}\,,\ \beta_\nu(x_{\nu+1}) = x_{\nu+1}^{-1}\, x_\nu\, x_{\nu+1}\,,\ \beta_\nu(x_\mu) = x_\mu \text{ for } \mu \neq \nu, \nu+1\,, \tag{2.9}$$

satisfy the defining relations (2.1) *for the generators of* $\mathcal{B}_n$*. Moreover,* $\mathcal{B}_n$ *is isomorphic to the automorphism group of* F_n *while its quotient with its centre gives* Aut F_n^**:*

$$\mathcal{B}_n \cong \text{Aut}\, F_n\,,\ \mathcal{B}_n^* := \mathcal{B}_n / Z_n \cong \text{Aut}\, F_n^*\,. \tag{2.10}$$

The mapping class group $M(\mathbb{S}^2, n)$ of $\mathbb{S}^2 \backslash \{z_1, \dots, z_n\}$ – i.e., the group of (isotopy classes of) orientation preserving self-homeomorphisms of the sphere with n-punctures – has been studied for nearly a century, starting with the work of Fricke-Klein[11] (1897 – following that of Hurwitz,

[11]Karl Emmanuel Robert Fricke (1861-1930), professor of Higher Mathematics at the Technische Hochschule in Braunschweig, and Felix Christian Klein (1849-1925) (known for his influential Erlangen Program, 1872, and for his role in creating the model research centre at the University of Göttingen from 1886 on) wrote a four volume treatise on automorphic and elliptic modular functions over a period of about 20 years.

mentioned above), followed by contributions by Artin, Magnus, Fadell, Van Buskirk, Arnold [8], Birman [20] among others. In the formulation of the main result below we follow the survey [114] (containing over 60 references).

Theorem 2.4. *The braid group $\mathcal{B}_n(\mathbb{S}^2)$ of the 2-sphere arises from $\mathcal{B}_n$ by adjoining the single relation $\Omega = 1$ (where Ω is the generator of $\mathcal{N}_n$ defined in* (2.7)*). It has a single element θ* (2.6) *of order two (for $n > 2$) that generates its centre $\mathbb{Z}/2$. The mapping class group $M(\mathbb{S}^2, n)$ is obtained from $\mathcal{B}_n(\mathbb{S}^2)$ by setting $\theta = 1$.*

Remark 2.1. It follows from Theorem 2.4 that $\mathcal{B}_n(\mathbb{S}^2)$ is a non-splitting central extension of $M(\mathbb{S}^2, n)$ (just like $SU(2)$ is such an extension of $SO(3)$).

For quantum deformations of unitary (say, $SU(k)$) 1-particle symmetry (to be considered in Section 4, below) with deformation parameter q (such that $q = 1$ corresponds to the undeformed case) the group algebra of the fundamental representation of $\mathcal{B}_n$ is a *Hecke*[12] *algebra* characterized by the following relations

$$b_i^2 + (q - q^{-1})\, b_i - 1 = (b_i + q)(b_i - q^{-1}) = 0\,. \tag{2.11}$$

(The normalization of b_i has been chosen for convenience, so that the products of its eigenvalues is -1. Introducing in the next section a quasi-triangular R-matrix we shall naturally come to a different normalization which involves half-integer powers of the parameter q. In both cases, for $q \to 1$, Eq. (2.11) and its counterpart in Section 11 reduce to the involutivity condition for the reflections generating the symmetric group.)

It is convenient to express b_i in terms of the (non-normalized) projectors (antisymmetrizers) e_i:

$$e_i = q^{-1} - b_i\,, \quad e_i^2 = (q + q^{-1})\, e_i\,, \tag{2.12}$$

which, in view of the braid relations (2.1) satisfy

$$e_i\, e_{i+1}\, e_i - e_i = e_{i+1}\, e_i\, e_{i+1} - e_{i+1} \tag{2.13}$$

$$e_i\, e_j = e_j\, e_i \quad \text{for} \quad |i - j| \geq 2\,. \tag{2.14}$$

We shall introduce (also for later applications) the *q-numbers* $[n]$ ($\equiv [n]_q$) setting

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad ([1] = 1\,,\ [2] = q + q^{-1}\,,\ [3] = q^2 + 1 + q^{-2}, \ldots)\,. \tag{2.15}$$

Exercise 2.1. Verify the relations
(a) $[2][n] = [n-1] + [n+1]\,,\ [3][n] = [n-2] + [n] + [n+2]$;
(b) if $q^N = -1$ then $[N] = 0$.

For *generic* q, i.e. for q not a root of unity, $[N] \neq 0$ for any non-zero natural number N. Define for such q, following [86], the series of antisymmetrizers

$$P_-^1 = \mathbb{I}\,,\ P_-^{k+1} = \frac{1}{[k+1]}\,(q^{-k}\, \mathbb{I} - q^{1-k}\, b_k + \ldots + (-1)^k\, b_1 \ldots b_k)\, P_-^k\,. \tag{2.16}$$

[12] Erich Hecke (1887-1947) studied in Göttingen with Hilbert, worked in Hamburg.

It is easy to check that

$$P_-^2 = \frac{e_1}{[2]}\,, \quad P_-^3 = \frac{1}{[3]!}\,(e_1\,e_2\,e_1 - e_1) = \frac{1}{[3]!}\,(e_2\,e_1\,e_2 - e_2) \tag{2.17}$$

where $[k]!$ is defined recursively by: $[0]! = 1$, $[k+1]! = [k]!\,[k+1]$ and $P_-^k = (P_-^k)_{12\ldots k}$.

Exercise 2.2. Prove that P_-^k are central projectors:

$$P_-^k\,b_i = b_i\,P_-^k = -q^{-1}\,P_-^k \quad \text{for} \quad 1 \le i \le k-1\,,$$

$$P_-^k\,P_{-j}^i = P_{-j}^i\,P_-^k = P_-^k \text{ for } P_{-j}^i = (P_-^i)_{j(j+1)\ldots(j+i-1)}\,,\ i+j-1 \le k\,; \tag{2.18}$$

furthermore,

$$P_-^k\,b_k\,P_-^k = \frac{q^k}{[k]}\,P_-^k - \frac{[k+1]}{[k]}\,P_-^{k+1} \tag{2.19}$$

(a relation that can be used as another recursive definition of P_-^k). It turns out that in applications to $2D$ conformal field theory q is precisely a root of 1. One has to deal with non-normalized projectors in that case – see Section 1.2 of [70] as well as Section 12 below.

We shall assume that the Hecke algebra of $\mathcal{B}_n$ is at most n-dimensional so that

$$\mathcal{P}_-^{n+1} = 0\,. \tag{2.20}$$

If k is the smallest positive integer for which $P_-^{k+1} = 0$ we say that we are dealing with an "even Hecke symmetry of rank k" in the terminology of [86]. Then P_-^k is an one-dimensional projector that can be written as a (tensor) product of two q-deformed Levi-Civita[13] tensors.

Assuming, on the other hand, that $P_-^3 = 0$ for $\mathcal{B}_\infty$ (i.e. that each of the expressions (2.13) vanish) we obtain the *Tempereley-Lieb algebra* which plays a prominent role in V.F.R. Jones theory of subfactors [96].

Here is an explicit realization of e_1 and e_2 (and hence of $\mathcal{B}_3$) satisfying

$$e_1\,e_2\,e_1 - e_1 = e_2\,e_1\,e_2 - e_2 = 0 \qquad (e_i^2 = [2]\,e_i) \tag{2.21}$$

in the triple tensor product of 2×2 matrices

$$(e_1)^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} = \varepsilon^{\alpha_1\alpha_2}\,\varepsilon_{\beta_1\beta_2}\,\delta^{\alpha_3}_{\beta_3}\,,\ (e_2)^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} = \delta^{\alpha_1}_{\beta_1}\,\varepsilon^{\alpha_2\alpha_3}\,\varepsilon_{\beta_2\beta_3} \tag{2.22}$$

where $\varepsilon^{\alpha\beta}$ is the (rank 2) q-deformed Levi-Civita tensor

$$(\varepsilon^{\alpha\beta}) = \begin{pmatrix} 0 & -q^{1/2} \\ q^{-1/2} & 0 \end{pmatrix} = (\varepsilon_{\alpha\beta}) \tag{2.23}$$

satisfying

$$\varepsilon^{\alpha\sigma}\,\varepsilon_{\sigma\beta} = -\delta^\alpha_\beta\,, \qquad (\varepsilon^{\alpha\sigma}\,\varepsilon_{\beta\sigma}) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}\,. \tag{2.24}$$

Exercise 2.3. Verify (2.21) using (2.22) and the properties of $\varepsilon^{\alpha\beta}$.

[13] Tullio Levi-Civita (1873-1941) published in 1900 "Méthodes de calcul différentiel absolu et leurs applications" together with his teacher Gregorio Ricci-Curbastro (1853-1925); Albert Einstein (1879-1955, Nobel Prize in Physics 1921) used this book to master tensor calculus in Riemannian geometry; Bernhard Riemann (1826-1866) introduced his geometry in his inaugural lecture at the University of Göttingen (1854).

Another solution of the braid relations of $\mathcal{B}_3$, in which the expression (2.13) is a 1-dimensional projector,

$$e_1\, e_2\, e_1 - e_1 = e_2\, e_1\, e_2 - e_2 = \varepsilon^{\alpha_1 \alpha_2 \alpha_3}\, \varepsilon_{\beta_1 \beta_2 \beta_3} \tag{2.25}$$

where

$$\varepsilon^{123} = -q^{3/2} = -q\, \varepsilon^{132} = -q\, \varepsilon^{213} = q^2 \varepsilon^{312} = q^2 \varepsilon^{231} = -q^3 \varepsilon^{321}\,,$$
$$\varepsilon^{\alpha\alpha\beta} = 0 = \varepsilon^{\alpha\beta\alpha} = \varepsilon^{\beta\alpha\alpha}\,, \quad (\varepsilon^{\alpha\beta\gamma}) = (\varepsilon_{\alpha\beta\gamma})\,, \tag{2.26}$$

is given by

$$(e_1)^{\alpha_1 \alpha_2 \alpha_3}_{\beta_1 \beta_2 \beta_3} = \varepsilon^{\alpha_1 \alpha_2 \sigma}\, \varepsilon_{\sigma \beta_1 \beta_2}\, \delta^{\alpha_3}_{\beta_3}\,, \quad (e_2)^{\alpha_1 \alpha_2 \alpha_3}_{\beta_1 \beta_2 \beta_3} = \delta^{\alpha_1}_{\beta_1}\, \varepsilon^{\sigma \alpha_2 \alpha_3}\, \varepsilon_{\beta_2 \beta_3 \sigma}\,. \tag{2.27}$$

Exercise 2.4. Use the identity

$$\varepsilon_{\beta \sigma_1 \sigma_2}\, \varepsilon^{\sigma_1 \sigma_2 \alpha} = [2]\, \delta^{\alpha}_{\beta} \tag{2.28}$$

to verify the relations $e_i^2 = [2]\, e_i\,,\ i = 1, 2$. Verify (2.25) for e_i given by (2.27).

Note that the order of indices of the quantum Levi-Civita tensor in (2.27) is important. It is easy to check, for instance that substituting the first product by $\varepsilon^{\alpha_1 \alpha_2 \sigma}\, \varepsilon_{\beta_1 \beta_2 \sigma}$ would violate the condition $e_1^2 = [2]\, e_1$.

We end up our brief survey of the braid group with a simple application to physically interesting new statistics in two dimensions.

The permutation group $\mathcal{S}_n$ has exactly two 1-dimensional representations (corresponding to the Young diagrams $\boldsymbol{Y}_a$ and $\boldsymbol{Y}_c$ of Figure A1 for $n = 3$ – see Appendix A): the fully symmetric (trivial) representation, corresponding to bosons and the totally antisymmetric one, describing fermions. By contrast the braid group $\mathcal{B}_n$ has a 1-parameter family of 1-dimensional (unitary) representations given by

$$\pi_q(b_i) = \bar{q} \qquad (\, \pi_q(b_i^{-1}) = q\ , \quad q\, \bar{q} = 1\,)\ . \tag{2.29}$$

(Note that $\pi_q(b_i)$ (2.29) trivially satisfy the Hecke algebra condition (2.11).) This representation describes (according to presently accepted theoretical models – see [63, 64]) the fractional quantum Hall effect. The story of how physicists got aware of the anyonic representations is told in [19].

3 Bialgebras and Hopf algebras: classical examples and definition

A natural way to arrive at the Hopf algebra generalization of the notion of a group G is to study the duality between an algebra U that could be either the *group algebra*, say $\mathbb{C}G$, or the *universal enveloping algebra* (UEA) $U(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ of G, and the algebra $\mathcal{F}(G)$ of functions on the group. (The appropriate topology of $\mathcal{F}(G)$ depends on the class of groups one is considering – see the introduction to [33]. As these introductory remarks are just ment as motivation we will not burden them with topological considerations.) We shall thus first explain why ordinary groups and Lie algebras can be viewed as Hopf algebras and only then will give the formal definition.

Remark 3.1. Mathematicians would often replace the field $\mathbb{C}$ of complex numbers in the definition of a group algebra by an arbitrary field K (having in mind, e.g., applications to algebraic groups – see [33]). Such generality may also be useful for some physical applications but again we refrain from complicating excessively this introductory note. The space of functions $\mathcal{F}(G)$ is sometimes denoted by $\mathbb{C}^G$ (or K^G – see [33]).

Usually $\mathbb{C}G$, $U(\mathcal{G})$ and $\mathcal{F}(G)$ are just viewed as *unital associative algebras* – in all these cases the product $m : U \otimes U \to U \;\; (m(X \otimes Y) \equiv X \cdot Y)$ and the unit element $\mathbb{I} \in U$ are defined in a natural way. It is important, however, that $\mathbb{C}G$ and $U(\mathcal{G})$ (as well as $\mathcal{F}(G)$ for finite groups G) are also *bialgebras*; to begin with, they are endowed with a coalgebra structure. A *coalgebra* is a linear space equipped with two linear maps, the *coproduct* $\Delta : U \to U \otimes U$ and the *counit* $\varepsilon : U \to \mathbb{C}$ such that

$$(\mathrm{id} \otimes \Delta)\, \Delta = (\Delta \otimes \mathrm{id})\, \Delta \tag{3.1}$$

$$(\mathrm{id} \otimes \varepsilon)\, \Delta\,(X) = (\varepsilon \otimes \mathrm{id})\, \Delta\,(X) = X \;,\; \forall\, X \in U \,. \tag{3.2}$$

In a bialgebra, the algebra and coalgebra structures also satisfy compatibility conditions which will be formulated later in Definition 3.1. It is the presence of the coproduct which allows to view the tensor product of any two representations of U again as a representation of U (rather than as a representation of $U \otimes U$ which is always possible for an associative algebra). The coproduct in U is related to the (pointwise) product of functions $f(g)$ in $\mathcal{F}(G)$ by

$$(A, f_1\, f_2) = (\Delta(A), f_1 \otimes f_2) = \sum_{(A)} (A_1, f_1)(A_2, f_2) \tag{3.3}$$

where we are using Sweedler's notation[14]

$$\Delta(A) = \sum_{(A)} A_1 \otimes A_2 \quad \text{for} \quad A \in U \quad (\Rightarrow A_1, A_2 \in U\,)\,. \tag{3.4}$$

Any element of $\mathbb{C}G$ is, by definition, a finite linear combination of elements of G with complex coefficients:

$$A = \sum_g a(g)\, g \qquad \Rightarrow \qquad (A, f) = \sum_g a(g)\, f(g) \quad (\in\, \mathbb{C}) \;. \tag{3.5}$$

Applying this to the left hand side of (3.3) with $(f_1\, f_2)(g) := f_1(g)\, f_2(g)$ and comparing the result with the right hand side, we deduce

$$\Delta\, g = g \otimes g \qquad \Rightarrow \qquad \Delta\, A = \sum_g a(g)\, g \otimes g \;. \tag{3.6}$$

[14] It is a self-explaining notation introduced by Moss E. Sweedler in his book *Hopf Algebras* (W.A. Benjamin, N.Y. 1969) of the pre-quantum groups' era.

Both $\mathbb{C}G$ and $\mathcal{F}(G)$ are *unital* associative algebras; in other words, they have unit elements: the group unit $\mathbb{I} \in G \subset \mathbb{C}G$ and the constant function $f_0(g) = 1 \in \mathcal{F}(G)$. This allows to define a *counit* in both $\mathbb{C}G$ and $\mathcal{F}$, setting

$$\varepsilon(A) = (A, 1) = \sum_g a(g) \ , \quad \text{i.e.} \ \ \varepsilon(g) = 1 \in \mathbb{C} \ , \quad \varepsilon_{\mathcal{F}}(f) = (\mathbb{I}, f) = f(\mathbb{I}) \ . \tag{3.7}$$

If G is a *finite group* we can define in this simple algebraic manner a coproduct in $\mathcal{F}(G)$ as well, setting

$$\Delta_{\mathcal{F}}\, f(g_1, g_2) = f(g_1\, g_2) \ . \tag{3.8}$$

Remark 3.2. Note that for a finite group G the tensor product $\mathcal{F}(G) \otimes \mathcal{F}(G)$ is naturally isomorphic to the space $\mathcal{F}(G \times G)$ of functions of two group variables. For G infinite the tensor square of $\mathcal{F}(G)$ is a proper subset of $\mathcal{F}(G \times G)$.

Exercise 3.1. Verify that the coproducts Δ (3.6) and $\Delta_{\mathcal{F}}$ (3.8) and the counits ε and $\varepsilon_{\mathcal{F}}$ (3.7) satisfy the coalgebra conditions (3.1) and (3.2).

We may view $\mathbb{C}G$ and $\mathcal{F}(G)$ as *Hopf algebras* by introducing in each of these bialgebras the *antipode* S:

$$S : \mathbb{C}G \to \mathbb{C}G\, , \ Sg = g^{-1}\, ; \quad S_{\mathcal{F}} : \mathcal{F}(G) \to \mathcal{F}(G)\, , \ (S_{\mathcal{F}}\, f)(g) = f(g^{-1})\, . \tag{3.9}$$

In both cases S is defined as an algebraic *antihomomorphism*: the map S is linear, and $S(A_1\, A_2) = S(A_2)\, S(A_1)$.

To end up with our classical examples of a Hopf algebra we display Δ, ε and S for the UEA $U(\mathcal{G})$ of a Lie algebra $\mathcal{G}$ defining them for elements of $\mathcal{G}$:

$$\Delta(X) = X \otimes \mathbb{I} + \mathbb{I} \otimes X\, , \quad \varepsilon(X) = 0\, , \quad S(X) = -X\, , \quad \forall\, X \in \mathcal{G}\, . \tag{3.10}$$

We observe that the (associative) algebras $\mathbb{C}G$ and $U(\mathcal{G})$ are, in general, non-commutative but the coproduct in both case equals the *permuted* (or *opposite*) one

$$\Delta'(X) := \sum_{(X)} X_2 \otimes X_1 = \sum_{(X)} X_1 \otimes X_2 = \Delta(X)\, . \tag{3.11}$$

We say in such a case that the algebra U is *co-commutative.* By contrast, the algebra $\mathcal{F}(G)$ dual to $\mathbb{C}G$ is commutative but not co-commutative. Here is, finally, the abstract definition of a Hopf algebra (over an arbitrary field K) in which one demands neither commutativity nor co-commutativity.

Definition 3.1. *An associative unital algebra $\mathcal{B}$ with multiplication m and unit $\mathbb{I}$ is called a* bialgebra *if it also a* coalgebra *(with coproduct $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ and counit $\varepsilon : \mathcal{B} \to \mathbb{C}$ satisfying (3.1), (3.2)) and, in addition, Δ and ε are* unital algebra homomorphisms *i.e.,*

$$\Delta(X \cdot Y) = \Delta(X)\, \Delta(Y)\, , \qquad \Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I}\, ,$$
$$\varepsilon(X \cdot Y) = \varepsilon(X)\, \varepsilon(Y)\, , \qquad \varepsilon(\mathbb{I}) = 1 \in \mathbb{C} \ . \tag{3.12}$$

(The product $\Delta(X)\, \Delta(Y)$ belongs again to the tensor square $\mathcal{B} \otimes \mathcal{B}$ which inherits an algebraic structure by $(X_1 \otimes X_2)(Y_1 \otimes Y_2) = (X_1 Y_1 \otimes X_2 Y_2)$.)

Exercise 3.2. Verify that the coproducts Δ (3.6) and $\Delta_{\mathcal{F}}$ (3.8) and the counits ε and $\varepsilon_{\mathcal{F}}$ (3.7) satisfy the algebraic conditions (3.12).

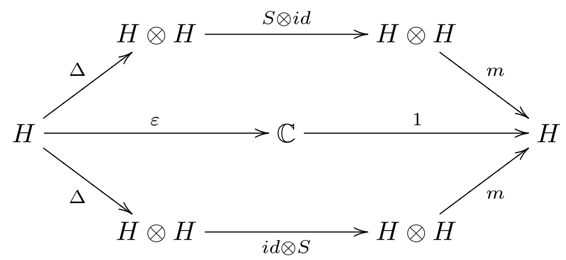
Exercise 3.3. Prove that the first relation in (3.12) implies the following compatibility condition between multiplication and comultiplication:

$$m \otimes m \, P_{23} \, \Delta(X) \otimes \Delta(Y) = \Delta \left(m(X \otimes Y) \right) \tag{3.13}$$

where P_{23} stands for the permutation of the factors 2 and 3 in the 4-fold tensor product, and we identify $X \cdot Y$ with $m(X \otimes Y)$. Using (3.4), the left-hand side of relation (3.13) can be written in more detail as

$$m \otimes m \, P_{23} (\sum_{(X)} X_1 \otimes X_2 \otimes \sum_{(Y)} Y_1 \otimes Y_2 \,) = \sum_{(X,Y)} (X_1 \cdot Y_1) \otimes (X_2 \cdot Y_2) \ .$$

Definition 3.2. *A* Hopf algebra *H is a bialgebra (over $\mathbb{C}$) equipped with a $\mathbb{C}$-linear antihomomorphism of algebras $S : H \to H$, the* antipode*, such that the following diagram commutes:*



(3.14)

Using the notation (3.4) *we can translate the content of* (3.14) *into the relation*

$$\sum_{(X)} S(X_1) \cdot X_2 = \sum_{(X)} X_1 \cdot S(X_2) = \varepsilon(X) \mathbb{1} \quad \forall \, X \in H \, . \tag{3.15}$$

The requirement that Δ and ε are algebra homomorphisms allows to introduce the bialgebra structure on $U(\mathcal{G})$ (or on its "quantum" deformation – see Section 4 below) by just defining them on the generators of the Lie algebra $\mathcal{G}$. The same remark applies to the algebra antihomomorphism S in the definition of a Hopf algebra structure.

As shown by L. Schwartz[15], the distributions supported by the unit element of a Lie group G with Lie algebra $\mathcal{G}$ form an algebra isomorphic to the UEA $U(\mathcal{G})$ (Theorem 3.7.1 of [33]). Thus endowing, as in (3.8), the smooth functions $C^\infty(G)$ on G with a coproduct (belonging to $C^\infty(G \times G)$) allows to view $U(\mathcal{G})$ as a subalgebra of a suitable *dual* of $C^\infty(G)$. This classical construction was generalized by Faddeev, Reshetikhin and Takhtajan [57] to provide an alternative approach to the quantum UEA (realized in the quantized chiral WZNW model by the Gauss[16] components of the monodromy matrix satisfying quadratic R-matrix relations, see Section 12).

[15]Laurent Schwartz (1915-2002) is best known for his theory of distributions. He is the Ph.D. adviser of Grothendieck and a member of the Bourbaki group (see M. Mashaal, *Bourbaki: une société secrète de mathématiciens*, Belin, 2002; A.D. Aczel, *The Artist and the Mathematician. The Story of Nicolas Bourbaki. The Genius Mathematician who Never Existed*, Thunder's Mouth Press, 2006, French transl. J.C. Lattès, 2009).

[16]Carl Friedrich Gauss (1777-1855), called the Prince of Mathematicians (see [18]) has left his imprint to nearly all domains of mathematics.

4 Quantum universal enveloping algebras: the $U_q(A_r)$ case

An important example, in which neither commutativity nor co-commutativity holds, is given by the *quantum universal enveloping algebra* (QUEA) $U_q(\mathcal{G})$ of a (semi)simple Lie algebra $\mathcal{G}$. We shall spell out its definition for $\mathcal{G} = A_r = s\ell_{r+1}$ (the rank r Lie algebra of the special linear group of $(r+1)\times(r+1)$ matrices) and (complex) parameter $q \neq 0, \pm 1$. It combines the properties of $\mathbb{C}G$ and $U_q(\mathcal{G})$ being generated by a mixture of group like and Lie algebra like elements.

The QUEA $U_q(A_r)$ has r group-like generators K_i (and their inverses K_i^{-1}) which correspond to the *Cartan*[17] *torus* and $2r$ Lie algebra-like ones (*raising and lowering operators*) E_i and F_i corresponding to simple roots. They obey the following commutation relations (CR):

$$K_i\, E_j\, K_i^{-1} = q^{(\alpha_i|\alpha_j)}\, E_j\ ,\ \ K_i\, F_j\, K_i^{-1} = q^{-(\alpha_i|\alpha_j)}\, F_j\ ,$$

$$[E_i, F_j] = \delta_{ij}\, \frac{K_i - K_i^{-1}}{q - q^{-1}}\ , \qquad i, j = 1, \ldots, r\ , \tag{4.1}$$

and the *Serre relations* (that are only non-trivial for $r > 1$):

$$E_i^{(2)}\, E_{i+1} + E_{i+1}\, E_i^{(2)} = E_i\, E_{i+1}\, E_i\ ,\ \ E_i\, E_{i+1}^{(2)} + E_{i+1}^{(2)}\, E_i = E_{i+1}\, E_i\, E_{i+1}\ ,$$

$$F_i^{(2)}\, F_{i+1} + F_{i+1}\, F_i^{(2)} = F_i\, F_{i+1}\, F_i\ ,\ \ F_i\, F_{i+1}^{(2)} + F_{i+1}^{(2)}\, F_i = F_{i+1}\, F_i\, F_{i+1}$$

$$\text{for}\quad X^{(n)} = \frac{1}{[n]!}\, X^n\ ;\quad [E_i, E_j] = 0 = [F_i, F_j] \quad \text{for} \quad |i-j| > 1\, . \tag{4.2}$$

Here (in the first relation (4.1)) α_i are the *simple roots*, normalized to have square 2, so that $((\alpha_i \mid \alpha_j))$ is the A_r *Cartan matrix* :

$$(\alpha_i \mid \alpha_i) = 2\ , \quad (\alpha_i \mid \alpha_{i+1}) = -1\ , \quad (\alpha_i \mid \alpha_j) = 0 \quad \text{for} \quad |i-j| > 1\, . \tag{4.3}$$

It is simple to display the "classical" ($q \to 1$) limit of these relations. Setting

$$K_i = q^{H_i} \qquad (K_i^{-1} = q^{-H_i})\, , \quad i = 1, \ldots, r \tag{4.4}$$

we find, at least formally, that the first two CR (4.1) are equivalent to the classical ones

$$[H_i, E_j] = (\alpha_i \mid \alpha_j)\, E_j\ , \quad [H_i, F_j] = -(\alpha_i \mid \alpha_j)\, F_j\ ,$$

while the third one has a classical limit:

$$[E_i, F_j] = [H_i]\, \delta_{ij} \qquad (\to H_i\, \delta_{ij} \quad \text{for} \quad q \to 1\,) \tag{4.5}$$

where we have extended the notation $[n]$ (2.15) for a q-number to operator valued entries ($n \to H_i$). Note however, that Eq. (4.4) is not algebraic (it involves the exponential function). That's why purists only use $K_i^{(\pm 1)}$ in dealing with $U_q(A_r)$.

We define the coproduct, the counit, and the antipode on the generators of $U_q(A_r)$ as follows

$$\Delta(K_i) = K_i \otimes K_i\ , \quad \Delta(E_i) = E_i \otimes K_i + \mathbb{1} \otimes E_i\ ,$$

[17] The French mathematician Élie Joseph Cartan (1869-1951) has introduced the general notion of antisymmetric differential forms (1894-1904) and the theory of spinors (1913) besides his major contribution to Lie algebras (his doctoral thesis of 1894) in which he completed Killing's work on the classification of semi-simple Lie algebras over the complex field. An introduction to this result in the simply laced case and a detailed treatment of the A_r algebra and its finite dimensional IRs is given in Appendix F.

$$\Delta(F_i) = F_i \otimes 1\!\!1 + K_i^{-1} \otimes F_i \ , \quad i = 1, \ldots, r \, ; \tag{4.6}$$

$$\varepsilon(K_i) = 1 \ , \quad \varepsilon(E_i) = \varepsilon(F_i) = 0 \, ; \tag{4.7}$$

$$S(E_i) = -E_i \, K_i^{-1} \ , \quad S(F_i) = -K_i \, F_i \ , \quad S(K_i) = K_i^{-1} \ . \tag{4.8}$$

Exercise 4.1. Verify (3.15) which can also be written in the form

$$m(\mathrm{id} \otimes S) \, \Delta(X) = m(S \otimes \mathrm{id}) \, \Delta(X) = \varepsilon(X) \, 1\!\!1 \tag{4.9}$$

for the generators of $U_q(A_r)$.

Exercise 4.2. Verify (on the generators) the relation $(\varepsilon \otimes \mathrm{id}) \, \Delta(X) = (\mathrm{id} \otimes \varepsilon) \, \Delta(X) = X$ (3.12).

Remark 4.1. It helps understanding both the origin and the meaning of quantum groups to observe that it is the coproduct that determines the deformation of the Lie algebra structure. To this end we note that the fundamental (undeformed) representation of A_r, given in terms of the Weyl matrices

$$(e_{ij})^k_\ell = \delta^k_i \, \delta_{j\ell} \quad (\Rightarrow e_{ij} \, e_{k\ell} = \delta_{jk} \, e_{i\ell}) \qquad i, j, k, \ell = 1, \ldots, r+1 \, , \tag{4.10}$$

by

$$E_i = e_{i\, i+1} \ , \quad F_i = e_{i+1\, i} \ , \quad H_i = e_{ii} - e_{i+1\, i+1} \ , \tag{4.11}$$

is also a representation of $U_q(A_r)$; in particular,

$$[E_i, F_j] = \delta_{ij} [H_i] = \delta_{ij} \, H_i \quad \text{for} \quad H_i = e_{ii} - e_{i+1\, i+1} \, . \tag{4.12}$$

More generally, if H is a hermitean matrix with eigenvalues $0, \pm 1$ then $[H] = H$. Furthermore, the Serre relations (4.2) (which involve the deformation parameter q) are also satisfied by the q-independent matrices of the defining $(r+1)$-dimensional representation of A_r since each term is separately equal to zero:

$$E_i^2 = 0 = F_i^2 = E_i \, E_{i+1} \, E_i = F_i \, F_{i+1} \, F_i$$

$$\text{for} \quad E_i = e_{i\, i+1} \ , \quad F_i = e_{i+1\, i} \; (= E_i^*) \, . \tag{4.13}$$

It is the non-cocommutative coproduct which replaces the symmetric tensor product of representations, that yields modified higher dimensional representations and forces us to use the q-deformed CR (4.1) (4.2). (As we shall see below the coproduct is also directly related to the appearance of braid group representations in the q-deformed Schur-Weyl duality.)

Remark 4.2. It is clear from the above CR that the "raising operators" E_i generate a Lie subalgebra. Since $\Delta(E_i)$ (4.6) also involves the Cartan generator K_i, they do not generate a Hopf subalgebra. (Only $\{E_i, K_j; i, j = 1, \ldots, r\}$ do give rise to a Hopf subalgebra.) The same remark holds for the "lowering operators", F_i. Thus, in general, there are fewer Hopf subalgebras in the deformed case than in the undeformed (cocommutative) one.

It is instructive to see how 2- and higher point $U_q(A_r)$ invariants appear in tensor products of finite dimensional representations. We shall work out the solution to this problem for the simplest case of $U_q(A_1) \equiv U_q$.

For *generic* q ($q \neq 0, q$ not a root of unity) the theory of finite dimensional representations of U_q is essentially the same as that of the undeformed algebra $A_1 \simeq su(2)$. The irreducible representations (IRs) of U_q are again labeled by the isospin I, or by the dimension $p := 2I+1$. An explicit realization of the U_q module $\mathcal{F}_p$ is given in terms of the *weight basis* $\{q^{\alpha_{pm}} \mid p, m\rangle\}$ for any choice of the (integer) exponents α_{pm}. Instead of the q-deformed Casimir operator $C_2^{(q)} := EF + FE + [2] \left[\frac{H}{2}\right]^2$ it is more convenient to use its rescaled version, $C = \frac{1}{2}(q - q^{-1})^2 \, C_2^{(q)} + [2]$:

$$C := \lambda^2 EF + q^{H-1} + q^{1-H} = \lambda^2 FE + q^{H+1} + q^{-H-1}\,, \ \lambda := q - q^{-1} \tag{4.14}$$

we have, as part of the definition of the weight basis

$$(C - q^p - q^{-p})\, q^{\alpha_{pm}} \mid p, m\rangle = 0 = (q^H - q^{2m-p+1})\, q^{\alpha_{pm}} \mid p, m\rangle\,. \tag{4.15}$$

Exercise 4.3. Check that (4.14) (4.15) imply

$$(EF - [m][p-m])\, q^{\alpha_{pm}} \mid p, m\rangle = 0 = (FE - [m+1][p-m-1])\, q^{\alpha_{pm}} \mid p, m\rangle \tag{4.16}$$

(independent of the choice of α_{pm}).

We shall single out the *canonical basis* $\{\mid p, m\rangle\}$ by the (α_{pm}-dependent) relations

$$E \mid p, m\rangle = [p-m-1] \mid p, m+1\rangle\,, \quad F \mid p, m\rangle = [m] \mid p, m-1\rangle\,. \tag{4.17}$$

It follows, in particular, that $\mathcal{F}_p$ has both a lowest and a highest weight vector, $\mid p, 0\rangle$ and $\mid p, p-1\rangle$, a property that is independent of the choice of α_{pm}:

$$E\, q^{\alpha_{p\,p-1}} \mid p, p-1\rangle = 0 = F\, q^{\alpha_{p0}} \mid p, 0\rangle\,. \tag{4.18}$$

What singles out the canonical basis (among the weight bases which satisfy (4.18)) is the reality of the coefficients in the right hand side of each of the two equations (4.17).

Another remarkable weight basis, that will be used shortly, is what we shall call an *E-basis*, $\{\mid p, m)\}$ (satisfying, by definition, Eq. (4.23) below for ϕ_I given by (4.22)), such that

$$E \mid p, m) = (p-m-1)_+ \mid p, m+1)\,, \quad F \mid p, m) = q^{2-p} (m)_+ \mid p, m-1)\,; \tag{4.19}$$

here the (complex for $q\,\bar{q} = 1$) q-numbers $(n)_+$ and $(n)_-$ (that will appear later) are defined by

$$(n)_+ := [n]\, q^{n-1} = 1 + q^2 + \dots + q^{2n-2} = \frac{1 - q^{2n}}{1 - q^2}\,, \tag{4.20}$$
$$(n)_- = [n]\, q^{1-n} = 1 + q^{-2} + \dots + q^{2-2n}\,.$$

Exercise 4.4. Verify that the vectors

$$\mid p, m) = q^{\alpha_{p,m}} \mid p, m\rangle\,, \quad \text{with} \quad \alpha_{pm} = \frac{m(m+3)}{2} - mp \tag{4.21}$$

satisfy (4.19) as a consequence of (4.17).

The E-basis allows to introduce U_q *coherent states* [75] which are vector valued polynomials of degree $p-1=2I$ in a formal variable u:

$$\Phi_I(u) := \sum_{m=0}^{2I} \binom{2I}{m}_+ u^m \mid 2I+1, m) \, , \qquad \binom{n}{m}_+ = \frac{(n)_+!}{(m)_+!\,(n-m)_+!} \tag{4.22}$$

($(0)_+! = (1)_+! = 1$, $(n+1)_+! = (n)_+!\,(n+1)_+$).

Exercise 4.5. Verify the relations

$$(E - D_+)\,\Phi_I(u) = 0 \quad \text{for} \quad (D_\pm\, f)(u) = \frac{f(q^{\pm 2}\,u) - f(u)}{(q^{\pm 2}-1)\,u} \, ; \tag{4.23}$$

$$K\phi_I(u) = q^{-2I}\,\phi_I(q^2 u) \, ;$$

$$F\phi_I(u) = u^{2I+2}\,D_-(u^{-2I}\,\Phi_I(q^{-2}u)) \, . \tag{4.24}$$

(*Hint* : use the relation $D_\pm\, u^m = (m)_\pm\, u^{m-1}$.)

A function $J^{(I)}(u_1,\ldots,u_n)$ on the n-fold tensor product of $\Phi_I(u)$'s is U_q invariant if it is homogeneous – as a consequence of $K(=q^H)$ invariance,

$$K : q^{-2nI}\,J^{(I)}(q^2\,u_1,\ldots,q^2\,u_n) = J^{(I)}(u_1,u_2,\ldots,u_n) \, , \tag{4.25}$$

and E- and F-invariant:

$$E : \sum_{k=1}^{n} D_{k+}\,J^{(I)}(u_1,\ldots,u_k,q^2\,u_{k+1},\ldots,q^2\,u_n)\,q^{2I(k-n)} = 0 \tag{4.26}$$

$$F : \sum_{k=1}^{n} u_k^{2I+2}\,q^{2I(k-1)}\,D_{k-}(u_k^{-2I}\,J^{(I)}(q^{-2}\,u_1,\ldots,q^{-2}\,u_{k-1},u_k,\ldots,u_n)) = 0 \, . \tag{4.27}$$

Eqs. (4.25–4.27) are obtained from (4.23, 4.24) by applying multiple coproduct formulae, like

$$\Delta^{(n-1)}(E) = \sum_{k=1}^{n} \mathbb{I}^{\otimes(k-1)} \otimes E \otimes K^{\otimes(n-k)} \tag{4.28}$$

where $A^{\otimes\nu} = A\otimes\ldots\otimes A$ (ν factors), $A^{\otimes\,0} = 1$.

Exercise 4.6. Prove using just (4.25) and (4.26) that the general 2-point invariant is proportional to

$$\begin{aligned} J^{(I)}(u_1,u_2) = w_{2I}\left(u_1,u_2;\frac{1}{2}\right) &= \prod_{n=0}^{2I-1}(q^{I-n}\,u_1 - q^{n-I}\,u_2) \\ &= q^{-I}\sum_{m=0}^{2I}\begin{bmatrix}2I\\m\end{bmatrix}(q\,u_1)^m\,(-u_2)^{2I-m} \, , \end{aligned} \tag{4.29}$$

where

$$w_k(u,v;\rho) = w_k(q^\rho\,u, q^{-\rho}\,v) = \prod_{\nu=0}^{k-1}\left(q^{\rho+\frac{k-1}{2}-\nu}\,u - q^{\nu-\rho-\frac{k-1}{2}}\,v\right) \, ,$$

$$w_k(x,y)(=w_k(x,y;0)) := \sum_{n=0}^{k} \begin{bmatrix} k \\ n \end{bmatrix} x^n (-y)^{k-n} = \prod_{\nu=0}^{k-1} (x - q^{k-1-2\nu}\, y)\,. \tag{4.30}$$

Proposition 4.1. *The space of* 4*-point* U_q*-invariants in the tensor product* $\mathcal{F}_p^{\otimes 4}$ *is* p*-dimensional and is spanned by*

$$J_\lambda^{(I)}(u_1,\dots,u_4) = w_{2I-\lambda}\left(u_1,u_2;\frac{\lambda+1}{2}\right) w_{2I-\lambda}\left(u_3,u_4;\frac{\lambda+1}{2}\right) \times \tag{4.31}$$
$$\times\, w_\lambda\left(u_2,u_3;I-\frac{\lambda-1}{2}\right) w_\lambda\left(u_1,u_4;\frac{\lambda+1}{2}-I\right)\,,\qquad \lambda=0,1,\dots,2I(=p-1)\,.$$

(For a proof see Appendix C. A more general result is established in [75].)

Clearly, the QUEA $U_q(A_r)$ is neither commutative nor co-commutative. The violation of co-commutativity however is not arbitrary: $U_q(A_r)$ is an *almost co-commutative quasi-triangular Hopf algebra.*

A Hopf algebra H is called *almost co-commutative* if there exists an invertible element R of $H\otimes H$ which intertwines the coproduct Δ (3.2) with its permuted one, Δ' (3.13)

$$R\,\Delta(X) = \Delta'(X)\,R\,,\qquad \forall\, X\in H\,. \tag{4.32}$$

It is called quasi-triangular if R satisfies, in addition

$$(\Delta\otimes \mathrm{id})(R) = R_{13}\,R_{23}\,,\quad (\mathrm{id}\otimes\Delta)(R) = R_{13}\,R_{12}\,. \tag{4.33}$$

Here we are using Faddeev's notation (see, e.g., [57]) R_{ij} for the action of R on the triple tensor product $H\otimes H\otimes H$: define the algebra morphisms $\phi_{ij} : H\otimes H \to H\otimes H\otimes H$ ($i,j=1,2,3$, $i\neq j$) by

$$\phi_{12}(a\otimes b) = a\otimes b\otimes 1\!\!1\,,\quad \phi_{23}(a\otimes b) = 1\!\!1\otimes a\otimes b\,,$$
$$\phi_{13}(a\otimes b) = a\otimes 1\!\!1\otimes b\,;\quad \text{then}\quad R_{ij} := \phi_{ij}\,R\,. \tag{4.34}$$

Applying $\varepsilon\otimes\mathrm{id}$ to both sides of the first equation (4.33) and $\mathrm{id}\otimes\varepsilon$ to the second one and using Eq. (3.2) we find

$$(\varepsilon\otimes\mathrm{id})\,R = (\mathrm{id}\otimes\varepsilon)\,R = 1\!\!1\,. \tag{4.35}$$

Indeed, if we apply $\varepsilon\otimes\mathrm{id}\otimes\mathrm{id}$ to both sides of the first equation (4.33) and use (3.2) we obtain

$$R_{23} = (\varepsilon\otimes\mathrm{id}\otimes\mathrm{id})\,R_{13}\,R_{23}\,;$$

since R_{23} is invertible this gives

$$(\varepsilon\otimes\mathrm{id}\otimes\mathrm{id})\,R_{13} = 1\!\!1\otimes 1\!\!1$$

in accord with (4.35).

Exercise 4.7. Prove, using quasi-triangularity, the relations

$$R^{-1} = (S\otimes\mathrm{id})\,R\,,\quad R = (\mathrm{id}\otimes S)\,R^{-1} = (S\otimes S)\,R\,. \tag{4.36}$$

The quasi-triangularity further implies the *Yang-Baxter equation* (YBE) for the R matrix:

$$R_{12}\,R_{13}\,R_{23} = R_{23}\,R_{13}\,R_{12}\,. \tag{4.37}$$

A natural construction of the R-matrix, which allows to verify the above properties, requires yet another concept, the *quantum double*, and is given in Section 5 below. Here we shall reproduce instead a simple example of a space of non-commutative matrices which allows to understand the meaning of the YBE (4.37) and its connection to the basic relation among braid group generators.

Let $T = (T^\alpha_\beta\,,\ \alpha, \beta = 1, \dots, n)$ be an $n \times n$ matrix whose entries do not commute but obey the *RTT relation* (see [57])

$$R_{12}\, T_1\, T_2 = T_2\, T_1\, R_{12} \quad \text{where} \quad T_1 = T \otimes 1\!\!1\ , \quad T_2 = 1\!\!1 \otimes T\,. \tag{4.38}$$

Natural examples of such T-matrices are provided by the Borel components of $U_q(A_r)$ (see Section 5). We then apply both sides of (4.37) to the triple product $T_1\, T_2\, T_3$:

$$\begin{aligned} & R_{12}\, R_{13}\, R_{23}\, T_1\, T_2\, T_3 = R_{12}\, R_{13}\, T_1\, T_3\, T_2\, R_{23} = \\ = \quad & R_{12}\, T_3\, T_1\, T_2\, R_{13}\, R_{23} = T_3\, T_2\, T_1\, R_{12}\, R_{13}\, R_{23}\,; \end{aligned}$$

$$\begin{aligned} & R_{23}\, R_{13}\, R_{12}\, T_1\, T_2\, T_3 = R_{23}\, R_{13}\, T_2\, T_1\, T_3\, R_{12} = \\ = \quad & R_{23}\, T_2\, T_3\, T_1\, R_{13}\, R_{12} = T_3\, T_2\, T_1\, R_{23}\, R_{13}\, R_{12}\,, \end{aligned} \tag{4.39}$$

where we have used the relations $R_{23}\, T_1 = T_1\, R_{23}$, $R_{13}\, T_2 = T_2\, R_{13}$, $R_{12}\, T_3 = T_3\, R_{12}$. Thus both sides of (4.37) when commuted with $T_1\, T_2\, T_3$ intertwine it with $T_3\, T_2\, T_1$, permuting the T_i in different order. Eq. (4.37) thus reflects the associativity of multiplication (of the elements) of T-matrices. (More on this interpretation of the YBE the reader will find in [116].)

Eq. (4.37) reminds us the Artin braid relation (2.1). To obtain the exact relation between the two we multiply both sides of (4.37) by the product of permutations $P_{12}\, P_{23}\, P_{12} = P_{13} = P_{23}\, P_{12}\, P_{23}$ and set

$$P_{i\, i+1}\, R_{i\, i+1} = \hat{R}_{i\, i+1}\ (= b_i)\,. \tag{4.40}$$

This gives for the left hand side of (4.37)

$$\begin{aligned} & P_{23}\, P_{12} (P_{23}\, R_{12}\, P_{23}) (P_{23}\, R_{13}\, P_{23})\, \hat{R}_{23} = P_{23}\, P_{12}\, R_{13}\, P_{12}^2\, R_{12}\, \hat{R}_{23} = \\ = \quad & P_{23}\, R_{23}\, \hat{R}_{12}\, \hat{R}_{23} = \hat{R}_{23}\, \hat{R}_{12}\, \hat{R}_{23}\,, \end{aligned}$$

where we have used $P_{23}\, R_{13}\, P_{23} = R_{12}$ etc. Similarly, the right hand side of (4.37) is reduced to $\hat{R}_{12}\, \hat{R}_{23}\, \hat{R}_{12}$. Thus the YBE (4.15) is equivalent to the braid relation

$$\hat{R}_{12}\, \hat{R}_{23}\, \hat{R}_{12} = \hat{R}_{23}\, \hat{R}_{12}\, \hat{R}_{23}\ , \quad \text{or} \quad b_1\, b_2\, b_1 = b_2\, b_1\, b_2 \tag{4.41}$$

for $\hat{R}_{i\, i+1}$ given by (4.40).

Appendix B. Multiple tensor products, Faddeev's notation, applications

Faddeev's notation, introduced in (4.34) and used sporadically throughout these notes, is simplifying substantially operations on multiple tensor products. For the benefit of the readers who are not familiar with it, we work out in this Appendix some examples and applications. We derive, in particular, basic properties of the polarized Casimir invariants that will be needed in the sequel. (To those who would sniff at the importance of notation we offer to multiply, say 137 with 374, using Roman figures only...)

Introduce, to begin with, the m-fold tensor product of a finite dimensional vector space V with itself:

$$V \otimes V \otimes \cdots \otimes V =: V_1\, V_2 \ldots V_m \ . \tag{B.1}$$

Here V_i indicates the i-th factor in the tensor product. (The notation is flexible: the right hand side may also stand for the tensor product of different spaces – depending on the context.) Let further $A : V \to V$ be a linear operator (a square matrix) acting in V; mathematicians would use the notation $A \in \mathrm{End}\, V$ (End standing for *endomorphisms*). Then

$$A_i = \mathbb{1} \otimes \ldots \mathbb{1} \otimes A \otimes \mathbb{1} \otimes \ldots \mathbb{1} \tag{B.2}$$

(A in the i-th place, unit operators acting on V_j for $1 \le j \le m\,,\ j \ne i$) is a shorthand for the natural embedding of $\mathrm{End}\, V_i$ into $\mathrm{End}\,(V_1\, V_2 \ldots V_m)$.

Given a basis in $V = \mathbb{C}^n$, we can think of the operators in V as $n \times n$ matrices. Then $A_1 B_2$ can be viewed as an $n^2 \times n^2$ matrix that is the *Kronecker product* of the matrices A and B. For instance, for $n = 2$ and

$$A = \begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix} \ , \qquad B = \begin{pmatrix} b^1_1 & b^1_2 \\ b^2_1 & b^2_2 \end{pmatrix}$$

we write

$$A_1 B_2 = \begin{pmatrix} a^1_1 B & a^1_2 B \\ a^2_1 B & a^2_2 B \end{pmatrix} = \begin{pmatrix} a^1_1 b^1_1 & a^1_1 b^1_2 & a^1_2 b^1_1 & a^1_2 b^1_2 \\ a^1_1 b^2_1 & a^1_1 b^2_2 & a^1_2 b^2_1 & a^1_2 b^2_2 \\ a^2_1 b^1_1 & a^2_1 b^1_2 & a^2_2 b^1_1 & a^2_2 b^1_2 \\ a^2_1 b^2_1 & a^2_1 b^2_2 & a^2_2 b^2_1 & a^2_2 b^2_2 \end{pmatrix} ; \tag{B.3}$$

in particular,

$$A_1 \equiv A \otimes \mathbb{1} = \begin{pmatrix} a^1_1 \mathbb{1} & a^1_2 \mathbb{1} \\ a^2_1 \mathbb{1} & a^2_2 \mathbb{1} \end{pmatrix} = \begin{pmatrix} a^1_1 & 0 & a^1_2 & 0 \\ 0 & a^1_1 & 0 & a^1_2 \\ a^2_1 & 0 & a^2_2 & 0 \\ 0 & a^2_1 & 0 & a^2_2 \end{pmatrix} ,$$

$$B_2 \equiv \mathbb{1} \otimes B = \begin{pmatrix} B & \mathbb{0} \\ \mathbb{0} & B \end{pmatrix} = \begin{pmatrix} b^1_1 & b^1_2 & 0 & 0 \\ b^2_1 & b^2_2 & 0 & 0 \\ 0 & 0 & b^1_1 & b^1_2 \\ 0 & 0 & b^2_1 & b^2_2 \end{pmatrix} . \tag{B.4}$$

The double index enumeration of elements of a Kronecker product thus follows the *lexicographic* ordering: $im < jn$ iff $i < j$ or, for $i = j\,,\ m < n$. In the case (B.3) considered above it is $11\,,\, 12\,,\, 21\,,\, 22\,$.

We now proceed to examples of operators acting in the tensor product of two spaces which display the above compact notation at work. Let us first consider the application announced in the beginning. Postponing a more systematic introduction to simple Lie algebras for Appendix F, we shall recall the definition of a polarized Casimir operator and will derive the basic identities for the commutators of such operators.

Let $\mathcal{G}$ be a real Lie algebra and let $\{T_a\,,\ a=1,\dots,d_\mathcal{G}=\dim(\mathcal{G})\}$ be a basis in $\sqrt{-1}\,\mathcal{G}$. The polarized Casimir operator C_{12} is an element of the tensor product $\mathcal{G}\otimes\mathcal{G}$ which satisfies a Lie algebraic invariance condition:

$$C_{12}=\eta^{ab}\,T_{a1}\,T_{b2}\ , \qquad [\,T_{a1}+T_{a2}\,,\,C_{12}]=0\ , \quad a=1,\dots,d_\mathcal{G}\ . \tag{B.5}$$

If $\mathcal{G}$ is simple, then C_{12} is determined by (B.5) up to a non-zero multiplicative factor. Then η^{ab} can be identified with the inverse *Killing metric tensor* whose normalization is specified in Appendix F. (In particular, its sign is fixed by the requirement that (η^{ab}) is a positive definite matrix, for $\mathcal{G}$ the Lie algebra of a compact Lie group.) Let ${f_{ab}}^c$ be the structure constants of $\mathcal{G}$ and η_{ab} (such that $\eta^{as}\eta_{sb}=\delta^a_b$), the Killing metric; we shall just need the following property which is independent of the normalization of η_{ab} :

$$\text{if} \qquad \frac{1}{i}\,[T_a\,,\,T_b]={f_{ab}}^c\,T_c \qquad \text{and} \qquad f_{abc}:={f_{ab}}^s\eta_{sc}\ , \tag{B.6}$$

then f_{abc} is totally antisymmetric.

Exercise B.1. Using the antisymmetry of f_{abc} (B.6), prove the invariance property (B.5).

The operators $\{t^a=\eta^{ab}T_b\,,\ a=1,...,d_\mathcal{G}\}$ form a *basis in* $\mathcal{G}$ *dual to* $\{T_a\}$.

Exercise B.2. Prove the relation

$$[C_{12},C_{13}]=[C_{13},C_{23}]=-[C_{12},C_{23}]=f_{abc}\,t^a_1\,t^b_2\,t^c_3\ . \tag{B.7}$$

(*Hint*: use the fact that only the repeated index corresponds to a nontrivial commutator.)

Exercise B.3. Verify that the consequence of (B.7)

$$[C_{12},C_{13}+C_{23}]=0=[C_{12}+C_{13},C_{23}] \tag{B.8}$$

follows directly from the invariance condition (B.5).

Our next example concerns the permutation operators P_{ij} , encountered in Appendix A. A basis of operators in an n-dimensional vector space V is given by the Weyl matrices e^j_i characterized by the property

$$e^j_i\,e^\ell_k=\delta^j_k\,e^\ell_i \qquad \text{i.e.,} \quad (e^j_i)^k{}_\ell=\delta^k_i\,\delta^j_\ell \tag{B.9}$$

(we shall identify e^j_i with e_{ij} , the order of indices being dictated by the first equation in (B.9)). The permutation operator P_{12} can be written in the form

$$P_{12}=\sum_{i,j}(e^j_i)_1(e^i_j)_2\ . \tag{B.10}$$

Using Faddeev's notation, its action is very simple to describe: P_{12} just exchanges the indices 1 and 2 , e.g. $P_{12}A_{2431}=A_{1432}P_{12}$.

Exercise B.4. For P_{12} given by (B.10), verify the equations

$$P_{12}\,(e^\ell_k)_2\,P_{12}=(e^\ell_k)_1\ , \qquad (P_{12})^2=1\!\!1\ . \tag{B.11}$$

For $n = 2$ it is easy to see, using (B.3) and (B.10), that

$$P_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \qquad \text{(B.12)}$$

The relation $A_1 B_2 = B_2 A_1$ implies that the entries of the matrices A and B commute; for $A = B$ this means that the entries of A commute among themselves. A non-commutative deformation of this property when exchanging the order is equivalent to a similarity transformation is expressed by the (quadratic) "RTT relations" (4.38) in which R is the quantum R-matrix. *Warning*: Note that $P_{12}\, A_1 B_2 P_{12} = A_2 B_1$ which is equal to $B_1 A_2$ (but not to $B_2 A_1$!) iff the entries of A and B commute. As one more application of Faddeev's notation we invite the reader to work out the derivation of Eqs. (4.39) and (4.41).

Appendix C. General form of n-point U_q-invariants

We shall construct in this Appendix a privileged basis of n-point (in particular, 4-point) U_q-invariants – something peculiar for the q-deformed case. We shall provide on the way the main ingredients of the proof of Proposition 4.1. We first comment on the meaning of the U_q-invariants (4.29)–(4.31) comparing them with the corresponding $SU(2)$ invariants. Then we discuss the role of the different U_q-invariance conditions (4.25)–(4.27) – proving on the way Proposition 4.1. Finally, we comment on the properties which distinguish the basis (4.31) of 4-point invariants. The Appendix may be viewed as a pedagogical introduction to the paper [76] which gives a system of n-point U_q-invariants in the tensor product of irreducible U_q-modules $\mathcal{F}_{p_i}$ corresponding to different isospins I_i (and dimensions $p_i = 2I_i + 1$) (see Proposition B.1 below).

The basic 2-point invariant with respect to $SU(2)$ is the skew symmetric tensor $\epsilon^{AB} = -\epsilon^{BA}$ $(A, B = 1, 2)$. If we introduce the undeformed coherent states $\Phi_I(\zeta)$ (obtained from (4.22) in the limit $q \to 1$ – *cf.* the book [126]), it is given (for $I = \frac{1}{2}$) by the difference $\zeta_{12} = \zeta_1 - \zeta_2$ of formal variables. Its generalization to higher isospins I is nothing but the power ζ_{12}^{2I}. The product $J^{(I)}(u_1, u_2)$ (4.29) is a deformation of this simple monomial and can be obtained as follows.

K invariance (4.25) (for generic q) implies that $J^{(I)}(u_1, u_2)$ is a homogeneous polynomial in u_1, u_2 of degree $2I$:

$$J^{(I)}(u_1, u_2) = \sum_{m=0}^{2I} a_{Im}\, u_1^m (-u_2)^{2I-m} \,.$$

Applying to it the condition (4.26) of E invariance we find the recursive relation

$$(m)_+\, q^{2I-2m}\, a_{Im} = (2I - m + 1)_+\, a_{Im-1} \Leftrightarrow a_{Im} = \frac{[2I - m + 1]}{[m]}\, q\, a_{Im-1} \,.$$

Solving the recurrence for $a_0 = q^{-I}$ we obtain the right hand side of (4.29). In verifying F-invariance of the 2-point function so obtained one uses the identity $(-n)_- = -q^2 (n)_+$.

A 3-point invariant in the tensor product of three U_q-modules $\boldsymbol{I}_1, \boldsymbol{I}_2, \boldsymbol{I}_3$ of isospins I_1, I_2, I_3 only exists if $\boldsymbol{I}_3$ enters the tensor product expansion of $\boldsymbol{I}_1 \otimes \boldsymbol{I}_2$,

$$|I_1 - I_2| \le I_3 \le I_1 + I_2 \,, \quad I_1 + I_2 - I_3 \in \mathbb{N} \,, \tag{C.1}$$

and then it is unique (up to normalization). Similar existence conditions hold for n-point invariants which may depend on at most $n - 3$ (discrete) parameters. In particular, there are $p = 2I + 1$ 4-point invariants J_λ, $\lambda = 0, 1, \ldots, 2I$ in the 4-fold tensor product $\boldsymbol{I}^{\otimes 4} (= \mathcal{F}_p^{\otimes 4})$. The expressions (4.31) clearly obey the homogeneity condition (4.25). In order to verify that they are also E-invariant one uses the relations

$$\begin{aligned} D_{1+}\, w_k(q^\rho\, u_1, q^{-\rho}\, u_2) &= q^{\rho + \frac{k-1}{2}}\, [k]\, w_{k-1}(q^{\rho+\frac{1}{2}}\, u_1, q^{-\rho-\frac{1}{2}}\, u_2) \\ D_{2+}\, w_k(q^\rho\, u_1, q^{-\rho}\, u_2) &= -q^{-\rho - \frac{k-1}{2}}\, [k]\, w_{k-1}(q^{\rho-\frac{1}{2}}\, u_1, q^{\frac{1}{2}-\rho}\, u_2) \,. \end{aligned} \tag{C.2}$$

The following more general result is established in [75].

Proposition C.1. *There exists a basis* $J^{(I_1,\ldots,I_n)}_{\{\mu_{ij}\}}$ *of* U_q*-invariant monomials in the* n*-fold tensor product* $\boldsymbol{I}_1 \otimes \boldsymbol{I}_2 \otimes \ldots \otimes \boldsymbol{I}_n$*,*

$$J^{(I_1,\ldots,I_n)}_{\{\mu_{ij}\}} = \prod_{1 \le i < j \le n} w_{ij} \,, \quad w_{ij} = w_{k_{ij}}(u_i, u_j; \rho_{ij}) \tag{C.3}$$

where $w_k(u,v;\rho)$ is defined by (4.30)*. The parameters $k_{ij}(=k_{ji})$ and ρ_{ij} have to satisfy (as a consequence of the invariance of* (C.3)*) the following conditions:*

$$\sum_j k_{ij} = 2\,I_i \qquad (k_{ii}=0)\,, \tag{C.4}$$

$$k_{ij}\,k_{\ell m} = 0 \quad for \quad i<\ell<j<m \quad (or \;\; \ell<i<m<j)\,; \tag{C.5}$$

if $k_{ij}>0$, then

$$\rho_{ij} + \frac{1}{2}\,k_{ij} = \sum_{s=i+1}^{j} I_s - \sum_{\substack{i\le\ell<m\le j\\(\ell,m)\neq(i,j)}} k_{\ell m}\,, \qquad 1\le i<j\le n\,. \tag{C.6}$$

There are $n-3$ (integer valued) parameters among the k_{ij} $(0\le k_{ij}\le \min(2I_i,2I_j))$ which label the general solution of (C.3)–(C.6).

Exercise C.1. For $n=4$, $I_1=I_2=I_3=I_4=:I$ set $k_{14}=\lambda$ and determine the remaining k_{ij} that reproduce the solution (4.31).

We observe that the *selection rule* (C.5) has no counterpart in the undeformed case. There are, so to speak, fewer U_q- than $SU(2)$-invariants. The basis of invariant monomials expressed as products of elementary 2-point invariants is essentially unique. The invariants (4.31) (in contrast to other choices used in the literature) are well defined and linearly independent also for q a root of unity.

5 Quantum Gauss decomposition and the Drinfeld double

Every element g of a dense open neighbourhood of the group unit of the general linear group $GL(n,\mathbb{C})$ admits a *Gauss decomposition* $g = b \cdot f$, where b is a lower triangular matrix while f is upper triangular with units on the diagonal. This corresponds to splitting of the Lie algebra into a Borel subalgebra generated (in the $s\ell_n$ case) by E_i and H_i and a nilpotent one, generated by F_i. In the q-deformed case, we see that such a splitting does not lead to Hopf subalgebras, since $\Delta(F_i)$ (4.4) also involves $K_i^{-1}(= q^{-H_i})$. A way out is to include the diagonal (Cartan) elements in both parts of the decomposition and then impose a relation among them. This allows to introduce the notion of quantum double and yields a streamlined construction of a universal quasi-triangular R-matrix. We shall outline this construction for the rank one case, $U_q(A_1)$ (the QUEA generated by K, E, F) and will indicate at the end how to extend it to the higher rank case.

We introduce a pair of *quantum Borel*[18] *algebras* $U_q\, b_\pm$ of two generators each: (k_-, E) and k_+, F, such that $k_+\, k_- = K$ satisfying

$$k_-\, E = q\, E\, k_- \,, \qquad F\, k_+ = q\, k_+\, F \tag{5.1}$$

and the mixed relations

$$[k_-, k_+] = 0\,, \quad F\, k_- = q\, k_-\, F\,, \quad k_+\, E = q\, E\, k_+\,, \quad [E, F] = \frac{k_-^2 - k_+^{-2}}{q - q^{-1}}\,. \tag{5.2}$$

(Ultimately, we shall set $k_- = k_+$.)

Introduce the triangular matrices

$$M_- = D_-^{-1} N_-^{-1} = \begin{pmatrix} k_- & 0 \\ \lambda\, k_-^{-1} E & k_-^{-1} \end{pmatrix}, \quad M_+ = N_+ D_+ = \begin{pmatrix} k_+^{-1} & -\lambda\, F\, k_+ \\ 0 & k_+ \end{pmatrix}, \tag{5.3}$$

$$\lambda = q - q^{-1}\,,$$

where $D_\pm$ are diagonal matrices, $D_\pm = \begin{pmatrix} k_\pm^{-1} & 0 \\ 0 & k_\pm \end{pmatrix}$ and $\mathbb{1} - N_\pm$ are nilpotent.

Exercise 5.1. Verify that the CR (5.1) are equivalent to the "RTT relations"

$$R_{12}\,(M_\pm)_2\,(M_\pm)_1 = (M_\pm)_1\,(M_{\pm 2})\,R_{12} \tag{5.4}$$

where R is expressed in terms of the Weyl matrices, $(e_{ij})^\alpha_\beta = \delta^\alpha_i\, \delta_{j\beta}$, as follows:

$$\begin{aligned} R &= q^{-\frac{1}{2}}(e_{11}\otimes e_{11} + e_{22}\otimes e_{22}) + q^{\frac{1}{2}}(e_{11}\otimes e_{22} + e_{22}\otimes e_{11} - \lambda\, e_{21}\otimes e_{12}) \\ &= q^{\frac{1}{2}} \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}. \end{aligned} \tag{5.5}$$

The coproduct in each $U_q\, b_\pm$ defined by

$$\Delta\,(M^\alpha_{\epsilon\beta}) = M^\alpha_{\epsilon\,\sigma} \otimes M^\sigma_{\epsilon\,\beta}\,, \quad \epsilon = \pm\,, \tag{5.6}$$

[18] The Swiss mathematician Armand Borel (1923-2003) studied at ETH, Zürich; since 1957 he was a professor at the Institute for Advanced Study, Princeton. He is one of the creators of the theory of linear algebraic groups.

is equivalent to

$$\Delta(k) = k_\pm \otimes k_\pm\,, \quad \Delta(E) = E \otimes k_-^2 + \mathbb{1} \otimes E\,, \quad \Delta(F) = F \otimes \mathbb{1} + k_+^{-2} \otimes F\,. \tag{5.7}$$

Similarly, the counit and the antipode acquire the form they have in the group algebra (Section 3) when expressed in terms of the matrices (5.3):

$$\varepsilon(M^\alpha_{\epsilon\,\beta}) = \delta^\alpha_\beta\,, \quad S(M^\alpha_{\epsilon\,\beta}) = (M_\epsilon^{-1})^\alpha_\beta\,. \tag{5.8}$$

The introduction of the pair of Borel Hopf algebras, $U_q\, b_\pm$, is justified by the following two facts.

(i) *There exists a unique bilinear pairing*

$$\langle Y, X\rangle \quad (\in \mathbb{C} \;\; for \;\; X \in U_q\, b_+\,,\; Y \in U_q\, b_-)$$

such that

$$\langle Y Y', X\rangle = \langle Y \otimes Y', \Delta(X)\rangle = \sum_{(X)} \langle Y, X_1\rangle\, \langle Y', X_2\rangle$$

for

$$Y, Y' \in U_q\, b_-\,, \quad \Delta(X) = \sum_{(X)} X_1 \otimes X_2 \in U_q\, b_+ \otimes U_q\, b_+\,, \tag{5.9}$$

$$\langle \Delta(Y), X \otimes X'\rangle = \sum_{(Y)} \langle Y_1, X\rangle\, \langle Y_2, X'\rangle = \langle Y, X' X\rangle \tag{5.10}$$

$$\langle \mathbb{1}, X\rangle = \varepsilon(X)\,, \quad \langle S(Y), X\rangle = \langle Y, S^{-1}(X)\rangle\,, \quad \varepsilon(Y) = \langle Y, \mathbb{1}\rangle\,. \tag{5.11}$$

It is given by

$$\langle E^\mu\, k_-^m, f_{\nu n}\rangle = \delta_{\mu\nu}\, \frac{[\mu]!}{(-\lambda)^\mu}\; q^{\frac{\mu(\mu-1)-mn}{2}} \tag{5.12}$$

where $\{f_{\nu n}\}$ *is a Poincaré*[19]*-Birkhoff-Witt*[20] (PBW) *basis in* $U_q\, b_+$*:*

$$f_{\nu n} := F^\nu\, k_+^n\ . \tag{5.13}$$

The mixed relations (5.2) *are recovered provided the product* XY *is constrained by*

$$XY(\cdot) = \sum_{(X)} Y(S^{-1}(X_3)\cdot X_1)\, X_2 \tag{5.14}$$

for

$$\Delta^{(2)}(X) = (\mathrm{id} \otimes \Delta)\, \Delta(X) = (\Delta \otimes \mathrm{id})\, \Delta(X) = \sum_{(X)} X_1 \otimes X_2 \otimes X_3\,. \tag{5.15}$$

The dot $(\cdot)$ *in* (5.14) *stands for the argument (say* $Z \in U_q\, b_+$*) of the functional* $Y(Z) \equiv \langle Y, Z\rangle$*.*

(ii) *The universal R-matrix of the quantum double* $(U_q\, b_-, U_q\, b_+)$ *is given by*

$$\mathcal{R} = \sum_{n,\nu} f_{\nu n} \otimes e_{\nu n} \qquad \text{for} \qquad \langle e_{\mu m}, f_{\nu n}\rangle = \delta_{\mu\nu}\, \delta_{mn}\ . \tag{5.16}$$

[19]Jules-Henri Poincaré (1854-1912), more than anybody else may be called the prophet of 20^{th} century mathematics. He is the founder of topology (called by him analysis situs), introducing, in particular, the concept of fundamental group, used in Section 2. He preceded Einstein in analyzing the relativity of time and simultaneity. Poincaré stated the PBW theorem in 1900.

[20]Garrett Birkhoff (1911-1996) son of the Harvard mathematician George David Birkhoff (1884-1944) is known for his contributions to abstract algebra. He and the German mathematician Ernst Witt (1911-1991) published independent proofs of Poincaré's statement in 1937.

(In the case of the restricted QUEA $\bar{U}_q$ for $q^h = -1$ – see Section 14 – the double construction has been worked out in [58] and [74].)

Remark 5.1. The matrices $M_\pm^{\pm 1}$ (5.3) provide the Gauss decomposition of the monodromy matrix M. The corresponding *quantum matrices* appearing in the context of the $su(2)$ current algebra model (Eq. (9.11)) satisfy

$$M = q^{-\frac{3}{2}} M_+ M_-^{-1} \qquad \text{for} \quad q = e^{-\frac{i\pi}{h}} \,, \quad h = 2, 3, \dots \tag{5.17}$$

(see Section 9 below; the choice of the phase factor is dictated by the result of Exercise 9.1 and 9.4).

The *double cover* D_q of $U_q(A_1)$ is obtained from the above quantum double by setting

$$k_+ = k_- \equiv k \qquad (k^2 = K)\,. \tag{5.18}$$

Its CR are then obtained from (5.1) and (5.2). Its significance stems from the fact that the universal R-matrix (5.16) of U_q belongs, in fact, to (a completion of) $D_q \otimes D_q$. It plays an important role in the physically interesting case of q a root of unity – see Section 14 below.

Writing the *Drinfeld-Jimbo universal R-matrix* requires introducing H (instead of K) and using transcendental functions, thus leaving the algebraic framework. We have (see [34])

$$\mathcal{R} = \sum_{\nu=0}^{\infty} \frac{q^{-\binom{\nu}{2}} (-\lambda)^\nu}{[\nu]!} F^\nu \otimes E^\nu \, q^{-\frac{1}{2} H \otimes H} \,, \qquad \binom{\nu}{2} = \frac{\nu(\nu-1)}{2} \,. \tag{5.19}$$

We shall see in Remark 14.1 below that the 4×4 matrix (5.5) is related to a finite dimensional counterpart of $\mathcal{R}$.

It turns out that an expression of the type (5.16)

$$R = \sum_\sigma f_\sigma \otimes e^\sigma \,, \quad f_\sigma \in U_q\, b_+ \,, \quad e^\sigma \in U_q\, b_- \,, \quad \langle e^\rho , f_\sigma \rangle = \delta^\rho_\sigma \tag{5.20}$$

using a pair of dual bases of the quantum double is easier to handle (than (5.19)) for extracting properties of the R-matrix – even without knowing the explicit form of the basis $\{f_\sigma\}$ (a formula like (5.13)) of $U_q\, b_+$. To give an example, we shall establish the quasi-triangularity relation $(\Delta \otimes \mathbb{I})\, R = R_{13}\, R_{23}$ (4.33) for R given by (5.20).

As $\{f_\sigma\}$ form a basis in the Hopf algebra $U_q\, b_+$ one can expand the coproduct of f_σ into tensor products $f_\rho \otimes f_\tau$ and use the first equation (5.9) to determine the coefficients:

$$\Delta(f_\sigma) = \sum_{\rho,\tau} g_\sigma^{\rho\tau} f_\rho \otimes f_\tau \,, \quad g_\sigma^{\rho\tau} = \langle e^\rho \cdot e^\tau , f_\sigma \rangle \,. \tag{5.21}$$

Inserting in the left side of the above quasi-triangularity relation we find

$$(\Delta \otimes \mathrm{id})\, R = \sum_{\sigma,\rho,\tau} g_\sigma^{\rho\tau} f_\rho \otimes f_\tau \otimes e^\sigma = \sum_{\rho,\tau} f_\rho \otimes f_\tau \otimes e^\rho \cdot e^\tau = R_{13}\, R_{23} \tag{5.22}$$

where we have used the relation

$$\sum_\sigma g_\sigma^{\rho\tau} e^\sigma = e^\rho \cdot e^\tau \tag{5.23}$$

which follows from (5.21) and the last equation (5.20). The relation $(\mathrm{id} \otimes \Delta)\, R = R_{13}\, R_{12}$ is established similarly.

Remark 5.2. If (U_q, R) is a quasi-triangular Hopf algebra, so is $(U_q, \tilde{R})$ for $\tilde{R} = R_{21}^{-1}$ (see [34], p. 123). The second universal R-matrix $\tilde{\mathcal{R}}$ can be obtained from the "transposed quantum double" $(U_q\, b_-, U_q\, b_+)$ by the same procedure which allowed us to construct $\mathcal{R}$ from $(U_q\, b_+, U_q\, b_-)$. The result is

$$\tilde{\mathcal{R}} = q^{\frac{1}{2} H \otimes H} \sum_{\nu=0}^{\infty} \frac{q^{\binom{\nu}{2}} \lambda^\nu}{[\nu]!}\, E^\nu \otimes F^\nu \,. \tag{5.24}$$

Exercise 5.2. Prove that Eq.(4.32) is equivalent to $R_{21}\, \Delta'(X) = \Delta(X)\, R_{21}$ (and hence, $\tilde{R} = R_{21}^{-1}$ satisfies (4.32) together with R).

All results of this section extend to $U_q(A_r)$ for $r > 1$ on the expense of writing somewhat more complicated formulae. We shall only spell out the first step: the definition of the Borel algebras $U_q\, b_\pm^r$ of rank r and the Gauss decomposition of the monodromy matrix.

The higher rank extension of (5.3) reads

$$M_+ = D_+\, N_+ \,, \quad M_-^{-1} = N_-\, D_- \tag{5.25}$$

where $D_\pm$ are diagonal matrices,

$$(D_\pm)_{ij} = d_i^\pm\, \delta_{ij} \,, \quad \prod_{i=1}^{r+1} d_i^\pm = 1 \qquad (i, j = 1, \dots, r+1) \tag{5.26}$$

while $\mathbb{1} - N_\pm$ are nilpotent:

$$N_+ = \begin{pmatrix} 1 & n_1^+ & n_{12}^+ & \dots & n_{1r}^+ \\ 0 & 1 & n_2^+ & \dots & n_{2r}^+ \\ 0 & 0 & 1 & \dots & n_r^+ \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} ,$$

$$N_- = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ n_1^- & 1 & 0 & \dots & 0 \\ n_{21}^- & n_2^- & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ n_{r1}^- & n_{r2}^- & n_{r3}^- & \dots & 1 \end{pmatrix} . \tag{5.27}$$

In the quantum case the $d_i^\pm$ and $n^\pm$ are expressed in terms of the generators $k_i^\pm$, E_i and F_i of the quantum double $(U_q\, b_+^r, U_q\, b_-^r)$:

$$d_1^\varepsilon = (k_1^\varepsilon)^{-1} \,,\ d_2^\varepsilon = k_1^\varepsilon (k_2^\varepsilon)^{-1}, \dots ,\ d_{r+1}^\varepsilon = k_r^\varepsilon \,;\ \varepsilon = \pm \,;$$

$$n_i^- = -\lambda\, E_i \,,\ n_i^+ = -\lambda\, F_i\, k_i^2\, k_{i-1}^{-1}\, k_{i+1}^{-1} \quad (i = 1, \dots, r,\ k_0 = k_{r+1} = \mathbb{1})$$

$$n_{21}^- = -\lambda (E_1\, E_2 - q\, E_2\, E_1), \text{ etc.}; \quad d_1\, n_{12}^+\, d_3^{-1} = -\lambda (F_2\, F_1 - q\, F_1\, F_2), \text{ etc.} \tag{5.28}$$

For

$$k_i^+ = k_i^- =: k_i \,, \quad K_i = k_{i-1}^{-1}\, k_i^2\, k_{i+1}^{-1} \quad (k_0 = \mathbb{1} = k_{r+1}) \tag{5.29}$$

(giving rise to the double cover of $U_q(s\ell_n)$) the counterpart of the (quantum) relation (5.17) reads:

$$M = q^{\frac{1}{n} - n}\, M_+\, M_-^{-1} \,. \tag{5.30}$$

6 Conformally invariant QFT in D space-time dimensions

Historically quantum field theory (QFT) arose (in the late 1920s) in an attempt to unify quantum mechanics with special relativity using the canonical Lagrangian (or Hamiltonian) approach and perturbation theory[21]. We shall base our treatment, instead, on the axiomatic framework developed in the second half of 20th century (see, e.g. [144, 97, 24, 87, 28]) with the dual aim (i) to separate sense from nonsense in the formal manipulations with divergences and (ii) to clarify the basic principles of relativistic local quantum theory and their general implications. Adding the requirement of conformal invariance to the physically justified Wightman axioms [144] (for a summary – see Appendix D) makes for the first time the axiomatic approach truly constructive (for surveys of axiomatic conformal field theory (CFT) in two and four dimensions – see [113, 148, 75] and [146]).

Let us first recall the concept of conformal transformations and conformal invariance in any number D of space-time dimensions.

A transformation $g : x \to y$ of an open set O of space-time into another open set, gO is said to be conformal if the infinitesimal square interval dx^2 gets just multiplied by a (positive) factor: if $g : x\,(\in O) \to y(x,g)(\in gO)$ then

$$dy^2(x,g) = \frac{dx^2}{\omega^2(x,g)}\,, \quad \omega(x,g) \in \mathbb{R}\,, \quad \omega(x,g) \neq 0 \;\text{ for }\; x \in O\,. \tag{6.1}$$

Thus a conformal transformation is a generalization of an isometry (that would correspond to $\omega = 1$). To fix the ideas we shall consider conformal transformations of Minkowski[22] space M, setting

$$dx^2 = d\boldsymbol{x}^2 - (dx^0)^2\ , \qquad d\boldsymbol{x}^2 = \sum_{i=1}^{D-1} (dx^i)^2\ . \tag{6.2}$$

One should, however, keep in mind that our discussion applies equally well to all conformally flat metrics (such that $ds^2 = \frac{dx^2}{\Omega^2(x)}$). In particular, all spaces of constant curvature – the positive curvature de Sitter[23] space and the negative curvature anti de Sitter space (on top of the zero curvature Minkowski space) – are conformally flat. The assumption that the choice of metric within a given conformal class should not affect the physics in a CFT thus forces us to adopt a more general point of view on QFT.

Typically conformal transformations develop singularities: they cannot be defined on the whole of M. That is the reason we speak of open (sub)sets of M in the definition (6.1). By contrast, the *conformal Killing*[24] *vector* $K^\mu(x)(= K^\mu(x,g))$ corresponding to an infinitesimal

[21] We are unable to choose a single "best" textbook on QFT. An authoritative 3-volume treatise is Weinberg's [152]. For a selection of original papers on quantum electrodynamics reflecting the development up to the 1950s – see [132]; a clear and concise exposition of later developments in renormalization theory including the use of Becchi-Rouet-Stora cohomology is contained in [128]. Different in style and purpose is the (often entertaining) book [12] which surveys the inter-relations between gauge theory and modern mathematics.

[22] Hermann Minkowski (1864-1909) introduced the 4-dimensional space-time (in 1908 in Göttingen), thus completing the special theory of relativity of Hendrik Antoon Lorentz (1853-1928, Nobel Prize in Physics, 1902), Henri Poincaré and Albert Einstein (1879-1955, Nobel Prize in Physics, 1921).

[23] The Dutch mathematician, physicist and astronomer Willem de Sitter (1872-1934), who was interested in the concept of inertia in general relativity, introduced (as an alternative to Einstein's static universe) his constant curvature space (with a zero mass density) with a positive cosmological constant in 1917. Presently, it is believed that our universe is approaching a de Sitter space-time. Einstein and de Sitter wrote a joint paper in 1932 on what came to be called *dark matter* (whose presence is only detected by its gravitational field).

[24] Wilhelm Karl Joseph Killing (1847-1923), a student of Weierstrass and Kummer in Berlin, became a professor at the seminary college in Braumsberg. He invented Lie algebras, independently of Sophus Lie, around 1880. In 1888-1890 Killing classified (essentially) the complex simple Lie algebras, inventing the notions of a Cartan

conformal transformation

$$y^\mu(x) = x^\mu + \varepsilon\, K^\mu(x) + O(\varepsilon^2)\,, \quad \omega(x) = 1 - \varepsilon\, f(x) + O(\varepsilon^2) \tag{6.3}$$

is well defined in M and satisfies the *conformal Killing equation*

$$\partial_\mu\, K_\nu + \partial_\nu\, K_\mu = 2f\eta_{\mu\nu} \quad (\eta_{\mu\nu} = \mathrm{diag}\,(-,++\ldots))\,. \tag{6.4}$$

Exercise 6.1. Writing $dx^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu$, $dy^2 = \eta_{\mu\nu}\, dy^\mu\, dy^\nu$ and inserting (6.3) in (6.1) derive (6.4) by equating the terms of order ε.

Exercise 6.2. Demonstrate, using (6.4), that

$$\partial \cdot K (\equiv \partial_\mu\, K^\mu) = Df\,, \quad (D-2)\,\partial_\lambda\,\partial_\mu\, f = 0\,. \tag{6.5}$$

Use the result to derive the following

Proposition 6.1. (Liouville[25] theorem) *The general form of the conformal Killing vector for $D > 2$ is given by*

$$K^\mu(x) = a^\mu + \alpha\, x^\mu + \lambda_{\mu\nu}\, x^\nu - 2(c\cdot x)\, x^\mu + x^2\, c^\mu\,, \quad \lambda_{\mu\nu} = -\lambda_{\nu\mu}\,. \tag{6.6}$$

Exercise 6.3. Verify that the conformal group C of M is spanned by Poincaré transformations $y^\mu = \Lambda^\mu_\nu\, x^\nu + a^\mu$, uniform dilation $y^\mu = \rho\, x^\mu$, $\rho > 0$, and *special conformal transformations* which can be defined as translations $T_c : x \to x + c$ sandwitched between two *conformal inversions* $R : x \to \frac{x}{x^2}$:

$$y(x,c) = R\, T_c\, R\, x = \left(\frac{x}{x^2} + c\right) \left[\left(\frac{x}{x^2} + c\right)^2\right]^{-1} = \frac{x + cx^2}{\omega(x,c)}\,,$$
$$\omega(x,c) = 1 + 2c\cdot x + c^2 x^2\,. \tag{6.7}$$

Clearly, the special conformal transformations (6.7) are singular (for $c \neq 0$) on the cone $\omega(x,c) = 0$ (that degenerates into a hyperplane for $c^2 = 0$). One can define, following Dirac[26] [44], the conformal compactification of space-time $\bar{M}$ as a projective quadric in $D{+}2$ dimensions:

$$\bar{M} = Q/\mathbb{R}^* \simeq \mathbb{S}^{D-1} \times \mathbb{S}^1 / \pm 1\,,$$

$$Q = \left\{ \vec{\xi} \in \mathbb{R}^{D,2}\,;\ \vec{\xi}^{\,2} = \sum_{\alpha=1}^{D} \xi_\alpha^2 - \xi_0^2 - \xi_{-1}^2 = 0 \right\}\,, \qquad \mathbb{R}^* = \mathbb{R} \setminus \{0\}\,. \tag{6.8}$$

M is embedded in a dense open set of $\bar{M}$ in which $\kappa := \xi^D + \xi^{-1} (= \xi_D - \xi_{-1}) \neq 0$:

$$x^\mu = \frac{1}{\kappa}\, \xi^\mu \qquad \left(x^2 = \frac{\xi^{-1} - \xi^D}{\kappa} \right)\,.$$

subalgebra and a Cartan matrix. He introduced the root systems and discovered the exceptional Lie algebra $\mathcal{G}_2$ (in 1887). For a popular article about Killing and his work on Lie algebras, see A. John Coleman, *The greatest mathematical paper of all time*, The Mathematical Intelligencer **11**:3 (1889) 29-38; see also, T. Hawkins, Wilhelm Killing and the structure of Lie algebras, Archive for History of Exact Science **26** (1982) 126-192.

[25] Joseph Liouville (1809-1882) published his theorem (for 3-dimensional Euclidean space) in a Note to the 5th edition of Gaspard Monge (1746-1818), *Application de l'analyse à la géométrie* (Paris, 1850) entitled "Extension au cas des trois dimensions de la question du tracé géographique" (pp. 609-616).

[26] Paul Adrien Maurice Dirac (1902-1984), Nobel Prize in Physics 1933, known for his equation and for the prediction of antiparticles, recollects (in his Varenna 1977 lecture) of his great appreciation of projective geometry since his student years at Bristol.

The quadric Q (6.8) is, clearly, invariant under the full orthogonal group $O(D,2)$. The reflection $(-1\!\!1) : \vec{\xi} \to -\vec{\xi}$ acts however as the identity transformation on $Q/\mathbb{R}^*$ so it is only the quotient group $O(D,2)/\pm 1$ which acts effectively on $\bar{M}$ and should be identified with the conformal group C (including reflections) of compactified Minkowski space. It is natural, following Segal[27] [136], to identify the *conformal energy* operator with the (hermitean) generator H of the centre of the Lie algebra $so(2) \times so(D)$ of the maximal compact subgroup of C, i.e., with the infinitesimal rotation in the $(-1,0)$-plane. It can be expressed in terms of the Minkowski space energy operator P_0 (the zeroth component of the energy momentum vector) and its conjugate by the conformal inversion R as

$$H = \frac{1}{2}\,(P_0 + R\,P_0\,R)\,. \tag{6.9}$$

Here $R\,P_0\,R$ is a physical (hermitean) generator of the special conformal transformation (6.7) (in other words, the vector field corresponding to the Lie algebra element $i\,R\,P_0\,R$ is $\left[\frac{\partial}{\partial c^0}\, y^\nu(x,c)\right]_{c=0} \frac{\partial}{\partial x^\nu}$). In a unitary representation of (a covering of) the conformal group H (6.9) is positive whenever the Minkowski energy P_0 is positive.

The following exercise shows that for $D = 2$ the conformal group is infinite dimensional.

Exercise 6.4. Let $f_\pm(z)$ be a pair of (non-constant) meromorphic functions (taking real values on the real line). Demonstrate that both changes of variables $x \to y$ such that

$$y^0 + y^1 = f_+(x^0 \pm x^1)\,, \quad y^0 - y^1 = f_-(x^0 \mp x^1)\,, \tag{6.10}$$

satisfy the condition (6.1) for a conformal mapping. Show that the upper sign in (6.10) corresponds to the connected component of the identity of the (infinite dimensional) group of meromorphic mappings, while the lower one belongs to the connected component of space reflections.

Exercise 6.5.
(a) Show that the only complex conformal transformations which transform circles into circles or straight lines are the non-singular fractional linear (also called *Möbius*[28]) transformations

$$z \to z' = gz \equiv \frac{az+b}{cz+d} \qquad ad - bc \neq 0\,. \tag{6.11}$$

(b) They preserve the real line if the matrix entries of the 2×2 matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are real. They preserve the upper half plane if the determinant of g is positive (then one can set $\det g = ad - bc = 1$, thus identifying the *real Möbius group* with $SL(2,\mathbb{R})$).

(c) The transformation (6.11) preserves the unit circle iff $d = \bar{a}$, $c = \bar{b}$. For $g \in SU(1,1)$ (i.e. for $g = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right)$, $\det g = |a|^2 - |b|^2 = 1$) the Möbius transformation (6.11) preserves the interior (as well as the exterior) of the unit circle. For $|a|^2 - |b|^2 < 0$ it exchanges $|z| < 1$ with $|z| > 1$.

There is a complex Möbius map g_c of the upper half plane τ ($\mathrm{Im}\,\tau > 0$) onto the unit disk ($|z| < 1$) intertwining the $SL(2,\mathbb{R})$ and the $SU(1,1)$ actions. Choosing $g_c\, i = 0$, $g_c\, 0 = 1$ we find

$$g_c : \tau \to z = \frac{1+i\tau}{1-i\tau} \qquad \left(\tau = i\,\frac{1-z}{1+z}\right)\,. \tag{6.12}$$

[27] Irving Ezra Segal (1918-1998), Professor in mathematics at MIT since 1960, realized the physical relevance of C^* algebras – see the emotional obituary [11].

[28] August Ferdinand Möbius (1790-1868). The Möbius group $SL(2,\mathbb{C})$ is a double cover of the (connected component of the) Lorentz group $SO^\uparrow(3,1)$.

It maps the real light ray $\tau = t\,(= x^0 + x^1)$ onto the unit circle, sending the point at infinity to -1. Thus g_c plays the role of a *compactification map* for the light ray.

Exercise 6.6. Demonstrate that the non-singular conformal transformation $z \to f(z)$ is a Möbius transformation iff the Schwarz[29] derivative

$$\{f, z\} := \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{6.13}$$

vanishes.

We shall now exhibit a 4-dimensional quaternionic analogue of (6.12). Consider the Lie algebra $u(2)$ of 2×2 anti-hermitean matrices

$$i\,\tilde{x} = i\,x^0\,\sigma_0 - i\,x^j\,\sigma_j \qquad (\in u(2))$$

where σ_j are the Pauli[30] matrices, $\sigma_0 = \mathbb{I}$ is the 2×2 unit matrix. The Cayley[31] map from the Lie algebra $u(2)$ to the group $U(2)$ of 2×2 unitary matrices,

$$i\,\tilde{x} \to u = \frac{1 + i\,\tilde{x}}{1 - i\,\tilde{x}} \in U(2) \quad \text{for} \quad x^\mu \in \mathbb{R}\,, \tag{6.14}$$

can be viewed as an alternative of the conformal compactification (6.8).

Exercise 6.7. Writing u (6.14) in the form $u = u^4\,\mathbb{I} - i\,u^j\,\sigma_j$ prove that u^α are related to $\vec{\xi}$ in (6.8) and (6.9) by

$$u^\alpha = \frac{\xi^\alpha}{\xi^{-1} + i\,\xi^0}\,, \quad \alpha = 1, 2, 3, 4 \quad \left(\sum_{\alpha=1}^{4} u^\alpha\,\bar{u}^\alpha = 1\right). \tag{6.15}$$

Exercise 6.8. Prove that the Lie algebra $su(2,2)$ of the pseudo-unitary group $SU(2,2)$ coincides with the conformal Lie algebra $so(4,2)$.

(*Hint* : use the realization of Appendix C to [123] for $D = 4$.)

Exercise 6.9. Prove that $SU(2,2)$ is a 4-fold cover of the connected conformal group $C_0 \simeq SO_0(4,2)/\pm 1$ of 4-dimensional Minkowski space or, equivalently, a 2-fold cover of the connected pseudo-orthogonal group $SO_0(4,2)$:

$$SO_0(4,2) \simeq SU(2,2)/\pm 1\,. \tag{6.16}$$

The complex variable parametrization (6.15) of $\bar{M}$ admits an obvious extension for arbitrary space-time dimensions D (just) letting the index α run, from 1 to D). Denoting the complex

[29] Karl Hermann Amadeus Schwarz (1843-1921) a student of Karl Weierstrass (1815-1897); introduced his derivative in 1872.

[30] Wolfgang Ernest Pauli (born in Vienna 1900, died in 1958 in room 137 of a hospital in Zürich). During his stay in Hamburg (1923-1928) he discovered the exclusion principle (1925), for which he was awarded the Nobel Prize in Physics in 1945, and introduced the Pauli matrices (in 1927).

[31] Arthur Cayley (1821-1895) after studying at Trinity became (at 25) a lawyer for 14 years in London writing during that period over 200 mathematical papers. He was first to define the modern way the concept of a group. The Cayley transform originally appeared (1846) as a mapping between skew symmetric and special orthogonal matrices.

D-vector by z (leaving u to denote either an element of $u(2)$ or a real unit D-vector) we can write

$$\bar{M} = \left\{ z_\alpha = e^{it} u_\alpha\,,\ \alpha = 1, \ldots, D\,,\ \ t, u_\alpha \in \mathbb{R}\,,\ \sum_{\alpha=1}^{D} u_\alpha^2 = 1 \right\} = \mathbb{S}^1 \times \mathbb{S}^{D-1} / \pm 1\,. \tag{6.17}$$

Flat Minkowski space M is embedded as a dense open set in $\bar{M}$ by setting

$$z_i = \frac{x_i}{\omega(x)}\,,\ i = 1, \ldots, D-1\,; \qquad z_D = \frac{1 - x^2}{2\,\omega(x)}\,,$$
$$2\,\omega(x) = 1 + x^2 - 2ix^0 \qquad (\, x^2 = \boldsymbol{x}^2 - (x^0)^2\,)\,. \tag{6.18}$$

Discrete masses of atoms and elementary particles violate "the great principle of similitude"[32] (i.e. scale and, *a fortiori*, conformal invariance). The situation in QFT is still more involved – and more interesting: dimensional parameters arise in the process of renormalization even if they are absent in the classical theory. Dilation and conformal invariance can only be preserved for a renormalization group fixed point, i.e., for a *critical theory*, the QFT counterpart of a point of phase transition. One may hope that the study of an idealized critical theory with no dimensional parameters will prove to be an essential step in understanding QFT – just as Galilei's[33] law of inertia, that neglects friction, has been crucial in formulating and understanding classical mechanics. (For a more comprehensive discussion of the relevance of conformal invariance see the Introduction to [146].)

The case of 2-*dimensional conformal field theory* (2D CFT), to which are devoted the next two sections, is attractive from several points of view. It not only provides soluble QFT models satisfying the axioms, but the euclidean version of such models applies to 2D critical phenomena. String vacua are also described by a class of 2D CFT. (For a survey of QFT and strings addressed to mathematicians – see [131].)

Before going to the discussion of a class of 2D CFT models we shall make a general remark pertinent to a CFT in any even number of space-time dimensions.

It is important to distinguish in axiomatic QFT between *local observables*, such as the stress-energy tensor and conserved local currents on one hand, and gauge dependent *charged fields* which intertwine among different representations (or *superselection sectors*) of the algebra of observables, on the other. (This is stressed, in particular, in Haag's approach to local quantum physics, [87], in which a compact gauge group of the first kind is derived from intrinsic properties of the observable algebra.) In the framework of axiomatic CFT we postulate that local observables are *globally conformal invariant* (GCI) – i.e., invariant under finite conformal transformations in Minkowski spaces, [121]. This is a highly non-trivial requirement since a finite interval $(x_1 - x_2)^2$ goes under special conformal transformations (6.7) into

$$[y_1(x_1, c) - y_2(x_2, c)]^2 = \frac{(x_1 - x_2)^2}{\omega(x_1, c)\,\omega(x_2, c)}\,. \tag{6.19}$$

The product of ω-factors (unlike the square in the infinitesimal law (6.1)) may change sign. The local commutativity for space-like separations implies Huygens[34] locality: the commutator of local fields has support on light-like separations (it vanishes for both space-like and time-like

[32] See Lord Rayleigh, *The principle of similitude*, Nature **95**:2368 (March 1915) 66-68 and 644. John William Strutt – Lord Rayleigh (1842-1919) was awarded the 1904 Nobel Prize for his discovery of the inert gas argon.

[33] Galileo Galilei (1564-1642) amplified his views on mechanics in his last dialogue (1638) written when exiled to his villa at Arcetri.

[34] The Dutch physicist, mathematician and astronomer Christian Huygens (1629-1695) is the originator of the wave theory of light.

$x_1 - x_2$). Moreover, one can express the strong (Huygens) locality between two observable Bose fields by the algebraic relation

$$[(x_1 - x_2)^2]^N \, [\phi(x_1), \psi(x_2)] = 0 \quad \text{for} \quad N \gg 0 \tag{6.20}$$

($N \gg 0$ meaning "for sufficiently large N"). This allows a formulation of GCI QFT in terms of formal power series (instead of distributions), [120, 13]. Combined with the remaining Wightman axioms it implies rationality of correlation functions of observable fields [121], long believed to be a peculiarity of chiral observable fields in $1+1$ dimension (for a review – see [123]). It should be noted, however, that canonical free fields and the stress energy tensor in odd space-time dimensions violate Huygens locality (and hence, GCI).

Exercise 6.10. Use the Schwinger[35] α-representation $\left(\frac{1}{p^2} = \int_0^\infty e^{-\alpha p^2} \, d\alpha\right)$ to derive for euclidean p and x in D-dimensional space-time the relation

$$\int \frac{e^{ipx}}{p^2} \, \frac{d^D p}{(2\pi)^{D/2}} = \frac{\Gamma\left(\frac{D}{2} - 1\right)}{4\pi^{D/2}} \, (x^2)^{1-\frac{D}{2}} \, . \tag{6.21}$$

Deduce from here, using energy positivity that the Minkowski space 2-point function $w(x_{12}) = \langle 0 \mid \varphi(x_1) \, \varphi(x_2) \mid 0 \rangle$ for a free massless field in $D = 3$ space time dimensions (with euclidean propagator $\frac{1}{p^2}$) is

$$w(x) = \frac{1}{4\pi(x^2 + i0x^0)^{1/2}} \, . \tag{6.22}$$

Thus, the GCI postulate is only appropriate for even D. A survey of both standard (infinitesimal) and global conformal invariance in QFT in four dimensions is contained in [146] (see also the introduction to [122]).

[35] Julian Seymour Schwinger (1918-1994) shared the 1965 Nobel Prize in Physics with Richard Feynman (1918-1988) and Sin-Itiro Tomonaga (1906-1979) for his work in quantum electrodynamics.

Appendix D. Informal summary of Wightman axioms

Pure quantum states are described by *unit rays* in a *complex* (positive metric) *Hilbert space* $\mathcal{H}$ which carries a *unitary positive energy ray representation* of the proper orthochronous Poincaré ($\equiv$ inhomogeneous Lorentz) group $\mathcal{P}_+^\uparrow$. A *ray* (or *projective*) representation of $\mathcal{P}_+^\uparrow$ is equivalent to a *single valued representation* $U(A,a)$ of its *universal covering group* (which is, by definition, simply connected). For $D > 2$ the covering of $\mathcal{P}_+^\uparrow$ is obtained by substituting the Lorentz group $SO^\uparrow(D-1,1)$ by its double cover, the spin group Spin $(D-1,1)$. For $D = 4$ this double cover is isomorphic to the group $SL(2,\mathbb{C})$ of complex 2×2 matrices of determinant 1. We have, denoting the 2×2 unit matrix by σ_0, $A\,\sigma_\mu\, x^\mu A^* = \sigma_\mu\, \Lambda_\nu^M x^\nu$ for $A \in SL(2,\mathbb{C}) \simeq \text{Spin}\,(3,1)$

$$\Lambda = \Lambda(A) \in SO^\uparrow(3,1) \simeq SL(2,\mathbb{C})/\pm 1\,. \tag{D.1}$$

Positive energy means that the hermitean generator of translation, the energy momentum vector P_μ has joint spectrum in the forward light cone; moreover the unique translation invariant state is the *vacuum* $|0\rangle$:

$$P_0 \geq |\boldsymbol{P}|\,, \quad |\boldsymbol{P}|^2 = \sum_{i=1}^{D-1} P_i^2\,, \qquad P_\mu \mid 0\rangle = 0\,, \quad \mu = 0,1,\dots,D-1\,. \tag{D.2}$$

Quantum fields $\phi(x)$ are *operator (spin-tensor) valued tempered distributions* which transform covariantly under $U(A,a)$:

$$U(A,a)\,\phi(x)\,U(A,a)^* = V(A^{-1})\,\phi(\Lambda(A)\,x+a) \qquad (U^* = U^{-1})\,, \tag{D.3}$$

V being a finite dimensional representation of the "quantum mechanical Lorentz group" Spin $(D-1,1)$.

It is a consequence of energy positivity that the vector valued function $\phi(x) \mid 0\rangle$ admits analytic continuation to complex $z^\mu = x^\mu + iy^\mu$ in the *forward tube* (noting that in our conventions $U(a,\mathbb{1}) = e^{iP\cdot a}$):

$$\frac{\partial}{\partial \bar z}\,\phi(z) \mid 0\rangle = 0 \quad \text{for} \quad z \in \mathcal{T}_+ = \{z = x+iy \in \mathbb{C}^D\,; \quad y^0 > |\boldsymbol{y}|\}\,. \tag{D.4}$$

Exercise D.1. Prove that $\mathcal{T}_+$ is invariant under the action of the connected component C_0 of the (real) conformal group. (*Hint* : verify that $\mathcal{T}_+$ is invariant under the Weyl inversion

$$z \to wz = \frac{I_s\, z}{z^2}\,, \qquad I_s(z^0,\boldsymbol{z}) = (z^0,-\boldsymbol{z}) \tag{D.5}$$

and notice that C_0 is generated by w and by real translations. For a stronger result, known to V. Glaser (1924-1984) – see [150].)

Exercise D.2. Extend the projective quadric construction to the conformal compactification of complexified Minkowski space $M_{\mathbb{C}}$ and verify that the stabilizer of $z = (i,\mathbf{0})$ ($\in \mathcal{T}_+$) is the maximal compact subgroup of C_0.

Observable (Bose) fields commute for space-like separations:

$$[\phi(x_1),\psi(x_2)] = 0 \quad \text{for} \quad (x_1-x_2)^2 > 0 \ (\textit{local commutativity}). \tag{D.6}$$

The vacuum is assumed to be a *cyclic vector* with respect to the set of (relativistic) local fields, so that every vector in $\mathcal{H}$ can be written as a strong limit of linear combinations of vectors of the form $\phi_1(x_1)\ldots\phi_n(x_n) \mid 0\rangle$ (smeared with test functions). It follows that the full content of the theory can be expressed in terms of *(Wightman) correlation functions* – vacuum expectation values of fields products.

Exercise D.3. Prove that the Cayley map (6.14) extends to points z of the tube domain $\mathcal{T}_+$. The image T_+ of $\mathcal{T}_+$ under this map is given by

$$T_+ = \left\{ z \in \mathbb{C}^4\,;\ |z^2| < 1\,,\ |z|^2 = \sum_{\alpha=1}^{4} |z_\alpha|^2 < \frac{1}{2}\,(1 + |z^2|^2) \right\}\,. \tag{D.7}$$

Extend the map (6.14) (and T_+) to any number D of space-time dimensions.

Remark D.1. The pre-compact domain (D.7) is biholomorphically equivalent to the *classical Cartan domain of type* IV (see e.g. [93] and [138] pp. 182-192).

7 Two-dimensional conformal field theory

7.1 $U(1)$ conformal current algebra

Basic objects in QFT are the *correlation functions* – vacuum expectation values of products of local fields which satisfy certain symmetry properties and can be viewed as boundary values of analytic functions as a consequence of the spectral conditions (energy positivity). Conformal invariance plays the role of a dynamical principle: it allows to determine 2-point functions (uniquely, up to normalization) and 3-point functions (up to a few constants). (Four point functions and higher can only be determined in a GCI QFT.) The 2-point function of two currents $j_\mu(x)$ of scale dimension $D-1$ in D space-time dimensions has the form ([148]):

$$W^\mu_\nu(x_{12}) := \langle 0 \mid j^\mu(x_1)\, j_\nu(x_2) \mid 0\rangle = N_J\, r^\mu_\nu(x_{12})\, \rho_{12}^{1-D}\,,\ \ \rho_{12} = x_{12}^2 + i\,0\,x_{12}^0 \tag{7.1}$$

where the $i\,0\,x^0_{12}$ defines the right hand side of (7.1) (cf. (6.2)) as a distribution,

$$x_{12} = x_1 - x_2\,, \quad x^2 = \boldsymbol{x}^2 - (x^0)^2\,, \quad \boldsymbol{x}^2 = \sum_{i=1}^{D-1} x_i^2\,, \quad r^\mu_\nu(x) = \delta^\mu_\nu - 2\,\frac{x^\mu x_\nu}{\rho} \tag{7.2}$$

($r^2 = \mathbb{I}\,,\ r^\mu_\nu\, x^\nu = -x^\mu$). $W^\mu_\nu(x)$ satisfies the conservation law

$$\partial_\mu\, W^\mu_\nu(x) = 0 \qquad (\text{for } \partial_\mu = \frac{\partial}{\partial\, x^\mu}) \tag{7.3}$$

implying (in view of Wightman positivity and the Reeh-Schlieder theorem [144, 24]) that the current itself is conserved (as an operator valued distribution):

$$\partial_\mu\, j^\mu(x) = 0\,. \tag{7.4}$$

For $D = 2$ we see, in addition, that W^μ_ν is a gradient:

$$W^\mu_\nu(x) = \partial_\nu\, N_J\, \frac{x^\mu}{\rho} \qquad (\rho = x^2 + i\,0\,x^0)\,, \tag{7.5}$$

hence the curl of j is also zero:

$$\partial_\mu\, j_\nu(x) - \partial_\nu\, j_\mu(x) = 0\,, \quad \mu,\nu = 0,1\,. \tag{7.6}$$

Exercise 7.1. Prove that Eqs. (7.4) and (7.6) imply that the current splits into two chiral components, depending on a single light cone variable $x^0 \pm x^1$ each:

$$\frac{1}{\sqrt{2}}\,(j^0 - j^1(x)) =: j(x^0 + x^1)\,, \quad \frac{1}{\sqrt{2}}\,(j^0 + j^1) =: \bar{j}(x^0 - x^1)\,. \tag{7.7}$$

As a consequence of energy positivity both vector valued function $j(t) \mid 0\rangle$ and $\bar{j}(\bar{t}) \mid 0\rangle$ are boundary values of functions analytic in the upper half plane. It is now convenient to use the compactification map g_c (6.12) from the upper half-plane onto the unit disk D_1. It gives rise to the *z-picture fields* $\phi(z)$ that are naturally identified with their formal Laurent[36] expansions

[36] Pierre Alphonse Laurent (Paris 1813-1854) introduced in 1843 the Laurent series in a memoir submitted for the “Grand Prix de l'Académie des Sciences”, but the submission was after the due date, and the paper was not published until after his death (at the age of 41).

yielding convergent in D_1 Taylor[37] series for the vector valued function $\phi(z) \mid 0\rangle$. The compact picture current, is identified by equating the corresponding 1-forms:

$$J(z)\,\frac{dz}{2\pi i} = j(t)\,dt\,, \tag{7.8}$$

i.e.

$$J(z) = 2\pi i\,\frac{dt}{dz}\,j(t(z)) = \frac{4\pi}{(1+z)^2}\,j\left(i\,\frac{1-z}{1+z}\right) \tag{7.9}$$

where we have devided by the length of the unit circle (i.e. of the compactified light ray) with respect to the (complex) measure $\frac{dz}{z}$. $J(z)$ is more convenient to work with (than $j(t)$), since its *mode expansion* is given by the (formal) Laurent series

$$J(z) = \sum_{n\in\mathbb{Z}} J_n\,z^{-n-1} \qquad (J_n = \mathrm{Res}_z(z^n\,J(z))) \tag{7.10}$$

(that replaces the integral Fourier[38] transform of $j(t)$).

Similarly, the conserved traceless stress energy tensor Θ for $D=2$ also splits into two chiral components,

$$\begin{aligned} \Theta(x^0+x^1) &= \frac{1}{\sqrt{2}}\,(\Theta_0^0 - \Theta_0^1) \quad (= \frac{1}{2\sqrt{2}}\,(\Theta_0^0 - \Theta_0^1 + \Theta_1^0 - \Theta_1^1)) \\ \overline{\Theta}(x^0-x^1) &= \frac{1}{\sqrt{2}}\,(\Theta_0^0 + \Theta_0^1)\,. \end{aligned} \tag{7.11}$$

Exercise 7.2. Use the conservation law, $\partial_\mu\,\Theta^\mu_\nu = 0$, and the tracelessness, $\Theta^\mu_\mu = 0$ of Θ^μ_ν to prove that $\frac{\partial\Theta}{\partial\bar{t}} = 0 = \frac{\partial\bar{\Theta}}{\partial t}$ for $t = x^0 + x^1$, $\bar{t} = x^0 - x^1$ and Θ and $\bar{\Theta}$ defined by the right hand side of the first and second equation (7.11).

Exercise 7.3. Equating the *quadratic differentials*

$$\Theta(t)\,dt^2 = T(z)\,\frac{dz^2}{2\pi} \tag{7.12}$$

express $T(z)$ in terms of $\Theta(t)$ for $t = i\,\frac{1-z}{1+z}$.

Remark 7.1. The (conserved) current $j^\mu(x)$ in D dimensions should have conformal dimension $D-1$ (in mass units) in order to allow interpreting $j^0(x)$ as the charge density of a dimensionless charge. Similarly, Θ^μ_ν has dimension D in D-dimensional space-time so that one may interpret Θ^0_0 as an energy density. This accounts for the difference between (7.9) and (7.12). The factor $(2\pi)^{-1}$ in (7.12) is chosen to simplify the 2-point function of $T(z)$ in the theory of a free Weyl fermion.

[37] The English mathematician Brook Taylor (1685-1731) proved a theorem about power series expansions (following ideas of Isaac Newton, 1642-1727) in a paper of 1715 which remained unrecognized until 1772 when Joseph-Louis Lagrange (1736-1813) proclaimed it the basic principle of differential calculus.

[38] The French mathematician and physicist Jean Baptiste Joseph Fourier (1768-1830) went with Napoleon Bonaparte on his Egyptian expedition in 1798; was governor of Lower Egypt (until 1801). In his "Théorie analytique de la chaleur" (1822) he introduced the Fourier series (exhibiting discontinuous functions with convergent Fourier series). His claims were made precise and proven by Johann Peter Gustav Lejeune Dirichlet (1805-1859).

Exercise 7.4.
(a) Given (7.1) for $D = 2$, compute the 2-point function for $j(t)$. (*Answer* :

$$\langle 0 \mid j(t_1)\, j(t_2) \mid 0 \rangle = \frac{-N_J}{(t_1 - t_2 - i0)^2}\,; \tag{7.13}$$

hint : use the fact that $\rho_{12} = i0(t_{12} + \bar{t}_{12}) - t_{12}\,\bar{t}_{12}$, for $t_{12} = t_1 - t_2$, $t_i = x_i^0 + x_i^1$.)

(b) Viewing the right hand side of (7.13) as a rational function of t_{12} (i.e. neglecting the $i\,0$ prescription) and setting $N_J = (2\pi)^{-2}$ prove that the 2-point function of $J(z)$ (7.9) is

$$\langle 0 \mid J(z_1)\, J(z_2) \mid 0 \rangle = \frac{1}{z_{12}^2}\,, \quad z_{12} = z_1 - z_2\,. \tag{7.14}$$

Remark 7.2. The solution of the $2D$ massless Dirac equation

$$(\gamma^0\, \partial_0 \,+\, \gamma^1\, \partial_1)\, \Psi = 0$$

for

$$\gamma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \left(\gamma_0\, \gamma_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\gamma^0\, \gamma^1 \right)$$

assumes the form $\Psi = \begin{pmatrix} \psi(t) \\ \bar{\psi}(\bar{t}) \end{pmatrix}$. If we define $j(t)$ in the theory of a free Weyl field $\psi(t)$ from the *operator product expansion*

$$\frac{1}{2}\, (\psi^*(t_1)\, \psi(t_2) - \psi(t_1)\, \psi^*(t_2)) = j\left(\frac{t_1 + t_2}{2} \right) + O(t_{12}^2) \tag{7.15}$$

then the 2-point function (7.13) of j will indeed involve the normalization constant $N_J = (2\pi)^{-2}$ (see [75] Appendix C).

If we write the mode expansion of the z-picture stress-energy tensor as

$$T(z) = \sum_{n \in \mathbb{Z}} L_n\, z^{-n-2} \qquad \left(\bar{T}(\bar{z}) = \sum_n \bar{L}_n\, \bar{z}^{-n-2} \right) \tag{7.16}$$

then the conformal energy H (6.9) is the sum of the left and right mover's zero modes:

$$H = L_0 + \bar{L}_0\,. \tag{7.17}$$

We shall identify accordingly the chiral energy operator with L_0. We have

$$[L_0, J(z)] = \left(z\, \frac{d}{dz} + 1 \right) J(z) = \frac{d}{dz}\, (z\, J(z)) \Rightarrow [L_0, J_n] = -n\, J_n\,; \tag{7.18}$$

$$[L_0, T(z)] = \left(z\, \frac{d}{dz} + 2 \right) T(z) \Rightarrow [L_0, L_n] = -n\, L_n\,. \tag{7.19}$$

More generally, if $W(z)$ is a chiral Bose field of integer dimension d,

$$W(z) = \sum_n W_n\, z^{-n-d}\,, \tag{7.20}$$

then

$$[L_0, W(z)] = \left(z\, \frac{d}{dz} + d \right) W(z) \Rightarrow [L_0, W_n] = -n\, W_n\,. \tag{7.21}$$

It follows from the analysis of Section 6 (see, in particular, Exercises 6.2 and 6.4) that there is an infinite parameter set of invertible local conformal transformations $z \to f(z)$ of a neighbourhood of the origin (in which $f'(z) \neq 0$). If the theory is assumed to be invariant under such an "infinite conformal group" then the correlation functions would have been independent of z which would mean that all chiral fields (including the stress-energy tensor) would vanish. What actually happens is that the vacuum state is not invariant under the infinite dimensional conformal group. Correlation functions of chiral fields, like (7.14) or

$$\langle 0 \mid T(z_1)\, T(z_2) \mid 0 \rangle = \frac{c}{2\, z_{12}^4} \qquad (c > 0) \tag{7.22}$$

are only invariant under the Möbius group of fractional linear transformations (see Exercise 6.6). Noting the Lie algebras $s\ell(2,\mathbb{R})$, $su(1,1)$ and $so(2,1)$ are isomorphic we can say that the correlation functions of a D-dimensional CFT are $so(D,2)$ invariant for all $D \geq 1$. As we shall see shortly the chiral Möbius Lie algebra is spanned by L_0, $L_{\pm 1}$; the CR (7.19) should be completed by

$$[L_1, L_{-1}] = 2\, L_0 \quad \text{so that} \quad [L_m, L_n] = (m-n)\, L_{m+n}\,, \quad m,n = 0, \pm 1\,. \tag{7.23}$$

The z-picture correlation functions (like (7.14) (7.22)) having the same form as the x-space ones are, in particular, translation invariant, the (non hermitean) generator of translations of the complex variable z being L_{-1} which should also annihilate the vacuum:

$$L_{-1} \mid 0 \rangle = 0\,, \quad [L_{-1}, W(z)] = \frac{dW(z)}{dz} \quad \Rightarrow \quad [W_n, L_{-1}] = (n+d-1)\, W_{n-1}\,. \tag{7.24}$$

The upper half plane, the analyticity domain of $\phi(\tau) \mid 0\rangle$ for any chiral field ϕ, is mapped by the complex Möbius transformation (7.8) onto the unit disk. Thus we expect that z-picture fields applied to the vacuum give rise to Taylor expansions convergent for $|z| < 1$. To formulate the precise statement we need the notion of z-picture conjugate of a hermitean chiral field $W(z)$ of dimension d and expansion (7.20):

$$(W(z))^* = \frac{1}{\bar{z}^{2d}}\, W\left(\frac{1}{\bar{z}}\right)\,. \tag{7.25}$$

Proposition 7.1.
(a) *The vector valued function $W(z) \mid 0\rangle$ for a hermitean scalar field W (7.20) of a positive integral dimension d has the form*

$$W(z) \mid 0 \rangle = \sum_{n=0}^{\infty} W_{-n-d}\, z^n \mid 0 \rangle\,, \quad \textit{i.e.} \quad W_n \mid 0 \rangle = 0 \quad \textit{for} \quad n + d > 0\,. \tag{7.26}$$

(b) *The norm square of this vector is given by a power series convergent for $\bar{z}\, z < 1$.*

Proof. (a) $W_n \mid 0\rangle = 0$ for $n > 0$ because $(L_0 + n)\, W_n \mid 0 \rangle = 0$ and we have assumed energy positivity. Hence $W(z) \mid 0\rangle$ may at most have a finite number (no more than d) negative powers of z in its Laurent expansion. Hence the formal power series

$$F(z,w) := e^{wL_{-1}}\, W(z) \mid 0 \rangle$$

can be written in the form $F(z,w) = \frac{v_0(w)}{z^N} + \frac{v_1(z,w)}{z^{N-1}}$ where v_0 and v_1 only involve non-negative powers of z and w in their (formal) Laurent expansions. On the other hand, Eq. (7.24) implies that $\frac{\partial F}{\partial z} = \frac{\partial F}{\partial w}$. This is only possible if $N = 0$, implying (7.26). Thus the lowest energy state generated by the W modes is $W_{-d} \mid 0\rangle$ of energy d.

Remark 7.3. We have thus proved that, under the assumption of energy positivity, any translation covariant formal power series $W(z) \mid 0\rangle$ involves no negative powers of z. Thus the vector $W(0) \mid 0\rangle$ is well defined (and determines $W(z)$ – see Appendix D). A more general result of this type, applicable to higher dimensional GCI theories, is contained in Proposition 3.2 (a) of [122].

(b) the 2-point function of W is determined from translation and dilation invariance to have the form

$$\langle 0 \mid W(z_1)\, W(z_2) \mid 0\rangle = \frac{N_W}{z_{12}^{2d}}\,. \tag{7.27}$$

(Hilbert space positivity demands $N_W > 0$.) It follows from here and from the conjugation rule (7.25) that the norm square of the vector (7.26),

$$\begin{aligned} \|W(z) \mid 0\rangle\|^2 &= \frac{1}{\bar z^{2d}}\, \langle 0 \mid W\left(\frac{1}{\bar z}\right) W(z) \mid 0\rangle = \frac{N_W}{(1 - z\,\bar z)^{2d}} \\ &= N_W \sum_{n=0}^{\infty} \binom{2d+n-1}{n} (z\,\bar z)^n\,, \end{aligned} \tag{7.28}$$

indeed converges for $|z|^2 < 1$. □

Proposition 7.1 implies that the 2-point correlator (7.27) should be viewed as a boundary value of a function analytic in the domain $|z_2| < |z_1|$ where it is defined as a (convergent) power serie in $\frac{z_2}{z_1}$. The same rational function in the domain $|z_1| < |z_2|$ will be written as $(z_2 - z_1)^{-2d}$.

Proposition 7.2. *In a chiral theory satisfying both Hilbert space and energy positivity the modes J_n of a local current $J(z)$ with 2-point function* (7.14) *satisfy the Heisenberg* CR

$$[J_n, J_m] = n\, \delta_{n,-m}\,. \tag{7.29}$$

Proof. Local commutativity implies

$$[J(z_1), J(z_2)] = \sum_{n=0}^{n_J} A_n(z_2)\, \partial_2^n\, \delta(z_1 - z_2)\,. \tag{7.30}$$

Here the z picture δ-function is given by a formal Laurent series and obeys the defining property of a δ-function when applied to an analytic function f of z:

$$\delta(z_1 - z_2) = \sum_{n\in\mathbb{Z}} \frac{z_2^n}{z_1^{n+1}} \left(= \frac{1}{z_{12}} + \frac{1}{z_{21}}\right) \qquad \mathrm{Res}_{z_2}\, \delta(z_1 - z_2)\, f(z_2) = f(z_1)\,. \tag{7.31}$$

(Here $\frac{1}{z_{12}} = \frac{1}{z_1} \sum\limits_{n=0}^{\infty} \left(\frac{z_2}{z_1}\right)^n$, $\frac{1}{z_{21}} = \frac{1}{z_2} \sum\limits_{n=0}^{\infty} \left(\frac{z_1}{z_2}\right)^n$ have disjoint convergence domains. For a distribution F given by a formal Laurent series $F(z) = \sum\limits_n F_n\, z^n$, we set $\mathrm{Res}_z\, F = F_{-1}$.) Using conservation of scale dimension and the fact that $\partial_2^n\, \delta(z_1 - z_2)$ has dimension $n+1$ we conclude that the field A_n in (7.30) has dimension $1 - n$.

Lemma 7.1. *If the dimension d of the chiral field W with 2-point function (7.27) is a negative integer, $d = -N$, then W violates both energy and Hilbert space positivity.*

Proof (of Lemma). The 2-point function $(z_{12})^{2N}$ corresponds to a minimal energy state $W_N \mid 0\rangle \neq 0$ of energy $-N$. The norm square (7.28) then goes into

$$(1 - z\,\bar{z})^{2N} = \sum_{n=0}^{2N} (-1)^n \binom{2N}{n} (z\,\bar{z})^n = \sum_{n=0}^{2N} \| W_{N-n} \mid 0\rangle \|^2 \,(z\,\bar{z})^n$$

giving, in particular, $\|W_{N-1} \mid 0\rangle\|^2 = -2N \|W_N \mid 0\rangle\|^2$. □

Thus, our assumptions imply that $n_J = 1$: only two terms – with $n = 0$ and $n = 1$ – contribute to the sum (7.30). The uniqueness of the vacuum implies that $A_1(z)$ is a constant multiple of the identity. Comparison with (7.14) tells us that this constant is 1. The antisymmetry of the commutator under the exchange $z_1 \rightleftarrows z_2$ implies, on the other hand, $A_0 = 0$. Thus,

$$[J(z_1), J(z_2)] = \partial_2\, \delta(z_1 - z_2) = \sum_n n\, \frac{z_2^{n-1}}{z_1^{n+1}} = \sum_{m,n} [J_m, J_{-n}]\, \frac{z_2^{n-1}}{z_1^{m+1}} \tag{7.32}$$

which yields (7.29).

Exercise 7.5. Let $T(z)$ (7.16) with 2-point function (7.22) satisfy locality, energy and Hilbert space positivity. Derive the Virasoro CR:

$$[L_n, L_m] = (n - m)\, L_{n+m} + \frac{c}{12}\,(n^3 - n)\, \delta_{n,-m}\,. \tag{7.33}$$

(This is the *Lüscher-Mack theorem* – see [113]; for a complete proof – see [75]; for related work, done in Brazil, see [132, 133].)

It is the *central charge* c in (7.33), the *conformal anomaly*, which expresses the violation of the infinite dimensional conformal symmetry by the vacuum state. Note that its coefficient vanishes for $n = 0, \pm 1$, so that (7.33) reproduces the Möbius CR (7.23) as a special case.

7.2 Primary fields. The critical Ising model

A field ϕ is said to be *primary* if it transforms homogeneously (without anomaly) with respect to commutations with the *chiral algebra* $\mathcal{A}$. For instance *the current $J(z)$ is a Virasoro primary field.* It is covariant under infinitesimal reparametrizations:

$$[L_n, J(z)] = \frac{d}{dz}\,(z^{n+1}\, J(z)) \tag{7.34}$$

(J is however not primary with respect to the current algebra since (7.29) is inhomogeneous). More generally, a $2D$ field $\phi(z, \bar{z})$ is said to be primary of weight $(\Delta, \bar{\Delta})$ with respect to the direct sum of (left and right) *Virasoro algebras* if

$$[L_n, \phi] = z^n \left(z\, \frac{\partial}{\partial z} + (n+1)\, \Delta \right) \phi\,, \quad [\bar{L}_n, \phi] = \bar{z}^n \left(\bar{z}\, \frac{\partial}{\partial \bar{z}} + (n+1)\, \bar{\Delta} \right) \phi\,. \tag{7.35}$$

The difference $s = \Delta - \bar{\Delta}$ is called the *spin* (or the *helicity*) of ϕ. Usually only fields with $2s \in \mathbb{Z}$ are encountered. Such fields live on a cylinder – i.e. their x-space counterparts satisfy

$$\phi(x^0, x^1 + 2\pi) = (-1)^{2s}\, \phi(x^0, x^1)\,. \tag{7.36}$$

Primary fields are relatively local to the observables. To check the locality of ϕ with respect to T we note that (7.35) is essentially equivalent to the *operator product expansion* (OPE)

$$T(z_1)\,\phi(z_2) = \Delta\, \frac{\phi(z_2)}{z_{12}^2} + \frac{1}{z_{12}}\,\phi'(z_2) + O(1) \tag{7.37}$$

($O(1)$ standing for non-singular terms in z_{12}). This indeed amounts to local CR since

$$\frac{1}{z_{12}} + \frac{1}{z_{21}} = \delta(z_{12})\,, \quad \frac{1}{z_{12}^2} - \frac{1}{z_{21}^2} = \frac{\partial}{\partial z_2}\,\delta(z_{12})\,. \tag{7.38}$$

A $2D$ CFT is called *rational* if the chiral algebra $\mathcal{A}$ has a finite number of *unitary positive energy irreducible representations* (UPEIR) related to the (defining) *vacuum representation* of $\mathcal{A}$ by primary fields, relatively local to the observables. An example of a *rational conformal field theory* (RCFT) is provided by the Virasoro *minimal models* [17] corresponding to central charge $c = c(m) = 1 - \frac{6}{(m+2)(m+3)}$, $m = 1, 2, \ldots$. The first chiral theory of this series, $c(1) = \frac{1}{2}$, can be viewed as generated by a free real fermion field, the Majorana[39]-Weyl field

$$\psi(z) = \sum_n \psi_{n-\frac{1}{2}}\, z^{-n}\ ,\ \ [\psi_\rho, \psi_\sigma]_+ = \delta_{\rho,-\sigma}\ ,\ \ \psi_\rho^* = \psi_{-\rho}\,, \quad \rho, \sigma \in \mathbb{Z} + \frac{1}{2}\,. \tag{7.39}$$

Exercise 7.6. Prove that

$$T(z) = -\frac{1}{4} \lim_{z_{1,2}\to z} \frac{\partial^2}{\partial z_1\, \partial z_2}\,\{z_{12}\, \psi(z_1)\, \psi(z_2)\} = \frac{1}{2} : \psi'(z)\, \psi(z) : \tag{7.40}$$

has all properties of the stress energy tensor with central charge $c = \frac{1}{2}$; in particular,

$$T(z_1)\,\psi(z_2) = \frac{\psi'(z_1)}{2\,z_{12}} + \frac{\psi(z_1)}{2\,z_{12}^2} + O(1) = \frac{\psi'(z_2)}{z_{12}} + \frac{1}{2}\,\frac{\psi(z_2)}{z_{12}^2} + O(1)$$

$$\langle 0 \mid T(z_1)\, T(z_2) \mid 0 \rangle = \frac{1}{4\, z_{12}^4}\,. \tag{7.41}$$

Define the *Ramond sector* of the (chiral) Ising model by its lowest weight vector $\mid \Delta \rangle$ such that $\langle \Delta \mid \Delta \rangle = 1$ and

$$\psi(z) \mid \Delta \rangle = \sum_{n=0}^{\infty} \psi_{-n} z^{n-\frac{1}{2}} \mid \Delta \rangle\ ,$$

$$L_n \mid \Delta \rangle = \Delta\, \delta_{n0} \mid \Delta \rangle\ , \quad \langle \Delta \mid L_{-n} = \Delta\, \delta_{n0} \langle \Delta \mid \quad \text{for} \quad n \geq 0\ . \tag{7.42}$$

[39] Ettore Majorana (1906-1938?), one of the "ragazzi di via Panisperna"; their leader, Enrico Fermi, compares his genius with that of Galileo and Newton, adding: "Ettore had what nobody else in the world has. Unfortunately, he lacked what is instead easy to find in other men: simple common sense." Majorana disappeared on March 25, 1938 (listed among the passengers in a boat trip from Palermo to Napoli). The (real) $4D$ Majorana spinors, introduced in his last paper (of 1937), are now used to describe a massive neutrino.

More precisely, the Ramond sector is the irreducible unitary module $\mathcal{H}_\Delta$ of lowest weight Δ defined as follows (for a more detailed and precise treatment – see, e.g., [101]). Let the *Verma module* $\mathcal{V}_\Delta$ be the set of all vectors obtained from $|\Delta\rangle$ by applying a polynomial in the Virasoro generators L_n (due to (7.42) this is equivalent to applying only polynomials of L_{-n}, $n > 0$). $\mathcal{H}_\Delta$ is then defined as the quotient space of $\mathcal{V}_\Delta$, factored by the invariant subspaces generated by all null vectors v satisfying, by definition, $L_n\, v = 0$ for all $n > 0$ (and having as a consequence zero norm square). The study of *Kac's determinant* (for a review – see [101]) shows that there are finitely many null vectors in each minimal model. (We shall exhibit such a null vector in the present case in Exercise 7.8 below.)

Exercise 7.7. Prove that

$$\langle \Delta \mid \psi(z_1)\psi(z_2) \mid \Delta \rangle = \frac{z_1 + z_2}{2 z_{12} \sqrt{z_1 z_2}} \ . \tag{7.43}$$

(*Hint*: Use (7.42) to prove that $\sqrt{z_1 z_2}\, \langle \Delta \mid \psi(z_1)\psi(z_2) \mid \Delta \rangle$ is a rational function of z_1 and z_2; also use $\lim_{z_1 \to z_2} (z_{12}\, \psi(z_1)\psi(z_2)) = 1$.) Deduce that

$$\psi_0^2 \mid \Delta \rangle = \frac{1}{2} \mid \Delta \rangle \ . \tag{7.44}$$

Exercise 7.8. Use (7.40) and the relation $\langle \Delta \mid T(z) \mid \Delta \rangle = \Delta\, z^{-2}$ to prove that

$$\Delta = \frac{1}{16} \ . \tag{7.45}$$

Prove that $(3L_{-2} - 4L_{-1}^2) \mid \frac{1}{16} \rangle$ is a *null vector*, i.e. that

$$L_n (3L_{-2} - 4L_{-1}^2) \mid \frac{1}{16} \rangle = 0 \qquad \text{for} \quad n = 1, 2, \dots \ , \tag{7.46}$$

and a similar relation for the bra null vector $\langle \frac{1}{16} \mid (3L_2 - 4L_1^2)$ (cf. (7.42)).

Exercise 7.9. (a) Let $\phi(z)$ be a conformal field of dimension d. Prove that in general

$$\langle \Delta \mid \phi(z) \mid \Delta \rangle = C z^{-d} \tag{7.47}$$

where C is a constant.

(b) Postulating that $\langle \frac{1}{16} \mid (3L_2 - 4L_1^2) = 0$ (i.e. factoring the null vector), derive from

$$\langle \frac{1}{16} \mid (3L_2 - 4L_1^2)\, \phi(z) \mid \frac{1}{16} \rangle = 0 \tag{7.48}$$

(using (7.35)) a second order differential equation for the right hand side of (7.47). Deduce that $C = 0$ unless $d = 0$ or $d = \frac{1}{2}$.

Let $\sigma(z, \bar z)$ be the *spin* (also called *magnetization*) *field* of weight $(\frac{1}{16}, \frac{1}{16})$ which interpolates between the vacuum and the Ramond sector of the $2D$ Ising model, while $\epsilon(z, \bar z) = \psi(z)\, \bar\psi(\bar z)$ be the corresponding *energy field* of weight $(\frac{1}{2}, \frac{1}{2})$. Evaluating the 3-point function

$$\langle 0 \mid \sigma(z_1, \bar z_1)\, \sigma(z_2, \bar z_2)\, \epsilon(z_3, \bar z_3) \mid 0 \rangle \ ,$$

using (7.44) and noting the vanishing of the correlation function of three σ's that follows from Exercise 7.9, one can deduce the OPE

$$\sigma(z_1, \bar{z}_1)\, \sigma(z_2, \bar{z}_2) = (z_{12}\bar{z}_{12})^{-\frac{1}{8}}\, [1 + \frac{1}{2}\, z_{12}\bar{z}_{12}\, \epsilon(z_2, \bar{z}_2) + \ldots] \ . \qquad (7.49)$$

Thus, the product of two σ involves two primary fields (or two *conformal families* in the terminology of [17]), $\mathbb{I}$ and ϵ. Combined with a similar derivation of the OPE $\epsilon(z_1, \bar{z}_1)\, \sigma(z_2, \bar{z}_2)$, this yields the *fusion rules* [151]:

$$[\sigma][\sigma] = [\mathbb{I}] + [\epsilon] \ , \quad [\epsilon][\sigma] = [\sigma] \ , \quad [\epsilon][\epsilon] = [\mathbb{I}] \ . \qquad (7.50)$$

These rules resemble the tensor product expansions of (finite dimensional) IRs of compact groups. In the latter case the product of the corresponding dimensions satisfies the same relation.

Exercise 7.10. Verify the validity of the tensor product expansion formula of two IRs of $SU(2)$ for their dimensions:

$$[I_1] \otimes [I_2] = \bigoplus_{I=|I_1-I_2|}^{I_1+I_2} [I] \quad \Rightarrow \quad (2I_1+1)(2I_2+1) = \sum_{I=|I_1-I_2|}^{I_1+I_2} (2I+1) \ . \qquad (7.51)$$

If we *define*, in accord with [151], the *quantum dimensions* $[\phi]$ as solutions of the corresponding fusion rule equations, we will find, in the case of (7.50), a non-integer dimension $[\sigma]$:

$$[\epsilon] = 1 = [\mathbb{I}] \ , \quad [\sigma]^2 = 2 \ . \qquad (7.52)$$

The appearance of non-integer quantum dimensions is a clear sign that the symmetry of the corresponding primary fields is not implemented by a group.

Exercise 7.11. Verify that for $q^4 = -1$, the q-numbers [2] and [3] satisfy the equations for $[\sigma]$ and $[\epsilon]$:

$$[2]^2 = [3] + 1 \quad (\text{for any } q) \ , \quad [3] = 1 \quad (\text{for } q^4 = -1) \ . \qquad (7.53)$$

The quantum dimension is a low-dimensional counterpart (applicable to configuration spaces with a non-trivial homotopy group) of the notion of *statistical dimension* of Doplicher, Haag and Roberts ([87, 48]; for a brief elementary overview and more references – see [147]). They appear as characteristic invariants of representations of not just the (quasitriangular) Hopf algebras (or quantum groups considered in these lectures) but also for their generalizations like the *weak Hopf algebras* [23], inspired by the "quantum groupoids" of the (mostly unpublished) work of A. Ocneanu (see [127] where Ocneanu's ideas are demonstrated to imply a far reaching generalization of the Verlinde algebra).

For a general study of RCFT – see [119] (as well as the text [42]).

Appendix E. Axioms for a chiral vertex algebra

Chiral CFT has become, starting with the work of Borcherds [25], a domain in pure mathematics under the name of *vertex algebras*, that is already a subject of several books – see, e.g. [60, 100, 59, 77]. Our brief survey, following [82], [47] and [100], should be viewed as a formalization and extension of the discussion of Section 7. Accordingly, we shall formulate the axioms for bosonic graded vertex algebras only, mentioning the fermionic (and superalgebra) case in a subsequent remark.

A *graded vertex algebra* consists of a $\mathbb{Z}_+$ graded pre-Hilbert vector space,

$$\mathcal{V} = \bigoplus_{n=0}^{\infty} \mathcal{V}_n \,, \quad \dim \mathcal{V}_0 = 1 \,, \quad \dim \mathcal{V}_n < \infty \tag{E.1}$$

equipped with a *translation operator* T $(= L_{-1})$ and a *state field correspondence* $Y : \mathcal{V} \to (\mathrm{End}\, V)[[z, z^{-1}]]$ (read: Y is a map from $\mathcal{V}$ to the space of formal Laurent series $Y(v, z)$, $v \in \mathcal{V}$ whose coefficients are endomorphisms – i.e. linear operators from $\mathcal{V}$ to $\mathcal{V}$) satisfying the following axioms.

(i) *Vacuum* : the 1-dimensional space $\mathcal{V}_0$ is spanned by the *vacuum vector* $\mid 0\rangle$ such that

$$T \mid 0\rangle = 0 \,, \quad \langle 0 \mid 0\rangle = 1 \,. \tag{E.2}$$

(ii) *Translation covariant fields* : to each vector $v \in \mathcal{V}$ there corresponds a formal Laurent series $Y(v, z)$ with operator valued coefficients such that
(a) *the vector valued function*

$$Y(v, z) \mid 0\rangle = e^{zT}\, v \,, \tag{E.3}$$

is analytic (in the norm topology) for $|z| < 1$; furthermore

$$[T, Y(v, z)] = \frac{d}{dz}\, Y(v, z) \,. \tag{E.4}$$

(b) Assuming *linearity* in the vector argument v, i.e. requiring

$$Y(c_1\, v_1 + c_2\, v_2, z) = c_1\, Y(v_1, z) + c_2\, Y(v_2, z) \quad \text{for} \quad v_1, v_2 \in \mathcal{V} \,,\ c_1, c_2 \in \mathbb{C} \,, \tag{E.5}$$

we can define Y by first displaying its properties for *homogeneous elements*, $v_\ell \in \mathcal{V}_\ell$; then

$$Y(v_\ell, z) = \sum_n Y_n(v_\ell)\, z^{-n-\ell} \tag{E.6}$$

where
(c) $Y_n(v_\ell)$ *changes the grading* by $-n$:

$$Y_n(v_\ell) : \mathcal{V}_k \to \mathcal{V}_{k-n} \qquad (\mathcal{V}_{k-n} = \{0\} \ \text{for} \ \ k < n) \,. \tag{E.7}$$

Eq. (E.7), together with (E.1), is our *energy positivity* requirement. We identify the chiral vertex algebra Hamiltonian with the *Virasoro energy* L_0 satisfying

$$(L_0 - n)\, \mathcal{V}_n = 0 \,, \quad [L_0, T] = T \,. \tag{E.8}$$

Formal Laurent series of different arguments, $Y(v_1, z_1)$, $Y(v_2, z_2)$, can be multiplied giving a formal Laurent series $Y(v_1, z_1) \cdot Y(v_2, z_2)$ of two variables.

(iii) *Local commutativity* :

$$(z_{12})^N [Y(v_1, z_1), Y(v_2, z_2)] = 0 \quad \text{for} \quad N \gg 0 \,. \tag{E.9}$$

We denote by $\mathcal{A}(\mathcal{V})$ the set of formal power series Y satisfying the axioms (i–iii). The following Proposition, singled out by Goddard (see [47]), justifies the notation $Y(v, z)$.

Proposition E.1. *If two formal Laurent series $Y_1(v, z)$ and $Y_2(v, z)$ belong to $\mathcal{A}(\mathcal{V})$ (and hence satisfy* (E.3) *with the same v) then they coincide.*

Sketch of proof. Using locality one finds that the difference $Y_1(v, z) - Y_2(v, z)$ vanishes not just on the vacuum but on any other vector $v_1 \in \mathcal{V}$. □

This uniqueness result has a number of applications. We single out the following

Corollary E.1. *It follows from Proposition* E.1 *that*

(a) $Y(\mid 0\rangle, z) = \mathbb{1}$;

(b) $Y(Tv, z) = \frac{d}{dz} Y(v, z)$.

Exercise E.1. Prove, using energy positivity, that the Laurent series $Y(v_1, z)\, v_2$ has a finite number of negative powers of z. Demonstrate that for energy eigenstates, $(L_0 - d_i)\, v_i = 0$ for $i = 1, 2$, the leading negative power does not exceed $d_1 + d_2$.

Studying OPE of products of elements of $\mathcal{A}(\mathcal{V})$, it is useful to extend the definition of $Y(v, z)$ to v of the form $Y(v_1, w)\, v_2$ (which is not a finite energy state).

Exercise E.2. Demonstrate that both sides of the equality

$$Y(v_1, z_1)\, Y(v_2, z_2) \mid 0\rangle = Y(Y(v_1, z_{12})\, v_2, z_2) \mid 0\rangle \tag{E.10}$$

define analytic (in the Hilbert norm topology) vector valued functions for $|z_2| < |z_1| < 1$ and sufficiently small $|z_{12}|$ and that the equality (E.10) holds.

The stress-energy tensor $T(z)$ can be identified with $Y(L_{-2} \mid 0\rangle, z)$.

A field $Y(v, z)$ is (Virasoro) primary (i.e., satisfies (7.35)) if v is a *ground state*:

$$L_n\, v = 0 \,, \quad \text{for} \quad n > 0 \,, \quad (L_0 - d)\, v = 0 \,. \tag{E.11}$$

8 Extensions of the $u(1)$ current algebra and their representations

Given a finite dimensional Lie algebra $\mathcal{G}$ with (real) *structure constants* $f_c^{ab} = -f_c^{ba}$ (satisfying the *Jacobi*[40] *identity* $f_s^{ab} f_d^{sc} + f_s^{ca} f_d^{sb} + f_s^{bc} f_d^{sa} = 0$), we can define a *Kac-Moody algebra* generated by (hermitean) currents $J^a(z)$ given by formal power series

$$J^a(z) = \sum_{n\in\mathbb{Z}} J_n^a \, z^{-n-1} \; (J_n^{a*} = J_{-n}^a) \, , \quad a = 1, \dots , d_{\mathcal{G}} := \dim \mathcal{G} \, , \tag{8.1}$$

where the modes J_n^a satisfy the CR

$$[J_m^a, J_n^b] = i \, f^{ab}{}_c \, J_{m+n}^c + k \, m \, \delta_{m,-n} \, g^{ab} \tag{8.2}$$

(g^{ab} standing for a $\mathcal{G}$ invariant positive metric).

The representation theory of the $u(1)$ current algebra (7.29) (which appears as a special case of (8.2) for $d_{\mathcal{G}} = 1$ and $f_c^{ab} = 0$) is relatively simple – this is, in fact, an infinite Heisenberg[41] algebra whose positive energy representations are labeled by the eigenvalues of the charge operator J_0.

We define the normal product $: J(z_1) \, J(z_2) :$ of two $u(1)$ currents through their OPE

$$J(z_1) \, J(z_2) = \frac{1}{z_{12}^2} + : J(z_1) \, J(z_2) : \, . \tag{8.3}$$

Normal products $: J^n(z) :$ belong to the chiral algebra $\mathcal{A}(\mathcal{V}_J)$ where $\mathcal{V}_J$ is the space generated by polynomial of the current's negative modes J_{-n} acting on the vacuum.

Exercise 8.1. The *Sugawara stress tensor* of $\mathcal{A}(\mathcal{V}_J)$,

$$T(z) = \frac{1}{2} : J^2(z) : , \tag{8.4}$$

satisfies the defining OPEs for a J, T system (*cf.* (7.37)):

$$T(z_1) \, J(z_2) = \frac{1}{z_{12}^2} \, J(z_1) + O(1) = \frac{1}{z_{12}^2} \, J(z_2) + \frac{1}{z_{12}} \, J'(z_2) + O(1) \, , \tag{8.5}$$

$$\begin{aligned} T(z_1) \, T(z_2) &= \frac{1}{2} \frac{1}{z_{12}^4} + \frac{: J(z_1) \, J(z_2) :}{z_{12}^2} + O(1) \\ &= \frac{1}{2} \frac{1}{z_{12}^4} + 2 \, \frac{T(z_2)}{z_{12}^2} + \frac{T'(z_2)}{z_{12}} + O(1) \, . \end{aligned} \tag{8.6}$$

Deduce that for a chiral CFT generated by $J(z)$ the Virasoro central charge is $c = 1$.

[40] The German mathematician Carl Gustav Jacob Jacobi (1804-1851) was considered to be the most inspiring teacher of his time. Bourbaki, in particular, Jean Dieudonné (1906-1992), have taken as a motto his words (from a letter to Legendre of 1830, deploring the fact that Fourier introduces his series just as an application to the heat equation): "le but unique de la science c'est l'honneur de l'esprit humain". Most of his papers were published post humously.

[41] Werner Heisenberg (1901-1976) has been awarded in 1932 the Nobel Prize in Physics for the creation of quantum mechanics (1925). The CR $[q, p] = i\hbar$ first appeared in work of Born-Jordan and of Dirac.

Exercise 8.2. Prove that Eq. (8.4) allows to write the Virasoro modes in terms of J_ℓ:

$$L_0 = \frac{1}{2}\, J_0^2 + \sum_{\ell=1}^{\infty} J_{-\ell}\, J_\ell\,, \quad L_n = \frac{1}{2} \sum_{\ell \in \mathbb{Z}} J_{n-\ell}\, J_\ell \quad \text{for} \quad n \neq 0\,. \tag{8.7}$$

Verify the CR (7.33) (with $c = 1$) and (7.34) for these expressions.

Proposition 8.1. *The unitary irreducible positive energy representations (UIPERs) of $\mathcal{A}(\mathcal{V}_J)$ correspond to ground states $\mid g\rangle$ labeled by a real number g such that*

$$(J_0 - g) \mid g\rangle = 0 = J_n \mid g\rangle \quad \text{for} \quad n > 0 \tag{8.8}$$

($g = 0$ corresponding to the defining vacuum UIPER of $\mathcal{A}(\mathcal{V}_J)$). To each $g \neq 0$ corresponds a pair $\psi(z, \pm g)$ of hermitean conjugate primary fields of dimension g^2 such that for each of them

$$\psi(z,g) \mid 0\rangle = e^{zL_{-1}} \mid g\rangle\,, \quad [J(z_1), \psi(z_2, g)] = g\, \delta(z_{12})\, \psi(z_2, g)\,. \tag{8.9}$$

(Thus all $\psi(z,g)$ are relatively local to the current.) *$\psi(z, \pm g)$ locally commute among themselves iff g^2 is an even integer; then*

$$(z_{12})^{g^2} [\psi(z_1, g), \psi(z_2, -g)] = 0 (= [\psi(z_1, \pm g), \psi(z_2, \pm g)])\,. \tag{8.10}$$

Sketch of proof (for a comprehensive discussion – see [29]). Introduce the *abelian*[42] (i.e. commutative) group $\{E^{ng}, n \in \mathbb{Z}\}$ of unitary operators such that

$$E^g \mid 0\rangle = \mid g\rangle \quad (E^g)^* = E^{-g} = (E^g)^{-1}\,, \quad [J(z), E^g] = \frac{g}{z}\,. \tag{8.11}$$

Introduce further the integrals of the frequency parts of the current:

$$\varphi_+(z) = \sum_{n=1}^{\infty} \frac{1}{n}\, J_{-n}\, z^n\,, \quad \varphi_-(z) = -\sum_{n=1}^{\infty} \frac{1}{n}\, J_n\, z^{-n}\,; \tag{8.12}$$

then the *chiral vertex operator* (CVO) $\psi(z,g)$ can be written in the form

$$\psi(z,g) = E^g\, e^{g\varphi_+(z)}\, z^{gJ_0}\, e^{g\varphi_-(z)}\,. \tag{8.13}$$

To verify the current-field CR (8.9) we use (8.11) and

$$[J(z_1), \varphi_+(z_2)] = \frac{1}{z_{12}} - \frac{1}{z_1}\,, \quad [J(z_1), \varphi_-(z_2)] = \frac{1}{z_{21}}\,. \tag{8.14}$$

We have $\left(L_0 - \frac{g^2}{2}\right) \mid g\rangle = 0$ as a consequence of (8.7) and (8.8); hence, $\psi(z, \pm g)$ have scale dimension $\frac{g^2}{2}$, so that the (non-vanishing) 2-point function is

$$\langle 0 \mid \psi(z_1, g)\, \psi(z_2, -g) \mid 0\rangle = (z_{12})^{-g^2}\,. \tag{8.15}$$

As the sign of g is not fixed and g^2 is even the 2-point function is symmetric (viewed as a rational function) with respect to the exchange of factors.

[42] Named after the Norwegian mathematician Niels Henrik Abel (1802-1829) who proved in 1824 the impossibility to solve the general fifth degree equation in radicals and created (in 1825, in Freiburg) the theory of elliptic, hyperelliptic (and, more generally, *abelian*) functions.

Exercise 8.3. Verify (8.15) using (8.13).

Eq. (8.15) and the remark that the singularity of the 2-point function dominates those of higher point correlation functions as a consequence of Wightman positivity imply the strong locality condition (8.10). □

Exercise 8.4. Use the CR $[J_0, E^g] = g\, E^g$ to prove

$$\langle 0 \mid E^g \mid 0 \rangle = \delta_{g0} \qquad (\text{i.e. } \langle 0 \mid E^g \mid 0 \rangle = 0 \quad \text{for} \quad g \neq 0\,, \quad E^0 = 1\!\!1)\,. \tag{8.16}$$

Exercise 8.5. Use (8.13) to compute the n-point function

$$\langle 0 \mid \psi(z_1, g_1) \dots \psi(z_n, g_n) \mid 0 \rangle = \prod_{1 \leq i < j \leq n} (z_{ij})^{g_i g_j}\,. \tag{8.17}$$

It follows from Proposition 8.1 that the CFT of the chiral algebra $\mathcal{A}(\mathcal{V}_J)$ has a continuum of inequivalent UIPERs and hence, is *not* rational (recall the definition at the end of Section 7). On the other hand, if g^2 is a (positive) even integer then the algebra $\mathcal{A}(g^2)$ generated by the pair of oppositely charged Bose fields $\psi(z, \pm g)$ provides a *local extension* of $\mathcal{A}(\mathcal{V}_J)$. Indeed, the current J is contained in the OPE of the product $\psi(z_1, g)\, \psi(z_2, -g)$ which defines a bilocal field ([75]):

$$\begin{aligned} z_{12}^{g^2}\, \psi(z_1, g)\, \psi(z_2, -g) &=: e^{g \int_{z_2}^{z_1} J(z) dz} := 1 + g \int_{z_2}^{z_1} J(z)\, dz \\ +\quad 6g^2 \int_{z_2}^{z_1} & \frac{(z_1 - z)(z - z_2)}{z_{12}}\, T(z)\, dz + g^3\, z_{12}^3\, R_3(z_1, z_2; g)\,, \end{aligned}$$

$$\begin{aligned} \langle 0 \mid J(z_1)\, T(z_2) \mid 0 \rangle = 0 &= \langle 0 \mid J(z_1)\, R(z_1, z_2; g) \mid 0 \rangle \\ &= \langle 0 \mid T(z_1)\, R(z_1, z_2; g) \mid 0 \rangle\,. \end{aligned} \tag{8.18}$$

The following set of exercises is designed to establish (8.18) (explaining on the way its meaning).

Exercise 8.5. Prove the equivalence of the following two definitions of the normal exponent in the first equation (8.18):

$$: e^{g \int_{z_2}^{z_1} J(z)\, dz} := e^{g(\varphi_+(z_1) - \varphi_+(z_2))}\, z_{12}^{g J_0}\, e^{g(\varphi_-(z_1) - \varphi_-(z_2))} \tag{8.19}$$

$$: e^{g \int_{z_2}^{z_1} J(z) dz} := \frac{e^{g \int_{z_2}^{z_1} J(z) dz}}{\langle 0 \mid e^{g \int_{z_2}^{z_1} J(z) dz} \mid 0 \rangle}\,. \tag{8.20}$$

(Eq. (8.20) should be understood as an expansion in powers of g defining the corresponding normal products iteratively.) Use (8.19) to verify the first equation (8.18).

Exercise 8.6. Prove that the CR (8.9) is equivalent to the following CR between the frequency parts of the current and the charged field ψ:

$$[\psi(z_1, g), J_{(+)}(z_2)] = -\frac{g}{z_{12}}\, \psi(z_1, g)\,,\ [J^{(-)}(z_1), \psi(z_2, g)] = \frac{g}{z_{12}}\, \psi(z_2, g) \tag{8.21}$$

for

$$J_{(+)}(z) = \sum_{n=1}^{\infty} J_{-n}\, z^{n-1} = \varphi'_+(z)\,, \; J^{(-)}(z) = \sum_{n=0}^{\infty} \frac{J_n}{z^{n+1}} = \varphi'_-(z)\,. \tag{8.22}$$

Exercise 8.7. Use (8.21) and the vacuum conditions

$$J^{(-)}(z) \mid 0\rangle = 0 = \langle 0 \mid J_{(+)}(z) \tag{8.23}$$

to prove the *Ward*[43] *identity* for current-field correlation functions:

$$\begin{aligned} &\langle 0 \mid \psi(z_1, g_1) \ldots \psi(z_k, g_k)\, J(z)\, \psi(z_{k+1}, g_{k+1}) \ldots \psi(z_n, g_n) \mid 0\rangle \\ = \; &\left(\sum_{j=k+1}^{n} \frac{g_j}{z - z_j} - \sum_{i=1}^{k} \frac{g_i}{z_i - z} \right) \langle 0 \mid \psi(z_1, g_1) \ldots \psi(z_n, g_n) \mid 0\rangle\,. \end{aligned} \tag{8.24}$$

Thus the Ward identities allow to express current-charge fields correlation functions in terms of charged fields correlations. We find, in particular, the 3-point function

$$\langle 0 \mid \psi(z_1, g)\, \psi(z_2, -g)\, J(z_3) \mid 0\rangle = \frac{g}{z_{13}\, z_{23}}\, z_{12}^{1-g^2}\,. \tag{8.25}$$

Exercise 8.8. Derive in a similar manner the Ward-Takahashi identities for correlation functions of ψ's with the stress energy tensor; deduce from this the expression for the 3-point function

$$\langle 0 \mid \psi(z_1, g)\, \psi(z_2, -g)\, T(z_3) \mid 0\rangle = \frac{g^2}{2}\, \frac{z_{12}^{2-g^2}}{z_{13}^2\, z_{23}^2}\,. \tag{8.26}$$

Exercise 8.9. Use (8.25) (8.26) and the orthogonality relations in Eq. (8.18) to verify the third equation (8.18).

Remark 8.1. The algebra $\mathcal{A}(g^2)$ contains charged fields $\psi(z, ng)$ of all charges multiple of g ($n \in \mathbb{Z}$). They appear as "composite fields" in OPEs of $\psi(z, \pm g)$. We have the following iterative rule:

$$\psi(z_1, g)\, \psi(z_2, ng) = z_{12}^{ng^2} \{\psi(z_2, (n+1)g) + O(z_{12})\}\,. \tag{8.27}$$

Thus the (isomorphic to $\mathbb{Z}$) group of all powers of U_g, introduced in the "Sketch of proof" of Proposition 8.1, is realized in the vacuum representation of $\mathcal{A}(g^2)$. It follows that $\mathcal{A}(m^2\, g^2)$ (for $m > 1$ integer) is a proper subalgebra if $\mathcal{A}(g^2)$.

One can, sure, also consider the CVO $\psi(z, g)$ for any (positive) integer g^2; the odd g^2 then correspond to Fermi fields. The local commutativity property (8.10), extends in this case to a *graded local commutativity*:

$$z_{12}^{g^2}\, \psi(z_1, g)\, \psi(z_2, -g) = z_{21}^{g^2}\, \psi(z_2, -g)\, \psi(z_1, g) \quad \text{for} \quad g^2 \in \mathbb{N}\,. \tag{8.28}$$

The chiral algebra $\mathcal{A}(4(2\nu + 1))$ appears as the bosonic part of the ($\mathbb{Z}_2$ graded) chiral superalgebra $\mathcal{A}(2\nu + 1)$, $\nu = 0, 1, \ldots$.

[43] John Clive Ward (1924-2000), British physicist; the Ward identity in quantum electrodynamics relates the renormalization of the wave function of the electron to its vertex function renormalization.

Exercise 8.10. Let $G^{\pm}(z) = \sqrt{\frac{2}{3}}\,\psi(z, \pm\sqrt{3})$ and normalize the associated $u(1)$ current $J(z)$ so that to exclude irrationalities in the OPE (8.18), setting $3\, z_{12}^2 \langle 0 \mid J(z_1)\, J(z_2) \mid 0\rangle = 1$. Prove the anticommutation relations

$$[G^+(z_1), G^-(z_2)]_+ = 2\, T(z_2)\, \delta(z_{12}) + (J(z_1) + J(z_2))\, \partial_2\, \delta(z_{12}) + \frac{1}{3}\, \partial_2^2\, \delta(z_{12})$$

$$\left(\langle 0 \mid G^+(z_1)\, G^-(z_2) \mid 0 \rangle = \frac{2}{3\, z_{12}^3} \right) . \tag{8.29}$$

Setting further

$$G^{\pm}(z) = \sum_n G^{\pm}_{n-\frac{1}{2}}\, z^{-n-1} , \tag{8.30}$$

deduce the modes' (anti)commutation relations

$$[G^{\pm}_{n-\frac{1}{2}}, G^{\mp}_{\frac{1}{2}-m}]_+ = 2\, L_{n-m} \pm (n+m-1)\, J_{n-m} + \frac{n(n-1)}{3}\, \delta_{nm}$$

$$[J_n, J_m] = \frac{n}{3}\, \delta_{n,-m} , \quad [J_n, G^{\pm}_{\rho}] = \pm\, G^{\pm}_{n+\rho} . \tag{8.31}$$

This is the (vacuum) *Neveu-Schwarz sector* of the $N = 2$ (extended) superconformal model ([26, 80]).

The chiral algebras $\mathcal{A}(g^2)$ for integer $g^2 > 0$ provide te simplest examples of *rational CFT*.

Proposition 8.2. *The algebra $\mathcal{A}(g^2)$ for $g^2 = 2, 4, 6, \ldots$ has g^2 UIPERs generated by primary CVO $\psi(z, e_k)$, relatively local to $\psi(z, g)$. They correspond to $g\, e_k = k$, $1 - \frac{g^2}{2} \le k \le \frac{g^2}{2}$. The fusion rules for the primary fields $\psi(z, e_k)$ are given by the multiplication rules of the finite cyclic group of g^2 elements*

$$\frac{\mathbb{Z}\, e_1}{\mathbb{Z}\, g} \simeq \frac{\mathbb{Z}}{g^2\, \mathbb{Z}} . \tag{8.32}$$

Sketch of proof (see [29]). Any UIPER of $\mathcal{A}(g^2)$ gives rise to a fully reducible unitary positive energy representation of the $u(1)$ current algebra $\mathcal{A}(\mathcal{V}_J)$ whose spectrum of J_0 is contained in the set $e + \mathbb{Z}\, g$ for some (real) e. The OPE

$$\psi(z, g)\, \psi(0, e) \mid 0\rangle = \psi(z, g) \mid e\rangle = z^{ge}(1 + O(z)) \mid g + e\rangle \tag{8.33}$$

only corresponds to a relatively local ground state $\mid e\rangle$ if it is single valued, i.e. if the power ge of z is an integer. Noting that e is determined $\bmod\ ng$ $(n \in \mathbb{Z})$ we can choose $|e| \le \frac{g}{2}$. The rest is straightforward. □

The *field algebra* $\mathcal{F}\left(\frac{1}{g^2}\right)$ $(\supset \mathcal{A}(g^2))$ generated by the pair of charged primary fields $\psi\left(z, \pm\frac{1}{g}\right)$ admits a finite cyclic *group of global gauge transformations* acting on the state space by powers of the operator

$$U (= U_{1/g}) = e^{2\pi i \frac{J_0}{g}} , \quad (U^{g^2} - 1)\, \mathcal{F}\left(\frac{1}{g^2}\right) = 0 . \tag{8.34}$$

It generates an automorphism of the field algebra such that

$$U\, \psi(z, e)\, U^{-1} = e^{2\pi i \frac{e}{g}}\, \psi(z, e) \Rightarrow U A\, U^{-1} = A \quad \text{for} \quad A \in \mathcal{A}(g^2) . \tag{8.35}$$

Remark 8.2. The primary chiral vertex operator $\psi(z,e)$ is a multivalued function of z. In fact, the extension of the relation (8.33) to two primary charges e_1, e_2 ($e_i\, g \in \mathbb{Z}$),

$$\psi(z,e_1) \mid e_2\rangle = z^{e_1 e_2}(1+O(z)) \mid e_1 + e_2\rangle$$

is a multivalued function of z unless $e_1 e_2$ is also an integer. Setting $z = e^{it}$ we find a charge dependent *twisted periodicity* condition for ψ as a function of t:

$$\psi(e^{i(t+2\pi)}, e_1) \mid e_2\rangle = e^{2\pi i e_1 e_2}\ \psi(e^{it}, e_1) \mid e_2\rangle\,. \tag{8.36}$$

The exchange relations of $\psi(z,e)$ with itself give rise to a nontrivial one-dimensional representation of the braid group which defines for non-integer e^2 an *anyonic statistics*. (The idea for such statistics appears already in [109]. More on the ancestry of the "anyon" can be found in [19] − *cf.* Section 2.) For $eg \in \mathbb{Z}$ such an anyonic representation of $\mathcal{B}_2$ is isomorphic to a finite cyclic group. (A bound state of g^2 anyons obeys the Bose-Fermi alternative.) More general lattice vertex algebras yielding anyonic statistics are applied to the description of the fractional quantum Hall effect, [63, 75] (see also [31] where the intriguing plateau with Hall conductivity $\sigma_H = \frac{5}{2}\,\frac{e^2}{h}$ is considered). A concise unified description of (extended) abelian and of non-abelian (Kac-Moody) current algebras is given in Section 1 of [102].

9 The $su(2)$ current algebra model. Knizhnik-Zamolodchikov equation

9.1 The affine Kac-Moody algebra $\widehat{su}(2)_k$. The current $J(z,\zeta)$. Operator Ward identities

The simplest models associated with a (non-abelian) braid group statistics are the *affine Kac-Moody current algebra models* with a chiral algebra $\mathcal{A}_k(\mathcal{G})$ determined by a simple Lie algebra $\mathcal{G}$ and an *integer level*[44] k $(= 1, 2, \ldots)$. The simplest among the simple Lie algebras (corresponding to a compact Lie group) is $\mathcal{G} = su(2)$ spanned by three generators J_0^a, $a = 1, 2, 3$ satisfying

$$[J_0^a, J_0^b] = i\,\varepsilon^{abc}\,J_0^c \tag{9.1}$$

where ε^{abc} is the totally antisymmetric Levi-Civita tensor ($\varepsilon^{123} = 1 = \varepsilon^{312} = -\varepsilon^{321} = \ldots$). The corresponding local currents are given by (8.1) (with $d_{su(2)} = 3$), the current modes satisfying the CR (8.2) with

$$f^{ab}_{\ \ c} = \varepsilon^{abc}\,, \qquad g^{ab} = \frac{1}{2}\,\delta^{ab}\,. \tag{9.2}$$

The resulting infinite dimensional Lie algebra is denoted by $\widehat{su}(2)_k$.

Exercise 9.1. Verify that the $\widehat{su}(2)_k$ Sugawara chiral stress-energy tensor

$$T(z) = \frac{1}{h} : \vec{J}^2(z) : \equiv \frac{1}{h}\sum_{a=1}^{3} : (J^a(z))^2 : \,, \qquad h = k+2 \tag{9.3}$$

gives rise to the OPE (8.5) with $J(z)$ substituted by $J^a(z)$ while the second equation (8.6) is replaced by

$$T(z_1)\,T(z_2) = \frac{c_k}{2\,z_{12}^4} + \frac{T(z_1) + T(z_2)}{z_{12}^2} + O(1)\,, \qquad c_k = \frac{3k}{k+2}\,. \tag{9.4}$$

The *renormalized level* $h = k+2$ is also called the *height* of $\mathcal{A}_k(A_1) = \widehat{su}(2)_k$.

Exercise 9.2. Prove that for $k = 1$ $(= c_1)$ and $J(z) = \sqrt{2}\,J^3(z)$, the stress tensor (9.3) coincides with (8.4) while the "charged components" of the current are reproduced by the vertex operator construction (8.13),

$$J^{\pm}(z) := J^1(z) \pm i\,J^2(z) = E^{\pm\sqrt{2}}\,e^{\pm\sqrt{2}\,\varphi_+(z)}\,z^{\pm\sqrt{2}\,J_0}\,e^{\pm\sqrt{2}\,\varphi_-(z)} \tag{9.5}$$

with $\varphi_\pm$ given by (8.12).

Hint : verify that $T(z)$ (8.4) (for $J(z) = \sqrt{2}\,J^3(z)$) and $J^{\pm}(z)$ satisfy the correct OPE (or CR); deduce furthermore the OPE

$$J^+(z_1)\,J^-(z_2) = \frac{1}{z_{12}^2}\left\{1 + 2\int_{z_2}^{z_1} J^3(z)\,dz + z_{12}(T(z_1) + T(z_2)) + O(z_{12}^2)\right\} \tag{9.6}$$

[44] Speaking of an associative chiral algebra $\mathcal{A}_k$ characterized by a natural number k (or of the corresponding infinite dimensional Lie algebra) we depart from the terminology of the theory of affine Kac-Moody algebras $\hat{\mathcal{G}}$ ([99]) in which the central charge, say K, is an operator commuting with the current modes. The algebra $\mathcal{A}_k$ would then correspond to a representation of $\hat{\mathcal{G}}$ in which we have chosen a particular eigenvalue k of the central charge K (thus specifying the vacuum of the theory).

for $[J^1(z_1), J^2(z_2)] = J^3(z_2)\,\delta(z_{12})\,, \quad \langle 0 \mid J^a(z_1)\, J^b(z_2) \mid 0\rangle\, z_{12}^2 = \frac{1}{2}\,\delta^{ab}\,.$

Exercise 9.2 demonstrates that we may only expect to encounter non-abelian braid group statistics in a unitary CFT for $k \geq 2$.

The representation theory of affine Kac-Moody algebras [99] tells us that $\widehat{su}(2)_k$ admits $k+1$ UIPERs labeled by the values I of the isospin of the ground states of *integrable representations*, such that $2I \leq k$. We use here the physicist terminology: a *ground state* is a lowest energy state with respect to the *conformal energy operator* $H_c = L_0 + \bar{L}_0$. As L_0 and $\bar{L}_0$ commute, it factorizes into a product a ground states for the left and right movers' current algebras. We shall only mention *diagonal representations* (with $I = \bar{I}$) in this brief synopsis of the $\widehat{su}(2)_k$ CFT model and will spell out the properties of the chiral (say, left movers') representations.

Exercise 9.3.
(a) Prove, using (7.21), that the chiral energy operator L_0 commutes with the currents' zero modes: $[L_0, J_0^a] = 0$, $a = 1, 2, 3$. Deduce, as a corollary that the subspace of ground states of isospin I has dimension $p \equiv 2I + 1$. A basis $|k, II_3\rangle$ of ground states in which J_0^3 is diagonal is characterized by

$$J_n^a \mid k, II_3\rangle = 0 \quad \text{for} \quad n > 0\,, \quad (J_0^3 - I_3) \mid k, II_3\rangle = 0\,,$$
$$I_3 = -I, 1-I, \ldots, I \qquad (2I = 0, 1, \ldots, k)\,. \tag{9.7}$$

(b) Prove as a consequence of (9.7) and the Sugawara formula (9.3) that

$$L_0 = \frac{1}{h}\left(\vec{J}_0^{\,2} + 2\sum_{n=1}^{\infty} \vec{J}_{-n}\,.\,\vec{J}_n\right)\,, \quad (L_0 - \Delta_I) \mid k, II_3\rangle = 0\,,$$
$$\Delta_I = \frac{I(I+1)}{h} \qquad (h = k+2)\,. \tag{9.8}$$

The 2D primary field $\phi_I(z,\bar{z})$ which intertwines the vacuum representation of $\widehat{su}(2)_k \oplus \widehat{su}(2)_k$ with the one of weight (I, I) is thus a $(2I+1)\times(2I+1)$ component isospin tensor. In particular, the *step operator* $\phi_{\frac{1}{2}}$ can be viewed as an $SU(2)$ "group valued" field $g(z,\bar{z}) = \{g(z,\bar{z})^A_B\,,\ A, B = 1, 2\}$. The quotation marks should remind us that the quantum field $g(z,\bar{z})$ is actually a 2×2 matrix of operator valued distributions; only its classical counterpart can be assumed to belong to $SU(2)$. The 2-point function of g factorizes:

$$\langle 0 \mid g(z_1,\bar{z}_1)^{A_1}_{B_1}\, g(z_2,\bar{z}_2)^{A_2}_{B_2} \mid 0\rangle = \frac{\epsilon^{A_1A_2}}{(z_{12})^{2\Delta}}\ \frac{\epsilon_{B_1B_2}}{(\bar{z}_{12})^{2\Delta}}$$
$$(\epsilon^{12} = \epsilon_{12} = 1 = -\epsilon_{21})\,, \quad \Delta = \Delta_{\frac{1}{2}} = \frac{3}{4h}\,.$$

Its 4-point function, however, does not (for $k > 1$) but can be represented, in general, as a sum of two factorized expressions. This suggests writing $g(z,\bar{z})$ as a matrix product of chiral fields:

$$g(z,\bar{z})^A_B = g(z)^A_\alpha\, \bar{g}^{-1}(\bar{z})^\alpha_B \left(\equiv \sum_{\alpha=1}^{2} g(z)^A_\alpha\, \bar{g}^{-1}(\bar{z})^\alpha_B\right) \tag{9.9}$$

(a splitting that can be also justified in the classical canonical theory – see Section 10 below).

The 2D field $g(z,\bar{z})$ provides an example of a conformal but not GCI field, as its correlation functions are not rational. It is also locally commuting but violates the stronger Huygens locality (6.20). Note that for Euclidean compactified space-time $t \to it$, z and $\bar{z}$ are complex conjugate ($z = e^{-t+ix}$, $\bar{z} = e^{-t-ix}$); locality then implies that $g(z,\bar{z})$ should be periodic in x (i.e. single valued in z):

$$e^{2\pi i(L_0-\bar{L}_0)}\, g(z,\bar{z})\, e^{2\pi i(\bar{L}_0-L_0)} = g(e^{2\pi i}\, z, e^{-2\pi i}\,\bar{z}) = g(z,\bar{z})\,,$$
$$e^{2\pi i L_0}\, g(z,\bar{z})\, e^{-2\pi i L_0} = e^{2\pi i\Delta}\, g(e^{2\pi i}\, z,\bar{z})\,. \tag{9.10}$$

This only implies that the chiral components of $g(z,\bar{z})$ appearing in the right hand side of (9.9), should have the same *monodromy* M:

$$e^{2\pi i L_0}\, g(z)\, e^{-2\pi i L_0} = e^{2\pi i\Delta}\, g(z\, e^{2\pi i}) = g(z)\, M \;\left(= g(z)^A_\sigma\, M^\sigma_\alpha\right),$$
$$e^{-2\pi i\Delta}\, \bar{g}(\bar{z}\, e^{-2\pi i}) = \bar{g}(\bar{z})\, M \tag{9.11}$$

which will then cancel in the product (9.9).

Exercise 9.4. Use (9.11) to prove $(M - q^{-3/2}) \mid 0\rangle = 0$ for $q = e^{-\frac{i\pi}{h}}$.

The chiral fields $g(z)$ and $\bar{g}(\bar{z})$ satisfy a differential equation involving the $\widehat{su}(2)$ currents which follow from the Ward(-Takahashi) identities and from the Sugawara formula. In order to write it down it is convenient to combine the three components $J^a(z)$ of the current into a second degree polynomial in a formal variable ζ:

$$J(z,\zeta) = J^-(z) + 2\,\zeta\, J^3(z) - \zeta^2\, J^+(z) \quad (J^\pm = J^1 \pm i\, J^2)\,. \tag{9.12}$$

We leave it to the reader to verify that then the 2- and 3-point functions of the current assume the form:

$$\langle 0 \mid J(z_1,\zeta_1)\, J(z_2,\zeta_2) \mid 0\rangle = -k\, \frac{\zeta_{12}^2}{z_{12}^2} \quad (\zeta_{12} = \zeta_1 - \zeta_2)$$
$$\langle 0 \mid J(z_1,\zeta_1)\, J(z_2,\zeta_2)\, J(z_3,\zeta_3) \mid 0\rangle = k\, \frac{\zeta_{12}\,\zeta_{23}\,\zeta_{13}}{z_{12}\, z_{23}\, z_{13}}\,. \tag{9.13}$$

A chiral $\widehat{su}(2)_k$ primary field ϕ_I (of isospin I) has both $SU(2)$ and U_q indices (like $g(z)^A_\alpha$ for $I = \frac{1}{2}$) and can be viewed, alternatively, as a polynomial (of degree $2I$) in two formal variables ζ and u, respectively. We shall expand its 4-point function in the U_q-invariant amplitudes $\mathcal{J}^{(I)}_\lambda(u_1,\ldots,u_4)$ (4.31):

$$\langle 0 \mid \phi_I(z_1,\zeta_1;u_1)\, \phi_I(z_2,\zeta_2;u_2)\, \phi_I(z_3,\zeta_3;u_3)\, \phi_I(z_4,\zeta_4;u_4) \mid 0\rangle$$
$$= \sum_{\lambda=0}^{2I} w_\lambda(z_1,\zeta_1;z_2,\zeta_2;z_3,\zeta_3;z_4,\zeta_4)\, \mathcal{J}^{(I)}_\lambda(u_1,u_2,u_3,u_4)\,. \tag{9.14}$$

The properties of the primary field ϕ_I are determined by its commutation relations with the modes $J_n(\zeta)$ of the current encoded in:

$$[J^{(-)}(z_1,\zeta_1), \phi_I(z_2,\zeta_2;u)] = -\frac{1}{z_{12}}\, (\zeta_{12}^2\, \partial_{\zeta_2} + 2I\, \zeta_{12})\, \phi_I(z_2,\zeta_2;u)$$
$$[\phi_I(z_1,\zeta_1;u), J_{(+)}(z_2,\zeta_2)] = \frac{1}{z_{12}}\, \left(\zeta_{12}^2\, \frac{\partial}{\partial\zeta_1} - 2I\, \zeta_{12}\right) \phi_I(z_1,\zeta_1;u)\,, \tag{9.15}$$

where the frequency parts of the current, $J^{(-)}$ and $J_{(+)}$ are defined as in (8.22); setting similarly $T^{(-)}(z) = \sum_{n=0}^{\infty} \frac{L_{n-1}}{z^{n+1}}$, $T_{(+)}(z) = \sum_{n=0}^{\infty} L_{-n-2}\, z^n$ we find

$$[T^{(-)}(z_1), \phi_I(z_2,\zeta;u)] = \frac{\Delta_I}{z_{12}^2}\,\phi_I(z_2,\zeta;u) + \frac{1}{z_{12}}\,\partial_{z_2}\,\phi_I(z_2,\zeta;u)$$
$$[\phi_I(z_1,\zeta;u), T_{(+)}(z_2)] = \frac{\Delta_I}{z_{12}^2}\,\phi_I(z_1,\zeta;u) - \frac{1}{z_{12}}\,\partial_{z_1}\,\phi_I(z_1,\zeta;u)\,. \tag{9.16}$$

9.2 The $\widehat{su}(2)_k$ Knizhnik-Zamolodchikov equation

Proposition 9.1. *Let $\phi_I(z,\zeta;u)$ be an $\widehat{su}(2)_k$ primary fields, that is a field satisfying* (9.15) *and* (9.16). *Then Δ_I is given by* (9.8) *and ϕ_I satisfies the Knizhnik*[45]*-Zamolodchikov (KZ) equation [105]*

$$h\,\partial_z\,\phi_I(z,\zeta;u) = I :\partial_\zeta\, J(z,\zeta)\,\phi_I(z,\zeta;u): - :J(z,\zeta)\,\partial_\zeta\,\phi_I(z,\zeta;u): \tag{9.17}$$

where the normal product is defined by the non-singular term in the current-field OPE and is expressed simply in terms of the frequency parts of J:

$$:J(z,\zeta_1)\,\phi_I(z,\zeta_2) := J_{(+)}(z,\zeta_1)\,\phi_I(z,\zeta_2) + \phi_I(z,\zeta_2)\,J^{(-)}(z,\zeta_1)\,. \tag{9.18}$$

Sketch of proof. Eq. (9.17) folows from the known CR $[L_n, \phi_I(z)]$, $[J_m, \phi_I(z)]$ derived from (9.15) (9.16) and from the Sugawara expression (9.3) for T. (See [145] and [75] Chapter 5 for details.) □

It is instructive to display the KZ equation for the basic group valued field $g(z,\bar{z})$ in a matrix form, spelling out in this case the meaning of the right hand side of (9.17). To this end we introduce the matrix valued current $\underset{\sim}{J}(z) = J^i(z)\,\sigma_i$ related to $J(z,\zeta)$ (9.12) by

$$J(z,\zeta) = (\zeta,1)\,\underset{\sim}{J}(z)\begin{pmatrix}1\\-\zeta\end{pmatrix} = (\zeta,1)\begin{pmatrix}J^3(z) & J^+(z)\\ J^-(z) & -J^3(z)\end{pmatrix}\begin{pmatrix}1\\-\zeta\end{pmatrix}. \tag{9.19}$$

Exercise 9.5. Prove that Eq. (9.17) (for $I = \frac{1}{2}$) is equivalent to

$$h\,\frac{\partial}{\partial z}\,g(z,\bar{z})^A_B = - :\underset{\sim}{J}(z)^A_S\,g(z,\bar{z})^S_B:\,. \tag{9.20}$$

(*Solution* : setting $g(z,\bar{z};\zeta)_B = \zeta\,g(z,\bar{z})^1_B + g(z,\bar{z})^2_B$ we find

$$\begin{aligned}
&\frac{1}{2} :\frac{\partial J(z,\zeta)}{\partial\zeta}\,g(z,\bar{z};\zeta)_B: - :J(z,\zeta)\,\frac{\partial}{\partial\zeta}\,g(z,\bar{z};\zeta)_B:\\
=&\ :(J^3(z)\,g(z,\bar{z})^2_B - J^-(z)\,g(z,\bar{z})^1_B - (J^3(z)\,g(z,\bar{z})^1_B + J^+(z)\,g(z,\bar{z})^2_B)\,\zeta):\\
=&\ - :(\zeta\,\underset{\sim}{J}(z)^1_S + \underset{\sim}{J}(z)^2_S)\,g(z,\bar{z})^S_B:\ .)
\end{aligned}$$

Proposition 9.2. *The operator KZ equation* (9.17) *and the (operator) Ward identities* (9.15) (9.16) *allow to write down the KZ equation for any correlation function of ϕ_I. In particular,*

[45] Vadim Genrikhovich Knizhnik (Kiev 1962-Moscow 1987) was a student of A.M. Polyakov.

$SU(2)$ and conformal invariant amplitudes $f_\lambda(\xi,\eta)$ of the 4-point functions w_λ (9.14), defined by

$$w_\lambda(z_1,\zeta_1;\ldots;z_4,\zeta_4) = P_I(z_{ij},\zeta_{ij})\, f_\lambda(\xi,\eta)\,,\ \xi = \frac{\zeta_{12}\,\zeta_{34}}{\zeta_{13}\,\zeta_{24}}\,,\ \eta = \frac{z_{12}\,z_{34}}{z_{13}\,z_{24}}\,, \tag{9.21}$$

$$p_I(z_{ij},\zeta_{ij}) = \left(\frac{z_{13}\,z_{24}}{z_{12}\,z_{23}\,z_{34}\,z_{14}}\right)^{2\Delta_I} (\zeta_{13}\,\zeta_{24})^{2I}\,, \tag{9.22}$$

satisfy the KZ equation

$$\left(h\,\frac{\partial}{\partial\eta} - \frac{(\vec{I}_1+\vec{I}_2)^2}{\eta} + \frac{(\vec{I}_2+\vec{I}_3)^2}{1-\eta}\right) f_\lambda = 0\,, \tag{9.23}$$

where the Casimir invariants can be expressed as second order differential operators in ξ:

$$(\vec{I}_1+\vec{I}_2)^2 = 2I(2I+1-2I\xi) - [4I(1-\xi)-\xi]\,\xi\,\frac{\partial}{\partial\xi} + \xi^2(1-\xi)\,\frac{\partial^2}{\partial\xi^2}\,,$$

$$(\vec{I}_2+\vec{I}_3)^2 = 2I(2I+1-2I(1-\xi)) + (4I\xi+1-\xi)(1-\xi)\,\frac{\partial}{\partial\xi} + \xi(1-\xi)^2\,\frac{\partial^2}{\partial\xi^2}\,. \tag{9.24}$$

Sketch of proof. Applying Eq. (9.17) to $\phi_I(z_2,\zeta_2;u_2)$ and moving $J^{(-)}(z_2,\zeta_2)$ to the right and $J_{(+)}(z_2,\zeta_2)$ to the left, using in both cases Eq. (9.15) as well as $J^{(-)}(z,\zeta)\mid 0\rangle = 0 = \langle 0\mid J_{(+)}(z,\zeta)$, we find for the full 4-point function (9.14) and hence for each w_λ the equation

$$\left(h\,\partial_{z_2} + \frac{C_{12}}{z_{12}} - \frac{C_{23}}{z_{23}} - \frac{C_{24}}{z_{24}}\right) w_\lambda(z_1,\zeta_1;\ldots;z_4,\zeta_4) = 0\,, \tag{9.25}$$

where $C_{ij} = 2\vec{I}_i\cdot\vec{I}_j = 2\sum_{a=1}^{3} I_i^a\,I_j^a$ is the *polarized $su(2)$ Casimir operator* that can be expressed as a differential operator in ζ_i and ζ_j. Inserting (9.19) into (9.23) and using the identity

$$(\vec{I}_1+\vec{I}_2+\vec{I}_3+\vec{I}_4)\,w_\lambda = 0 = (2\vec{I}_1\cdot\vec{I}_2 + 2\vec{I}_2\cdot\vec{I}_3 + 2\vec{I}_2\cdot\vec{I}_4 + 2I(I+1))\,w_\lambda\,, \tag{9.26}$$

we obtain (9.21). Using further the relations

$$C_{ij} = 2I\,[I + \zeta_{ij}(\partial_j - \partial_i)] - \zeta_{ij}^2\,\partial_i\,\partial_j \tag{9.27}$$

for $\vec{I}_i^2 = I(I+1)\,,\ \partial_i = \frac{\partial}{\partial\zeta_i}$, we find (9.24). □

Exercise 9.6 Derive the relation

$$[C_{13},C_{23}] = [C_{12},C_{13}] = -[C_{12},C_{23}] = 4I^2(\zeta_{12}\partial_{23} - \zeta_{23}\partial_{12}) +$$

$$+\,2I\{\zeta_{13}(\zeta_{12}-\zeta_{23})\partial_1\partial_3 + \zeta_{23}(\zeta_{12}+\zeta_{13})\partial_2\partial_3 - \zeta_{12}(\zeta_{13}+\zeta_{23})\partial_1\partial_2\} -$$

$$-\,2\,\zeta_{12}\zeta_{23}\zeta_{13}\,\partial_1\partial_2\partial_3 = \varepsilon^{abc}I_{a1}I_{b2}I_{c3}\,, \qquad \partial_{ij} = \partial_i - \partial_j\,, \tag{9.28}$$

in accord with (B.7).

The basis $\{w_\lambda\}$ (or $\{f_\lambda\}$) of solutions to (9.25) (or (9.23)) is fixed by the requirement that the full 4-point function (9.14) is invariant under the diagonal action of the braid group $\mathcal{B}_4$

on w_λ and $\mathcal{J}_\lambda^{(I)}$. We shall write down this solution expanding w_λ in a set $\{J_\ell^{(I)}\}$ of $SU(2)$ invariants obtained from $\mathcal{J}_\lambda^{(I)}$ in the limit $q \to 1$:

$$J_\ell^{(I)}(\zeta_1, \ldots, \zeta_4) = (\zeta_{12}\,\zeta_{34})^{2I-\ell}(\zeta_{14}\,\zeta_{23})^{\ell} = (\zeta_{13}\,\zeta_{24})^{2I}\,\xi^{2I-\ell}\,(1-\xi)^{\ell}\,. \tag{9.29}$$

The result is ([143])

$$f_\lambda(\xi, \eta) = \sum_{\ell=0}^{2I} \xi^{2I-\ell}\,(1-\xi)^{\ell}\,\eta^{\ell}(1-\eta)^{2I-\ell}\,g_\lambda^{\ell}(\eta) \tag{9.30}$$

where g_λ^ℓ is given by the $2I$-fold integral

$$g_\lambda^{\ell}(\eta) = \int_0^{\eta} dt_1 \int_0^{t_1} dt_2 \quad \ldots \quad \int_0^{t_{\lambda-1}} dt_\lambda \int_\eta^1 dt_{\lambda+1} \int_{t_{\lambda+1}}^1 dt_{\lambda+2}$$
$$\ldots \quad \int_{t_{2I-1}}^1 dt_{2I}\, P_\lambda^{\ell}(\eta, t_i)\,, \tag{9.31}$$

$$\begin{aligned} P_\lambda^{\ell}(\eta, t_i) \quad &= \quad \prod_{i=1}^{2I} t_i^{\frac{1}{h}}(1-t_i)^{\frac{1}{h}} \prod_{i=1}^{\lambda}(\eta - t_i)^{\frac{1}{h}-1} \prod_{j=\lambda+1}^{2I}(t_j-\eta)^{\frac{1}{h}-1} \prod_{1\le i<j\le 2I}(\epsilon_{\lambda j}\, t_{ij})^{\frac{2}{h}} \\ &\times \quad \sum_{\sigma\in\mathcal{S}_{2I}} \prod_{i=1}^{\ell} t_{\sigma(i)}^{-1} \prod_{j=\ell+1}^{2I}(1-t_{\sigma(j)})^{-1}\,, \end{aligned} \tag{9.32}$$

the sum being spread over all permutations $\sigma : (1, \ldots, 2I) \to (\sigma(1), \ldots, \sigma(2I))$. In order to verify the braid invariance of the resulting 4-point function (9.14) one computes separately the $\mathcal{B}_4$ action on the U_q invariants $\mathcal{J}_\lambda^{(I)}(u_1, \ldots, u_4)$ (using the braid operator $\hat{R}$ (4.40)) and of the (analytically continued) functions (9.21)–(9.23), (9.30)–(9.32), taking into account the transformation properties of the $SU(2)$ invariants $J_\ell^{(I)}(\zeta_1, \ldots, \zeta_4)$ under permutation:

$$1 \rightleftarrows 2 : J_\ell^{(I)}(\zeta_2, \zeta_1, \zeta_3, \zeta_4) = (-1)^{2I-\ell} \sum_{s=0}^{\ell} \binom{\ell}{s}\, J_s^{(I)}(\zeta_1, \zeta_2, \zeta_3, \zeta_4)\ ,$$

$$2 \rightleftarrows 3 : J_\ell^{(I)}(\zeta_1, \zeta_3, \zeta_2, \zeta_4) = (-1)^{\ell} \sum_{s=\ell}^{2I} \binom{2I-\ell}{2I-s}\, J_s^{(I)}(\zeta_1, \zeta_2, \zeta_3, \zeta_4)\ . \tag{9.33}$$

We shall not work out here the details (see [143] and [142] where the case of different isospins is also outlined) but will write down the resulting lower and upper triangular braid matrices in Section 13 below.

9.3 Generalizations: the KZ operator as a flat connection; coset space models

The form (9.25) of the KZ equation admits a straightforward generalization to n-point correlation functions and to (compact form $\mathcal{G}$ of) an arbitrary simple Lie algebra:

$$h\nabla_i w := \left(h\frac{\partial}{\partial z_i} - \sum_{\substack{j=1\\ j\neq i}} \frac{C_{ij}}{z_{ij}} \right) w(z_1, \Lambda_1; \ldots; z_n, \Lambda_n) = 0\ ,$$
$$h = k + g^{\vee}\ , \qquad i = 1, \ldots, n\ , \qquad w \in \mathrm{Inv}(\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n)\ , \tag{9.34}$$

where C_{ij} is the polarized Casimir invariant regarded as an operator in the tensor product $\otimes_{\ell=1}^{n} \mathcal{V}_\ell$ of irreducible $\mathcal{G}$-modules $\mathcal{V}_\ell \equiv \mathcal{V}(\Lambda_\ell)$ with non-trivial entries in $\mathcal{V}_i \otimes \mathcal{V}_j$ and acting as the unit operator in the remaining $\mathcal{V}_\ell$):

$$C_{ij} = \eta^{ab}\, T_{ai}\, T_{bj} \tag{9.35}$$

(cf. (B.5)), $g^\vee$ is the dual Coxeter number of $\mathcal{G}$, $\{T_a\,,\ a = 1, \dots, d_{\mathcal{G}}\}$ is a (hermitean) basis in the vector space $\sqrt{-1}\,\mathcal{G}$, η^{ab} is the inverse Killing metric on $\sqrt{-1}\,\mathcal{G}$, T_{ai} being the (matrix) representation of T_a in $\mathcal{V}_i$.

Proposition 9.3. *The connection* 1*-form*

$$\mathbf{A} = \sum_{1\le i<j\le n} r_{ij}(z_{ij})\, dz_{ij} \qquad (\, z_{ij} = z_i - z_j\,) \tag{9.36}$$

where r_{ij} act non-trivially on $\mathcal{V}_i \otimes \mathcal{V}_j$, is flat,

$$d\,\mathbf{A} + \mathbf{A} \wedge \mathbf{A} = 0\ , \tag{9.37}$$

iff the r-matrix satisfies the classical Yang-Baxter equation with spectral parameter

$$[r_{12}(z_{12})\,,\, r_{13}(z_{13})] + [r_{12}(z_{12})\,,\, r_{23}(z_{23})] + [r_{13}(z_{13})\,,\, r_{23}(z_{23})] = 0 \tag{9.38}$$

($r_{ij}(z_{ij}) = -r_{ji}(z_{ji})$).

Proof. Obviously, $d\,\mathbf{A} = 0$; it can be verified that the exterior product $\mathbf{A}\wedge\mathbf{A}$ vanishes if (9.38) takes place (see Proposition 16.2.1 of [34]).

Exercise 9.7.
(a) Verify that

$$r_{ij}(z_{ij}) = \frac{C_{ij}}{z_{ij}} \tag{9.39}$$

satisfies (9.38) if C_{ij} obey (B.8).

(b) Prove that the operators (9.34) commute: $[\nabla_i, \nabla_j] = 0$.

Kac-Moody current algebras are also used as a starting point for constructing *coset space models.* We shall illustrate this sketching the Goddard-Kent-Olive (GKO) construction [83, 84] of the minimal ([17]) conformal models (cf. Section 7.2). The GKO construction is associated with the observation that the central charge $c(m)$ of an unitary minimal conformal model can be presented as a linear combination of central charges (of type c_k (9.4)) of the $su(2)$ current algebra:

$$c(k+2) = c_k + c_1 - c_{k+1} = 1 - \frac{6}{(k+2)(k+3)}\ . \tag{9.40}$$

Consider the tensor product $\widehat{su}(2)_k \times \widehat{su}(2)_1$ of representations of the $su(2)$ affine Kac-Moody algebra corresponding to levels k and 1. The associated $su(2)$ currents $\vec{J}(z;k)$ and $\vec{J}(z;1)$ commute (by definition). The linear combination $T(z;k,1)$ of Sugawara stress tensors corresponding to the coset

$$\widehat{su}(2)_k \times \widehat{su}(2)_1 \,/\, \widehat{su}(2)_{k+1}\ , \tag{9.41}$$

$$T(z;k,1) := \\ = \frac{1}{k+2}\ :\vec{J}^2(z;k):\ + \frac{1}{3}\ :\vec{J}^2(z;1):\ - \frac{1}{k+3}\ :(\vec{J}(z;k) + \vec{J}(z;1))^2: \tag{9.42}$$

can be demonstrated to satisfy all conditions for a Virasoro stress energy tensor with central charge $c(k+2)$ (9.40).

In a remarkable subsequent development [68] (under the general heading of *Hamiltonian reduction*) that uses the above ζ-variables' parametrization of solutions of the KZ equation for rational values of the isospin one also constructs correlation functions of primary fields for minimal models by identifying ζ_i with the corresponding world sheet variables z_i .

For a survey of the KZ equation in the framework of representation theory – see [51].

10 Canonical approach to the classical WZNW model

We have been using throughout the axiomatic approach to conformal current algebras (combined with the representation theory of affine Kac-Moody algebras). A canonical approach to the Wess-Zumino-Novikov-Witten (WZNW) model has been developed around the same time, starting with a classical theory and its subsequent quantization. The canonical action principle was set forth by Witten [155] (developing ideas of [153] and [124]) – a few months before the KZ equation was published. It required a few more years before Babelon [9], Blok [22], the Faddeev group [55, 3, 1] and Gawedzki [78] related it to the r- and R-matrix formalism and to the Yang-Baxter equation.

We begin with an outlook of the *first order Lagrangian* (also called *covariant Hamiltonian*) formalism following chiefly [78] and taking some hints from [98].

In general, a field theory lives on a fibre bundle locally equivalent to $\mathcal{M} \times \mathcal{F}$ with a D-dimensional base space-time manifold $\mathcal{M}$ and a fiber $\mathcal{F}$ of field configurations. We shall use, correspondingly, two exterior differentials, a *horizontal* one, d, acting on (the tangent space to) $\mathcal{M}$, and a *vertical* one, δ, acting on $\mathcal{F}$, so that the total exterior differential $\boldsymbol{d}$ on $\mathcal{M}\times\mathcal{F}$ appears as their sum:

$$\boldsymbol{d} = d + \delta \ , \quad d^2 = \delta^2 = 0 = [\, d, \delta]_+ \,(\equiv d\delta + \delta d\,) \quad \Rightarrow \quad \boldsymbol{d}^2 = 0 \ . \tag{10.1}$$

Whenever an action density (Lagrangian) exists it gives rise to a D-form $\boldsymbol{L}$ on $\mathcal{M} \times \mathcal{F}$ that will be assumed linear in the field differentials. Adding a closed D-form (in particular, a full differential) to $\boldsymbol{L}$ does not change the equations of motion. The $(D + 1)$-form

$$\omega := \boldsymbol{d}\boldsymbol{L} \qquad (\Rightarrow \boldsymbol{d}\,\omega = 0\,) \tag{10.2}$$

provides an invariant characterization of the system: the *pull-back* (the horizontal projection) of its contraction with verticle vector fields $\frac{\delta}{\delta\phi}$ reproduces the equations of motion, while the integral of ω over a $(D - 1)$-dimensional space-like (say, equal time) surface in $\mathcal{M}$ defines a symplectic form on the fields.

To give a simple example we shall first consider the case of *particle dynamics* in which space-time is reduced to the time axis ($D = 1$). We shall write the Langrangian 1-form $\boldsymbol{L}$ as the Legendre[46] transform of a Hamiltonian H:

$$\boldsymbol{L} = p\,\boldsymbol{d}\,q - H(p, q)\,dt \qquad (\,p\,\boldsymbol{d}\,q := \sum_i p_i\,\boldsymbol{d}\,q^i\,) \ , \tag{10.3}$$

$$\omega = \boldsymbol{d}\,\boldsymbol{L} = \boldsymbol{d}\,p\,\boldsymbol{d}\,q \ - \delta H(p, q)\,dt = \delta p\,\delta q + \left((\dot q - \frac{\partial H}{\partial p})\,\delta p - (\dot p + \frac{\partial H}{\partial q})\,\delta q \right) \,dt \ .$$

(Here and in what follows we omit the wedge sign $\wedge$ for exterior products.) Clearly, ω reduces for $dt = 0$ to the canonical symplectic form $\delta p\,\delta q$. Varying with respect to the vertical variables, i.e. applying (separately) $\frac{\delta}{\delta p}$ and $\frac{\delta}{\delta q}$ following the rules

$$\frac{\delta}{\delta q}\,(\,\delta p\,\delta q\,) = -\delta p\,\frac{\delta}{\delta q}\,\delta q = -\delta p \ , \qquad \frac{\delta}{\delta p}\,(\,\delta p\,\delta q\,) = \delta q \qquad \text{etc.,} \tag{10.4}$$

and equating to zero the pull-back (i.e., the terms proportional to dt) of the corresponding 1-forms, we obtain the Hamiltonian equations of motion:

$$\dot q = \frac{\partial H}{\partial p} \ , \qquad \dot p = -\frac{\partial H}{\partial q} \ . \tag{10.5}$$

[46] The French mathematician Adrian-Marie Legendre (1752-1833) is also known for (among other things) his work on elliptic integrals and for the simplest spherical polynomials which carry his name.

The following exercise is designed to justify the term *covariant* Hamiltonian for a relativistic particle.

Exercise 10.1. Set

$$q = x = (x^\mu\,,\ \mu = 0,1,2,3)\,, \quad H = \frac{1}{2\lambda}\,(p^2 + m^2) \qquad (\,p^2 = \boldsymbol{p}^2 - p_0^2\,)\,.$$

(a) Derive the free particle equations of motion: $p = \lambda\,\dot{x}\,,\ \dot{p} = 0$.

(b) Vary with respect to the Lagrange multiplier λ to obtain the mass-shell constraint $p^2+m^2 = 0$.

(c) Prove that the gauge condition $\dot{x}^2 = -1$ (equivalent to setting $t = \tau$, the proper time) implies $\lambda = m$.

A closed $(D+1)$-form may exist also when there is no single valued local action. This is precisely the case with the (classical) Wess[47]-Zumino-Novikov-Witten (WZNW) *model* which we proceed to describe for the $su(2)$ current algebra.

Space-time is taken to be the 2-dimensional ($2D$) cylinder

$$\widetilde{\mathcal{M}} = \mathbb{R}\times\mathbb{S}^1 = \{x := (x^0, x^1) \equiv (t, \boldsymbol{x})\,,\ t\in\mathbb{R}\,,\ \boldsymbol{x}\in\mathbb{R}/2\pi\,\mathbb{Z}\}\,, \tag{10.6}$$

that is an infinite covering of the $2D$ compactified Minkowski space (6.17)

$$\mathcal{M} = \bar{M}_2 = \{e^{it}(\sin\boldsymbol{x}, \cos\boldsymbol{x})\,,\ t, \boldsymbol{x}\in\mathbb{R}\} = \mathbb{S}^1\times\mathbb{S}^1/\pm 1 \tag{10.7}$$

where, in the two-element group in the quotient, the non-trivial one, -1, corresponds to the equivalence $(t, \boldsymbol{x}) \sim (t+\pi, \boldsymbol{x}+\pi)$.

Remark 10.1. For $D > 2$ the cylinder $\tilde{\mathcal{M}} = \mathbb{R}\times\mathbb{S}^{D-1}$ is simply connected and represents the universal cover of $\bar{M}$ $(=\bar{M}_d)$ (6.20). For $D = 2$ the universal cover of $\mathcal{M}$ (and $\tilde{\mathcal{M}}$) is $\mathbb{R}\times\mathbb{R}$.

In the first order formalism $\mathcal{F}$ is taken to consist of a pair of (periodic in $\boldsymbol{x}$) maps (g, J) such that

$$g(x)\in SU(2)\,, \quad g(t, \boldsymbol{x}+2\pi) = g(t, \boldsymbol{x})\,, \tag{10.8}$$

$$J(x) = j_\mu(x)\,dx^\mu\,, \quad j_\mu(x)\in su(2)\,, \quad j_\mu(t, \boldsymbol{x}+2\pi) = j_\mu(t, \boldsymbol{x})\,. \tag{10.9}$$

(Note that the $su(2)$-valued 1-form $J(x)$ is horizontal.) The basic 3-form ω is defined by

$$4\pi\,\omega = \boldsymbol{d}\,\mathrm{tr}\left(\left(i\,g^{-1}\,\boldsymbol{d}\,g + \frac{1}{2k}\,J\right)\ {}^*J\right) + k\,\theta(g)\,, \tag{10.10}$$

where ${}^*J$ is the Hodge[48] dual to J,

$${}^*J(x) = j^0(x)\,dx^1 - j^1(x)\,dx^0 \equiv \epsilon_{\mu\nu}\,j^\mu(x)\,dx^\nu\,, \tag{10.11}$$

while the *Wess-Zumino* (WZ) *form*

$$\theta(g) = \frac{1}{3}\,\mathrm{tr}(g^{-1}\,\boldsymbol{d}g)^3 \qquad (\,(g^{-1}\boldsymbol{d}g)^3 \equiv g^{-1}\boldsymbol{d}g\wedge g^{-1}\boldsymbol{d}g\wedge g^{-1}\boldsymbol{d}g\,) \tag{10.12}$$

[47] Julius Wess (1934-2007), Austrian physicist, a student of Hans Thirring (1888-1976).

[48] The Scottish mathematician William V.D. Hodge (1903-1975) discovered topological relations between algebraic and differential geometry. (See M. Atiyah, *William Valance Douglas Hodge*, Bull. London Math. Soc. **9** (1) (1977) 99-118.)

coincides, if restricted to vertical directions, with the canonical 3-form on the group manifold G, which we shall display shortly.

The trace tr in (10.10), (10.12) can serve to define the Killing form on the (abstract) Lie algebra (independent of its matrix realization – see Appendix F). This is true for $\mathcal{G} = su(n)$ (in particular, for $n = 2$, the case of chief interest). Let $\{T_a\ ,\ a = 1, \ldots, n^2 - 1\}$ be a *hermitean basis* in $\sqrt{-1}\, su(n)$ (the physicists' convention) and let $\pi_f(T_a)$ be the corresponding $n \times n$ hermitean traceless matrices in the *fundamental representation* of $su(n)$. Then the *Killing metric*

$$\eta_{ab} \equiv (T_a, T_b) = \mathrm{tr}(\pi_f(T_a)\,\pi_f(T_b))\ (= \eta_{ba}) \tag{10.13}$$

is *positive definite*. (This last statement remains true for any compact Lie group.) Let further ${f_{ab}}^c$ be the (real) $su(n)$ structure constants in the basis $\{T_a\}$ (cf. (B.6)):

$$\frac{1}{i}\,[T_a, T_b] = {f_{ab}}^c\, T_c\ . \tag{10.14}$$

(Note that $\frac{1}{i}[\ ,\ ]$ is a natural operation, being a deformation of the Poisson brackets.) The cyclicity of the trace then implies that the tensor

$$f_{abc} := \frac{1}{i}\,\mathrm{tr}([T_a, T_b]\, T_c) = {f_{ab}}^s\, \eta_{sc} \tag{10.15}$$

is totally antisymmetric. Expanding the $\sqrt{-1}\,\mathcal{G}$-valued 1-form $i\, g^{-1}\, \delta g$ in the basis $\{T_a\}$,

$$i\, g^{-1}\, \delta g = Y^a\, T_a \qquad (\Rightarrow\ \delta\, Y^c = -\frac{1}{2}\, Y^a\, Y^b\, {f_{ab}}^c)\,, \tag{10.16}$$

we can write the canonical 3-form on G (for g independent of x) as

$$\theta(g) = \frac{1}{3}\,\mathrm{tr}(g^{-1}\,\delta g)^3 = -\frac{1}{3!}\, f_{abc}\, Y^a\, Y^b\, Y^c\,. \tag{10.17}$$

In order to make clear the meaning of Eqs. (10.16) and (10.17) we shall work out the simplest case, $n = 2$, viewing $G = SU(2)$ (whose group manifold coincides with the 3-sphere $\mathbb{S}^3$) as the group of unit quaternions,

$$g = \sum_{\alpha=1}^{4} \xi^\alpha\, Q_\alpha^\dagger \qquad \text{for} \quad \xi^2 := \sum_{\alpha=1}^{4} (\xi^\alpha)^2 = 1\ , \quad \xi^\alpha \in \mathbb{R}\ . \tag{10.18}$$

Here $Q_4 = 1$ and $Q_a\ ,\ a = 1, 2, 3$ are the imaginary quaternion units, satisfying

$$Q_1 Q_2 = -\, Q_2 Q_1 = Q_3 \quad \text{etc.,} \quad Q_a^2 = -1\ , \quad Q_a^\dagger = -Q_a\ . \tag{10.19}$$

The defining representation of $SU(2)$ corresponds to setting

$$Q_a = -i\,\sigma_a\ , \quad a = 1, 2, 3\ , \qquad Q_4 = \mathbb{I}\ . \tag{10.20}$$

Exercise 10.2. Derive the relations

$$\begin{aligned} &Q_\alpha Q_\beta^\dagger = \delta_{\alpha\beta} + i\,\sigma_{\alpha\beta}\ , \qquad \sigma_{\alpha\beta} = -\sigma_{\beta\alpha}\ , \\ &\sigma_{ab} = e_{abc}\,\sigma_c\ , \quad \sigma_{4a} = \sigma_a = -\sigma_{a4}\ , \quad a, b, c = 1, 2, 3 \end{aligned} \tag{10.21}$$

(e_{abc} being the totally antisymmetric Levi-Civita symbol normalized by $e_{123} = 1$) and use them to deduce

$$g^{-1} = \xi^\alpha Q_\alpha\ , \quad i g^{-1}\delta g \equiv i\,\xi^\alpha \delta\xi^\beta Q_\alpha Q_\beta^\dagger = -\xi^\alpha \delta\xi^\beta \sigma_{\alpha\beta} = Y^a(g)\sigma_a\ , \tag{10.22}$$

where

$$Y^a(g) = \xi^a \delta\xi^4 - \xi^4 \delta\xi^a - e_{abc}\, \xi^b \delta\xi^c \ . \tag{10.23}$$

Verify that $\delta Y^c(g)$ satisfies the (second) relation (10.16). (*Hint*: use that
i) for $T_a = \sigma_a$,

$$\eta_{ab} = 2\,\delta_{ab} \ , \quad f_{ab}{}^c = 2\, e_{abc} \ , \quad f_{abc} = 4\, e_{abc} \ ; \tag{10.24}$$

ii) on the tangent space to the sphere $\mathbb{S}^3$, the differentials satisfy $\sum_{\alpha=1}^4 \xi^\alpha \delta\xi^\alpha = 0$.)

Exercise 10.3. (a) Compute the exterior product of 1-forms

$$\begin{aligned} & Y^1(g)\, Y^2(g)\, Y^3(g) = \\ & = -\xi^4 \delta\xi^1 \delta\xi^2 \delta\xi^3 + \xi^1 \delta\xi^2 \delta\xi^3 \delta\xi^4 + \xi^2 \delta\xi^3 \delta\xi^1 \delta\xi^4 + \xi^3 \delta\xi^1 \delta\xi^2 \delta\xi^4 = \\ & = -\,\frac{1}{\xi^4}\, \delta\xi^1 \delta\xi^2 \delta\xi^3 \ . \end{aligned} \tag{10.25}$$

(b) Prove that the result (10.25) reproduces (up to a sign) the normalized volume form on the unit 3-sphere

$$\begin{aligned} & \delta V = \frac{1}{|\xi^4|}\, \delta\xi^1 \delta\xi^2 \delta\xi^3 = \frac{r^2}{\sqrt{1-r^2}}\, \delta r \sin\theta\, \delta\theta\, \delta\phi \ , \\ & \int_{\mathbb{S}^3} dV = 8\pi \int_0^1 \frac{r^2}{\sqrt{1-r^2}}\, dr = 4\pi \int_0^1 \sqrt{\frac{z}{1-z}}\, dz = 4\pi B(\frac{3}{2}, \frac{1}{2}) = 2\pi^2 \ . \end{aligned} \tag{10.26}$$

Exercise 10.4. The trace of the product of Lie algebra valued 1-forms $a_1 \ldots a_n$ obeys the graded cyclic property $\mathrm{tr}(a_1 a_2 \ldots a_n) = (-1)^{n-1}\, \mathrm{tr}(a_n a_1 \ldots a_{n-1})$. Deduce from here that the 3-form (10.17) is closed,

$$\delta\, \theta(g) = 0\, . \tag{10.27}$$

Statement 10.1. *The form $\theta(g)$ is not exact; properly normalized, it generates the third homology group $H_3(G) \simeq \mathbb{Z}$ of G.*

The statement follows from the fact that for any compact simple Lie group G , $H_3(G)$ coincides with the *third homotopy group* $\pi_3(G) \simeq \mathbb{Z}$ (see e.g. [134]; for a historical survey – see [41]) and is generated by a cycle represented by a specific embedding of $\mathbb{S}^3 (\simeq SU(2)) \stackrel{g}{\to} G$. All such cycles satisfy $\int_{g(\mathbb{S}^3)} \theta(g) = \lambda\, N(g)$ with $N(g) \in \mathbb{Z}$, and a generating one is characterized by $N(g) = 1$. For $G = SU(n)$, in particular, it corresponds to the standard embedding $SU(2) \subset SU(n)$; the normalization constant λ does not depend on n and can be thus calculated for $n = 2$ and g given by (10.18) (i.e. by the identity map from $SU(2)$ to $SU(2)$). The role of the 3-form $\theta(g)$ is understood from a dual (cohomological) point of view. It turns out that, computing the cohomology groups $H^k(G)$ of a simple compact Lie group G , we can restrict our attention to (two-sided) *invariant* k-forms (every invariant form on G is closed). Any such form determines (and is determined by) an ad-invariant totally antisymmetric tensor of rank k on the tangent space at the identity of G , i.e. on its Lie algebra $\mathcal{G}$. For $k = 3$ the unique, up to normalization, such tensor is f_{abc} , corresponding to the invariant (and hence, closed) form (10.17).

For $G = SU(2)$, taking into account $Y^a Y^b Y^c = e_{abc}\, Y^1 Y^2 Y^3$ and (10.24), we find

$$\theta(g) = -\frac{1}{3!}\, f_{abc}\, Y^a(g)\, Y^b(g)\, Y^c(g) = -\frac{1}{3!}\, 4\, e_{abc}\, (\, -\, e_{abc}\, \delta V\,) = 4\, \delta V \ , \tag{10.28}$$

so that the value of the integral over $\mathbb{S}^3$ is

$$(\lambda N(g) =) \int_{\mathbb{S}^3} \theta(g) = 4\, vol(\mathbb{S}^3) = 8\pi^2 \ . \tag{10.29}$$

As $N(g) = 1$, we thus obtain $\lambda = 8\pi^2$.

The parametrization (10.18) and the ensuing basis of 1-forms (10.23) is, sure, not the only one in which the above calculations can be performed. Another convenient basis is provided by the complex variable parametrization of $SU(2)$, an extension of which will be used in describing the chiral zero modes in Section 11 below. Setting $z_1 = \xi^4 + i\xi^3$, $z_2 = \xi^2 + i\xi^1$ $(|z_1|^2 + |z_2|^2 = 1)$, we can introduce a complex basis of $su(2)$-valued right invariant forms,

$$\delta g g^{-1} = \Theta\sigma_3 + \Psi\sigma_+ - \bar{\Psi}\sigma_- \qquad (\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)) \, , \tag{10.30}$$

where

$$\begin{aligned} &\Theta = \bar{z}_1\delta z_1 + \bar{z}_2\delta z_2 \, (= -\bar{\Theta}) \ , \quad \Psi = z_1\delta z_2 - z_2\delta z_1 \\ &\Rightarrow \quad \delta\Theta = \bar{\Psi}\,\Psi \ , \quad \delta\Psi = 2\,\Theta\Psi \ . \end{aligned} \tag{10.31}$$

Exercise 10.5. Demonstrate that

$$\delta V \, (= \frac{1}{12} \operatorname{tr}(\delta g g^{-1})^3 \,) \, = \frac{1}{2} \bar{\Theta} \, \delta\Theta = \rho_1 \delta\rho_1 \delta\varphi_1 \delta\varphi_2 \ , \tag{10.32}$$

where $0 \le \rho_1 \le 1, \; 0 \le \varphi_j < 2\pi$,

$$z_j = \rho_j \, e^{i\varphi_j} \ , \ j = 1, 2 \, , \qquad \rho_1^2 + \rho_2^2 = 1 \ . \tag{10.33}$$

Exercise 10.6. (a) Verify that replacing g by g^{-1} one just changes the sign of θ and hence, of its integral, so that $N(g^{-1}) = -1$.

(b) Demonstrate that for g substituted by g^2 the product of basic 1-forms becomes

$$-Y^1(g^2)\, Y^2(g^2)\, Y^3(g^2) = 8\, \xi^4 \delta\xi^1 \delta\xi^2 \delta\xi^3 \ . \tag{10.34}$$

Deduce from here that the integral of $\theta(g^2)$ is twice the integral of $\theta(g)$.

(c) Is the equation

$$\frac{1}{8\pi^2} \int_{\mathbb{S}^3} \theta(g^m) =: N(g^m) = m \tag{10.35}$$

valid for any integer m? (*Hint*: to answer (c), deduce the relation

$$\theta(gh) = \theta(g) + \theta(h) - d \operatorname{tr}(g^{-1}dg\, dh h^{-1}) \tag{10.36}$$

that is the Polyakov-Wiegmann formula [129].)

Exercise 10.7. Using the relation $dx^\mu \, dx^\nu = -\epsilon^{\mu\nu} \, dx^0 \, dx^1$ $(\epsilon^{\mu\nu} = -\epsilon_{\mu\nu})$ derive the following expressions for $J\,{}^*J$ and its exterior differential:

$$J\,{}^*J = j_\mu \, j^\mu \, dx^0 \, dx^1 (= -\,{}^*J\, J) \, , \quad \boldsymbol{d}\, J\,{}^*J = 2\, j_\mu \, \delta j^\mu \, dx^0 \, dx^1 \, . \tag{10.37}$$

Varying ω with respect to ${}^*J$ and "pulling back" (i.e. projecting on horizontal differentials) we find the *classical KZ equation* :

$$i\, g^{-1}\, dg + \frac{1}{k}\, J = 0\,. \tag{10.38}$$

To see the precise relation between (9.20) and (10.38) we set $J = \underset{\sim}{J}(z)\, dz + \underset{\sim}{\bar{J}}(\bar{z})\, d\bar{z}$ and multiply both sides of (10.38) by $k\, g(z,\bar{z})$. The effect of quantization then consists in replacing operator products with normal products and the level k with (its renormalized value) the height h.

Varying further the 3-form (10.10) with respect to g we find the second equation of motion

$$d\,{}^*J = -ik(g^{-1}\, dg)^2 = \frac{i}{k}\, J^2 \quad \Leftrightarrow \quad \partial_\mu\, j^\mu \equiv \partial_1\, j_1 - \partial_0\, j_0 = \frac{i}{k}\,[j_0, j_1]\;. \tag{10.39}$$

Taking the exterior derivative of (10.38) and comparing with (10.39) we deduce

$$d(J + {}^*J) = 0 \quad \Leftrightarrow \quad \partial_+\, j_R = 0\;, \qquad \partial_\pm = \frac{1}{2}(\partial_1 \pm \partial_0)\;; \tag{10.40}$$

the left- and right-movers' currents are given by

$$j_R = \frac{1}{2}(j^0 + j^1)\,,\; j_L = \frac{1}{2}\, g(j^1 - j^0)\, g^{-1}(= -ik(\partial_+\, g)\, g^{-1})\,,\; \partial_-\, j_L = 0\,. \tag{10.41}$$

We can also substitute the Hodge dual 1-forms $\mathcal{J}$ and ${}^*\mathcal{J}$ by the left and right current forms

$$\mathcal{J}_L = j_L(x^+)\, dx^+ = -ik(\partial_+\, g)\, g^{-1}\, dx^+\;,\; \mathcal{J}_R = j_R(x^-)\, dx^- = -ik(g^{-1}\, \partial_-\, g)\, dx^-\,. \tag{10.42}$$

We have

$$\mathcal{J} = \mathcal{J}_R + g^{-1}\, \mathcal{J}_L\, g\;, \quad {}^*\mathcal{J} = \mathcal{J}_R - g^{-1}\, \mathcal{J}_L\, g\;; \tag{10.43}$$

each of the chiral 1-forms is closed:

$$d\, \mathcal{J}_L = 0 = d\, \mathcal{J}_R\,. \tag{10.44}$$

Eqs. (10.42), (10.44) summarize the equations of motion of the WZNW model.

Our next task is to introduce a covariant Hamiltonian and the stress energy tensor. To this end we first recall the Legendre transform (10.3) in particle dynamics. If we associate $\boldsymbol{d}q$ with $i\, g^{-1}\, \boldsymbol{d}\, g$ (in other words, if we identify $i\, g^{-1}\, \partial_\mu\, g$ with the velocity on the group manifold) then the current $j^\mu(x)$ will play the role of covariant canonical momentum, and the coefficient to the space-time volume form $\frac{dx^0 dx^1}{2\pi}$ (with a minus sign) should be identified with the covariant Hamiltonian H:

$$\frac{1}{8\pi k}\, \mathrm{tr}(\mathcal{J}\, {}^*\mathcal{J}) = \frac{1}{8\pi k}\, \mathrm{tr}(j_\mu\, j^\mu)\, dx^0 dx^1 =: -H(j)\, \frac{dx^0\, dx^1}{2\pi}\,. \tag{10.45}$$

The stree energy tensor T^μ_ν is expressed in terms of H and its functional derivatives, reproducing the Sugawara formula:

$$T^\mu_\nu(x) = \mathrm{tr}\left(\frac{\delta H}{\delta j_\mu(x)}\, j_\nu(x)\right) - H\, \delta^\mu_\nu = \frac{1}{2k}\, \mathrm{tr}\left(\frac{1}{2}\, j^2(x)\, \delta^\mu_\nu - j^\mu(x)\, j_\nu(x)\right)\,. \tag{10.46}$$

This expression becomes particularly simple for the chiral components of T:

$$T_L := \frac{1}{2}\,(T^0_0 - T^1_0) = \frac{1}{8k}\, \mathrm{tr}(j^1 - j^0)^2 = \frac{1}{2k}\, \mathrm{tr}\, j_L^2\,,$$
$$T_R := \frac{1}{2}\,(T^0_0 + T^1_0) = \frac{1}{8k}\, \mathrm{tr}(j^1 + j^0)^2 = \frac{1}{2k}\, \mathrm{tr}\, j_R^2\,. \tag{10.47}$$

While the 3-form ω (10.10) is single valued on the entire group manifold G, the corresponding Lagrangian density

$$4\pi\, \boldsymbol{L} = \mathrm{tr}((ig^{-1}\,\boldsymbol{d}\,g + \frac{1}{2k}\,\boldsymbol{J})\,{}^*\mathcal{J}) + k\,\boldsymbol{d}^{-1}\,\theta(g) \tag{10.48}$$

cannot be globally defined on G since the WZ form θ, albeit closed, is not exact (Statement 10.1). Accordingly, the WZ term in the WZNW action in the second order formalism,

$$S = -\frac{k}{8\pi}\int_{\mathcal{M}} \mathrm{tr}(g^{-1}\,\partial_\mu\, g\, g^{-1}\,\partial^\mu\,\partial)\, dx^0\, dx^1 + \frac{k}{12\pi}\int_{\mathcal{M}} d^{-1}\,\mathrm{tr}(g^{-1}\,dg)^3 \tag{10.49}$$

is multivalued (the coefficient $\frac{1}{12\pi}$ being derived from (10.29)). The possible continuations of the form $\theta(g)$ from the 2D compactified Minkowski space $\mathcal{M}$ (10.8) to the (real) 3-dimensional *bulk* $\bar{\mathcal{B}}$, the closure of

$$\mathcal{B} := \{z_\alpha = \rho\, e^{it}\, u_\alpha\,,\ \alpha = 1,2\,;\ (e^{it}\, u_\alpha) \in \mathcal{M}\,,\ 0 \le \rho < 1\}\,,\ \partial\bar{\mathcal{B}} = \mathcal{M} \tag{10.50}$$

split into equivalence classes labeled by elements of the third homotopy group $\pi_3(G) \simeq \mathbb{Z}$. For integer k, the action S is only determined modulo $2\pi\,\mathbb{Z}$ (so that the exponential e^{iS} is single-valued).

The symplectic form of the model can be written in three equivalent forms:

$$\begin{aligned}
\Omega^{(2)} &= \frac{1}{4\pi}\int_{-\pi}^{\pi} dx^1\, \mathrm{tr}\,(k\, g^{-1}\, g' - i\, j^0)(g^{-1}\,\delta g)^2 + i\,\delta j^0\, g^{-1}\,\delta g) \\
&= -\frac{1}{2\pi}\int_{-\pi}^{\pi} dx^1\, \mathrm{tr}\left(i\,\delta(j_L\,\delta g\, g^{-1}) + \frac{k}{2}\,\delta g\, g^{-1}(g^{-1}\,\delta g)'\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} dx^1\, \mathrm{tr}\left(i\,\delta(j_R\, g^{-1}\,\delta g) + \frac{k}{2}\, g^{-1}\,\delta g(g^{-1}\,\delta g)'\right)
\end{aligned} \tag{10.51}$$

where $f'(x^0, x^1)$ stands for partial derivative of f in x^1.

Exercise 10.8. Use the relations

$$j^0 = 2j^R + ik\, g^{-1}\, g' = 2\, g^{-1}\, j_L\, g - ik\, g^{-1}\, g' = j_R - g^{-1}\, j_L\, g\,,$$

and

$$ik\,\mathrm{tr}(\delta g\, g^{-1}\,(\delta g\, g^{-1})') = \mathrm{tr}(\delta j_1\, g^{-1}\,\delta g)$$

to verify the equivalence of the three expressions (10.51).

The general solution of the equations of motion (10.38)–(10.41),

$$\partial_+(g^{-1}\,\partial_-\, g) = 0 \quad \Leftrightarrow \quad \partial_-((\partial_+\, g)\, g^{-1}) = 0 \tag{10.52}$$

can be written in a factorized form

$$g(x^0, x^1) \equiv g(x^+, x^-) = g_L(x^+)\, g_R^{-1}(x^-)\,, \quad x^\pm = x^1 \pm x^0 \tag{10.53}$$

(cf. (9.9)). The following result of Gawedzki [78] allows to split the symplectic form (10.51) into a left- and right-movers' part as well.

Proposition 10.1. *One can split $\Omega^{(2)}$ into a sum of two closed chiral forms which only differ in sign,*

$$\Omega^{(2)} = \Omega(g_L, M) - \Omega(g_R, M)\,,\ \Omega(g, M) = \Omega_c(g) - \frac{k}{4\pi}\,\rho(M)\,, \tag{10.54}$$

$$\Omega_c(g) = \frac{k}{4\pi}\,\mathrm{tr}\left\{\int_{-\pi}^{\pi} dx\, g^{-1}\,\delta g (g^{-1}\,\delta g)' + b^{-1}\,\delta b\,\delta M M^{-1}\right\}, \tag{10.55}$$

where

$$b := g(-\pi)\ (g = g_L \text{ or } g_R)\,, \quad M = b^{-1}\, g(\pi)\,, \quad \rho(M) = \mathrm{tr}(M_+^{-1}\,\delta M_+\, M_-^{-1}\,\delta M_-) \tag{10.56}$$

with M *independent of the chirality,* $M = b_L^{-1}\, g_L(\pi) = b_R^{-1}\, g_R(\pi)$*. Furthermore, one has*

$$\boldsymbol{d}\,\Omega_c(g) = \delta\,\Omega_c(g) = \frac{k}{4\pi}\,\theta(M)\,, \quad \delta\rho(M) = \theta(M)\,. \tag{10.57}$$

In verifying the last equation (10.57) one uses the relations

$$\mathrm{tr}(M_\pm^{-1}\,\delta M_\pm)^3 = 0\ , \quad \delta\rho(M) = \frac{1}{3}\,\mathrm{tr}(M_+^{-1}\,\delta M_+ - M_-^{-1}\,\delta M_-)^3 \quad (= \theta(M))\,. \tag{10.58}$$

Remark 10.1. Note that the 2-form $\rho(M)$ appearing in (10.56) is not globally defined on $G(\subset \tilde{G})$ since the Gauss decomposition of the type (5.30) only exists if

$$M_n^n \neq 0 \neq \det\begin{pmatrix} M_{n-1}^{n-1} & M_n^{n-1} \\ M_{n-1}^n & M_n^n \end{pmatrix} \quad \text{etc.} \tag{10.59}$$

This is in accord with our observation (Statement 10.1) that the 3-form θ, albeit closed, is not exact.

As we have seen in our survey of the axiomatic approach to the $su(2)$ current algebra model (Section 9) only the chiral components of g ($g(z)$ and $\bar{g}(\bar{z})$ in the z-picture, $g_\pm(x_\pm)$ in the present context) display a quantum group symmetry. In the canonical approach the splitting (10.53) is suggested by the fact that the chiral (left and right) currents j_C are periodic in their arguments

$$j_C(x + 2\pi) = j_C(x) \quad \text{for} \quad C = L, R \tag{10.60}$$

(thus appearing as *chiral observables*) and *Poisson commute*

$$\{j_L(x^+), j_R(y^-)\} = 0\,. \tag{10.61}$$

(Computing Poisson brackets from a given symplectic form $\Omega = \frac{1}{2}\,\Omega_{ij}\,\delta\xi^i\delta\xi^j$ amounts to inverting the skew symmetric matrix (Ω_{ij}). In the infinite dimensional case at hand this requires, in general, some work – see [78]. The trivial Poisson bracket relations (10.61) follow however simply from the splitting (10.54)–(10.56) of the form $\Omega^{(2)}$ into chiral parts and from the fact that j_L and j_R are periodic and hence commute with the monodromy M. They are also a consequence of the observation that j_L and j_R appear as Noether[49] currents for two commuting, left and right, symmetries.)

Eq. (10.61) is the classical counterpart of the *local commutativity* of observable Bose fields.

The chiral group valued fields $g_L(x^+)$ and $g_R(x^-)$ are determined by the corresponding currents and the *classical chiral KZ equations* (the chiral counterparts of (10.38)):

$$k\,\partial_+\, g_L(x^+) = i\, j_L(x^+)\, g_L(x^+)\ , \qquad k\,\partial_-\, g_R(x^-) = -i\, j_R(x^-)\, g(x^-)\ . \tag{10.62}$$

[49] (Amalie) Emmy Noether (1882-1935) became in 1919 the first woman professor at the University of Göttingen.

11 Bloch waves and zero-modes for $G = SU(n)$

11.1 Splitting of the chiral symplectic form between Bloch waves and zero modes

A sufficient condition for the periodicity in x^1 of $g(x^0, x^1)$ (10.53) is the *twisted periodicity* of its chiral components

$$g_C(x + 2\pi) = g_C(x)\, M\,, \quad M \in \tilde{G}\,, \quad C = L, R\,, \tag{11.1}$$

where $\tilde{G}$ is a (possible) extension of the group $G = SU(n)$ and the *monodromy* M is independent of the chirality $C\,(= L, R)$. We shall now demonstrate that the chiral phase space can be split into relatively simple x-dependent (hence, infinite dimensional) *Bloch waves* [10] with a diagonal monodromy and a finite dimensional symplectic submanifold [5] which exhibits a Poisson-Lie symmetry. This amounts to expanding the covariant group-valued chiral variable $g_C(x)\,(= (g^A_\alpha(x)),\ A, \alpha = 1, \dots, n)$ in terms of classical chiral vertex operators (or Bloch waves) $u(x) = (u^A_j(x))$ which are, by definition, quasiperiodic fields with diagonal monodromy:

$$g^A_\alpha(x) = u^A_j(x)\, a^j_\alpha \left(\equiv \sum_{j=1}^{n} u^A_j(x)\, a^j_\alpha \right)\,, \quad u(x + 2\pi) = u(x)\, M_p\,. \tag{11.2}$$

Here M_p is a diagonal unitary matrix that will be parametrized as

$$M_p = \exp\left(\frac{2\pi i}{k} p\!\!\!/\right)\,, \qquad (p\!\!\!/)^j_\ell = p_j\, \delta^j_\ell\,, \quad j, \ell = 1, \dots, n \tag{11.3}$$

where $p = (p_1, \dots, p_n)$ $(\,P := \sum_{i=1}^n p_i = 0\,)$ belongs to the level k (dual) positive Weyl alcove,

$$0 < p_{ij} < k \quad \text{for} \quad 1 \le i < j \le n\,. \tag{11.4}$$

Remark 11.1. As noted in the discussion of the classical KZ equation (10.38), in the *quantum* case (surveyed axiomatically in Section 9 for $\mathcal{G} = su(2)$), the level k is substituted by the *height* $h = k + n$ (the renormalized level – cf. (9.11) and Exercise 9.4; n is the dual Coxeter number of $s\ell_n$). On the other hand, p becomes a set of commuting operators which, acting on an IR of $su(n)$, give the shifted weight: $p = \Lambda + \rho$, where $\Lambda = (\lambda_1, \dots, \lambda_{n-1})$ is a dominant $su(n)$ weight and ρ is the half sum of positive roots (cf. Appendix F). In other words, in the quantum case

$$p_{i i+1} := p_i - p_{i+1} = \lambda_i + 1\,, \quad P = 0\,. \tag{11.5}$$

In particular, for $n = 2$, we have

$$p_1 = \frac{1}{2}\, p = -p_2\,, \quad p\, (= p_{12}) = 2I + 1 \quad \text{for } \mathcal{G} = su(2)\,, \tag{11.6}$$

I being the isospin ($\frac{1}{2}$ the $su(2)$ weight).

We now proceed to introducing individual symplectic forms on the infinite dimensional manifold of Bloch waves and on the zero modes' phase space $\mathcal{M}_q$, spanned by a^i_α and p_j. We shall be particularly interested in the symplectic structure on $\mathcal{M}_q$. The subsequent somewhat technical construction has a twofold motivation: it gives, at the pre-quantum level, a canonical derivation of a *dynamical* (classical) *r-matrix*; upon quantization it yields a remarkable quantum matrix algebra, studied in [88] and [70].

There is an ambiguity in splitting the chiral symplectic form $\Omega_c(g)$ into a Bloch wave and a finite dimensional part which affects the phase space properties of a^i_α. The following statement is verified by a straightforward computation.

Proposition 11.1. *For every choice of the closed* 2*-form* $\omega_q(p)$ *and* $g(x)$ *given by* (11.2), *the chiral symplectic form* $\Omega(g, M)$ (10.54) *splits into a Bloch wave form*

$$\Omega_B(u, M_p) = \Omega(u, M_p) + \omega_q(p) \ , \tag{11.7}$$
$$\Omega(u, M_p) = \frac{k}{4\pi}\,\mathrm{tr}\left\{\int_{-\pi}^{\pi} dx\, u^{-1}(x)\,\delta u(x)(u^{-1}(x)\,\delta u(x))' + b^{-1}\delta b\,\delta M_p\, M_p^{-1}\right\}$$

and a finite dimensional one,

$$\Omega_q(a, M_p) = \Omega(a, M_p) - \omega_q(p) \ , \tag{11.8}$$
$$\Omega(a, M_p) = \frac{k}{4\pi}\left\{\mathrm{tr}\,(\delta a\, a^{-1}(2\,\delta M_p\, M_p^{-1} + M_p\,\delta a\, a^{-1}\, M_p^{-1})) - \rho(a^{-1}\, M_p\, a)\right\} \ .$$

To determine the ambiguity (the "gauge freedom") in the choice of symplectic form $\Omega_q(a, M_p)$, we shall view $\mathcal{M}_q$ as a submanifold of co-dimension 2 of the $n(n+1)$ dimensional space $\mathcal{M}_q^{\mathrm{ex}}$ of all a^j_α , p_i. The constraint $P \approx 0$ (11.5) will be supplemented by a "gauge condition" fixing the determinant $D(a) := \det(a^j_\alpha)$. We shall use in what follows the relations

$$\epsilon_{i_n\ldots i_1} a^{i_n}_{\alpha_n}\ldots a^{i_1}_{\alpha_1} = D(a)\,\epsilon_{\alpha_n\ldots\alpha_1} \ , \qquad a^{i_n}_{\alpha_n}\ldots a^{i_1}_{\alpha_1}\epsilon^{\alpha_n\ldots\alpha_1} = \epsilon^{i_n\ldots i_1} D(a) \tag{11.9}$$

(summation over equal upper and lower indices is assumed, and $\epsilon_{n\ldots 1} = 1 = \epsilon^{n\ldots 1}$). The corresponding cofactor (signed minor) matrix $A = (A^\alpha_j)$ such that

$$a^i_\alpha A^\alpha_j = D(a)\,\delta^i_j \ , \quad A^\alpha_i a^i_\beta = D(a)\,\delta^\alpha_\beta \qquad \text{i.e.,} \quad (a^{-1})^\alpha_i = \frac{A^\alpha_i}{D(a)} \tag{11.10}$$

is determined from either one of the following equations:

$$a^{i_n}_{\alpha_n}\ldots \widehat{a^{i_\ell}_{\alpha_\ell}}\ldots a^{i_1}_{\alpha_1}\epsilon^{\alpha_n\ldots\alpha_\ell\ldots\alpha_1} = \epsilon^{i_n\ldots i_\ell\ldots i_1} A^{\alpha_\ell}_{i_\ell} \ ,$$
$$\epsilon_{i_n\ldots i_\ell\ldots i_1} a^{i_n}_{\alpha_n}\ldots \widehat{a^{i_\ell}_{\alpha_\ell}}\ldots a^{i_1}_{\alpha_1} = A^{\alpha_\ell}_{i_\ell}\,\epsilon_{\alpha_n\ldots\alpha_\ell\ldots\alpha_1} \tag{11.11}$$

(the hat meaning omission; note that missing indices in the left hand side, e.g. α_ℓ in the second equation, correspond to "dummy" (summation) ones in the right hand side).

We shall choose $D(a)$ to be equal to the p-dependent quantity

$$\mathcal{D}_q(p) := \prod_{i<j}\,[p_{ij}] \ , \quad p_{ij} = p_i - p_j \ , \quad [p] = \frac{q^p - q^{-p}}{q - q^{-1}} \tag{11.12}$$

with $q = q_k = e^{-\frac{i\pi}{k}}$ (instead to 1, as it looks natural for $G = SU(n)$). Due to (11.4), all $[p_{ij}] > 0$ for $i < j$ and hence, $\mathcal{D}_q(p)$ is also positive. The $su(n)$ Weyl group is simply the group of permutations of p_j. One can define its action on the matrix $a = (a^j_\alpha)$ as well as permuting the rows labelled by the upper index j. In general, $D(a)$ and $D_q(p)$ are both pseudo-invariant with respect to the corresponding action (they only change sign for odd permutations, and it is consistent to assume that the determinant $D(a)$ is also *positive* in the extended space), so that their ratio is Weyl invariant. Additional justification of the choice $D(a) \approx \mathcal{D}_q(p)$ will be given after we find the PB for the zero modes in an extended phase space.

It is convenient to expand the form $\delta a\, a^{-1}$ in the extended phase space into n^2 basic right-invariant forms Θ^j_k. Introducing the $n\times n$ Weyl matrices e^j_i satisfying

$$e^j_i\, e^\ell_k = \delta^j_k\, e^\ell_i \quad (\Leftrightarrow\ (e^j_i)^k{}_\ell = \delta^k_i\, \delta^j_\ell\,) \tag{11.13}$$

we shall write

$$-i\,\delta\, a\, a^{-1} = e^\ell_j\, \Theta^j_\ell \left(\equiv \sum_{j,\ell=1}^{n} e^\ell_j\, \Theta^j_\ell \right) \quad \Leftrightarrow \quad \Theta^j_\ell = -i\,\mathrm{tr}(e^j_\ell\, \delta\, a\, a^{-1})\,. \tag{11.14}$$

Taking further into account the Maurer[50]-Cartan equations

$$\delta(\delta\, a\, a^{-1}) = (\delta\, a\, a^{-1})^2 \qquad \Rightarrow \qquad \delta\,\Theta^j_\ell = i\,\Theta^j_s\,\Theta^s_\ell\ , \tag{11.15}$$

we can rewrite the extended form $\Omega(a, M_p)$ (11.8) as

$$\Omega^{\rm ex}(a, M_p) = \sum_{s=1}^{n} \delta\, p_s\, \Theta^s_s - \frac{k}{4\pi} \left\{ (q-\bar q) \sum_{j<\ell} [2p_{j\ell}]\, \Theta^j_\ell\, \Theta^\ell_j + \rho(a^{-1} M_p\, a) \right\}\,. \tag{11.16}$$

The second and the third term in the right hand side are not sensitive to the extension (the 2-form ρ is only restricted by (10.57), and θ is invariant, hence ρ can be left unchanged). The first (k-independent) term can be rewritten singling out the "total momentum" P (11.5):

$$\sum_{s=1}^{n} \delta\, p_s\, \Theta^s_s = \sum_{j=1}^{r} \delta\, p_{jj+1}\, \Theta^{(j)} + \delta\, P\, \Theta^{(n)}\ , \quad n = r+1\,, \tag{11.17}$$

where

$$\Theta^{(j)} = \frac{1}{n} \left\{ (n-j) \sum_{s=1}^{j} \Theta^s_s - j \sum_{s=j+1}^{n} \Theta^s_s \right\}\ , \quad \Theta^{(n)} = \frac{1}{n} \sum_{s=1}^{n} \Theta^s_s = -\frac{i}{n} \frac{\delta D(a)}{D(a)}\,. \tag{11.18}$$

Hence,

$$\Omega^{\rm ex}(a, M_p) = \Omega(a, M_p) - \frac{i}{n}\, \delta P \frac{\delta D(a)}{D(a)}\,. \tag{11.19}$$

The (arbitrary) 2-form $\omega_q(p)$ is by definition P-independent. We shall look for a closed Weyl invariant 2-form

$$\omega^{\rm ex}_q(p) = \sum_{1\le j<\ell\le n} f_{j\ell}(p)\, \delta p_j\, \delta p_\ell = \sum_{j<\ell} c_{j\ell}(p)\, \delta p_{j\ell}\, \delta P + \sum_{j<\ell<m} f_{j\ell m}(p)\, \delta p_{j\ell}\, \delta p_{\ell m} \tag{11.20}$$

where

$$n \sum_{j<\ell} c_{j\ell}(p)\, \delta p_{j\ell} = \sum_{j<\ell} f_{j\ell}(p)\, \delta p_{j\ell}\ , \quad n f_{j\ell m}(p) = f_{j\ell}(p) + f_{\ell m}(p) - f_{jm}(p) \tag{11.21}$$

such that the terms proportional to δP cancel in the difference

$$\begin{aligned} &\Omega^{\rm ex}_q(a, M_p) := \Omega^{\rm ex}(a, M_p) - \omega^{\rm ex}_q(p) = \Omega_q(a, M_p) - i\, \delta P\, \delta\chi\,, \\ &\chi := \frac{1}{n} \log \frac{D(a)}{\mathcal{D}_q(p)} \end{aligned} \tag{11.22}$$

[50]Ludwig Maurer (1859-1927) – German mathematician whose dissertation at the University of Strassburg was devoted to the theory of matrix groups.

(implying $\Omega_q^{\rm ex}(a,M_p) = \Omega_q(a,M_p)$ for $D(a) = \mathcal{D}_q(p)$). It is easy to verify that all these conditions are satisfied if

$$
\begin{aligned}
& c_{j\ell}(p) = i\,\frac{\pi}{nk}\cot\left(\frac{\pi}{k}\,p_{j\ell}\right)\ , \\
& f_{j\ell m}(p) = f(p_{j\ell}) + f(p_{\ell m}) - f(p_{jm})\ , \quad f(p) = -f(-p)\ . \qquad (11.23)
\end{aligned}
$$

In computing $c_{j\ell}(p)$ we have used (11.12) which gives

$$
\frac{\delta\mathcal{D}_q(p)}{\mathcal{D}_q(p)} = \frac{\pi}{k}\sum_{j<\ell}\cot\left(\frac{\pi}{k}\,p_{j\ell}\right)\delta p_{j\ell}\,. \qquad (11.24)
$$

As the number of terms in the two expressions (11.20) for $\omega_q^{\rm ex}(p)$ do not appear to match, we shall help the reader to display the relation between them.

Exercise 11.1. Prove (11.20), (11.21) by deriving the identities

$$
\begin{aligned}
& np_\ell = P + P_\ell\ , \quad P_\ell := \sum_s p_{\ell s}\ , \\
& \sum_{j<\ell} f_{j\ell}(p)\,\delta p_{j\ell}\delta P_\ell = n\sum_{j<\ell<m} f_{j\ell m}(p)\,\delta p_{j\ell}\delta p_{\ell m} \qquad (11.25)
\end{aligned}
$$

for $f_{j\ell}(p)$ and $f_{j\ell m}(p)$ related by (11.21).

A necessary and sufficient condition for the form $\omega_q^{\rm ex}(p)$ (11.20) to be closed is the (second Maxwell) equation for the skew-symmetric coefficients $f_{j\ell}(p) = -f_{\ell j}(p)$:

$$
\frac{\partial}{\partial p_m}f_{j\ell} + \frac{\partial}{\partial p_j}f_{\ell m} + \frac{\partial}{\partial p_\ell}f_{mj} = 0\ . \qquad (11.26)
$$

It is straightforward to verify that the Weyl invariant expressions (11.23) for the coefficients in the right hand side of the form (11.20) do satisfy this condition.

Exercise 11.2. Verify that for

$$
f_{j\ell}(p) = n\,c_{j\ell}(p) + \sum_s f_{j\ell s}(p) \qquad (11.27)
$$

where $f_{j\ell s}(p)$ is totally antisymmetric, we have

$$
\sum_{j<\ell}\sum_s f_{j\ell s}(p)\,\delta p_{j\ell} = 0 \qquad \Rightarrow \qquad \sum_{j<\ell} f_{j\ell}(p)\,\delta p_{j\ell} = n\sum_{j<\ell} c_{j\ell}(p)\,\delta p_{j\ell}\ . \qquad (11.28)
$$

Whenever the necessity arises to write down an explicit $f_{j\ell}(p)$, we shall use the special *covariant gauge*

$$
\begin{aligned}
& f(p_{j\ell}) = (\lambda+1)\,c_{j\ell}(p) = i\,(\lambda+1)\,\frac{\pi}{nk}\,\cot\left(\frac{\pi}{k}\,p_{j\ell}\right)\ , \\
& \omega_{q,\lambda}^{\rm ex}(p) = i\,\frac{\pi}{k}\sum_{j<\ell}\cot\left(\frac{\pi}{k}\,p_{j\ell}\right)\delta p_j\delta p_\ell + \lambda\,\omega_q^{(0)}(p)\ , \qquad (11.29) \\
& \omega_q^{(0)}(p) = i\,\frac{\pi}{nk}\sum_{j<\ell<m}\left(\cot\left(\frac{\pi}{k}\,p_{j\ell}\right) + \cot\left(\frac{\pi}{k}\,p_{\ell m}\right) - \cot\left(\frac{\pi}{k}\,p_{jm}\right)\right)\delta p_{j\ell}\delta p_{\ell m}\ .
\end{aligned}
$$

As we shall see in what follows, choosing $\lambda = 0$ leading to $f_{j\ell}(p) = n\, c_{j\ell}(p)$ turns out to be technically simpler.

11.2 Computing Poisson and Dirac brackets

Our next task is to derive the *Poisson bracket* (PB) relations among a^i_α and p_j inverting the symplectic form (11.22), (11.16), (11.20) and taking into account the second class constraint (in Dirac's terminology [45])

$$P \left(= \sum_{j=1}^{n} p_j \right) \approx 0 \ , \qquad \chi \left(= \frac{1}{n} \log \frac{D(a)}{\mathcal{D}_q(p)} \right) \approx 0 \ . \tag{11.30}$$

If we regard $P \approx 0$ as a natural constraint, then $\chi \approx 0$ plays the role as associated (Weyl invariant) gauge condition.

We recall that given a symplectic form Ω and a *Hamiltonian vector field* $\hat{X}_f$ obeying the defining relation $\hat{X}_f\, \Omega = \delta f$, we can compute the PB $\{f, g\}$ by setting

$$\{f, g\} = X_f\, g \qquad \left(\text{if } X = X_i \frac{\partial}{\partial x_i} \ , \ \text{then } \hat{X} = X_i \frac{\delta}{\delta x_i} \ , \ [\, \frac{\delta}{\delta x_i} \, , \, \delta x_j]_+ = \delta^i_j \right) \ . \tag{11.31}$$

As the dependence of $\Omega^{\rm ex}_q$ (11.22) on P and χ is split, the corresponding Hamiltonian vector fields are

$$\hat{X}_\chi = i\, \frac{\delta}{\delta P} \ , \quad \hat{X}_P = -i\, \frac{\delta}{\delta \chi} \quad \Rightarrow \quad \{\chi, P\} = i \ . \tag{11.32}$$

The merit of using the extended phase space $\mathcal{M}^{\rm ex}_q$ and the constraint formalism stems from the relative simplicity of $\Omega^{\rm ex}_q$ (11.22),

$$\Omega^{\rm ex}_q = \sum_{s=1}^{n} \delta p_s\, \Theta^s_s - \omega^{\rm ex}_q(p) - \frac{k}{4\pi} \left((q - \bar{q}) \sum_{j<\ell} [2\, p_{j\ell}]\, \Theta^j_\ell\, \Theta^\ell_j + \rho(a^{-1} M_p\, a) \right) \ . \tag{11.33}$$

The PB on $\mathcal{M}_q$ is reproduced by the Dirac bracket on $\mathcal{M}^{\rm ex}_q$:

$$\{f, g\}_D = \{f, g\} + \frac{1}{\{P, \chi\}} \left(\{f, P\}\{\chi, g\} - \{f, \chi\}\{P, g\} \right) \qquad \left(\frac{1}{\{P, \chi\}} = i \right) \ . \tag{11.34}$$

In fact, the second term in the right-hand side of (11.34) vanishes in most cases of interest since χ is central for the zero modes' Poisson algebra restricted to the hypersurface of the first constraint $P = 0$:

$$\{\chi, a^j_\alpha\} = 0 = \{\chi, p_{j\ell}\} \tag{11.35}$$

(as we shall verify it by a direct computation below).

In spite of the resulting simplification the computation of PB (in particular, among a^j_α) remains quite involved (see [71, 73]) and we shall replace it by studying a limiting case (with an interest of its own) and then stating without proof the general result.

The symplectic form Ω_q (11.8), as well as its extension $\Omega^{\rm ex}_q$ (11.33), can be viewed as a deformation of a simpler limit form Ω_1, respectively $\Omega^{\rm ex}_1$ (in which the deformation parameter q is set to one, corresponding to an infinite level, $k \to \infty$). To find this limit we use the fact that

$$\lim_{k \to \infty} \left\{ \frac{\pi}{k} \cot \left(\frac{\pi}{k}\, p_{j\ell} \right) \right\} = \frac{1}{p_{j\ell}} \ , \quad \lim_{k \to \infty} \left\{ \frac{k}{4\pi}\, (q - \bar{q})\, [2 p_{j\ell}] \right\} = -i p_{j\ell} \quad (\text{for } q = e^{-\frac{i\pi}{k}}) \ ,$$

$$\lim_{k\to\infty}\left\{\frac{k}{4\pi}\,\rho(a^{-1}M_p\,a)\right\}=0 \tag{11.36}$$

with the result

$$\begin{aligned}\Omega_1^{\rm ex} &= \delta\sum_{s=1}^{n}p_s\,\Theta_s^s - i\sum_{1\le j<\ell\le n}\frac{\delta p_j\,\delta p_\ell}{p_{j\ell}} - \\ &-\frac{i\lambda}{n}\sum_{1\le j<l<m\le n}\left(\frac{1}{p_{j\ell}}+\frac{1}{p_{\ell m}}-\frac{1}{p_{jm}}\right)\delta p_{j\ell}\delta p_{\ell m} = \\ &= \Omega_1+\delta P\,\Theta^{(n)}-\omega_{1,\lambda}^{\rm ex}(p)\ ,\qquad \Omega_1=\delta\sum_{j=1}^{n-1}p_{j\,j+1}\Theta^{(j)}\ .\end{aligned} \tag{11.37}$$

(The main simplification in the undeformed limit comes from the fact that $\Omega_1^{\rm ex}$ is closed, $\delta\Omega_1^{\rm ex}=0$, without any ambiguous WZ term.)

In order to display the role of the second (determinant) constraint (11.30) (without mixing it with the freedom in the choice of $\omega_q(p)$) we start with the simplest case, $n=2$, in which the gauge term $-\lambda\,\omega_q^{(0)}(p)$ vanishes identically and the form Ω_1 in (11.37) can be expressed in terms of the variables z of (10.30), (10.31) as

$$\Omega_1(p,a)=\delta\left(\frac{p}{2}(\Theta_1^1-\Theta_2^2)\right)=\frac{1}{2i}\,\delta\left(p\,\frac{\bar z\delta z-z\delta\bar z}{\bar z z}\right)\ ,\quad p\equiv p_{12}\ (>0)\,. \tag{11.38}$$

Here we have used

$$a=\begin{pmatrix}z_1 & z_2\\ -\bar z_2 & \bar z_1\end{pmatrix}\ ,\quad a^{-1}=\frac{1}{D(a)}\begin{pmatrix}\bar z_1 & -z_2\\ \bar z_2 & z_1\end{pmatrix}\ ,\quad D(a)=\bar z z:=\bar z_1 z_1+\bar z_2 z_2\ . \tag{11.39}$$

Two definitions of the symplectic structure, one given by the (unextended) symplectic form Ω_1 (11.38) on the real 4-dimensional manifold

$$\mathcal{M}_1(p)=\{p\,,\,z_\alpha\,,\,\bar z_\alpha\,;\ \bar z z=p\,\}\ ,\quad 2i\Theta^{(2)}=i\Theta_s^s=\frac{\delta p}{p} \tag{11.40}$$

and a second, starting with a 6-dimensional extended phase space $\mathcal{M}_1^{\rm ex}$ (of $p_1,p_2,z,\bar z$) with the 2-form $\Omega_1^{\rm ex}$ (11.37) and the second class constraints (11.30), yield the same PB among the dynamical variables $p=p_{12}\,,z_\alpha\,,\bar z_\beta\,$.

To begin with, the form Ω_1 on $\mathcal{M}_1(p)$ reduces to the standard *Kähler form* on $\mathbb{C}^2$:

$$\Omega_1(p,a)=-i\,\delta\bar z\,\delta z\ . \tag{11.41}$$

Given a non-degenerate skew-symmetric matrix Ω_{ij} (that defines a finite dimensional symplectic form) and its inverse

$$\Omega=\frac{1}{2}\,\Omega_{ij}\,\delta\xi^i\,\delta\xi^j\ ,\qquad \Omega^{is}\,\Omega_{sj}=\delta^i_j\ , \tag{11.42}$$

the PB are expressed in terms of the *Poisson bivector* $\mathcal{P}$:

$$\{f(\xi),g(\xi)\}=\mathcal{P}(f(\xi)\otimes g(\xi))=\Omega^{ij}\,\frac{\partial f}{\partial\xi^i}\,\frac{\partial g}{\partial\xi^j}\ . \tag{11.43}$$

Applied to the Kähler form (11.41), this gives the canonical PB (we only reproduce the nontrivial ones):

$$\begin{aligned}&\{z_\alpha,\bar z_\beta\}=i\,\delta_{\alpha\beta}\ ,\quad \alpha,\beta=1,2\ ,\\ \Rightarrow\quad &\{z_\alpha,p\}=\{z_\alpha,\bar z z\}=iz_\alpha\ ,\quad \{\bar z_\alpha,p\}=\{\bar z_\alpha,\bar z z\}=-i\bar z_\alpha\ .\end{aligned} \tag{11.44}$$

Coming to the second approach, we shall introduce a basis of right invariant vector fields

$$\frac{\delta}{\delta p_\ell} \ , \quad \hat{V}^\ell_j = i \, \mathrm{tr} \left(e^\ell_j \, a \, \frac{\delta}{\delta a} \right) = i \, a^\ell_\sigma \, \frac{\delta}{\delta a^j_\sigma} \tag{11.45}$$

dual to δp_ℓ and Θ^j_ℓ ,

$$\frac{\delta}{\delta p_\ell} \, \delta p_j = \delta^\ell_j \ , \quad \frac{\delta}{\delta p_\ell} \, \Theta^j_m = 0 \ ; \quad \hat{V}^\ell_j \, \Theta^{j'}_{\ell'} = \delta^{j'}_j \, \delta^\ell_{\ell'} \ , \quad \hat{V}^\ell_j \, \delta p_m = 0 \ , \tag{11.46}$$

that will be useful to invert $\Omega^{\rm ex}_1$ for an arbitrary n. For $n = 2$ the Poisson bivector on the 6-dimensional phase space $\mathcal{M}^{\rm ex}_1$ has the form

$$\mathcal{P} = \sum_{s=1}^{2} V^s_s \wedge \frac{\partial}{\partial p_s} + \frac{i}{p_{12}} \left(V^2_1 \wedge V^1_2 - V^1_1 \wedge V^2_2 \right) . \tag{11.47}$$

Exercise 11.3. Derive for $n = 2$, using (11.47), the PB relations

$$\{ p_j, p_\ell \} = 0 \ , \quad \{ a^j_\alpha, p_\ell \} = i \, \delta^j_\ell \, a^j_\alpha \quad (\Rightarrow \{ a^j_\alpha, p_{\ell m} \} = i (\delta^j_\ell - \delta^j_m) \, a^j_\alpha \,) \, , \tag{11.48}$$

$$\{ a_1, a_2 \} = r^{(1)}_{12} \, (p) \, a_1 \, a_2 \ , \quad \text{i.e.} \quad \{ a^j_\alpha, a^\ell_\beta \} = r^{(1)} (p)^{j\ell}_{j'\ell'} \, a^{j'}_\alpha \, a^{\ell'}_\beta \tag{11.49}$$

where the (undeformed) *classical dynamical r-matrix* is given by

$$r^{(1)} (p)^{j\ell}_{j'\ell'} = \begin{cases} \frac{i}{p_{j\ell}} \, (\delta^j_{j'} \, \delta^\ell_{\ell'} - \delta^j_{\ell'} \, \delta^\ell_{j'}) & \text{for } j \neq \ell \\ 0 & \text{for } j = \ell \end{cases} . \tag{11.50}$$

It is now easy to check, for $n = 2$, that the PB obtained in $\mathcal{M}^{\rm ex}_1$ imply the relations (11.44) derived in $\mathcal{M}_1$ for $D(a) = p$. To this end, one rewrites (11.48) and (11.49) in terms of z and $\bar{z}$ using (11.39):

$$z_\alpha = a^1_\alpha \ , \quad \bar{z}_\beta = a^2_\gamma \epsilon^{\gamma\beta} \qquad (\, \epsilon^{\alpha\beta} = \epsilon_{\alpha\beta} \, , \ \epsilon_{21} = 1 \,) \tag{11.51}$$

leading, in particular, to

$$\{ z_\alpha, p_{12} \} = i z_\alpha \ , \qquad \{ \bar{z}_\alpha, p_{12} \} = -i \bar{z}_\alpha \qquad (\ \Rightarrow \ \{ \bar{z} z, p_{12} \} = 0 \,) \ . \tag{11.52}$$

On the other hand, the nontrivial PB (11.49) reads

$$\begin{aligned} \{ a^1_\alpha, a^2_\beta \} &= \frac{i}{p_{12}} \, (a^1_\alpha a^2_\beta - a^2_\alpha a^1_\beta) = - \frac{i}{p_{12}} \, \epsilon_{\alpha\beta} D(a) \\ \Rightarrow \quad \{ z_\alpha, \bar{z}_\beta \} &= -i \, \epsilon_{\alpha\gamma} \epsilon^{\gamma\beta} \frac{D(a)}{p_{12}} = i \, \delta_{\alpha\beta} \frac{D(a)}{p_{12}} \end{aligned} \tag{11.53}$$

which coincides with the first PB of (11.44) for $D(a) = p_{12}$ (or $\chi := \frac{1}{2} \log \frac{D(a)}{p_{12}} = 0$). Using $D(a) = \epsilon^{\beta\gamma} a^2_\beta a^1_\gamma$, we obtain

$$\begin{aligned} \{ a^1_\alpha, D(a) \} &= \epsilon^{\beta\gamma} \{ a^1_\alpha, a^2_\beta \} a^1_\gamma = - \frac{i}{p_{12}} \, \epsilon_{\alpha\beta} \epsilon^{\beta\gamma} a^1_\gamma D(a) = \frac{i}{p_{12}} \, a^1_\alpha \, D(a) \ , \\ \{ a^2_\alpha, D(a) \} &= \epsilon^{\beta\gamma} \{ a^2_\alpha, a^1_\gamma \} a^2_\beta = \frac{i}{p_{12}} \, \epsilon^{\beta\gamma} \epsilon_{\gamma\alpha} \, a^2_\beta D(a) = - \frac{i}{p_{12}} \, a^2_\alpha \, D(a) \ . \end{aligned} \tag{11.54}$$

On the surface of the constraint $\chi = 0$, the last two relations reproduce the PB of $\bar{z} z$ in (11.44). Unlike $D(a)$, the constraint χ belongs to the centre of the Poisson algebra for $P = 0$, as

$$\{ p_{12}, D(a) \} = 0 = \{ p_{12}, \chi \} \qquad (\, \{ P, D(a) \} = 2 i D(a) \,) \tag{11.55}$$

and

$$\{a^i_\alpha, \frac{D(a)}{p_{12}}\} = \frac{1}{p_{12}}\{a^i_\alpha, D(a)\} - \frac{D(a)}{p_{12}^2}\{a^i_\alpha, p_{12}\} = 0 \quad \Rightarrow \quad \{a^i_\alpha, \chi\} = 0\ . \tag{11.56}$$

Identifying the quantized z with (an isodoublet of) creation operators and equating, for linear functions of z and $\bar{z}$ the commutator with i times the PB, we recover the canonical commutation relations (CCR) among the a's and thereby, the Schwinger model for $SU(2)$.

Exercise 11.4. Prove that for $D(a) = 1$ (instead of $D(a) = p$) one finds the (non-trivial) PB

$$\{z_1, \bar{z}_1\} = \frac{i}{p}\,|z_2|^2\ , \qquad \{z_2, \bar{z}_2\} = \frac{i}{p}\,|z_1|^2 \tag{11.57}$$

which disagree with the CCR.

Proceeding now to the general case ($n > 2$), we find that one should just replace the Poisson bivector (11.47) by

$$\mathcal{P} = \sum_{s=1}^{n} V^s_s \wedge \frac{\partial}{\partial p_s} + \sum_{1\le j<\ell\le n} (\frac{i}{p_{j\ell}} V^\ell_j \wedge V^j_\ell - f_{j\ell}(p)\, V^j_j \wedge V^\ell_\ell)\,. \tag{11.58}$$

We see that in the $q = 1$ case all PB are covariant under the right $SU(n)$ action; in fact, the second index (α or β) of the a's is unaffected by the PB.

This property is violated when we proceed to the q-deformed case. The PB (11.48) then remain unchanged while (11.49) is replaced by

$$\{a_1, a_2\} = r_{12}(p)\, a_1\, a_2 - \frac{\pi}{k}\, a_1\, a_2\, r_{12}\ . \tag{11.59}$$

Here $r_{12}(p)$ is a solution of the (modified) *classical dynamical Yang-Baxter equation* (YBE)

$$\begin{aligned}&[r_{12}(p), r_{13}(p)] + [r_{12}(p), r_{23}(p)] + [r_{13}(p), r_{23}(p)] + \mathrm{Alt}\,(dr(p)) = \frac{\pi^2}{k^2}\,[C_{12}, C_{23}]\ ,\\ &\mathrm{Alt}\,(dr(p)) := -i\sum_{s=1}^{n}\frac{\partial}{\partial p_s}\,(\,(e^s_s)_1 r_{23}(p) - (e^s_s)_2 r_{13}(p) + (e^s_s)_3 r_{12}(p)\,)\end{aligned} \tag{11.60}$$

(see [53]). It is obtained from (11.50) by substituting $\frac{1}{p_{j\ell}}$ by $\frac{\pi}{k}\cot\left(\frac{\pi}{k}\,p_{j\ell}\right)$ and taking into account the gauge freedom in choosing $\omega^{\rm ex}_q(p)$ (11.20),

$$r\,(p)^{j\ell}_{j'\ell'} = \begin{cases} f_{j\ell}(p)\,\delta^j_{j'}\,\delta^\ell_{\ell'} - i\,\frac{\pi}{k}\cot\left(\frac{\pi}{k}\,p_{j\ell}\right)\delta^j_{\ell'}\,\delta^\ell_{j'} & \text{for } j\ne\ell\\ 0 & \text{for } j=\ell\end{cases} \tag{11.61}$$

(as we shall see, the choice $\lambda = 0$ in (11.29) leading to $f_{j\ell}(p) = i\,\frac{\pi}{k}\cot\left(\frac{\pi}{k}\,p_{j\ell}\right)$ gives some technical advantages in the quantization procedure), while r_{12} is a solution of the modified *classical YBE*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = [C_{12}, C_{23}] \tag{11.62}$$

(cf. [137]) where the polarized Casimir operator $C_{12}\,(= C_{21}) = \eta^{ab}\,T_{a1}\,T_{b2}$, see (B.5), is an element of the symmetric tensor product $\mathrm{Sym}\,(\mathcal{G}\otimes\mathcal{G})$ and the Killing metric η_{ab} ($\eta^{as}\eta_{sb} = \delta^a_b$) is defined in (10.13). The essential difference between (11.60) and (11.62) is in the term $\mathrm{Alt}\,(dr(p))$ containing derivatives in the dynamical variables p_s .

Exercise 11.5. Verify that if r_{12} satisfies the modified equation (11.62), then

$$r^{\epsilon}_{12} = r_{12} + \epsilon\, C_{12}\ , \quad \epsilon = \pm \tag{11.63}$$

satisfy the classical YBE

$$[r^{\epsilon}_{12}, r^{\epsilon}_{13}] + [r^{\epsilon}_{12}, r^{\epsilon}_{23}] + [r^{\epsilon}_{13}, r^{\epsilon}_{23}] = 0\,. \tag{11.64}$$

(*Hint*: Use the invariance of C_{12} (B.5) to show that the sum of the mixed terms vanishes.)

There are many solutions of Eq. (11.62) (respectively, of the classical YBE (11.64)) corresponding to different (local) choices of $\rho(M)$ satisfying (10.58). We shall use the *standard r-matrix* that corresponds to $\rho(M) = \mathrm{tr}(M_+^{-1}\,\delta M_+\, M_-^{-1}\,\delta M_-)$ (10.56):

$$r_{12} = \sum_{i<j} ((e^j_i)_1\,(e^i_j)_2 - (e^i_j)_1\,(e^j_i)_2)\ , \qquad \text{i.e.}$$

$$r^{\alpha'\beta'}_{\alpha\beta} = \epsilon_{\alpha\beta}\,\delta^{\alpha'}_{\beta}\,\delta^{\beta'}_{\alpha}\ , \quad \epsilon_{\alpha\beta} = \begin{cases} -1\,, & \alpha<\beta \\ \ \ 0\,, & \alpha=\beta \\ \ \ 1\,, & \alpha>\beta \end{cases}\ . \tag{11.65}$$

Exercise 11.6. Show that the polarized Casimir for $s\ell_n$ is expressed in terms of the permutation operator P_{12} as

$$C^{s\ell_n}_{12} = P_{12} - \frac{1}{n}\,\mathbb{I}_{12}\ , \quad P_{12} = \sum_{i,j=1}^{n} (e^j_i)_1 (e^i_j)_2\ , \quad \mathbb{I}_{12} = \mathbb{I}_1\,\mathbb{I}_2 = \sum_{i,j=1}^{n} (e^i_i)_1 (e^j_j)_2\ . \tag{11.66}$$

In order to prove that (11.35) take place, one derives from (11.59), (11.61), (11.65)

$$\begin{aligned} \{a^j_\beta, a^n_{\alpha_n}\ldots a^1_{\alpha_1}\} &= i\frac{\pi}{k}\sum_{\ell\neq j}\cot\left(\frac{\pi}{k}p_{j\ell}\right)\,a^j_\beta\,a^n_{\alpha_n}\ldots\,a^\ell_{\alpha_\ell}\,\ldots a^1_{\alpha_1} - \\ -i\frac{\pi}{k}&\sum_{\ell\neq j}\cot\left(\frac{\pi}{k}p_{j\ell}\right)\,a^\ell_\beta a^j_{\alpha_\ell}\,a^n_{\alpha_n}\ldots\,\widehat{a^\ell_{\alpha_\ell}}\,\ldots a^1_{\alpha_1} - \\ -\frac{\pi}{k}&\sum_{\ell}\epsilon_{\beta\alpha_\ell}\,a^\ell_\beta a^j_{\alpha_\ell}\,a^n_{\alpha_n}\ldots\,\widehat{a^\ell_{\alpha_\ell}}\,\ldots a^1_{\alpha_1}\ . \end{aligned} \tag{11.67}$$

The second and the third terms in (11.67) vanish when multiplied by $\epsilon^{\alpha_n\ldots\alpha_\ell\ldots\alpha_1}$ and summed over repeated indices, due to

$$\begin{aligned} &\sum_{\ell\neq j}\cot\left(\frac{\pi}{k}p_{j\ell}\right)\,a^\ell_\beta a^j_{\alpha_\ell}\,a^n_{\alpha_n}\ldots\,\widehat{a^\ell_{\alpha_\ell}}\,\ldots a^1_{\alpha_1}\,\epsilon^{\alpha_n\ldots\alpha_\ell\ldots\alpha_1} = \\ &= \sum_{\ell\neq j}\cot\left(\frac{\pi}{k}p_{j\ell}\right)\,a^\ell_\beta\,a^j_{\alpha_\ell}A^{\alpha_\ell}_\ell = \sum_{\ell\neq j}\cot\left(\frac{\pi}{k}p_{j\ell}\right)\,a^\ell_\beta\,D(a)\,\delta^j_\ell = 0\ , \end{aligned} \tag{11.68}$$

$$\begin{aligned} &\sum_{\ell}\epsilon_{\beta\alpha_\ell}\,a^\ell_\beta\,a^j_{\alpha_\ell}\,a^n_{\alpha_n}\ldots\,\widehat{a^\ell_{\alpha_\ell}}\,\ldots a^1_{\alpha_1}\,\epsilon^{\alpha_n\ldots\alpha_\ell\ldots\alpha_1} = \\ &= \sum_{\ell}\epsilon_{\beta\alpha_\ell}\,a^j_{\alpha_\ell}\,A^{\alpha_\ell}_\ell\,a^\ell_\beta = \epsilon_{\beta\alpha_\ell}\,a^j_{\alpha_\ell}\,D(a)\,\delta^{\alpha_\ell}_\beta = 0\ . \end{aligned} \tag{11.69}$$

Hence, (cf. (11.9))

$$\{a^j_\beta, \log D(a)\} = \frac{1}{D(a)}\,\{a^j_\beta, D(a)\} = i\frac{\pi}{k}\sum_{\ell\neq j}\cot\left(\frac{\pi}{k}p_{j\ell}\right)\,a^j_\beta\ . \tag{11.70}$$

On the other hand, the PB (11.48) imply

$$\{D(a), p_\ell\} = i\, D(a) \quad \Rightarrow \quad \{D(a), p_{j\ell}\} = 0 \quad \Rightarrow \quad \{\chi, p_{j\ell}\} = 0\,, \tag{11.71}$$

as well as

$$\{a^j_\beta, U(p)\} = \{a^j_\beta, p_\ell\} \frac{\partial U}{\partial p_\ell}(p) = i \frac{\partial U}{\partial p_j}(p)\, a^j_\beta \tag{11.72}$$

(there is *no* summation in j). In particular, the calculation of the PB (11.72) for $U(p) = \log D_q(p) = \sum_{i<\ell} \log[p_{i\ell}]$, gives the same result as (11.70),

$$\{a^j_\alpha,\, \log D_q(p)\} = i\frac{\pi}{k} \left(\sum_{\ell>j} \cot\left(\frac{\pi}{k} p_{j\ell}\right) - \sum_{\ell<j} \cot\left(\frac{\pi}{k} p_{\ell j}\right) \right) a^j_\alpha =$$
$$= i\frac{\pi}{k} \sum_{\ell\neq j} \cot\left(\frac{\pi}{k} p_{j\ell}\right) a^j_\alpha \ ,$$

implying $\{\chi, a^j_\alpha\} = 0$.

The passage to the $(n+2)(n-1)$-dimensional (unextended) phase space $\mathcal{M}_q$ using the Dirac brackets (11.34) is straightforward; only the second PB (11.48) is changing:

$$\{a^j_\alpha, p_\ell\}_D = i\, a^j_\alpha \left(\delta^j_\ell - \frac{1}{n}\right) \quad \Rightarrow \quad \{a^j_\alpha, p_{\ell m}\} = i(\delta^j_\ell - \delta^j_m)\, a^j_\alpha\,. \tag{11.73}$$

The classical counterpart of the quantum group transformation law for a^i_α is expressed simply in terms of r^ϵ:

$$\{M_{\epsilon 1}, a_2\} = \frac{\pi}{k}\, a_2\, r^\epsilon_{12}\, M_{\epsilon 1} \ , \quad \epsilon = \pm \ . \tag{11.74}$$

11.3 PB for Bloch waves

To find the PB for the Bloch waves $u(x)$, we need to invert the symplectic form (11.7). To this end, we shall introduce *loop group* (periodic) variables

$$\ell(x) = u(x)\, e^{-i\frac{\not p}{k}x} \quad (\not p = \sum_{s=1}^n p_s e^s_s\,) \ , \qquad \ell(x+2\pi) = \ell(x) \tag{11.75}$$

(the exponential factor compensating the nontrivial monodromy (11.3) of $u(x)$), in terms of which the left invariant, matrix valued Bloch waves' 1-forms are expressed as

$$u^{-1}(x)\, \delta u(x) \ = \ e^{-i\frac{\not p}{k}x}\, \ell^{-1}(x)\, \delta\ell(x)\, e^{i\frac{\not p}{k}x} + i\frac{\delta \not p}{k} x \ . \tag{11.76}$$

The mode expansion of the periodic matrix valued 1-forms

$$-ik\, \ell^{-1}(x)\, \delta\ell(x) = \sum_{m\in\mathbb{Z}} \Xi_m\, e^{-imx} \ , \qquad \Xi_m = \sum_{i,j=1}^n (\Xi_m)^i_j\, e^j_i \tag{11.77}$$

allows to write the extended symplectic form simply as

$$\Omega^{\rm ex}(u, M_p) = \frac{1}{k}\, {\rm tr}\, \{\delta(\not p\, \Xi_0) + i \sum_{m=1}^{\infty} m\, \Xi_{-m}\Xi_m\} =$$
$$= \frac{1}{k} \sum_{s=1}^{n} \delta p_s\, (\Xi_0)^s_s + + \frac{i}{2k} \sum_{m\in\mathbb{Z}} \sum_{i,j=1}^{n} (m + \frac{p_{ij}}{k})\, (\Xi_{-m})^j_i (\Xi_m)^i_j \equiv$$
$$\equiv \frac{1}{k} \sum_{s=1}^{n} \delta p_s\, (\Xi_0)^s_s + \frac{i}{k} \sum_{m=1}^{\infty} \sum_{s=1}^{n} m (\Xi_{-m})^s_s (\Xi_m)^s_s +$$
$$+ \frac{i}{k} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (m + \frac{p_{ij}}{k})\, (\Xi_{-m})^j_i (\Xi_m)^i_j \ . \qquad (11.78)$$

(As p belongs to the interior of the positive level k Weyl alcove, $0 < \frac{p_{ij}}{k} < 1$ for $1 \le i < j \le n$.)

Exercise 11.7.
(a) From $\delta(\ell^{-1}\delta\ell) = -(\ell^{-1}\delta\ell)^2$, deduce that

$$\delta\, \Xi_n = \frac{1}{ik} \sum_m \Xi_{n-m}\Xi_m \qquad \Rightarrow \qquad \delta\, \Xi_0 = \frac{1}{ik} \sum_m \Xi_{-m}\Xi_m \ . \qquad (11.79)$$

(b) Prove that

$$[\not p\, ,\, e^j_i\,] = p_{ij}\, e^j_i \ , \qquad e^{i\frac{\not p}{k}x}\, e^j_i = e^{i\frac{p_{ij}}{k}x}\, e^j_i\, e^{i\frac{\not p}{k}x} \ . \qquad (11.80)$$

Exercise 11.8. Derive (11.78) from (11.7). *Hint*: Use the relation

$$\mathrm{tr}\, \{ \Big(\ell^{-1}(-\pi)\delta\ell(-\pi) - \int_{-\pi}^{\pi} \frac{dx}{2\pi}\, x\, (\ell^{-1}(x)\delta\ell(x))' \Big)\, \delta\not p\} =$$
$$= \mathrm{tr}\, \{ \int_{-\pi}^{\pi} \frac{dx}{2\pi}\, \ell^{-1}(x)\, \delta\ell(x)\, \delta\not p\, \} = \frac{1}{ik}\, \mathrm{tr}\, (\delta\not p\, \Xi_0) \ . \qquad (11.81)$$

The form $\Omega^{\rm ex}(u, M_p)$ (11.78) can be readily inverted in terms of the vector fields $(\hat V^m)^j_i$, $\frac{\hat\delta}{\delta p_\ell}$ whose only non-trivial contractions are

$$(\hat V^\ell)^j_i (\Xi_m)^r_s = \delta^\ell_m \delta^r_i \delta^j_s \ , \qquad \frac{\delta}{\delta p_\ell}\, \delta p_m = \delta^\ell_m \qquad (11.82)$$

(i.e., $(\hat V^m)^j_i$, $\frac{\delta}{\delta p_\ell}$ are dual to the 1-forms $(\Xi_m)^i_j$, δp_ℓ , respectively): the corresponding Poisson bivector is

$$\mathcal{P} = -k \sum_{\ell=1}^{n} \frac{\partial}{\partial p_\ell} \wedge (V^0)^\ell_\ell + \frac{ik}{2} \sum_{m\ne 0} \sum_{\ell=1}^{n} \frac{1}{m} (V^{-m})^\ell_\ell \wedge (V^m)^\ell_\ell +$$
$$+ \frac{ik}{2} \sum_m \sum_{1\le i\ne j\le n} \frac{1}{m + \frac{p_{ij}}{k}} (V^{-m})^i_j \wedge (V^m)^j_i \ . \qquad (11.83)$$

Finally, we obtain from Eq.(11.76) the contractions with $\delta u(x)$:

$$(\hat V^m)^j_i\, \delta u(x) = \frac{i}{k}\, u(x)\, e^j_i\, e^{-i(m+\frac{p_{ij}}{k})x} \ , \quad \frac{\delta}{\delta p_\ell}\, \delta u(x) = \frac{i}{k}\, x\, u(x) e^\ell_\ell \ . \qquad (11.84)$$

This gives (trivially) $\{p_\ell\ , p_m\} = 0\,,\ \ \ell, m = 1,\dots,n\,,$ and $\{u(x)\,,\, p_\ell\} = i\, u(x)\, e^\ell_\ell$ so that, by (11.3), (11.2),

$$\{u^A_j(x)\,,\, p_\ell\} = i\, u^A_j(x)\delta_{j\ell} \quad \Rightarrow \quad \{(M_p)^\ell_\ell\,,\, u^A_j(x)\} = \frac{2\pi}{k} u^A_j(x+2\pi)\delta_{j\ell} \tag{11.85}$$

The PB of two Bloch wave fields, on the other hand, is quadratic,

$$\{u_1(x_1)\,,\, u_2(x_2)\} \equiv \mathcal{P}\,(u(x_1)\otimes u(x_2)) = \tag{11.86}$$

$$= \frac{\pi}{k}\, u_1(x_1)u_2(x_2)\left(\varepsilon(x_{12})\sum_\ell (e^\ell_\ell)_1(e^\ell_\ell)_2 + \frac{1}{i\pi}\sum_{i\neq j}\sum_{m\in\mathbb{Z}} \frac{e^{i(m+\frac{p_{ij}}{k})x_{12}}}{m+\frac{p_{ij}}{k}}(e^i_j)_1(e^j_i)_2\right) =$$

$$= \frac{\pi}{k}\, u_1(x_1)u_2(x_2)\left(\varepsilon(x_{12})\sum_\ell (e^\ell_\ell)_1(e^\ell_\ell)_2 + \sum_{i\neq j} \varepsilon_{\frac{p_{ij}}{k}}(x_{12})\,(e^i_j)_1(e^j_i)_2\right) -$$

$$- \, u_1(x_1)u_2(x_2)\, r^{(s)}_{12}(p)\ .$$

Here the classical dynamical r-matrix $r^{(s)}_{12}(p)$ – the superscript $^{(s)}$ staying for "simple" (Lie algebra) – which appeared for the first time in the present context in [14], coincides with the off-diagonal part of (11.61),

$$r^{(s)}_{12}(p) = -i\frac{\pi}{k}\sum_{j\neq\ell}\cot\,(\frac{\pi}{k}p_{j\ell})(e^\ell_j)_1(e^j_\ell)_2\ , \qquad \text{i.e.}$$

$$r^{(s)}(p)^{j\ell}_{j'\ell'} = \begin{cases} -i\,\frac{\pi}{k}\cot\left(\frac{\pi}{k}\,p_{j\ell}\right)\delta^j_{\ell'}\,\delta^\ell_{j'} & \text{for } j\neq\ell \\ \qquad 0 & \text{for } j=\ell \end{cases}\ , \tag{11.87}$$

and the (discontinuous) functions $\varepsilon(x)\,,\ \varepsilon_z(x)$ are given by the series

$$\varepsilon(x) := \frac{1}{i\pi}\sum_{m\neq 0}\frac{e^{imx}}{m} + \frac{x}{\pi} = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\sin mx}{m} + \frac{x}{\pi}\ , \tag{11.88}$$

$$\varepsilon_z(x) := \frac{1}{i\pi}\sum_m \frac{e^{i(m+z)x}-1}{m+z} \qquad (\, z\notin\mathbb{Z}\,)\ , \tag{11.89}$$

respectively. Both of them have to be regarded as functionals; the first is just a quasiperiodic generalization of the sign function $sgn(x)$,

$$\begin{aligned} &\varepsilon(x+2\pi N) = \varepsilon(x) + 2N \quad (\,N\in\mathbb{Z}\,)\ , \qquad \varepsilon(0) = 0\ ,\\ &\varepsilon(x) = sgn(x) \quad \text{for} \quad -2\pi < x < 2\pi\ , \end{aligned} \tag{11.90}$$

and its derivative is twice the *periodic* δ-function

$$\delta_{per}(x) := \frac{1}{2\pi}\sum_m e^{imx} \equiv \sum_m \delta(x+2\pi m)\ . \tag{11.91}$$

The properties of $\varepsilon_z(x)$ follow from the Euler formula for $\cot(\pi z)$, yielding (for $x\in\mathbb{R}\,,\ z\notin\mathbb{Z}$)

$$\lim_{N\to\infty}\frac{1}{\pi}\sum_{m=-N}^{N}\frac{e^{i(m+z)x}}{m+z} \;=\; \cot(\pi z) + i\,\varepsilon_z(x)\ , \qquad \varepsilon_z(0) = 0\ . \tag{11.92}$$

The derivative of $\varepsilon_z(x)$ in x is, on the other hand, proportional to a twisted version of the periodic δ-function,

$$\frac{1}{2}\frac{\partial}{\partial x}\,\varepsilon_z(x) = e^{izx}\,\delta_{per}(x)\ , \tag{11.93}$$

which implies that for $-2\pi < x < 2\pi$, $\varepsilon_z(x) = sgn(x) = \varepsilon(x)$. Looking at (11.86), one concludes that for $-\pi < x_1, x_2 < \pi$ the two terms containing $\varepsilon(x)$ and $\varepsilon_z(x)$ combine to the sign function times the permutation matrix, cf. (11.66):

$$\{u_1(x_1)\,,\,u_2(x_2)\} = u_1(x_1)u_2(x_2)\,(\frac{\pi}{k}\,sgn(x_{12})P_{12} - r_{12}^{(s)}(p)\,) \quad \text{for} \quad -\pi < x_1, x_2 < \pi\ . \tag{11.94}$$

Due to the quasiperiodicity property of $u(x)$ (11.2), restricting the values of both arguments to intervals of length 2π still allows one to reconstruct the PB for general x_1 and x_2 – one has to use the expression for the diagonal monodromy M_p (11.3) and then the PB (11.85) of $u(x)$ with p_ℓ . Written in components, the PB (11.86) reads simply

$$\{u_j^A(x_1)\,,\,u_\ell^B(x_2)\} = \frac{\pi}{k}\,u_\ell^A(x_1)\,u_j^B(x_2)\begin{cases} \varepsilon(x_{12}) & ,\, j = \ell \\ \frac{1}{i\pi}\sum_m \frac{e^{i(m+\frac{p_{j\ell}}{k})x_{12}}}{m+\frac{p_{j\ell}}{k}} & ,\, j \neq \ell \end{cases}\ . \tag{11.95}$$

Exercise 11.9. Prove that, for any $x_1, x_2 \in \mathbb{R}$,

$$\{u_1(x_1+2\pi)\,,\,u_2(x_2)\} = \{(u(x_1)M_p)_1\,,\,u_2(x_2)\}\ .$$

Hint: Use (11.95), (11.85), the quasiperiodicity of $\varepsilon(x)$ (11.90) and the "twisted periodicity" property

$$\sum_m \frac{e^{i(m+z)(x+2\pi)}}{m+z} = e^{2\pi i z}\sum_m \frac{e^{i(m+z)x}}{m+z} \qquad (\,z \notin \mathbb{Z}\,)\ .$$

So far we have dealt with the extension of the symplectic form $\Omega(u, M_p)$ (11.7). To take into account the impact of adding $\omega^{\rm ex}(p)$ and hence, obtain the PB corresponding to $\Omega_B^{\rm ex}(u, M_p)$, one can perform the following trick (cf. [15]). One can easily prove that a redefinition $u(x)$ with a p-dependent *diagonal* matrix (which does not change, hence, the monodromy matrix M_p),

$$\tilde{u}(x) = u(x)\,e^{V(p)}\,, \quad V(p) = \sum_s V^s(p)\,e_s^s \quad (\Rightarrow\ \det\tilde{u}(x) = \det u(x)\,e^{\sum_s V^s(p)}\,) \tag{11.96}$$

results in the following change of the symplectic form:

$$\Omega^{\rm ex}(\tilde{u}, M_p) = \Omega^{\rm ex}(u, M_p) - i\sum_{j<\ell}\left(\frac{\partial V^j}{\partial p_\ell} - \frac{\partial V^\ell}{\partial p_j}\right)\delta p_j\,\delta p_\ell\ . \tag{11.97}$$

Choosing

$$V^s(p) = \log\prod_{r=1}^{s-1}[p_{rs}] \qquad (\,[p_{rs}] = \frac{\sin\frac{\pi}{k}p_{rs}}{\sin\frac{\pi}{k}} > 0 \quad \text{for} \quad r < s\,) \tag{11.98}$$

(so that $\det\tilde{u}(x) = \mathcal{D}_q(p)\,\det u(x)$, cf. (11.96), (11.12)), we find

$$\frac{\partial V^j}{\partial p_\ell} - \frac{\partial V^\ell}{\partial p_j} = \frac{\pi}{k}\sum_{r=1}^{j-1}\cot(\frac{\pi}{k}p_{rj})\,\delta_\ell^r - \frac{\pi}{k}\sum_{r=1}^{\ell-1}\cot(\frac{\pi}{k}p_{r\ell})\,\delta_j^r = -\frac{\pi}{k}\cot(\frac{\pi}{k}p_{j\ell}) \tag{11.99}$$

(only a single term in one of the sums can be nonzero). Thus, the additional term in (11.97) equals (11.20), (11.29) (for $\lambda = 0$), i.e. the 2-form $\omega^{\rm ex}(p)$ is absorbed by the rescaling (11.96) of u with $V(p)$ given by (11.98):

$$\Omega_B^{\rm ex}(u, M_p) = \Omega^{\rm ex}(\tilde{u}, M_p)\ , \tag{11.100}$$

Conversely, we can calculate the PB following from the symplectic form $\Omega_B^{\rm ex}(u, M_p)$ by assuming that $\tilde u$ satisfies (11.85), (11.86) and $u(x) = \tilde u(x) e^{-V(p)}$. One finds that the dynamical r-matrix $r_{12}^{(s)}(p)$ (11.87) (for $\tilde u$) gets in addition exactly the missing diagonal elements that make it different from $r_{12}(p)$ (11.61) (for $\lambda = 0$):

$$r_{12}(p) = i\frac{\pi}{k} \sum_{j\neq\ell} \cot(\frac{\pi}{k} p_{j\ell}) \, ((e_j^j)_1 \, (e_\ell^\ell)_2 - (e_j^\ell)_1 \, (e_\ell^j)_2) \ . \tag{11.101}$$

To conclude the classical description of the Bloch waves, it remains to impose the relevant constraints and thus obtain the Dirac brackets from the PB in the extended phase space. In order to maintain the unimodularity of the group-valued chiral field $g(x)$, it follows from its decomposition into Bloch waves and zero modes (11.2) and from $\det(a^j_\alpha) \equiv D(a) = \mathcal{D}_q(p)$ (11.12) that we should require

$$\mathcal{D}_q(p) \, \det(u_j^A(x)) = 1 \ . \tag{11.102}$$

Due to the x-dependence, this amounts to imposing an infinite number of constraints. We shall not go into the details of the calculation, the result of which is simple and easy to describe. The modification of the first relation in (11.85) is quite similar to that in the zero modes case, (11.73):

$$\{u_j^A(x)\, ,\, p_\ell\}_D = i\, u_j^A(x) (\delta_{j\ell} - \frac{1}{n}) \quad \Rightarrow \quad \{u_j^A(x)\, ,\, p_{\ell m}\} = i\, u_j^A(x) (\delta_{j\ell} - \delta_{jm}) \tag{11.103}$$

(again, the Poisson brackets of $p_{\ell m}$ do not change). Another $\frac{1}{n}$ factor gets subtracted from P_{12} to produce, due to (11.66),

$$\{u_1(x_1)\, ,\, u_2(x_2)\}_D = u_1(x_1)\, u_2(x_2)\, (\frac{\pi}{k}\, sgn(x_{12})\, C_{12} - r_{12}(p)\,) \tag{11.104}$$

for $-\pi < x_1, x_2 < \pi$, where C_{12} is the polarized Casimir operator for $s\ell_n$ and the classical dynamical r-matrix $r_{12}(p)$ is given by (11.101).

It is important to note that the Jacobi identities for the PB involving three Bloch waves, or two Bloch wave fields and p_j, are satisfied due to the (modified) classical dynamical YBE (11.60) and the *neutrality* (or *zero weight*) condition

$$[(e_j^j)_1 + (e_j^j)_2\, ,\, r_{12}(p)] = 0 \ , \tag{11.105}$$

respectively, satisfied by $r_{12}(p) = -r_{21}(p)$ (11.101).

The PB of the covariant group valued field $g(x) = u(x)\, a$ (11.2) are obtained by combining those (for the Dirac brackets) of the Bloch waves (11.104) and of the zero modes (11.59). For $-\pi < x_1, x_2 < \pi$, they read

$$\begin{aligned} \{g_1(x_1), g_2(x_2)\} &= \{u_1(x_1), u_2(x_2)\}\, a_1\, a_2 + u_1(x_1)\, u_2(x_2) \{a_1, a_2\} = \\ &= -\frac{\pi}{k}\, g_1(x_1)\, g_2(x_2)\, (r^-_{12}\, \theta(x_{12}) + r^+_{12}\, \theta(x_{21}))\ , \end{aligned} \tag{11.106}$$

where r^ϵ_{12} are defined in (11.63).

Appendix F. Simple Lie algebras; Lie algebras of compact groups

For a more complete and systematic treatment the reader is referred to his favourite text on Lie algebras – say [94], [67], [27] or [65].

F1. General properties of (semi)simple Lie algebras

A *semisimple Lie algebra* $\mathcal{G}$ is characterized by the property that its *commutator ideal* $C(\mathcal{G})$ coincides with $\mathcal{G}$:

$$C(\mathcal{G}) := [\mathcal{G}, \mathcal{G}] = \mathcal{G}\,. \tag{F.1}$$

In general, for an arbitrary Lie algebra $\mathcal{G}$, $C(\mathcal{G})$ is an ideal in the sense that $[X, C(\mathcal{G})] \subset C(\mathcal{G})$ for any X in $\mathcal{G}$.

Each Lie algebra $\mathcal{G}$ is a vector space of *dimension* $d_{\mathcal{G}} \equiv \dim(\mathcal{G})$. The $d_{\mathcal{G}}$-dimensional adjoint representation ad of the Lie algebra $\mathcal{G}(\ni X)$ acts on the vector space $\mathcal{G}(\ni Y)$ according to

$$ad\, X(Y) = [X, Y]\,. \tag{F.2}$$

Every semisimple Lie algebra $\mathcal{G}$ admits a non-degenerate symmetric, invariant, bilinear *Killing form*

$$(X, Y) = \frac{1}{C_2(ad)} \operatorname{tr}(ad\, X \cdot ad\, Y) \;(= (Y, X)\,)\,, \tag{F.3}$$

where $C_2(\pi)$ is the eigenvalue of the second order Casimir operator in the IR π (to be defined in F3 below). The invariance property,

$$([X, Y], Z) = (X, [Y, Z])\,, \tag{F.4}$$

follows from the cyclicity of the trace.

Let $\mathcal{G}_{\mathbb{C}}$ be a complex Lie algebra (the notation being geared to the case when it is the complexification of a real one, $\mathcal{G}$). Its structure is displayed by using a *Cartan-Weyl basis.* A *Cartan subalgebra* $\mathfrak{h}$ of $\mathcal{G}_{\mathbb{C}}$ is a maximal abelian subalgebra, such that each element of $ad\,\mathfrak{h}$ is diagonalizable. All Cartan subalgebras of $\mathcal{G}_{\mathbb{C}}$ are conjugate to each other (by inner automorphisms); they have, in particular, the same dimension, r, called the *rank* of $\mathcal{G}_{\mathbb{C}}$. We shall view $\mathfrak{h}$ as the complexification of an r-dimensional real Euclidean vector space $\mathbb{R}^r = \mathfrak{h}_{\mathbb{R}}$ of hermitean matrices in the adjoint representation of $\mathcal{G}_{\mathbb{C}}$ with a (positive definite) inner product given by (F.3). The common eigenspaces $\mathcal{G}_\alpha$ of all elements h of $\mathfrak{h}$, such that

$$[h, \mathcal{G}_\alpha] = \alpha(h)\, \mathcal{G}_\alpha \qquad \forall\, h \in \mathfrak{h} \tag{F.5}$$

where the linear functional $\alpha \in \mathfrak{h}^*$ called *root* is not identically zero, are 1-dimensional (the eigenvalue zero being r-fold degenerate). Then the vector space $\mathcal{G}_{\mathbb{C}}$ has the following decomposition:

$$\mathcal{G}_{\mathbb{C}} = \mathfrak{h} \bigoplus_{\alpha \neq 0} \mathcal{G}_\alpha\;. \tag{F.6}$$

In order to introduce a class of preferred bases in a given (semi)simple Lie algebra $\mathcal{G}_{\mathbb{C}}$ we shall first introduce the notion of a root system and of a basis of simple roots.

A subset Φ of $\mathbb{R}^r$ is called a *root system* if it has the following properties:

(R1) The set Φ is finite, its linear span is $\mathbb{R}^r$ and it does not contain 0.

(R2) If $\alpha \in \Phi$, then $k\,\alpha \in \Phi$ (for $k \in \mathbb{R}$) iff $k = \pm 1$.

(R3) If $\alpha \in \Phi$, then the reflection

$$r_\alpha(\beta) = \beta - c(\beta, \alpha)\,\alpha \tag{F.7}$$

where $c(\beta,\alpha) = 2\,\frac{(\beta|\alpha)}{(\alpha|\alpha)}$ and $(\alpha \mid \beta)$ $(= (\beta \mid \alpha))$ is the Euclidean scalar product in $\mathbb{R}^r$, transforms Φ into itself.

(R4) If $\alpha, \beta \in \Phi$, then $c(\beta, \alpha) \in \mathbb{Z}$.

Properties (R1) and (R3) imply that the *Weyl group* $\mathcal{W}$ generated by the reflections (F.7) is a finite group.

A subset Δ of Φ is called a *basis* if

(B1) Δ is a basis of $\mathbb{R}^r$;

(B2) Each root $\beta \in \Phi$ can be written in the form $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$, where either all $k_\alpha \geq 0$ or all $k_\alpha \leq 0$.

The roots belonging to Δ are then called *simple*. Given a root system Φ in $\mathbb{R}^r$, there always exists a basis Δ of Φ. Different bases are permuted by the elements of the Weyl group. A root β of Φ is called *positive* with respect to the basis Δ, if it can be expanded in simple roots with non-negative integral coefficients.

A simple Lie algebra of rank r has exactly r simple roots α_i, $i = 1, \ldots, r$. The basic raising and lowering operators, the *Chevalley*[51] *generators* e_i and f_i, associated with α_i and $-\alpha_i$, respectively, satisfy

$$[e_i, f_j] = \delta_{ij}\, h_i\,. \tag{F.8}$$

Then one assumes, in addition to (F.8), the CRs

$$[h_i, e_j] = c_{ji}\, e_j\,, \quad [h_i, f_j] = -c_{ji}\, f_j\,, \quad c_{ji} = 2\,\frac{(\alpha_j \mid \alpha_i)}{(\alpha_i \mid \alpha_i)}\,. \tag{F.9}$$

They involve the *Cartan matrix* (c_{ji}) whose entries can be viewed as the value of the functional α_j on the *coroot* $\alpha_i^\vee = \frac{2\,\alpha_i}{(\alpha_i|\alpha_i)}$. Note that $c_{ji} = (\alpha_j \mid \alpha_i^\vee)$ is independent of the normalization of the Killing form (F.3). We follow the convention of [65, 67, 94]; other authors use a different convention in which the transposed of the Cartan matrix appears instead. In each case the Cartan matrix elements are only linear in the simple root α_j associated with the raising operator e_j.

We shall interrupt our exposition of the general theory to work out the *example of the semisimple Lie algebra so*(4) that splits into two simple ones. The generators of the Lie algebra $so(n)$ of the compact orthogonal group $SO(n)$ can be identified with the $n \times n$ real skew symmetric matrices

$$X_{ij} = e_{ij} - e_{ji}\ (= -X_{ji})\,, \quad 1 \leq i < j \leq n\,, \tag{F.10}$$

where e_{ij} are the *Weyl matrices* $(e_{ij})^k{}_\ell = \delta_i^k\,\delta_{j\ell}$. (Note that in our conventions $e_{ij} = e_i^j$. Here we use lower indices as the antisymmetric X_{ij} then appear in a more natural form.) We start with a Cartan basis of $so(4, \mathbb{C})$ given by the commuting hermitean (pure imaginary) matrices $L_{12} = i\,X_{12}$ and $L_{34} = i\,X_{34}$ and set

$$e_1 = \frac{1}{2}\,(-L_{23} - i\,L_{13} - L_{14} + i\,L_{24}) = f_1^*$$

[51] Claude Chevalley (1909-1984) – a founding member of the Bourbaki group, see P. Cartier, Notices of the AMS **31** (1984) 775.

$$e_2 = \frac{1}{2}\,(-L_{23} - i\,L_{13} + L_{14} - i\,L_{24}) = f_2^* \,, \quad L_{jk} = i\,X_{jk} (= L_{jk}^*)\,. \tag{F.11}$$

We find

$$\begin{gathered}[L_{12}, e_j] = e_j \,, \quad [L_{12}, f_j] = -f_j \,, \quad j = 1, 2 \\ [L_{34}, e_j] = (-1)^{j-1}\, e_j \,, \quad [L_{34}, f_j] = (-1)^j\, f_j \,; \\ [e_i, f_j] = \delta_{ij}\, h_i \,, \quad [h_i, e_j] = 2\,\delta_{ij}\, e_i \,, \\ h_1 = L_{12} + L_{34} \,, \quad h_2 = L_{12} - L_{34} \,. \end{gathered} \tag{F.12}$$

The equations (F.12) display the splitting of $so(4,\mathbb{C})$ into two rank one algebras:

$$so(4,\mathbb{C}) \simeq A_1 \oplus A_1 \,. \tag{F.13}$$

This example illustrates a general property: every semisimple Lie algebra splits into a (finite) direct sum of simple ones.

The property (R4) in (F.7) restricts severely the possible root systems of simple complex Lie algebras and is the key to their complete classification. as finite sets of vectors in a Euclidean space. To see how it works, one writes

$$(\alpha|\beta^\vee) = c(\alpha,\beta) = 2\frac{|\alpha|}{|\beta|}\cos(\alpha,\beta) \qquad (\, c(\alpha_j,\alpha_i) \equiv c_{ji}\,)\;. \tag{F.14}$$

The integrality of $c(\alpha,\beta)$ for $\alpha \neq \beta$ implies $c(\alpha,\beta)c(\beta,\alpha) = 4\cos^2(\alpha,\beta) \in \{0,1,2,3\}$ as well as $\frac{c(\alpha,\beta)}{c(\beta,\alpha)} = \frac{|\alpha|^2}{|\beta|^2}$ (for $c(\beta,\alpha) \neq 0$ and hence, $c(\alpha,\beta) \neq 0$). Thus, there is only a finite list of possible configurations (angles and relative lengths) for any root system, which can be investigated further case by case. In particular, all elements of the Cartan matrix (c_{ij}) are integral and the non-diagonal ones (corresponding to $i \neq j$) are non-positive. More precisely, we have

$$c_{ii} = 2\;, \quad c_{ij} \in \{0,-1,-2,-3\} \quad \text{for} \quad i \neq j\;, \quad c_{ij} = 0 \;\Leftrightarrow\; c_{ji} = 0\;. \tag{F.15}$$

The symmetric matrix

$$(\alpha_i \mid \alpha_j) = \frac{(\alpha_j \mid \alpha_j)}{2}\, c_{ij} \tag{F.16}$$

is positive definite.

Returning to the general characterization of simple Lie algebras we should add to the CR (F.8), (F.9) the *Serre relations.* They can be expressed in terms of the adjoint representation as

$$(ad(e_i))^{1-c_{ji}}\, e_j = 0 = (ad(f_i))^{1-c_{ji}}\, f_j \qquad (\text{for } i \neq j)\,. \tag{F.17}$$

Thus, the Cartan matrix contains all the information needed to reconstruct in a unique way the root system as well as the commutation relations of $\mathcal{G}$.

In the example of chief interest for us, the case of the Lie algebra of all complex traceless matrices, we have

$$c_{ii+1} = -1 = c_{i+1i}\;,\; i = 1,\ldots,r-1\;; \quad c_{ij} = 0 \quad \text{if } |i-j| > 1 \;\text{ for }\; \mathcal{G} = A_r\,, \tag{F.18}$$

so that Eqs. (F.15) involve no more than double commutators in that case:

$$\begin{gathered}[e_i, e_j] = 0 = [f_i, f_j] \quad \text{for } |i-j| > 1\,, \\ [e_1,[e_1,e_2]] = 0 = [[e_1,e_2],e_2] = \cdots \quad \text{for } \mathcal{G} = A_r\,. \end{gathered} \tag{F.19}$$

If α and β are positive roots, then

$$[e_\alpha, e_\beta] = N_{\alpha+\beta}\, e_{\alpha+\beta} \;\text{ where }\; N_{\alpha+\beta} \in \mathbb{Z} \;\text{ and }\; N_{\alpha+\beta} = 0 \;\text{ if }\; \alpha+\beta \text{ is not a root}\,. \tag{F.20}$$

This 1955 result of Chevalley (whose proof was completed by Tits in 1966) opens the way to considering simple Lie algebras over an arbitrary field.

If $\mathcal{G}_{\mathbb{C}}$ is semisimple then there is a unique real form $\mathcal{G}$ for which the corresponding (say, connected and simply connected) group G is compact. It can be characterized by the fact that the Killing form (F.3) is negative definite on $\mathcal{G}$. If we define a hermitean conjugation on $\mathcal{G}_{\mathbb{C}}$ acting on the Chevalley generators as

$$h_i^* = h_i\,, \quad e_i^* = f_i \quad (f_i^* = e_i) \tag{F.21}$$

then $\mathcal{G}$ will consist of the antihermitean elements of $\mathcal{G}_{\mathbb{C}}$.

More generally, any compact Lie group is *reductive*: it splits into a semisimple factor and a product of $U(1)$ subgroups, possibly factored by a common finite central subgroup; for instance,

$$U(n) = SU(n) \times U(1) \,/\, C_n \tag{F.22}$$

where C_n is the n-element cyclic group of n-th roots of 1, the centre of $SU(n)$.

F2. Dynkin diagrams for simple Lie algebras

We are only interested (in the main body of these lectures) in the Lie algebras $A_r \simeq s\ell_{r+1}$ whose exponentiated compact forms coincide with the special unitary groups $SU(n)$, $n = r+1$. These belong to the class of *simply laced* Lie algebras for which all roots have the same length square (which we shall choose equal to two) and thus the Cartan matrix (F.8) is symmetric, $c_{ij} = (\alpha_i \mid \alpha_j) = c_{ji}$. We shall outline in what follows the classification of simple simply laced Lie algebras. This is still interesting and will allow us to simplify the exposition (by identifying, in particular, the roots α_i and the coroots $\alpha_i^\vee$). The results about non-simply laced algebras will be summarized at the end of this section.

In view of (F.15) (and of the condition $(\alpha \mid \alpha) = 2$ for all roots α) the matrix with elements

$$2\,\delta_{ij} - c_{ij} = n_{ij} \tag{F.23}$$

has non-negative integer entries and can hence be viewed as the adjacency matrix of a graph $\Gamma = \Gamma(\mathcal{G})$ of r vertices and n_{ij} edges between the vertices i and j. We shall call the graph Γ a (simply laced) *Dynkin diagram* if the associated quadratic form (c_{ij}) is positive definite.

Theorem F.1. *The graph Γ is a Dynkin diagram if it belongs to one of the types displayed on* Figure F1.

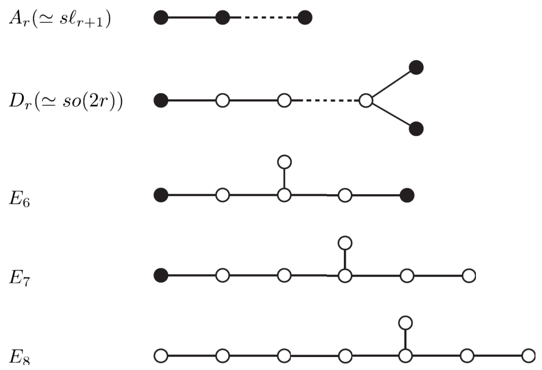


Figure F1: Simply laced Dynkin diagrams

(The number of vertices in the graphs of type A_r and D_r is r. The black vertices correspond to a Kac label $a_i = 1$ – see below.)

The *proof* of Theorem F.1 uses the following exercise (see [52], Problem 5.3).

Exercise F.1. The graphs (of $r+1$ vertices) displayed on Figure F2 have $\det(\tilde{C}_{\mu\nu}) = 0$ where $(\tilde{C}_{\mu\nu},\ \mu,\nu = 0,1,\ldots,r)$ is the (extended) *affine* Cartan matrix such that $\tilde{C}_{00} = 2$, $\tilde{C}_{0j_0} = \tilde{C}_{j_0 0} = -1$ for the (unique for $\hat{D}_r$ and $\hat{E}_r$) vertex j_0 connected with the new vertex 0.

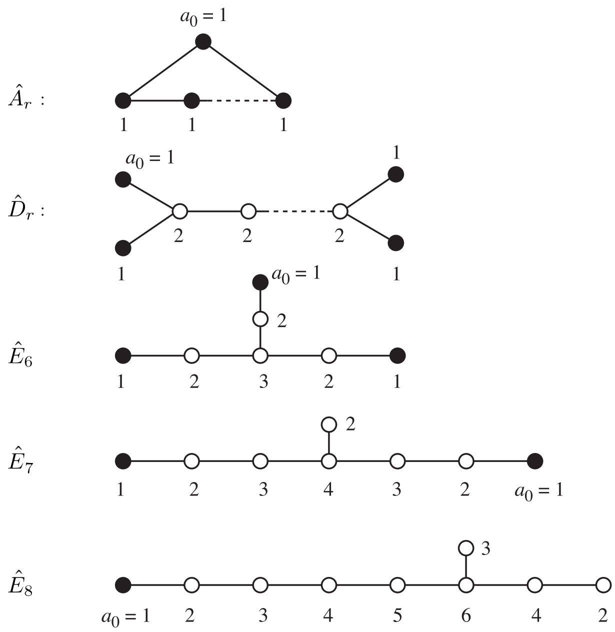


Figure F2: Affine Dynkin diagrams

(The positive numbers associated to each vertex determine the highest root θ and the (dual) Coxeter number – see below.)

We note that one only needs a symmetric, positive definite Cartan matrix (c_{ij}) satisfying the conditions of Theorem F.1 (in particular, Eq. (F.15), but not *a priori* a connection with a simple Lie algebra) in order to define a root system R_Γ, associated with the Dynkin diagram Γ. To see this, we consider lattice $Q = \mathbb{Z}^r$ of vectors x, y with integral components and integer valued scalar product:

$$(x \mid y) := \sum_{i,j=1}^{r} x^i\, c_{ij}\, y^i \Rightarrow (x \mid x) \in 2\mathbb{N}\,. \tag{F.24}$$

We then *define*, following Section 5.4 of [52], a *root* corresponding to the graph Γ with Cartan matrix (c_{ij}) as any vector α of Q such that $(\alpha \mid \alpha) = 2$, in other words any non-zero vector of minimal length. It is an easy exercise to see that such vectors form a finite set Φ_Γ which contains the *simple roots* α_i with components $(\alpha_i)^j = \delta_i^j$, and hence generates Q. It is slightly more difficult to prove that if $\alpha = \sum\limits_{i=1}^{r} k_i\, \alpha_i \in \Phi_\Gamma$ then all coefficients k_i $(\in \mathbb{Z})$ have the same sign (see Lemma 5.16 of [52]). This allows to define the concept of positive (and negative) roots.

The *Kac labels* (positive integers) a_ν, $\nu = 0, 1, \ldots, r$ associated with each vertex ν of an affine Dynkin diagram (see [99]) have the property

$$a_\mu = \frac{1}{2} \sum_\nu n_{\mu\nu}\, a_\nu \, . \tag{F.25}$$

In other words, since $n_{\mu\nu}$ (defined as in (F.23)) take values 0 and 1, a_μ is equal to the half sum of the labels a_ν of its neighbours. The *highest root* θ of a simply laced simple Lie algebra with labels a_ν is given by

$$\theta = \sum_{j=1}^{r} a_j\, \alpha_j \, . \tag{F.26}$$

The Coxeter number g (equal for a simply laced algebra to the dual Coxeter number $g^\vee$) is given by

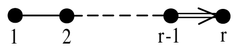
$$g = \sum_{\nu=0}^{r} a_\nu \; ; \quad g(A_r) = r+1 \; , \quad g(D_r) = 2r-2 \; ,$$
$$g(E_6) = 12 \; , \quad g(E_7) = 18 \; , \quad g(E_8) = 30 \; . \tag{F.27}$$

Remark F.1. The A-D-E Dynkin diagrams of Figure F1 also label the *quivers of finite type* (i.e. graphs with finitely many indecomposable representations) – see Section 5 of [52].

The affine Dynkin diagrams of Figure F2 serve to classify (not just the infinite dimensional affine Kac-Moody algebras – see [99]) but also the finite subgroups of $SU(2)$ – and hence the Platonic solids – through the McKay correspondence [118, 140, 107]. To quote [52] "if we needed to make contact with an alien civilization and show them how sophisticated our civilization is, perhaps showing them Dynkin diagrams would be the best choice".

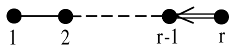
We shall now display the Dynkin diagrams for the remaining two series B_ℓ ($\ell \geq 2$) and C_ℓ ($\ell \geq 3$), corresponding to the odd orthogonal and to the symplectic Lie algebras and for the two non-simply laced exceptional Lie algebras, F_4 and G_2, expressing on the way the (simple) roots of all simple Lie algebras in terms of an orthonormal basis $\{\varepsilon_i\}$. Leaving the case of A_r (of chief interest for us), that is somewhat special, for a more detailed treatment in the next section we summarize the results as follows.

$\mathbf{B_r = so(2r+1)}$



$\alpha_i = \varepsilon_i - \varepsilon_{i+1}\,,\ i = 1, \ldots, r-1\ , \quad \alpha_r = \varepsilon_r\,,$
number of positive roots $\mid \Phi_+^{B_r} \mid = r^2\,,$
highest root $\theta^{B_r} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_r = \varepsilon_1 + \varepsilon_2\ .$

$\mathbf{C_r = sp(2r)}$



$\alpha_i = \varepsilon_i - \varepsilon_{i+1}\,,\ i = 1, \ldots, r-1\ , \quad \alpha_r = 2\varepsilon_r\,,$
$\mid \Phi_+^{C_r} \mid = r^2\,,$
$\theta^{C_r} = 2\alpha_1 + \cdots + 2\alpha_{r-1} + \alpha_r = 2\varepsilon_1\ .$

$\mathbf{D_r = so(2r)}$ (see Fig. F1)
$\alpha_i = \varepsilon_i - \varepsilon_{i+1}\,,\ i = 1, \ldots, r-1\ , \quad \alpha_r = \varepsilon_{r-1} + \varepsilon_r\,,$
$\mid \Phi_+^{D_r} \mid = r(r-1)\,,$
$\theta^{D_r} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r = \varepsilon_1 + \varepsilon_2\ .$

$\mathbf{E_8}$ (see Fig. F1)

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}\,,\ i = 1,\ldots,7\,, \quad \alpha_8 = \frac{1}{2}\,(-\sum_{k=1}^{5}\varepsilon_k + \sum_{k=6}^{8}\varepsilon_k)\,,$$

$$|\,\Phi_+^{E_8}\,| = 120\,, \quad \theta^{E_8} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 = \frac{1}{2}\,(\varepsilon_1 - \sum_{k=6}^{8}\varepsilon_k)\ .$$

$\mathbf{E_7}$ (see Fig. F1)

$$\alpha_i = \varepsilon_{i+1} - \varepsilon_{i+2}\,,\ i = 1,\ldots,6\,, \quad \alpha_7 = \frac{1}{2}\,(-\sum_{k=1}^{5}\varepsilon_k + \sum_{k=6}^{8}\varepsilon_k)$$

$$(\,(\alpha_i \mid \theta^{E_8}) = 0\,,\ i = 1,\ldots,7\,)\,, \qquad |\,\Phi_+^{E_7}\,| = 63\,,$$

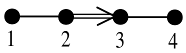
$$\theta^{E_7} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 = -\varepsilon_1 - \varepsilon_8\ .$$

$\mathbf{E_6}$ (see Fig. F1)

$$\alpha_i = \varepsilon_{i+2} - \varepsilon_{i+3}\,,\ i = 1,\ldots,5\,, \quad \alpha_6 = \frac{1}{2}\,(-\sum_{k=1}^{5}\varepsilon_k + \sum_{k=6}^{8}\varepsilon_k)$$

$$(\,(\alpha_i \mid \theta^{E_8}) = 0 = (\alpha_i \mid \Lambda_{E_7}^{(1)})\,,\ i = 1,\ldots,6\,)\,, \qquad |\,\Phi_+^{E_6}\,| = 36\,,$$

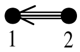
$$\theta^{E_6} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 = -\varepsilon_1 - \varepsilon_2\ .$$



$\mathbf{F_4}$

$$\alpha_1 = \varepsilon_1 - \varepsilon_2\,, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3\,, \quad \alpha_3 = \varepsilon_3\,, \quad \alpha_4 = \frac{1}{2}\,(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)\,,$$

$$|\,\Phi_+^{F_4}\,| = 24\,, \quad \theta^{F_4} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = \varepsilon_1 + \varepsilon_4\ .$$



$\mathbf{G_2}$

$$\alpha_1 = \varepsilon_1 - \varepsilon_2\,, \quad \alpha_2 = \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1 \qquad (\,(\alpha_i \mid \varepsilon_1 + \varepsilon_2 + \varepsilon_3) = 0\ ,\ i = 1,2\,)\,,$$

$$|\,\Phi_+^{G_2}\,| = 6\,, \quad \theta^{G_2} = 3\alpha_1 + 2\alpha_2 = 2\varepsilon_3 - \varepsilon_1 - \varepsilon_2\ .$$

Table F1

The labelling of the nodes of the non-simply laced Dynkin diagrams displayed in Table 1 corresponds to the chosen ordering of simple roots. (For simply laced Dynkin diagrams displayed on Figure F1 we use the following ordering of the simple roots: the labels start from 1 and increase from left to right along the horizontal lines; for D_r the two rightmost dots correspond to α_{r-1} and α_r, while for E_r , α_r corresponds to the unique node above this line.) In writing down the roots of the exceptional E-series in terms of an orthonormal basis $\{\varepsilon_i\}$ we have exploited the embeddings of the root systems $\Phi^{E_6} \subset \Phi^{E_7} \subset \Phi^{E_8}$. More generally, if a Dynkin diagram is a subdiagram of another one, than that is true for the corresponding root systems as well. To display this fact for the simple roots of E_r we note that with the above ordering convention we have

$$\alpha_i(E_r) = \alpha_{i+1}(E_{r+1}) \qquad \text{for} \quad r = 6,7\ . \tag{F.25}$$

To each root system Φ there corresponds a *root lattice* $Q = Q(\Phi)$ consisting of all vectors of the form $\sum_i n_i\alpha_i$ where $\{\alpha_1,\ldots,\alpha_r\}$ is a basis of simple roots in Φ and n_i are arbitrary integers. In order to determine a sublattice, say $Q(E_r) \subset Q(E_{r+1})$, it is enough to display a normal to $Q(E_r)$ in $\mathbb{R}^{r+1}$. An elegant way to do that uses the notion of *fundamental weight* which plays a basis role in the representation theory of the underlying Lie algebra. The *weight lattice* $\mathcal{P}$ dual to the (r-dimensional) root lattice Q is defined as the set of *all* vectors Λ in $\mathbb{R}^r$ such that

$$(\Lambda \mid \alpha^\vee) \in \mathbb{Z} \qquad \text{for all} \quad \alpha \in Q \qquad (\alpha^\vee = 2\,\frac{\alpha}{(\alpha \mid \alpha)}) \, . \tag{F.26}$$

Clearly, $\mathcal{P} \supset Q$.

To a fixed basis of simple roots $\{\alpha_i\}$ in Q there corresponds a dual basis of *fundamental weights* $\Lambda^{(i)}$ in $\mathcal{P}$ such that

$$(\Lambda^{(i)} \mid \alpha_j^\vee) = \delta^i_j \, . \tag{F.27}$$

Eq. (F.25) now implies that the normal to $Q(E_7)$ in $\mathcal{P}(E_8)$ is $\Lambda^{(1)}(E_8)$ while the normal to $Q(E_6)$ in $\mathcal{P}(E_7)$ is $\Lambda^{(1)}(E_7)$.

Exercise F.2. Verify that

$$2\Lambda^{(1)}_{E_7} = \theta^{E_8} \; (= \frac{1}{2}\,(\varepsilon_1 - \sum_{k=2}^{8} \varepsilon_k)\,) \tag{F.28}$$

while

$$2\Lambda^{(1)}_{E_7} = 3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6 + \alpha_7 = \frac{1}{2}\,(\,3(\varepsilon_2 - \varepsilon_1) - \sum_{k=3}^{8} \varepsilon_k\,) \, , \tag{F.29}$$

where α_i in (F.28) and (F.29) are the simple roots of E_8 and E_7 , respectively.

The non-simply laced Lie algebras, B_r , C_r , F_4 , G_2 , have roots of two different lengths. The double or triple line with an arrow is directed towards the short simple root while the number of lines in a pointed multiple line is equal to the ratio of the length squares of the long and the short roots:

$$\frac{\mid \alpha_L \mid^2}{\mid \alpha_S \mid^2} = \begin{cases} 2 & \text{for } B_r\,,\, C_r \text{ and } F_4 \\ 3 & \text{for } G_2 \end{cases} \qquad (\, |\alpha|^2 \equiv (\alpha \mid \alpha)\,) \, . \tag{F.30}$$

The positive roots of B_r , C_r and D_r have the form

$$B_r \,:\, \varepsilon_i \pm \varepsilon_j \,,\, \varepsilon_k \,; \quad C_r \,:\, \varepsilon_i \pm \varepsilon_j \,,\, 2\,\varepsilon_k \,, \quad 1 \le i < j \le r \,,\; 1 \le k \le r \,,$$

$$D_r \,:\, \varepsilon_i \pm \varepsilon_j \,, \quad 1 \le i < j \le r \, . \tag{F.31}$$

It is easy to compute, using Table F1, the Cartan matrices of all simple Lie algebras; for instance,

$$c\,(B_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \, , \qquad c\,(C_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} \, . \tag{F.32}$$

The Lie algebras B_r and C_r are obtained from one another by exchanging the short and long roots; B_2 and C_2 are isomorphic.

Exercise F.3. Compute the Cartan matrices and their determinants for the five exceptional Lie algebras E_r, F_4, G_2 .

Remark F.2. If one continues the exceptional series E_r to $r = 4, 5$, using again the prescription (F.25) one would obtain the Lie algebras of the so called Grand Unified Theories: $E_5 \simeq D_5 (= so(10))$, $E_4 \simeq A_4 (= su(5))$ (the compact forms of the complex Lie algebras).

F3. $\mathbf{A_r}$: orthogonal basis, roots and weights; Weyl group; Casimir invariants

As we have identified roots and coroots (see the beginning of F2) it is legitimate to consider the restriction of the Killing form to the root lattice $Q = \sum_i \mathbb{Z}\,\alpha_i$. It is convenient to view Q as embedded into an $n\,(\equiv r+1)$-dimensional orthogonal lattice $\mathbb{Z}^n$ with orthonormal basis $\{e_\nu,\, \nu = 1,\ldots,n\}$. We can identify e_ν with the diagonal Weyl matrices $e_{\nu\nu}$ with scalar product coinciding with the trace of the product:

$$(e_\mu \mid e_\nu) = \mathrm{tr}(e_{\mu\mu}\, e_{\nu\nu}) = \delta_{\mu\nu} \ , \quad \mu,\nu = 1,\ldots,n\,(\equiv r+1)\,. \tag{F.33}$$

Both the simple roots α_i and all positive roots, α_{ij}, are expressed as differences of two e_i's:

$$\alpha_i = e_i - e_{i+1} \ , \quad \alpha_{ij} := e_i - e_j = \sum_{s=i}^{j-1} \alpha_s \ , \quad 1 \le i < j\,. \tag{F.34}$$

The weight lattice $\mathcal{P}$ is spanned (over the ring of integers $\mathbb{Z}$) by the fundamental weights $\Lambda^{(i)}$ (F.27),

$$(\Lambda^{(i)} \mid \alpha_j) = \delta^i_j\,. \tag{F.35}$$

For any *dominant* weight (highest weight of a finite dimensional IR) Λ we shall write

$$\Lambda = \sum_{i=1}^{r} \lambda_i\, \Lambda^{(i)} = \sum_{\nu=1}^{n} \ell_\nu\, e_\nu \ , \quad n = r+1 \quad (\lambda_i \ge 0)\,. \tag{F.36}$$

Taking the scalar product of both sides with α_j (F.34) and using (F.33) and (F.35) we find

$$\lambda_i = \ell_i - \ell_{i+1}\,. \tag{F.37}$$

Postulating, in addition,

$$\sum_{\nu=1}^{n} \ell_\nu = 0 \tag{F.38}$$

we can express the ℓ_ν in terms of λ_i. We find, in particular, for the fundamental weights and for the highest root

$$\Lambda^{(i)} = \frac{n-i}{n} \sum_{s=1}^{i} e_s - \frac{i}{n} \sum_{s=i+1}^{n} e_s \ , \quad \theta = e_1 - e_n = \Lambda^{(1)} + \Lambda^{(r)}\,. \tag{F.39}$$

The Weyl group $\mathcal{W}_\Gamma$ of a Dynkin diagram Γ of r vertices is the automorphism group of the vector space $\mathbb{R}^r = \sum_i \mathbb{R}\,\alpha_i$ which preserves the root system Φ (cf. property (R3)). It is a subgroup of the group of permutations of all (positive and negative) roots and hence, is a finite group. It is generated by the reflections r_i of simple roots (cf. (F.7))

$$r_i\, x = x - 2\, \frac{(\alpha_i \mid x)}{(\alpha_i \mid \alpha_i)}\, \alpha_i\ . \tag{F.40}$$

For $\mathcal{G} = A_r\,,\ s_i$ is nothing but the substitution $e_i \leftrightarrow e_{i+1}$. The r_i are thus the standard generators (satisfying (A.2)) of the permutation group $\mathcal{S}_n$, $n = r+1$. According to (F.40) and (F.35) we have

$$r_i\, \Lambda^{(j)} = \Lambda^{(j)} - \alpha_i\, \delta^j_i\,. \tag{F.41}$$

The vector

$$p = \sum_{\nu=1}^{n} p_\nu\, e_\nu \quad (n = r+1) \tag{F.42}$$

of components p_ν satisfying (11.5) is related to the weight vector Λ (F.28) by

$$p = \Lambda + \rho \tag{F.43}$$

where ρ is the half sum of positive roots:

$$\rho = \frac{1}{2} \sum_{\alpha>0} \alpha = \sum_{j=1}^{r} \Lambda^{(j)} = \sum_{\nu=0}^{\left[\frac{r}{2}\right]} \frac{r-\nu}{2} \left(e_{\nu+1} - e_{r+1-\nu}\right). \tag{F.44}$$

For positive Λ (and p) we say that Λ (respectively p) belongs to the positive Weyl chamber. The A_r-Weyl group acts on Λ (and p) by just permuting ℓ_ν (and p_ν).

We are now ready to write the (Weyl invariant) eigenvalues $C_2(p,n)$ of the second order Casimir operator of $A_r \simeq s\ell_n$:

$$\begin{aligned} C_2(p,n) = (\Lambda \mid \Lambda + 2\rho) = (p - \rho \mid p + \rho) &= \sum_{\nu=1}^{n} p_\nu^2 - \frac{n(n^2-1)}{12} \\ &= \frac{1}{n} \sum_{1\le i<j\le n} p_{ij}^2 - \frac{n(n^2-1)}{12} \,. \end{aligned} \tag{F.45}$$

Exercise F.4. Verify, using (F.44), the "strange formula"

$$|\rho|^2 = \frac{n(n^2-1)}{12} \,. \tag{F.46}$$

Prove that the eigenvalue (F.45) can be expressed in terms of the components ℓ_ν and λ_i of (F.36)–(F.38) as

$$C_2(\Lambda,n) = \sum_{\nu=1}^{n} \ell_\nu(\ell_\nu - 2\nu) = \sum_{i=1}^{n-1} \frac{i(n-i)}{n} \lambda_i(\lambda_i + n) \,. \tag{F.47}$$

Verify that

$$\ell_\nu(\Lambda^{(1)}) = \delta_\nu^1 - \frac{1}{n} \,, \quad \ell_\nu(\Lambda^{(2)}) = \delta_\nu^1 + \delta_\nu^2 - \frac{2}{n} \,, \quad \ell_\nu(2\,\Lambda^{(1)}) = 2\,\ell_\nu(\Lambda^{(1)}) \,.$$

The dimension d_Λ ($\equiv d(p)$) of the IR of weight Λ is obtained as a special case of Weyl's character formula:

$$d_\Lambda = \prod_{\alpha>0} \frac{(\Lambda+\rho \mid \alpha)}{(\rho \mid \alpha)} = \prod_{i=1}^{n-1} \left(\frac{1}{i!} \prod_{j=i+1}^{n} p_{ij} \right) \,. \tag{F.48}$$

Exercise F.5. Compute the Casimir invariant and the dimension of the (irreducible) antisymmetric and the symmetric powers of the defining (n-dimensional) representation of $s\ell_n$.

(*Answer:* $C_2(\Lambda^{(2)},n) = \frac{2}{n}(n-2)(n+1)$, $C_2(2\,\Lambda^{(1)},n) = \frac{2}{n}(n-1)(n+2)$, $d_{\Lambda^{(2)}} = \binom{n}{2}$, $d_{2\Lambda^{(1)}} = \binom{n+1}{2}$.)

The *Dynkin index* $N(\pi)$ of a representation π of $\mathcal{G}$ is related to the Killing form (X,Y) on $\mathcal{G}$ by

$$(X,Y) = \frac{1}{N(\pi)}\, \mathrm{tr}(\pi(X)\, \pi(Y))\,. \tag{F.49}$$

Let $\{T_a\,,\ a=1,\dots,d_{\mathcal{G}}\}$, be a basis in $\mathcal{G}$,

$$\eta_{ab} = (T_a, T_b) \tag{F.50}$$

the corresponding Killing metric tensor, and η^{ab} its inverse: $\eta^{as}\,\eta_{sb} = \delta^a_b$. The second order Casimir operator

$$C_2(\mathcal{G}) = \eta^{ab}\, T_a\, T_b$$

has eigenvalue $C_2(\pi,\mathcal{G})$ in the IR π satisfying

$$\pi(C_2(\mathcal{G})) = \eta^{ab}\, \pi(T_a)\, \pi(T_b) = C_2(\pi,\mathcal{G})\, \mathbb{I}_\pi\,,$$

and hence

$$d_{\mathcal{G}} = \frac{1}{N(\pi)}\, C_2(\pi,\mathcal{G})\, \dim\pi\,. \tag{F.51}$$

In particular, if π is the adjoint representation of $\mathcal{G}$, $\pi = ad$, $\dim ad = d_{\mathcal{G}}$, we find

$$N(ad) = C_2(ad,\mathcal{G})\,, \tag{F.52}$$

in accord with (F.3).

Exercise F.6. Verify for $\mathcal{G} = su(n)$ the general formula (cf. (F.39))

$$(C_2(\theta,n) =)\ C_2(ad,\mathcal{G}) = (\theta, \theta+2\rho) = (\theta \mid \theta)\, g^\vee \quad (=2n \text{ for } \mathcal{G} = su(n))\,. \tag{F.53}$$

Use (F.51) to show that $N(\Lambda_1) = N(\Lambda_r) = 1$ for $\mathcal{G} = A_r$.

Question: Which Casimir eigenvalue appears in the power of q in (5.30)?

12 Quantum exchange relations

"Quantization is an art, not a science. This is best seen in quantum integrable models."

Ludwig Faddeev
Oral intervention after Witten's talk [85]

Lausanne, 16 March, 2009

The process of *quantization* has two distinct parts: (i) a *deformation of the algebra of dynamical variables* with leading term given by the PB; (ii) a *representation of the* resulting *quantum algebra* in terms of linear operators in an inner-product vector space. The two parts for the zero modes' algebra are realized in Sections 12.1 and 12.2, respectively. (Note that the term "deformation quantization" is often used in a more narrow sense involving the so called star product − see, e.g. [106] and [46] as well as references to the earlier work of Moshé Flato (1937-1998), André Lichnerowicz (1915-1998) and D. Sternheimer, among others, cited there.) Section 12.3 deals with the quantization of Bloch waves; it also combines the outcome with the results of the previuos subsections to display the properties of the quantum chiral field $g(x)$.

12.1. Quantization of group like quantities; the quantum matrix algebra

The naive idea that quantization means replacing PB by commutators is tailored to fit Lie algebras of Poisson brackets and their representations. For group valued functions yielding quadratic exchange relations this is no longer true. The simplest example is provided by the Weyl form of the canonical CR:

$$\{e^{i\alpha p}, e^{i\beta x}\} = \alpha\beta\, e^{i\alpha p}\, e^{i\beta x} \quad \Leftarrow \quad e^{i\alpha p}\, e^{i\beta x} = e^{i\hbar\alpha\beta}\, e^{i\beta x}\, e^{i\alpha p}\,. \tag{12.1}$$

(The arrow indicates that we can recover the PB as a (quasi-)classical limit of the quantum exchange relations, setting

$$\{e^{i\alpha p}, e^{i\beta x}\} = \lim_{\hbar\to 0} \frac{1}{i\hbar}\, [e^{i\alpha p}, e^{i\beta x}]\,,$$

while to obtain the Weyl CR from the PB one requires additional hypotheses.)

We shall now spell out the desiderata that will yield the quantum exchange relations which reproduce in the quasi-classical limit the PB (11.73), (11.59).

To make clear the meaning of a (quasi-)classical limit we shall introduce for a moment Planck's constant $\hbar$ setting

$$k = \frac{\bar{k}}{\hbar}\,, \quad p_{ij} = \frac{\bar{p}_{ij}}{\hbar} \quad \Rightarrow \quad \frac{n}{k} \to 0\,, \quad p_{ij} \to \infty\,, \quad \frac{p_{ij}}{k} \text{ finite for } \hbar \to 0\,. \tag{12.2}$$

Our first requirement is to renormalize under quantization the level k to the height $h = k + n\left(= \frac{\bar{k}+\hbar n}{\hbar}\right)$ (cf. Remark 11.1) changing accordingly the deformation parameter q from its value $q = q_k$ in (11.12) to

$$q\ (= q_h) = e^{-\frac{i\pi}{h}}\,, \qquad h = k + n \ \text{ for } \ \mathcal{G} = su(n)\,. \tag{12.3}$$

This is allowed by the quasi-classical limit (12.2) but is only justified (to the best of our knowledge) by the study of the Bloch wave part of the WZNW model where it is a consequence of having to use a normal product in the Sugawara formula (Section 9).

To quantize (11.73) we note that it displays an intertwining property of a^j_α between different representations of the Weyl group. We shall therefore postulate the (exponentiated) group theoretic version of (11.73):

$$q^{p_{j\ell}}\, a^i_\alpha = a^i_\alpha\, q^{p_{j\ell}+\delta^i_j-\delta^i_\ell}\,. \tag{12.4}$$

Further, we look for quantum exchange relations of the form

$$R_{12}(p)\, a_1\, a_2 = a_2\, a_1\, R_{12} \quad \Leftrightarrow \quad \hat{R}_{12}(p)\, a_1\, a_2 = a_1\, a_2\, \hat{R}_{12} \tag{12.5}$$

that would reproduce (11.59) in the quasi-classical limit. (In the $n = 2$ case, the exchange relations (12.5) appeared for the first time in [1], see also [30].) We demand that $R = P\hat{R}$ and $R(p) = P\hat{R}(p)$ (where $P = P_{12}$ stands for permutation, $P\, a_2\, a_1 = a_1\, a_2\, P\,,\ P^2 = \mathbb{1}$) satisfy the (quantum) YBE (4.35) (equivalent to the braid relation (4.39)) and its dynamical counterpart:

$$R_{12}(p)\, R_{13}(p - v_2)\, R_{23}(p) = R_{23}(p - v_1)\, R_{13}(p)\, R_{12}(p - v_3)\,. \tag{12.6}$$

Here $\{v^{(i)} = e^i_i\,,\ i = 1,\dots,n\}$ stand for the Weyl basis in the chosen Cartan subalgebra of $g\ell_n\,(\supset s\ell_n)$. Viewed as an operator in $\mathbb{C}^n$ the weight vector p is a diagonal matrix, expressed in terms of $v^{(i)}$ as

$$p = \sum_{i=1}^{n} p_i\, v^{(i)}\,. \tag{12.7}$$

To make once more explicit the meaning of Faddeev's tensor product notation we write

$$R_{23}(p - v_1)^{i_1 i_2 i_3}_{j_1 j_2 j_3} = \delta^{i_1}_{j_1}\, R(p - v^{(i_1)})^{i_2 i_3}_{j_2 j_3}\,. \tag{12.8}$$

Exercise 12.1. Following the path from (4.35) to (4.39) derive the braid group counterpart of the dynamical YBE:

$$\hat{R}_{12}(p)\, \hat{R}_{23}(p - v_1)\, \hat{R}_{12}(p) = \hat{R}_{23}(p - v_1)\, \hat{R}_{12}(p)\, \hat{R}_{23}(p - v_1)\,. \tag{12.9}$$

The braid operators $b_1 := q^{-\frac{1}{n}}\hat{R}$ and $b_1(p) := q^{-\frac{1}{n}}\hat{R}(p)$ are assumed to satisfy the Hecke condition (2.11), which is equivalent to setting

$$\begin{aligned} b_1 &= q^{-\frac{1}{n}}\, \hat{R}_{12} = q^{-1}\, \mathbb{1}_{12} - A_{12}\ , \\ b_1(p) &= q^{-\frac{1}{n}}\, \hat{R}_{12}(p) = q^{-1}\, \mathbb{1}_{12} - A_{12}(p)\ , \end{aligned} \tag{12.10}$$

where in both cases

$$\begin{gathered} A_{12}\, A_{23}\, A_{12} - A_{12} = A_{23}\, A_{12}\, A_{23} - A_{23}\ , \\ A_{12}^2 = [2]\, A_{12} \quad (\,[2] = q + q^{-1}\,)\,, \quad [A_{ii+1}, A_{jj+1}] = 0 \quad \text{for} \quad |i-j| > 1\ . \end{gathered} \tag{12.11}$$

The standard (constant) solution of the YBE (which can be written for any simple Lie algebra – see [57]) is given by (12.10) with

$$A^{\alpha\beta}_{\alpha'\beta'} = q^{\epsilon_{\beta\alpha}}\, \delta^\alpha_{\alpha'}\, \delta^\beta_{\beta'} - \delta^\alpha_{\beta'}\, \delta^\beta_{\alpha'} \tag{12.12}$$

where

$$q^{\epsilon_{\beta\alpha}} = \begin{cases} q & \text{for} \quad \alpha < \beta \\ 1 & \text{for} \quad \alpha = \beta \\ q^{-1} & \text{for} \quad \alpha > \beta\,. \end{cases} \tag{12.13}$$

Exercise 12.2. Verify that A_{ii+1} given by (12.12) satisfies (12.11) and hence can be identified with the antisymmetrizers e_i of Section 2.

We shall relate (12.11), (12.12) with the series of projectors P_-^k of Section 2 by introducing inductively higher rank antisymmetrizers:

$$A_{13} := A_{12}(\bar{q}^2 - \bar{q}\, b_2 + b_2\, b_1) = A_{12}\, A_{23}\, A_{12} - A_{12}\,,$$
$$A_{14} := A_{13}(\bar{q}^3 - \bar{q}^2\, b_3 + \bar{q}\, b_3\, b_2 - b_3\, b_2\, b_1)\ ,\ \text{etc.} \tag{12.14}$$

It is easy to verify that A_{1k} is the non-normalized counterpart of P_-^k (up to the substitution $q \to \bar{q} \equiv q^{-1}$). We then have (in accord with (2.20))

$$A_{1n} = (\varepsilon^{\alpha_1 \ldots \alpha_n}\, \varepsilon_{\beta_1 \ldots \beta_n}) \quad (\Rightarrow A_{1n+1} = 0\,)\,. \tag{12.15}$$

It is straightforward to verify that for q given by (12.3) (which implies $q = 1 - i\,\frac{\pi}{k} + O\left(\frac{\pi^2}{k^2}\right)$) the R-matrix $R = P\hat{R}$ satisfies the quasi-classical limit

$$R_{12} = 1\!\!\mathrm{I}_{12} - i\,\frac{\pi}{k}\, r_{12}^- + O\left(\frac{\pi^2}{k^2}\right)\,. \tag{12.16}$$

We stress that the (quantum) YBE together with the assumption of $U_q(s\ell_n)$ covariance for q given by (12.3) leave no additional ambiguity in the choice of R.

In writing down the full (x-dependent) exchange relations for the chiral field $g(x)$, it will be important to realize that the exchange relations (12.5) have two equivalent forms. Indeed, interchanging the labels 1 and 2 in the first relation (12.5), one can write

$$R_{21}(p)\, a_2\, a_1 = a_1\, a_2\, R_{21}\ .$$

Multiplying both sides with $R_{21}^{-1}(p)$ from the left and R_{21}^{-1} from the right, we find

$$R_{21}^{-1}(p)\, a_1\, a_2 = a_2\, a_1\, R_{21}^{-1}\ .$$

We shall write (12.5) and the last equation in an unified form as

$$R_{12}^{\pm}(p)\, a_1\, a_2 = a_2\, a_1\, R_{12}^{\pm} \qquad \Leftrightarrow \qquad \hat{R}_{12}^{\pm}(p)\, a_1\, a_2 = a_1\, a_2\, \hat{R}_{12}^{\pm} \tag{12.17}$$

with

$$R_{12}^- = R_{12}\ ,\quad R_{12}^+ = R_{21}^{-1}\ ;\qquad R_{12}^-(p) = R_{12}(p)\ ,\quad R_{12}^+(p) = R_{21}^{-1}(p)\ . \tag{12.18}$$

The dynamical YBE was introduced by Gervais and Neveu [81] in the early days of $2D$ CFT for the exchange algebra associated with the Liouville equation and applied to the study of the WZNW zero modes already in [1]. Its general solution satisfying the

$$\textit{ice condition}:\ R(p)^{ij}_{i'j'} = 0 \quad \text{unless} \quad i = i'\,,\ j = j' \text{ or } i = j'\,,\ j = i' \tag{12.19}$$

implying

$$q^{-\frac{1}{n}}\, \hat{R}(p)^{ij}_{i'j'}\ =\ a_{ij}(p)\, \delta^i_{j'} \delta^j_{i'} + b_{ij}(p)\, \delta^i_{i'} \delta^j_{j'} \tag{12.20}$$

as well as the Hecke relation (12.10), (12.11) is found in [95] (see also [54]). As shown in [88], it can be brought to a canonical form which we shall write as

$$a_{ij}(p) = q^{\alpha_{ij}(p_{ij})}\, \xi(p_{ij})\ ,\qquad b_{ij}(p) = q^{-1} - \xi(p_{ij})\ ,$$
$$\xi(p) = \frac{[p-1]}{[p]}\ ,\qquad \alpha_{ij}(p) = -\alpha_{ji}(-p) \quad (\Rightarrow\ \alpha_{ii}(0) = 0\,)\ , \tag{12.21}$$

yielding the second equation (12.10) with

$$A(p)^{ij}_{i'j'} = \xi(p_{ij})\,(\delta^i_{i'}\,\delta^j_{j'} - q^{\alpha_{ij}(p_{ij})}\,\delta^i_{j'}\,\delta^j_{i'})\ . \tag{12.22}$$

The ice condition provides a compact representation of the (i,j) block of $R(p)$ as a 4×4 matrix (using the ordering $(ii),(ij),(ji),(jj)$, for any given pair $(i,j)\,,\ i<j$) which assumes, in the general case, the form

$$R(p)^{ij}_{i'j'} \;=\; q^{\frac{1}{n}} \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & q^{-\alpha_{ij}(p_{ij})}\frac{[p_{ij}+1]}{[p_{ij}]} & -\frac{q^{p_{ij}}}{[p_{ij}]} & 0 \\ 0 & \frac{q^{-p_{ij}}}{[p_{ij}]} & q^{\alpha_{ij}(p_{ij})}\frac{[p_{ij}-1]}{[p_{ij}]} & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} . \tag{12.23}$$

Using the expansions

$$\frac{[p\pm 1]}{[p]} = 1 \pm \frac{\pi}{k}\cot\left(\frac{\pi}{k}p\right) + O(\frac{\pi^2}{k^2})\ ,\quad \frac{q^{\pm p}}{[p]} = \frac{\pi}{k}\cot\left(\frac{\pi}{k}p\right) \pm i + O(\frac{\pi^2}{k^2})\ , \tag{12.24}$$

one recovers in the quasi-classical limit (given by (12.16) with the substitutions $R_{12}\ \to\ R_{12}(p)\,,\ \frac{\pi}{k}\,r_{12}\ \to\ r_{12}(p)$) the classical dynamical r-matrix (11.61) for

$$f_{j\ell}(p) = i\frac{\pi}{k}\cot\left(\frac{\pi}{k}p_{j\ell}\right) + i\frac{\pi}{k}\,\alpha_{j\ell}(p_{j\ell}) + O(\frac{\pi^2}{k^2})\ . \tag{12.25}$$

(The expansion of the coefficient $q^{\frac{1}{n}}$ provides the $\frac{1}{n}$ term for C_{12} (11.66).) Setting $i\frac{\pi}{k}\,\alpha_{j\ell}(p) = \lambda\sum_s f_{j\ell s}(p)$ where $f_{j\ell s}(p)$ is given by (11.23) (so that, in particular, $\alpha_{j\ell}(p)=0$ for $n=2$) makes the connection with our analysis in Section 11 explicit.

In contrast with the constant $\hat{R}$ case, the representation of the braid group generated by $\hat{R}(p)$ is "nonlocal"; the i-th braid operator is defined as

$$b_i(p) := q^{-\frac{1}{n}}\,\hat{R}_{i i+1}(p - \sum_{\ell=1}^{i-1} v_\ell) =: q^{-1}\,\mathbb{I}_{i i+1} - A_{i i+1}(p) \tag{12.26}$$

(as suggested already by the quantum dynamical YBE (12.6)).

The Hecke condition (2.11), being equivalent to $A_{ii+1}(A_{ii+1}-[2])=0$, allows to derive both the q-deformed Levi-Civita tensors ε and the dynamical ones $\epsilon(p)$. They belong to the eigenspace corresponding to the nonzero eigenvalue of all $A_i\,,\ i=1,\dots,n$, for those with upper indices, or of their transpose, for lower indices, i.e.

$$\begin{aligned} A^{\alpha_i\alpha_{i+1}}_{\beta_i\beta_{i+1}}\,\varepsilon^{\alpha_1\dots\beta_i\beta_{i+1}\dots\alpha_n} &= [2]\,\varepsilon^{\alpha_1\dots\alpha_i\alpha_{i+1}\dots\alpha_n}\ ,\\ \varepsilon_{\alpha_1\dots\beta_i\beta_{i+1}\dots\alpha_n}A^{\beta_i\beta_{i+1}}_{\alpha_i\alpha_{i+1}} &= [2]\,\varepsilon_{\alpha_1\dots\alpha_i\alpha_{i+1}\dots\alpha_n}\ . \end{aligned} \tag{12.27}$$

The solutions of (12.27) normalized by $\varepsilon_{\alpha_1\dots\alpha_n}\varepsilon^{\alpha_1\dots\alpha_n}=[n]!$ are unique, up to a sign. They vanish whenever some of their indices coincide while, in our conventions

$$\varepsilon^{\alpha_1\dots\alpha_n} = \varepsilon_{\alpha_1\dots\alpha_n} = q^{-\frac{n(n-1)}{4}}\,(-q)^{\ell(\sigma)} \qquad \text{for}\quad \sigma = \begin{pmatrix} n\ \dots\ 1\\ \alpha_1\dots\alpha_n\end{pmatrix} \in \mathcal{S}_n\ , \tag{12.28}$$

where $\ell(\sigma)$ is the length of the permutation σ.

Exercise 12.3. Verify for $n=2,3$ that the expressions (12.28) for the q-deformed Levi-Civita tensors reproduce (2.23) and (2.26) and satisfy (12.27).

The dynamical Levi-Civita tensors found by the same procedure in [88] read

$$\epsilon^{i_1\dots i_n}(p) = \epsilon_{i_1\dots i_n} \prod_{(j,i)\in J(\sigma)} q^{-\alpha_{ij}(p_{ij})} \prod_{1\le\mu<\nu\le n} \frac{[p_{i_\mu i_\nu} - 1]}{[p_{i_\mu i_\nu}]} \,, \tag{12.29}$$

$$\epsilon_{i_1\dots i_n}(p) = \epsilon_{i_1\dots i_n} \prod_{(j,i)\in J(\sigma)} q^{\alpha_{ij}(p_{ij})} \,, \quad J(\sigma) := \{(i_\mu, i_\nu) \,:\, \mu<\nu\,,\ i_\mu < i_\nu\}$$

where $\epsilon_{i_1\dots i_n}$ is the ordinary (*undeformed*) Levi-Civita tensor normalized by $\epsilon_{n\dots 1} = 1$ and σ is now the permutation $\sigma = \binom{n\ \dots 1}{i_1\dots i_n} \in \mathcal{S}_n$.

We see, in particular, that for $\alpha_{ij}(p_{ij}) = 0$, $\epsilon_{i_1\dots i_n}$ is p-independent; this simplifies considerably the expressions involved so we shall require it in what follows.

In order to complete the definition of the *quantum matrix algebra*, which we again denote by $\mathcal{M}_q = \mathcal{M}_q(R(p), R)$, we define the *quantum determinant*

$$D_q(a) := \frac{1}{[n]!}\, \epsilon_{i_1\dots i_n}\, a^{i_1}_{\alpha_1} \dots a^{i_n}_{\alpha_n}\, \varepsilon^{\alpha_1\dots\alpha_n} \ . \tag{12.30}$$

In accord with (12.29) (for $\alpha_{ij}(p_{ij}) = 0$), here $\epsilon_{i_1\dots i_n}$ is the totally antisymmetric Levi-Civita tensor normalized by $\epsilon_{n\dots 1} = 1$ while $\varepsilon^{\alpha_1\dots\alpha_n}$ is its q-deformed counterpart (12.28).

The definition (12.30) of the quantum determinant is justified by the following statement.

Proposition 12.1. *The product* $a_1 \dots a_n$ *intertwines between the two Levi-Civita tensors appearing in* (12.30):

$$\epsilon_{i_1\dots i_n}\, a^{i_1}_{\alpha_1} \dots a^{i_n}_{\alpha_n} = D_q(a)\, \varepsilon_{\alpha_1\dots\alpha_n} \quad \Leftrightarrow \quad D_q(a) = q^{\frac{n(n-1)}{4}}\, \epsilon_{i_1\dots i_n}\, a^{i_1}_{n} \dots a^{i_n}_{1} \ , \tag{12.31}$$

$$a^{i_1}_{\alpha_1} \dots a^{i_n}_{\alpha_n}\, \varepsilon^{\alpha_1\dots\alpha_n} = \epsilon^{i_1\dots i_n}(p)\, D_q(a) \ . \tag{12.32}$$

The *proof* of this statement (see Proposition 5.1 of [88]) is based on the exchange relations (12.4), (12.5) and on the projector property (2.18), i.e. on

$$A_{1i}\, A_{1j} = [i]!\, A_{1j} \quad \text{for} \quad 1 < i \le j \ . \tag{12.33}$$

The consistency of the constraint

$$D_q(a) = \mathcal{D}_q(p) := \prod_{1\le i<j\le n} [p_{ij}] \tag{12.34}$$

relies on the following important implication of Proposition 12.1.

Corollary 12.1. *The ratio* $\frac{D_q(a)}{\mathcal{D}_q(p)}$ *belongs to the centre of the algebra* $\mathcal{M}_q(R(p), R)$ (see Corollary 5.1 of [88]).

The relation

$$a\, M = M_p\, a \,, \tag{12.35}$$

with

$$(M_p)^i_j = q^{1-\frac{1}{n}-2p_i}\,\delta^i_j \tag{12.36}$$

in the quantum case, together with the constraint (12.34) allows to express the monodromy matrix M in terms of a^i_α and p_j. It follows from (12.4) that the elements of M commute with p. In fact, they generate the full commutant of p. On the other hand, according to (5.25) – (5.30) M is expressed in terms of the generators of the QUEA $U_q(s\ell_n)$, that thus appears as the p-invariant subalgebra of $\mathcal{M}_q$.

For q given by (12.3) ($q^h = -1$) the dynamical R-matrix may be singular as $[nh] = 0$. Getting rid of the denominators we obtain the following regular form of (12.17) which makes sense for all integer p_{ij} :

$$\begin{aligned} &[a^i_\alpha, a^j_\alpha] = 0\ , \qquad a^i_\alpha a^i_\beta = q^{\epsilon_{\alpha\beta}}\, a^i_\beta a^i_\alpha\ , \\ &[p_{ij}-1]\, a^j_\alpha a^i_\beta = [p_{ij}]\, a^i_\beta a^j_\alpha - q^{\epsilon_{\beta\alpha} p_{ij}} a^i_\alpha a^j_\beta \quad \text{for} \quad \alpha\neq\beta \quad \text{and } i\neq j \end{aligned} \tag{12.37}$$

($q^{\epsilon_{\alpha\beta}}$ is defined in (12.13)).

For $n = 2$ the determinant condition (12.31) is also quadratic and can be combined with the exchange relations; the result is

$$\begin{aligned} &a^2_\alpha a^1_\beta = a^1_\alpha a^2_\beta + [p]\,\varepsilon_{\alpha\beta} \qquad (\,p\equiv p_{12}\,)\ , \\ &a^i_\alpha a^i_\beta\,\varepsilon^{\alpha\beta} = 0 \qquad (\,\text{i.e.,}\quad a^i_2 a^i_1 = q\, a^i_1 a^i_2\ ,\quad i=1,2\,)\ , \\ &a^2_\alpha a^1_\beta\,\varepsilon^{\alpha\beta} = [p+1]\ , \qquad a^1_\alpha a^2_\beta\,\varepsilon^{\alpha\beta} = -[p-1]\ . \end{aligned} \tag{12.38}$$

Eq. (12.4) now gives simply

$$q^p a^1_\alpha = a^1_\alpha q^{p+1}\ , \qquad q^p a^2_\alpha = a^2_\alpha q^{p-1}\ . \tag{12.39}$$

These relations are covariant (as devised by Pusz and Woronowicz [130] back in the late 1980's) with respect to the U_q ($\equiv U_q(s\ell_2)$) coaction:

$$\begin{aligned} &K a^i_1 = q\, a^i_1 K\ , \quad a^i_2 K = q\, K a^i_2\ , \\ &[E, a^i_1] = 0\ , \qquad [E, a^i_2] = a^i_1 K\ , \\ &F a^i_1 - q^{-1} a^i_1 F = a^i_2\ , \quad F a^i_2 - q\, a^i_2 F = 0\ , \qquad i=1,2\ . \end{aligned} \tag{12.40}$$

12.2. Fock space representation

We now proceed to introducing a Fock-space operator realization of $\mathcal{M}_q$. To this end we define the Fock space $\mathcal{F}$ and its dual $\mathcal{F}'$ as $\mathcal{M}_q$-modules with 1-dimensional $U_q(s\ell_n)$ invariant subspaces of multiples of (non-zero) *bra* and *ket* vacuum vectors $\langle\, 0 \mid$ and $\mid 0\,\rangle$ (such that $\langle\, 0 \mid \mathcal{M}_q = \mathcal{F}'$, $\mathcal{M}_q \mid 0\,\rangle = \mathcal{F}$) satisfying

$$\begin{aligned} &a^i_\alpha \mid 0\,\rangle = 0 \quad \text{for } i>1\ , \quad \langle\, 0 \mid a^i_\alpha = 0 \quad \text{for } i<n\ ; \\ &q^{p_{ij}+i-j} \mid 0\,\rangle = \mid 0\,\rangle\,, \quad \langle\, 0 \mid q^{p_{ij}+i-j} = \langle\, 0 \mid; \\ &(X-\varepsilon(X)) \mid 0\,\rangle = 0 = \langle\, 0 \mid (X-\varepsilon(X))\,, \quad \forall\, X\in U_q(s\ell_n)\,. \end{aligned} \tag{12.41}$$

The duality between $\mathcal{F}$ and $\mathcal{F}'$ is established following [73] by a bilinear pairing $\langle \cdot \mid \cdot \rangle$ such that

$$\langle\, 0 \mid 0\,\rangle = 1\ , \qquad \langle \Phi \mid A \mid \Psi \rangle = \langle \Psi \mid {}^t A \mid \Phi \rangle \tag{12.42}$$

where $A \to {}^tA$ is a linear anti-involution (*transposition*) on $\mathcal{M}_q$ defined for generic q (i.e. q *not* a root of unity) by

$$\mathcal{D}_i(p)\, {}^t a^i_\alpha = \tilde{a}^\alpha_i := \frac{1}{[n-1]!}\, \varepsilon^{\alpha\alpha_1\dots\alpha_{n-1}}\, \epsilon_{ii_1\dots i_{n-1}}\, a^{i_1}_{\alpha_1} \dots a^{i_{n-1}}_{\alpha_{n-1}}\, ,$$

$${}^t(q^{p_i}) = q^{p_i}\, ;\ \mathcal{D}_i(p) = \prod_{\substack{j<\ell \\ j\neq i\neq \ell}} [p_{j\ell}] \quad (\Rightarrow [\mathcal{D}_i(p), a^i_\alpha] = 0 = [\mathcal{D}_i(p), \tilde{a}^\alpha_i])\, . \tag{12.43}$$

This anti-involution extends to $\mathcal{M}_q$ the transposition in $U_q(s\ell_n)$ (introduced in [72]) determined by its action on the Chevalley generators:

$${}^tE_i = \bar{q}\, F_i\, K_i\, (= F_i\, q^{H_i - 1})\ , \quad {}^tF_i = q\, K_i^{-1}\, E_i\ , \quad {}^tK_i = K_i\ . \tag{12.44}$$

Exercise 12.4. Verify that the linear anti-involution defined by (12.44) corresponds to the (ordinary, matrix) transposition of the monodromy matrix (5.30).

The space $\mathcal{F}$ admits a basis of weight vectors whose inner products can be computed (see Section 3.2 of [70]). Moreover, it was proven in [70] that *for generic* q, $\mathcal{F}$ provides a *model* for $U_q(s\ell_n)$. In other words, $\mathcal{F}$ can be split into an (infinite) direct sum of finite dimensional irreducible $U_q(s\ell_n)$ modules, such that each IR of $U_q(s\ell_n)$ appears exactly once:

$$\mathcal{F} = \bigoplus_p \mathcal{F}_p := \bigoplus_{p_{12}=1}^{\infty} \dots \bigoplus_{p_{n-1\,n}=1}^{\infty} \mathcal{F}_p\ , \qquad p = (p_1, \dots, p_n)\ , \quad p_{ij} = p_i - p_j\ . \tag{12.45}$$

Here $\mathcal{F}_p$ is an ireducible $U_q(s\ell_n)$ module of dimension $d(p)$ (F.40) spanned by vectors of the form

$$h_{\lambda_{n-1}}(a^{n-1}_1, \dots, a^{n-1}_n)\, h_{\lambda_{n-2}+\lambda_{n-1}}(a^{n-2}_1, \dots, a^{n-2}_n) \dots h_{\lambda_1+\dots+\lambda_{n-1}}(a^1_1, \dots, a^1_n) \mid 0\rangle\ , \tag{12.46}$$

where $h_m(x_1, \dots, x_n)$ are homogeneous polynomials of degree m and the non-negative integers

$$\lambda_i + \lambda_{i+1} + \dots + \lambda_{n-1} \equiv p_{in} - n + i\ , \quad i = 1, \dots, n-1\ ,$$

see (11.5), correspond to the length of the i-th row of an admissible Young diagram for $s\ell_n$, cf. Appendix A. (We note that the degree of homogeneity of $h_m(a^i_1, \dots, a^i_n)$ does not change under permutation of the arguments following the exchange relations (12.17); cf., for example, the second equation (12.38).) It follows that operators a^i_α have the meaning of adding a box to the i-th row of the diagram; in particular, a^1_α increases $p_{1n} = \sum_{s=1}^{n-1} p_{ss+1}$ (and a^n_α decreases it) by 1, while a^j_α does not change p_{1n} for $1 < j < n$ but

$$h_m(a^j_1, \dots, a^j_n)\, \mathcal{F}_p = 0 \quad \text{for } m > \lambda_{j-1} \quad (\text{or, } m \geq p_{j-1\,j})\ , \quad j = 2, 3, \dots, n-1\ . \tag{12.47}$$

For $n = 2$ we shall prove that the weight basis

$$\mid p, m\,\rangle = (a^1_1)^m\, (a^1_2)^{p-1-m} \mid 1, 0\,\rangle \tag{12.48}$$

with $\mid 1, 0\,\rangle$ coinciding with the (ket-)vacuum, previously denoted by $\mid 0\,\rangle$, has the properties (4.15), (4.17) of the canonical one.

For q given by (12.3) and $p_{1n} > h$ the module $\mathcal{F}_p$ is indecomposable. In fact, it was conjectured in [70] that the inner square of the highest and lowest weight vectors are given by

$$\langle \lambda_1 \dots \lambda_{n-1} \mid \lambda_1 \dots \lambda_{n-1}\rangle = \prod_{i<j} [p_{ij} - 1]! = \langle -\lambda_{n-1} \dots - \lambda_1 \mid -\lambda_{n-1} \dots - \lambda_1\rangle\, . \tag{12.49}$$

Exercise 12.5. (a) Verify (12.49) for $n = 2$ and $n = 3$.

(b) Assuming (12.49), conclude that for $p_{1n} > h$ the inner product is degenerate.

The quantum exchange relations

$$M_{\pm 2}\, a_1 = a_1\, R_{12}^{\mp}\, M_{\pm 2} \tag{12.50}$$

where

$$R_{12}^{-} = R_{12}\ , \quad R_{12}^{+} = R_{21}^{-1} \tag{12.51}$$

which correspond to the PB (11.74), have the following simple interpretation as a $U_q := U_q(s\ell_2)$ transformation law. If we define the adjoint action of U_q by

$$\mathrm{Ad}_X(z) = \sum_X X_1\, z\, S(X_2) \text{ for } \Delta(X) = \sum_X X_1 \otimes X_2 \tag{12.52}$$

then we find

$$\mathrm{Ad}_X(a^i_\alpha) = a^i_\beta (X^f)^\beta_\alpha\ , \quad \forall\, X \in U_q\ . \tag{12.53}$$

Here the superscript f stands for the fundamental (two-dimensional) representation of U_q:

$$E^f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ , \quad F^f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ , \quad K^f = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}\ ,$$

$$[H^f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ (= H^f)\ . \tag{12.54}$$

Using Eqs. (4.6) and (4.8) (for $r = 1$) for the coproduct and the antipode, we thus reproduce (12.40).

Exercise 12.6. Prove that the transformation law (12.40) implies the relations (4.15), (4.17) for the vectors (12.48).

Exercise 12.7. Verify that the exchange relations (12.4), (12.5) and the determinant condition (12.34) assume, for $n = 2$, the form

$$q^p\, a^1_\alpha = a^1_\alpha\, q^{p+1}\ , \quad q^p\, a^2_\alpha = a^2_\alpha\, q^{p-1}\ , \tag{12.55}$$

$$a^2_\alpha\, a^1_\beta - a^1_\alpha\, a^2_\beta = [p]\, \varepsilon_{\alpha\beta}\ , \qquad a^i_\alpha\, a^i_\beta\, \varepsilon^{\alpha\beta} = 0\ , \quad i = 1, 2\ ,$$
$$a^2_\alpha\, a^1_\beta\, \varepsilon^{\alpha\beta} = [p+1]\ , \qquad a^1_\alpha\, a^2_\beta\, \varepsilon^{\alpha\beta} = -[p-1]\ . \tag{12.56}$$

The factor $\mathcal{D}_i(p)$ in the definition (12.43) of transposition is equal to 1 for $n = 2$ and we have

$${}^t a^1_1 = q^{\frac{1}{2}}\, a^2_2\ , \quad {}^t a^1_2 = -q^{-\frac{1}{2}}\, a^2_1\ , \quad {}^t a^2_2 = q^{-\frac{1}{2}}\, a^1_1\ , \quad {}^t a^2_1 = -q^{\frac{1}{2}}\, a^1_2\ . \tag{12.57}$$

This allows to write the bra vector $\langle p, m \mid$ in the form

$$\langle p, m \mid = \langle 1, 0 \mid (a^2_1)^{p-1-m}\, (a^2_2)^m\, q^{m - \frac{p-1}{2}}\, (-1)^{p-1-m}\,. \tag{12.58}$$

Combining (12.48) and (12.58), and using (12.56) to derive

$$(a^2_2)^m\ \mid p, m\rangle = q^{m(m-p+\frac{1}{2})}\, [m]!\ \mid p-m, 0\rangle\,,$$

$$(-a_1^2)^{p-1-m} \mid p-m,0\rangle = q^{\frac{1}{2}(p-m-1)}\,[p-m-1]!\ \mid 1,0\rangle\,, \tag{12.59}$$

we find

$$\langle p', m' \mid p, m\rangle = \bar{q}^{m(p-1-m)}\,[m]!\,[p-1-m]!\,\delta_{pp'}\,\delta_{mm'} \tag{12.60}$$

(recovering, for $m = p-1$ and $m = 0$, (12.49) for $n = 2$). In the undeformed limit (for $q \to 1$) we reproduce the Schwinger oscillator algebra model [135] of $SU(2)$ (see also its realization [16] in Bargmann's Hilbert space of analytic functions). The U_q covariant deformation of Schwinger's oscillators was first considered by Pusz and Woronowicz [130] and was related to chiral vertex operators in [89].

For $q = q_h$ (12.3) the algebra $U_q(A_r)$ – and, in particular, $U_q \equiv U_q(A_1)$ – has an infinite ideal J_h generated by E^h, F^h and $K^h - K^{-h}$ (see (14.7) below), such that the *restricted QUEA* $\bar{U}_q := U_q/J_h$ is finite dimensional (in fact, $2h^3$ dimensional, for $U_q = U_q(A_1)$). Furthermore, this ideal is represented trivially in $\mathcal{F}$; it indeed follows from (4.15) and (4.17) that

$$E^h\,\mathcal{F} = 0 = F^h\,\mathcal{F} = (K^h - K^{-h})\,\mathcal{F}\,. \tag{12.61}$$

We shall leave the study of the resulting representation of $\bar{U}_q$ and of the Lusztig extension $\tilde{U}_q$ of $\bar{U}_q$ to Section 14.

12.3. From Bloch waves to chiral vertex operators[52]. The covariant chiral quantum field

The exchange relations for the Bloch waves $u(x)$ corresponding to the PB (11.104) involve an x-dependent dynamical R-matrix:

$$u_1(x_1)\,u_2(x_2) = u_2(x_2)\,u_1(x_1)\,R_{12}(p,x_{12}) = P_{12}\,u_1(x_2)\,u_2(x_1)\,\hat{R}_{12}(p,x_{12})\ . \tag{12.62}$$

Here

$$R(p,x) = R^-(p)\,\theta(x) + R^+(p)\,\theta(-x)\ , \qquad \hat{R}(p,x) = P R(p,x) \tag{12.63}$$

and $R^\pm(p)$ are defined in (12.18). Since $u(x)$ should be viewed as an operator-valued distribution, we need to introduce its smeared components which can be defined as the Fourier modes $\ell(\nu)$ of the loop group variables $\ell(x)$ (11.75):

$$\ell(\nu) = \int_{-\pi}^{\pi} e^{-i\nu x}\ell(x)\,dx = \int_{-\pi}^{\pi} e^{-i(\nu+\frac{\not p}{h})x}u(x)\,dx\ , \quad n\in\mathbb{Z}\ . \tag{12.64}$$

It looks natural to try an operator realization of the resulting quantum algebra analogous to that of the zero modes, introducing vector spaces $\mathcal{V}_p$ as infinite dimensional lowest weight $\widehat{su}(n)_k$ current algebra modules. Each $\mathcal{V}_p$ can be fully characterized by its $d(p)$-dimensional lowest energy subspace $\mathcal{V}_p^{(0)}$ such that

$$J_n^a\,\mathcal{V}_p^{(0)} = 0 \quad \text{for } n>0\ , \qquad (L_n - \Delta(p)\,\delta_{n0})\,\mathcal{V}_p^{(0)} = 0 \quad \text{for } n\geq 0\ , \tag{12.65}$$

where

$$\Delta(p) = \frac{C_2(p)}{2h} \tag{12.66}$$

with $C_2(p)$ given by (F.37). We would then have the following analogue of (12.47):

$$h_m(\ell_j^{A_1}(\nu_1),\ldots,\ell_j^{A_n}(\nu_n))\,\mathcal{V}_p = 0 \quad \text{for} \quad m\geq p_{j-1j}\ , \quad j=2,3,\ldots,n-1\ . \tag{12.67}$$

[52]The notion of a chiral vertex operator has been introduced within the axiomatic approach to CFT by Tsuchiya and Kanie [149].

The idea is that $u_j^A(x)$ acts as an elementary *chiral vertex operator* (CVO) or, a step operator, i.e. intertwines

$$\mathcal{V}_{p_{12},\ldots,p_{j-1\,j},p_{j\,j+1},\ldots,p_{n-1\,n}} \qquad \text{with} \qquad \mathcal{V}_{p_{12},\ldots,p_{j-1\,j}-1,p_{j\,j+1}+1,\ldots,p_{n-1\,n}}$$

where we set by definition $\mathcal{V}_{p_{12},\ldots,0,\ldots,p_{n-1\,n}} = 0$; thus, u_1^A increases p_{1n} $(= \sum_{s=1}^{n-1} p_{s\,s+1})$ by 1, u_n^A decreases p_{1n} by 1, while u_j^A does not change p_{1n} for $1 < j < n$.

This idea, however, does not work for q given by (12.3) (q a root of unity). Indeed, consider a matrix element of the type

$$\langle \Phi_{p'} \mid u(z_1)\, u(z_2) \mid \Phi_p \rangle \qquad \text{for} \qquad \Phi_p \in \mathcal{V}_p\ , \quad \Phi_{p'} \in \mathcal{V}_{p'}\ . \tag{12.68}$$

The corresponding (reduced) KZ equation is of hypergeometric type, and the braiding of the corresponding solutions recovers the quantum dynamical R-matrix $R(p)$ (12.10), (12.22) [92]. The problem is that its entries are singular at $p_{ij} = nh$ (and the corresponding solutions of the KZ equation involve logarithms) so that (12.68) only makes sense for $p_{1n} < h$.

By contrast, the exchange relations of the chiral field $g(x)$ (11.2) corresponding to the PB (11.106) are independent of p and hence, do not exhibit singularities:

$$g_1(x_1)\, g_2(x_2) = g_2(x_2)\, g_1(x_1)\, R_{12}(x_{12})\ , \tag{12.69}$$

where

$$R(x) = R^- \theta(x) + R^+ \theta(-x)\ . \tag{12.70}$$

Accordingly, the solutions of the KZ equations for the analog of (12.68) (with $u(z_i)$ replaced by $g(z_i)$, $i = 1, 2$) – as well as their braiding described by (12.69) – are well defined for any p and p' [92]. An analysis performed in [79] (in the context of boundary CFT) demonstrates that the singularities in the exchange relations of the chiral vertex operators are actually cancelled by the zeros of the corresponding relations among the a's.

A stronger result about the regular solutions of the KZ equation has been obtained in [143] (and will be summarized in the following Section 13).

13 Monodromy representations of the braid group [149]

The following braid relations have been derived in [143] for the regular basis (9.21), (9.30)–(9.31). Let b_j stand for the exchange of the variables (z_j, ζ_j) with (z_{j+1}, ζ_{j+1}) along a path (in z-space) for which $z_{j\,j+1} \to e^{-i\pi}\, z_{j\,j+1}$; then

$$b_1\, f_\mu^{(I)}(\xi,\eta) = (1-\xi)^{2I}\,(1-\eta)^{4\Delta_I}\, f_\mu^{(I)}\left(\frac{\xi}{\xi-1}, \frac{\eta\, e^{-i\pi}}{1-\eta}\right) = f_\lambda^{(I)}(\xi,\eta)\, B_{1\mu}^{\lambda}\,,$$

$$b_2\, f_\mu^{(I)}(\xi,\eta) = \xi^{2I}\,\eta^{4\Delta_I}\, f_\mu^{(I)}\left(\frac{1}{\xi}, \frac{1}{\eta}\right) = f_\lambda^{(I)}(\xi,\eta)\, B_{2\mu}^{\lambda}\,, \tag{13.1}$$

where B_1 is a lower triangular, B_2 is an upper triangular matrix:

$$B_{1\,\mu}^{\lambda} = (-1)^{2I-\lambda}\, q^{\lambda(\mu+1)-2I(I+1)} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = B_{2\,2I-\mu}^{2I-\lambda}\,, \tag{13.2}$$

and we have $B_3 = B_1$. Here q may be any *primitive h-th root* of -1:

$$q^h = -1 \qquad (q^n \neq -1 \quad \text{if} \quad 0 < n < h)\,. \tag{13.3}$$

It follows, in particular, that q is a phase factor ($q\,\bar{q} = 1$).

Exercise 13.1. Verify that the inverse matrices to $B_i(q)$ for q satisfying (13.3) are obtained by complex conjugation:

$$B_i(q)\, B_i(\bar{q}) = \mathbb{1} \quad \text{for} \quad q\,\bar{q} = 1 \tag{13.4}$$

($\mathbb{1}$ standing for the $(2I+1)\times(2I+1)$ unit matrix).

Exercise 13.2. Verify the braid relation

$$B_1\, B_2\, B_1 = B_2\, B_1\, B_2 = (-1)^{2I}\, \bar{q}^{2I(I+1)}\, F\,, \quad F_\mu^\lambda = \delta_{2I-\mu}^{\lambda} \tag{13.5}$$

(the *fusion matrix* F is, thus, a permutation matrix satisfying $F^2 = \mathbb{1}$). Verify that

$$B_2 = F\, B_1\, F \qquad (B_1 = F\, B_2\, F\,, \quad F = F^{-1})\,. \tag{13.6}$$

Remark 13.1. It can be demonstrated that, for q satisfying (13.3), $B_i^{2h}\,,\ i = 1,2$ is a multiple of the unit matrix for $2I+1 < h$ but is not diagonalizable for $2I+1 \geq h$.

Exercise 13.3. Verify the statement of Remark 13.1 for small values of h and $2I$.

It follows from Remark 13.1 that the braid matrices B_1 and B_2 are not diagonalizable – and hence not unitarizable for non-integrable representations of the $su(2)$ current algebra (i.e. for representations violating the upper bound $2I \leq k$ (9.7)). Note that the eigenvalues of B_i have absolute value 1, hence the matrices B_i are unitarizable exactly when they are diagonalizable. This explains why the B_1-diagonal basis, used in most of the literature, is ill defined beyond the unitarity limit, and justifies the attribute "regular" for the above triangular basis which always makes sense.

In order to give the reader a better feeling of this monodromy representation of the braid group $\mathcal{B}_4$ we shall consider in more detail the simplest representation corresponding to $2I = 1$ (i.e. to the braiding properties of the 4-point function of the chiral group-valued field $g(z)$).

Exercise 13.4. Verify that the normalized 2×2 braid matrices (of determinant -1)

$$b_1 = q^{\frac{1}{2}} \, B_1^{(2I=1)} = \begin{pmatrix} -\bar{q} & 0 \\ 1 & q \end{pmatrix} , \quad b_2 = q^{\frac{1}{2}} \, B_2^{(2I=1)} = \begin{pmatrix} q & 1 \\ 0 & -\bar{q} \end{pmatrix} \tag{13.7}$$

satisfy (up to the substitution $q \to \bar{q} \equiv q^{-1}$, corresponding to a different convention) the Hecke algebra relations (2.11).

Remark 13.2. The general Hecke algebra representation of $\mathcal{B}_4$ realized on the 4-fold tensor product $(\mathbb{C}^2)^{\otimes 4}$ of the space $\mathbb{C}^2$ of 2-component isospinors is 16 dimensional. It can be constructed in terms of the Tempereley-Lieb projectors (2.12) as follows:

$$\begin{aligned} & b_i = q - e_i \, , \quad i = 1,2,3 \, ; \quad e_1 = (\varepsilon^{\alpha_1 \alpha_2} \, \varepsilon_{\beta_1 \beta_2} \, \delta^{\alpha_3}_{\beta_3} \, \delta^{\alpha_4}_{\beta_4}) \, , \\ & e_2 = (\delta^{\alpha_1}_{\beta_1} \, \varepsilon^{\alpha_2 \alpha_3} \, \varepsilon_{\beta_2 \beta_3} \, \delta^{\alpha_4}_{\beta_4}) \, , \quad e_3 = (\delta^{\alpha_1}_{\beta_1} \, \delta^{\alpha_2}_{\beta_2} \, \varepsilon^{\alpha_3 \alpha_4} \, \varepsilon_{\beta_3 \beta_4}) \end{aligned} \tag{13.8}$$

where $\varepsilon^{\alpha\beta} = \varepsilon_{\alpha\beta}$ is the q-deformed Levi-Civita tensor (2.23). The above 2-dimensional representation of $\mathcal{B}_4$ is a subrepresentation of this 16-dimensional one, spanned by the $SU(2)$ invariant tensors in $(\mathbb{C}^2)^{\otimes 4}$. We leave it to the reader to work out the details of this projection.

We shall end up our study of the 2-dimensional representation of $\mathcal{B}_4$ by answering the following question.

The Schwarz problem: for which values of h $(= 3, 4, \ldots)$ and q satisfying (13.3) is the matrix group generated by the 2×2 matrices b_i (13.7), a finite group?

The answer to this question determines when the KZ equation (for $2I = 1$) admits elementary (algebraic) solutions.

As $b_1^{2h} = b_2^{2h} = \mathbb{I}$ for $h = 3, 4, \ldots$, it is enough to study the *commutator subgroup*, generated by the pair

$$b = b_1^{-1} \, b_2 = b_2 \, b_1 \, b_2^{-1} \, b_1^{-1} = \begin{pmatrix} -q^2 & -q \\ q & 1 - \bar{q}^2 \end{pmatrix} , \quad \bar{b} = b_1, b_2^{-1} \, . \tag{13.9}$$

The argument we shall present in solving the problem (a special case of [141]) is interesting in that it applies some elementary number theoretic methods.

Proposition 13.1. *The real symmetric matrix*

$$A = \begin{pmatrix} [2]^2 & [2] \\ [2] & [2]^2 \end{pmatrix} = {}^t S \begin{pmatrix} [3] & 0 \\ 0 & [2]^2 \end{pmatrix} S \, , \quad S = \begin{pmatrix} 1 & 0 \\ \frac{1}{[2]} & 1 \end{pmatrix} \tag{13.10}$$

(S being the matrix diagonalizing b_1,

$$S \, b_1 S^{-1} = \begin{pmatrix} -\bar{q} & 0 \\ 0 & q \end{pmatrix} , \tag{13.11}$$

which is well defined for $h > 2$) is $\mathcal{B}_4$ invariant:

$$b^* A \, b = A \, , \quad \forall \, b \in \mathcal{B}_4 \quad \Leftrightarrow \quad {}^t b_i \, A = A \, b_i \, , \quad i = 1, 2 \, , \quad \textit{for} \quad \bar{b}_i = b_i^{-1} \, , \tag{13.12}$$

where ${}^t S$ (and ${}^t b$) denotes the transposed of S (and b).

Proof. For b_1 Eq. (13.12) is a consequence of (13.11):

$${}^t b_1 \, A = {}^t b_1 \, {}^t S \begin{pmatrix} [3] & 0 \\ 0 & [2]^2 \end{pmatrix} S = {}^t S \begin{pmatrix} -[3] \, \bar{q} & 0 \\ 0 & [2]^2 \, q \end{pmatrix} S = A \, b_1 \, ;$$

for b_2 both sides of (13.12) give $[2]\begin{pmatrix}1+q^2 & q\\ q & -\bar{q}^2\end{pmatrix}$. The equivalence of the two invariance conditions for the realization (13.7) of b_i follows from (13.4). $\square$

Remark 13.3. The eigenvalues $[2]^2 \pm [2]$ of A differ from those of the diagonal matrix diag $([3],[2]^2)$ of (13.10). However the positivity conditions for both are equivalent because of the *inertia law for non-degenerate quadratic forms.*

Corollary 13.1. *The above* 2*-dimensional representation of* $\mathcal{B}_4$ *is unitarizable provided*

$$([2]=)\ q+\bar{q}=2\cos\frac{\pi}{h}\,, \quad \text{i.e.} \quad q=e^{\pm i\frac{\pi}{h}} \quad (\text{for } h>3)\,. \tag{13.13}$$

(For $h=3$ *the form* A (13.10) *is degenerate since then* $[3]=0$*.) Indeed, for* $h\geq 4$ *the matrix* A *is positive definite since then* $[2]>1$ $([3]\geq 1)$.

Eq. (13.13) that guarantees the positivity of A is the only one which depends on the choice of a primitive root of (13.3). To stress this point we introduce the notion of a *Galois*[53] *automorphism* for the *cyclotomic field* defined by (13.3). The map $q\rightarrow q^n$ is a Galois automorphism of the field $\mathbb{Q}[q]$ of polynomials in q (obeying (13.3)) with rational coefficients, iff $(n,2h)=1$ – i.e. iff n is coprime with $2h$.

Exercise 13.5.
(a) Prove that the Galois group for $h=10$ is isomorphic to the product of cyclic groups of two and four elements, $\mathbb{Z}/(2)\times\mathbb{Z}/(4)$. (*Hint* : it is spanned by the exponents ±1, ±3, ±7, ±9 with multiplication mod 20.)

(b) Prove, similarly, that the Galois group for $h=30$ is a 16 element group isomorphic to $\mathbb{Z}/(2)\times\mathbb{Z}/(2)\times\mathbb{Z}/(4)$.

Remark 13.4. The solution of Exercise 11.5(b) is related to the Coxeter exponents $(1,7,11,13,17,19,23,29)$ of the exceptional group E_8. (For an application of the Coxeter exponents to the classification of the $\widehat{su}(2)_k$ conformal invariant theories – see [32].)

A form A with coefficients in $\mathbb{Q}[q]$ is said to be *totally positive* if it is positive for all Galois transforms $q\rightarrow q^n$, $(n,2h)=1$ of q. The relevance of this concept to our problem is revealed by the following crucial lemma.

Proposition 13.2. *If the form* (13.10) *is totally positive,* i.e. *if* $[3]=q^2+1+\bar{q}^2>0$ *for all primitive roots of* (13.3)*, then the* 2*-dimensional representation of* $\mathcal{B}_4$*, which leaves the non-degenerate form A invariant, is a finite matrix group. Conversely, if the invariant hermitean form is unique (or, equivalently, if the representation of* $\mathcal{B}_4$ *under consideration is irreducible), then the total positivity of A is necessary for its finiteness.*

[53]The legendary Evariste Galois (1811-1832) was only appreciated posthumously. His major work on algebraic equations was finally published in 1846 (following a positive review by Liouville 3 years earlier) – some 14 years after his fatal duel. In the night before the duel Galois, 20, composed a letter to his friend Auguste Chevalier outlining his mathematical ideas. Here is what Hermann Weyl had to say about this "testament": "This letter, if judged by the novelty and profundity of ideas it contains, is perhaps the most substantial piece of writing in the whole literature of mankind".

The *proof* is based on the fact that the invariance group of a totally positive form A over a cyclotomic field is compact. Since $\mathcal{B}_4$ is discrete it would follow that the matrix group generated by b_1, b_2 (13.7) is finite.

As any finite dimensional representation of a compact group is unitarizable the unique invariant form A should be positive together with all its Galois transforms. $\square$

Proposition 13.3. *The commutator subgroup of the* 2×2 *matrix* ($2I = 1$) *realization of* $\mathcal{B}_4$ *generated by the matrices* b *and* $\bar{b}$ (10.27) *is only finite for* $h = 4, 6, 10$. *It is isomorphic to:* (i) *the* 24 *element double cover* $\tilde{\mathcal{A}}_4$ *of the tetrahedral group for* $h = 4$*;* (ii) *the* 8 *element group of quaternion units for* $h = 6$*, and* (iii) *the* 120 *element double cover* $\tilde{\mathcal{A}}_5$ *of the icosahedral group for* $h = 10$. *(Here* $\mathcal{A}_n$ *stands for the alternating subgroup of even permutations of* $\mathcal{S}_n$ *represented by* 3×3 *orthogonal matrices,* $\tilde{\mathcal{A}}_n$ *is its double cover belonging to* $SU(2)$*.)*

Proof. For both $h = 4$ ($[3] = 1$) and $h = 6$ ($[3] = 2$) the q-number $[3]$ is independent of the choice of a primitive h-th root of -1 – and is positive. In general, we have to verify for which h

$$[3]_{q^n} = 1 + 2 \cos \frac{2\, n\pi}{h} \geq 0 \quad \text{for all } n \text{ such that } (2h, n) = 1\ . \tag{13.14}$$

For $h = 4m - 1$, $m = 2, 3, \ldots$, we can set $n = 2m - 1$, for $h = 4m + 1$, $m = 1, 2, \ldots$, we may choose $n = 2m+1$, violating in both cases the inequality (13.14) (the maximal value of $[3]_{q^{2m+1}}$ occuring for $m = 1$: $[3]_3 = 1 + 2\cos\frac{6\pi}{5} = 1 - 2\cos\frac{\pi}{5} = 1 - \frac{1+\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} < 0$). For $h = 4m$, $m \geq 2$ we have $[3]_{q^{2m+1}} = 1 - 2\cos\frac{\pi}{2m} \leq 1 - 2\cos\frac{\pi}{4} = 1 - \sqrt{2} < 0$. Finally, for $h = 4m + 2$, $m \geq 2$ we have $[3]_{q^{2m-1}} = 1 + 2\cos\left(\frac{2m-1}{2m+1}\,\pi\right) = 1 - 2\cos\frac{2\pi}{2m+1}$ which implies

$$[3]_{q^3} \stackrel{(h=10)}{=\!=} 1 - 2\cos\frac{2\pi}{5} = \frac{3-\sqrt{5}}{2} > 0\,, \quad [3]_{q^{2m-1}} \leq 1 - 2\cos\frac{2\pi}{7} < 0 \text{ for } m \geq 3\,. \tag{13.15}$$

We conclude that the exceptional properties of the *golden ratio* (i.e. of $x = 2\cos\frac{\pi}{5}$ ($= \frac{1+\sqrt{5}}{2}$) satisfying $x^2 = x + 1$) ensure the positivity of $[3]_{q^3}$ thus verifying total positivity for ($h = 6$ and) $h = 10$ only (among $h = 4m + 2$).

In order to identify the various finite groups we use

$$b^3 = \bar{b}^3 = -1 = (b^{-1}\,\bar{b})^2 \quad \text{for} \quad h = 4\,, \tag{13.16}$$

$$b^2 = \bar{b}^2 = (b^{-1}\,\bar{b})^2 = -1 \quad \text{for} \quad h = 6\,, \tag{13.17}$$

$$(b^{-1}\,\bar{b}^2)^2 = (b^{-1}\,\bar{b})^3 = \bar{b}^5 = -1 \quad \text{for} \quad h = 10\,. \tag{13.18}$$

$\square$

Remark 13.5. Propositions 13.2 and 13.3 are special cases of Lemma 3.2 and Theorem 3.3 of [141] where *all* monodromy representations of $\mathcal{B}_4$ (for the $su(2)$ current algebra) realized by finite matrix groups are classified. The results for the 2-dimensional representations displayed here have been derived earlier (by quite different methods) by V. Jones (see the first reference [96]). Note that the exceptional values 4, 6 and 10 of the height h correspond to levels $k = h-2$ equal to the (real) dimensions 2, 4, 8 of the field of complex numbers and of the division algebras of quaternions and octonions.

The following corollary of Proposition 13.3 (much as the end of the proof of that Proposition) require deeper familiarity with finite groups defined in terms of generators and relations than we have given here.

Exercise 13.6. Prove as a corollary of Proposition 11.3 that the groups generated by the matrices b_i are central extensions of (i) the 48 element binary octahedral group (the double cover of the permutation group $\mathcal{S}_4$, isomorphic to the symmetry group of the octahedron) – for $h = 4$; (ii) the 24 element binary tetrahedral group $\tilde{\mathcal{A}}_4$ – for $h = 6$; (iii) the binary icosahedral group $\tilde{\mathcal{A}}_5$ – the only one coinciding with its commutator subgroup – for $h = 10$. (For background on discrete groups defined by generators and relations – see [39].)

The knowledge of the $(2I + 1)$-dimensional realization of $\mathcal{B}_4$ in the space of $su(2)$ current algebra 4-point blocks allows to establish another type of duality relation between quantum group and braid group representations. In order to display its full content we need to say something more about the representation theory of $U_q(A_1)$ for q an even root of unity. This will be the starting point of the next section.

14 Restricted and Lusztig QUEA for $q^h = -1$ and their representations

The 3-dimensional complex Lie algebra $A_1 \simeq s\ell(2)$ is *simple*: it admits no non-trivial ideals. The same is true for its compact real form $su(2)$. Simple (associative) algebras are the universal enveloping algebra $U(A_1)$ as well as its deformation $U_q(A_1)$ for *generic* q (i.e. $q \neq 0$ and q *not* a root of unity). By contrast, if q satisfies (13.3) then $U_q(A_1)$ admits a huge proper ideal. Technically, this comes out because the q-numbers $[nh]$ vanish for $q^h = -1$.

Exercise 14.1. Prove the CR

$$[E, F^n] = [n]\, F^{n-1}\, [H + 1 - n]\,, \quad [E^n, F] = [n]\, E^{n-1}\, [H + n - 1]\,. \tag{14.1}$$

Deduce that these commutators vanish iff n is a multiple of h.

The result of Exercise 14.1 allows to prove that E^h and F^h generate an ideal $U_q(A_1)$ for $q^h = -1$. In order to find a *maximal ideal* which contains these two elements we shall first construct a *model space* of $U_q(A_1)$ for generic q. (We recall that a vector space $\mathcal{F}$ is a model space for a Lie algebra $\mathcal{G}$ or for its UEA $U(\mathcal{G})$ if $\mathcal{F}$ is the direct sum of its finite dimensional irreducible modules, each encountered with multiplicity one.) To this end, we introduce the direct sum $\mathcal{F}$ $(= \mathcal{F}(q))$ of p-dimensional $U_q(A_1)$ modules $\mathcal{F}_p$ defined in Section 4:

$$\mathcal{F} = \bigoplus_{p=1}^{\infty} \mathcal{F}_p\,, \quad \mathcal{F}_p = \mathrm{Span}\,\{|p,m\rangle\,,\ 0 \leq m \leq p-1\} \tag{14.2}$$

where the canonical basis $\{|p,m\rangle\}$ is defined by the relations (4.15) (4.17).

Exercise 14.2.
(a) Derive the relations

$$E^n \mid p,m\rangle = \frac{[p-m-1]!}{[p-m-n-1]!} \mid p, m+n\rangle\ , \quad F^n \mid p,m\rangle = \frac{[m]!}{[m-n]!} \mid p, m-n\rangle\ . \tag{14.3}$$

(b) Verify, using (10.46)–(10.48), that $\mathcal{F}$ appears as a Fock space for a^i_α:

$$a^2_\alpha \mid 1,0\rangle = 0\ , \qquad \mid p,m\rangle = (a^1_1)^m\, (a^1_2)^{p-1-m} \mid 1,0\rangle\ . \tag{14.4}$$

(c) Deduce that for $q^h = -1$ the following identities hold on $\mathcal{F}$:

$$E^h\, \mathcal{F} = 0 = F^h\, \mathcal{F} = (K^{2h} - 1)\, \mathcal{F}\ . \tag{14.5}$$

Exercise 14.3. Assuming the knowledge of the PBW basis of $U_q(A_1)$ (viewed as a vector space – cf. Section 5),

$$\{E^\mu\, K^n\, F^\nu\ ,\quad \mu,\nu = 0,1,\dots\ ,\quad n \in \mathbb{Z}\} \tag{14.6}$$

prove that the quotient space with respect to the two-sided ideal defined by the kernel (14.5) of the representation of $U_q(A_1)$ in $\mathcal{F}$,

$$\bar{U}_q := U_q(A_1)/J_h\ , \quad J_h = \{E^h, F^h, K^h - K^{-h}\}\ , \tag{14.7}$$

is $2\,h^3$-dimensional.

The quotient $\bar{U}_q$ is called the *restricted QUEA* in [58] and [74].

We can similarly define the $4\,h^3$-dimensional quotient $\bar{D}_q$ of the double cover D_q of U_q (introduced in Section 5) by the same ideal J_h expressed in terms of k instead of $K = k^2$:

$$J_h = \{E^h, F^h, k^{2h} - k^{-2h}\}\,, \quad \bar{D}_q = D_q / J_h\,. \tag{14.8}$$

It allows to give meaning to the *universal R-matrix* of type (5.16) as a polynomial in the $\bar{D}_q$ generators, without invoking topology and completion. The reader will find the proof of the following result in [58] (see also Sections 2.2 and 3.1 of [74], whose conventions we have adopted here).

Proposition 14.1. (a) *The PBW bases in $\bar{U}_q(b_+)$ and $\bar{U}_q(b_-)$,*

$$f_{\nu n} = F^\nu k_+^n\ , \quad e_{\mu m} = \frac{(-\lambda)^\mu\, q^{-\frac{\mu(\mu-1)}{2}}}{4h\,[\mu]!} \sum_{s=0}^{4h-1} q^{\frac{ms}{2}}\, E^\mu k_-^s\ ,$$

$$m, n = 0, \dots, 4h-1\,, \quad \mu, \nu = 0, \dots, h-1\,, \tag{14.9}$$

are dual to each other with respect to the bilinear form defined in Section 5 (see Eq. (5.12)):

$$\langle e_{\mu m}, f_{\nu n}\rangle = \delta_{\mu\nu}\,\delta_{mn}\ , \quad \mu, \nu = 0, 1, \dots, h-1\ , \quad m, n = 0, 1, \dots, 4h-1\ . \tag{14.10}$$

(b) *The R-matrix of the* ($16h^4$*-element) quantum double is given by*

$$\mathcal{R}^{\rm double} = \sum_{\nu=0}^{h-1} \sum_{n=0}^{4h-1} f_{\nu n} \otimes e_{\nu n}\,. \tag{14.11}$$

It reduces for $k_+ = k_- \equiv k$ (5.18) *to the R-matrix of the ($4h^3$-element) double cover $\bar{D}_q$ of $\bar{U}_q$:*

$$\mathcal{R} = \frac{1}{4h} \sum_{\nu=0}^{h-1} \frac{(-\lambda)^\nu}{[\nu]!}\, q^{-\binom{\nu}{2}}\, F^\nu \otimes E^\nu \sum_{m,n=0}^{4h-1} q^{\frac{mn}{2}}\, k^m \otimes k^n\ \in \bar{D}_q \otimes \bar{D}_q\,, \tag{14.12}$$

which satisfies the quasi-triangularity condition (4.33).

Exercise 14.4. Derive from (14.12) the expression (5.5) for the 2-dimensional representation of D_q for which

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & \bar{q}^{\frac{1}{2}} \end{pmatrix} \quad (E^2 = 0 = F^2)\,. \tag{14.13}$$

Remark 14.1. The second universal R-matrix (5.24) also has a finite dimensional counterpart $\tilde{\mathcal{R}}$. Exchanging the factors in (14.12) and using (4.36), we obtain

$$\tilde{\mathcal{R}} = \frac{1}{4h} \sum_{m,n=0}^{4h-1} q^{-\frac{mn}{2}}\, k^m \otimes k^n \sum_{\nu=0}^{h-1} \frac{\lambda^\nu}{[\nu]!}\, q^{\binom{\nu}{2}}\, E^\nu \otimes F^\nu\,. \tag{14.14}$$

Given the universal R-matrix $\mathcal{R}$, one defines the (universal) *M-matrix* as $\mathcal{M} = \mathcal{R}_{21}\mathcal{R}$. It turns out that the M-matrix obtained from $\mathcal{R}\,(\in \bar{D}_q \otimes \bar{D}_q)$ (14.12) only contains even powers of k and hence, cf. Eq. (5.18), belongs to $\bar{U}_q \otimes \bar{U}_q \subset \bar{D}_q \otimes \bar{D}_q$:

$$\mathcal{M} = \mathcal{R}_{21}\mathcal{R} \;=\; \frac{1}{2h}\sum_{\mu,\nu=0}^{h-1} \frac{q^{\frac{\nu(\nu+1)-\mu(\mu-1)}{2}}\,(-\lambda)^{\mu+\nu}}{[\mu]!\,[\nu]!}\, E^\mu\, F^\nu \otimes F^\mu\, E^\nu \times \sum_{m,n=0}^{2h-1} q^{mn+\nu(n-m)}\, K^m \otimes K^n \;. \tag{14.15}$$

It follows from (4.32) (see also Exercise 5.2) and from the definition of $\mathcal{M}$ that it *commutes* with the coproduct,

$$\mathcal{M}\,\Delta(X) = \Delta(X)\,\mathcal{M}\;, \quad \forall X \in \bar{D}_q\;. \tag{14.16}$$

Obviously, the M-matrix obtained from the second universal R-matrix (14.14) $\tilde{\mathcal{R}} = \mathcal{R}_{21}^{-1}$ is just the inverse of (14.15).

Exercise 14.5.
(a) Verify that Eq.(5.17) for the 2×2 monodromy matrix M (with M_+ and M_- given by (5.17), (5.18)) reduces to

$$M = q^{-\frac{1}{2}} \begin{pmatrix} \lambda^2\, F\, E + q^{-1}K^{-1} & -q^{-1}\lambda\, F\, K \\ -\lambda\, E & q^{-1}K \end{pmatrix}\;, \qquad \lambda = q - q^{-1}\;. \tag{14.17}$$

(b) Show that M is obtained (up to the "quantum prefactor" $q^{-3/2}$) from $\mathcal{M}$ (14.15) by evaluating the first factor in the tensor square of $\bar{U}_q$ in the 2-dimensional fundamental representation $\pi_f(X) \equiv X^f$:

$$q^{\frac{3}{2}}\, M = (\pi_f \otimes \mathrm{id})\,\mathcal{M}\,. \tag{14.18}$$

Hint : Use that, for $q^h = -1$,

$$\sum_{s=0}^{2h-1} q^{ms} = \begin{cases} 2h & \text{for } m \equiv 0 \; mod\; 2h \\ 0 & \text{otherwise} \end{cases}\;.$$

The reader may learn more about the monodromy matrix from [58] and [74].

In order to display a new duality relation between $\mathcal{B}_4$ and U_q representations for the non-unitary extended chiral $su(2)$ WZNW model we need the *Lusztig extension* of the restricted QUEA $\bar{U}_q$ (see [111]). We first introduce, following [111], the *divided powers*

$$E^{(n)} = \frac{1}{[n]!}\, E^n\;, \quad F^{(n)} = \frac{1}{[n]!}\, F^n \tag{14.19}$$

satisfying $X^{(m)}\, X^{(n)} = \left[{n+m \atop n}\right] X^{(n+m)}$ ($\left[{n+m \atop n}\right] = \frac{[n+m]!}{[n]!\,[m]!}$; $X = E, F$),

$$[E^{(m)}, F^{(n)}] = \sum_{s=1}^{\min(m,n)} F^{(n-s)} \begin{bmatrix} H + 2s - m - n \\ s \end{bmatrix} E^{(m-s)}\,. \tag{14.20}$$

The right hand side of (14.19) only has a clear meaning for $n < h$ (since $[h] = 0$). The subsequent relations, however, make sense for all positive integers m, n and can serve as an implicit definition for higher divided powers. It is sufficient to add two new elements $E^{(h)}$ and

$F^{(h)}$ in order to obtain an infinite extension U_h of $\bar{U}_q$. Indeed, their powers generate a sequence of new elements.

Exercise 14.6.
(a) Defining the ratio $\frac{[nh]}{[h]}$ as a polynomial in $q^{\pm 1}$, deduce

$$\frac{[nh]}{[h]} = \sum_{\nu=0}^{n-1} q^{(n-1-2\nu)h} = (-1)^{n-1}\, n\,, \quad \begin{bmatrix} nh \\ n \end{bmatrix} = (-1)^{(n-1)h}\, n\,. \tag{14.21}$$

(*Hint* : use the identity $[nh+m] = (-1)^n\,[m]$.)

(b) Derive the general formula

$$\begin{bmatrix} Mh+m \\ Nh+n \end{bmatrix} = (-1)^{(M-1)Nh/mN-nM} \begin{bmatrix} m \\ n \end{bmatrix} \begin{pmatrix} M \\ N \end{pmatrix} \tag{14.22}$$

for $M \in \mathbb{Z}$, $N \in \mathbb{Z}_+$, $0 \le m, n \le h-1$, $\begin{pmatrix} M \\ N \end{pmatrix} = \frac{M(M-1)\ldots(M-N+1)}{N!}$.

For $n < h$ it is easy, using exercise 14.2(a), to verify the formulae

$$E^{(n)} \mid p, m\rangle = \begin{bmatrix} p-m-1 \\ n \end{bmatrix} \mid p, m+n\rangle\,, \ F^{(n)} \mid p, m\rangle = \begin{bmatrix} m \\ n \end{bmatrix} \mid p, m-n\rangle \tag{14.23}$$

which allow to extend the action of $E^{(n)}$ and $F^{(n)}$ on the canonical basis to all positive n.

We shall now describe the *irreducible representations* (IRs) of $\bar{U}_q$ and will then single out the IRs of U_h in $\mathcal{F}$.

It is convenient to introduce an operator $q^{\hat{p}}$ (and its inverse, $\bar{q}^{\,\hat{p}}$) which is diagonal on the canonical basis and has $2h$ different eigenvalues (that fix, in particular, the Casimir invariant (4.14)):

$$(q^{\hat{p}} - q^p) \mid p, m\rangle = 0\,, \quad C = q^{\hat{p}} + \bar{q}^{\,\hat{p}}\,, \quad q^{h\hat{p}} = q^{-h\hat{p}}\,. \tag{14.24}$$

The IRs of $\bar{U}_q$ are classified in [58] (these authors do not use, however, the operator $q^{\hat{p}}$ and introduce bases inequivalent to ours).

Proposition 14.2. *The finite dimensional QUEA $\bar{U}_q$ has exactly $2h$ IRs $V_p^{\pm}$, labeled by their* dimension *p and* parity *ϵ such that*

$$(q^{\hat{p}} - \epsilon\, q^p)\, V_p^{\epsilon} = 0\,, \quad \dim V_p^{\epsilon} = p\,, \quad p = 1, \ldots, h\,, \quad \epsilon = \pm\,. \tag{14.25}$$

The $\bar{U}_q$ module V_p^{ϵ} can be equipped with a canonical basis $|p, m\rangle^{\epsilon}$, $0 \le m \le p-1$ $(1 \le p \le h)$ such that

$$(q^H - \epsilon\, q^{2m-p+1}) \mid p, m\rangle^{\epsilon} = 0\,, \quad E \mid p, p-1\rangle^{\epsilon} = 0 = F \mid p, 0\rangle^{\epsilon}\,. \tag{14.26}$$

Corollary. *Eqs.* (14.25), (14.26) *and* (14.4) *imply the relations*

$$(EF - \epsilon\,[m][p-m]) \mid p, m\rangle^{\epsilon} = 0 = (FE - \epsilon\,[m+1][p-m-1]) \mid p, m\rangle^{\epsilon}\,. \tag{14.27}$$

We shall identify in what follows the irreducible $\bar{U}_q$ modules V_p^{ϵ} in the ($U_q(A_1)$-model) space $\mathcal{F}$. We will not reproduce the proof of [58] that these representations exhausts the IRs of $\bar{U}_q$.

The identification $V_p^+ = \mathcal{F}_p$ for $1 \le p \le h$ is immediate.

Exercise 14.7. Prove that the spaces $\mathcal{F}_{h+p}$, $1 \le p \le h$ admit two p-dimensional $\bar{U}_q$-invariant subspaces isomorphic to V_p^-. Verify that $\mathcal{F}_{h+p}$ is indecomposable for $0 < p < h$ and that the quotient $\mathcal{F}_{h+p}/V_p^- \oplus V_p^-$ is isomorphic to V_{h-p}^+. (*Hint* : identify $V_p^- \oplus V_p^-$ with the invariant subspace of $\mathcal{F}_{h+p}$ spanned by $\{\mid h+p, m\rangle\} \oplus \{\mid h+p, h+m\rangle\}$, $0 \le m \le p-1$.)

Remark 14.2. The actions of E and F on the two copies of V_p^- are equivalent albeit not identical:

$$E \mid h+p, m\rangle = -[p-m-1] \mid h+p, m+1\rangle \,,$$
$$F \mid h+p, m\rangle = [m] \mid h+p, m-1\rangle \tag{14.28}$$
$$E \mid h+p, h+m\rangle = [p-m-1] \mid h+p, h+m+1\rangle \,,$$
$$F \mid h+p, h+m\rangle = -[m] \mid h+p, h+m-1\rangle \,, \tag{14.29}$$

both yielding (14.27). (We may identify $|p, m\rangle^-$ with either $\mid h+p, h+m\rangle$ or $(-1)^m \mid h+p, m\rangle$.)

Exercise 14.8. Prove that the $\bar{U}_q$-modules $\mathcal{F}_{2h+p}$ $(1 \le p \le h)$ admit three p-dimensional invariant subspaces, each isomorphic to V_p^+, while the quotient space $\mathcal{F}_{2h+p}/V_p^+ \oplus V_p^+ \oplus V_p^+$ (for $p < h$) is isomorphic to $V_{h-p}^- \oplus V_{h-p}^-$. Describe the structure of $\mathcal{F}_{Nh+p}$, $N \in \mathbb{N}$, $1 \le p < h$.

Using the term *subquotient* for either an $\bar{U}_q$ submodule or a quotient we have the following easily verifiable result.

Proposition 14.3. *The direct sum of irreducible $\bar{U}_q$-modules that appear as subquotient of $\mathcal{F}_{Nh+p}$ of a given parity ϵ spans a single IR $\mathcal{V}_p^\epsilon$ of U_h; we have the following exact sequence (for $0 < p < h$) of U_h modules:*

$$0 \to \mathcal{V}_p^{\epsilon_N} \to \mathcal{F}_{Nh+p} \to \mathcal{V}_{h-p}^{-\epsilon_N} \to 0 \;, \qquad \epsilon_N = (-1)^N \,,$$
$$\mathcal{V}_p^{\epsilon_N} = \bigoplus_0^N V_p^{\epsilon_N} \;, \qquad \mathcal{V}_{h-p}^{-\epsilon_N} = \bigoplus_1^N V_{h-p}^{-\epsilon_N} \;. \tag{14.30}$$

Sketch of proof. A straightforward application of (14.22) and (14.23) gives

$$E^{(h)} \mid Nh+p, nh+m\rangle = \begin{bmatrix} (N-n)\,h+p-m-1 \\ h \end{bmatrix} \mid Nh+p, (n+1)\,h+m\rangle$$
$$= (-1)^{(N-n-1)h+p-m-1}(N-n) \mid Nh+p, (n+1)\,h+m\rangle \;,$$
$$0 \le n < N \,, \quad 0 \le m < p \le h \;, \tag{14.31}$$

$$E^{(h)} \mid Nh+p, nh+p+m\rangle = \begin{bmatrix} (N-n)\,h-m-1 \\ h \end{bmatrix} \mid Nh+p, (n+1)\,h+p+m\rangle \;,$$
$$0 \le n < N-1 \,, \quad 0 \le m \le h-p-1 \tag{14.32}$$

and similar relations for $F^{(h)}$. These relations imply the irreducibility with respect to U_h of the direct sums $\mathcal{V}_p^{\epsilon_N}$ and $\mathcal{V}_{h-p}^{\epsilon_N}$ (14.30) of IRs of $\bar{U}_q \subset U_h$. □

On the other hand, the relations

$$E^{(h)}\,\mathcal{F}_p = 0 = F^{(h)}\,\mathcal{F}_p \quad \text{for} \quad 1 \le p \le h \tag{14.33}$$

tell us that the "Lusztig quantum group" U_h only plays a role in $\mathcal{F}_p$ for $p > h$. Our aim will be to establish a duality relation between the indecomposable representations of U_h in $\mathcal{F}_{Nh+p}$ displayed in Proposition 14.3 and the representations (13.2) of $\mathcal{B}_4$ for $2I+1 = Nh+p$. We denote the corresponding p-dimensional $\mathcal{B}_4$ module of 4-point blocks by $S_4(p)$.

Proposition 14.4. (see Theorem 4.1 of [74])
(a) *The $\mathcal{B}_4$ modules $\mathcal{S}_4(p)$ are irreducible for $0 < p < h$ and for $p = Nh$.*

(b) *For $N > 0$ and $0 < p < h$, $S_4(Nh+p)$ is indecomposable with structure dual to that of $\mathcal{F}_{Nh+p}$ displayed in Proposition* 14.3. *It has a $N(h-p)$-dimensional invariant subspace*

$$S(N, h-p) = \text{Span}\,\{f_\mu^{(Nh+p)}\,,\ \mu = nh+p, \ldots, (n+1)\,h-1\}_{n=0}^{N-1} \tag{14.34}$$

which carries an IR of $\mathcal{B}_4$. The $(N+1)p$-dimensional quotient space $\tilde{S}(N+1,p)$ also carries an IR of the braid group.

Proof. The $\mathcal{B}_4$-invariance of $S(N,h-p)$ (14.34) follows from the proportionality of the $(Nh+p)$-dimensional matrices (13.2) to the q-binomial coefficients:

$$\begin{aligned} B^{mh+\alpha}_{1\,nh+\beta} &\sim \begin{bmatrix} nh+\alpha \\ nh+\beta \end{bmatrix} \sim \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \binom{m}{n} = 0 \\ B^{mh+\alpha}_{2\,nh+\beta} &\sim \begin{bmatrix} (N-m)\,h+p-\alpha-1 \\ (N-n-1)\,h+h+p-\beta-1 \end{bmatrix} \\ &\sim \begin{bmatrix} p-\alpha-1 \\ h+p-\beta-1 \end{bmatrix} \binom{N-m}{N-n-1} = 0 \end{aligned}$$

for $m = 0, \ldots, N, 0 \le \alpha \le p-1$, $n = 0, \ldots, N-1$, $p \le \beta \le h-1$; they vanish since $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$ for $0 \le \alpha < \beta$ (α, β integers). An inspection of the same expression (13.2) allows to conclude that the space $S(N,h-p)$ has no $\mathcal{B}_4$-invariant complement in $\mathcal{S}_4(Nh+p)$ which is, thus, indeed indecomposable. It is also readily verified that the quotient space

$$\tilde{S}(N+1,p) = \mathcal{S}_4(Nh+p)/S(N,h-p)$$

carries an IR of $\mathcal{B}_4$. □

We thus see that the indecomposable representations $\mathcal{F}_{Nh+p}$ (of U_h) and $\mathcal{S}_4(Nh+p)$ (of $\mathcal{B}_4$) contain the same number (two) of irreducible components (of the same dimensions) but the arrows of the exact sequences are reversed. This sums up the meaning of duality for indecomposable representations.

Remark 14.2. Note that the difference of conformal dimensions

$$\Delta_{Nh+I} - \Delta_I = N(Nh+p) \qquad (0 < p = 2I+1 < h)$$

is a (positive) integer; this explains the similarity of the corresponding braid group representations $\mathcal{S}_4(p)$ and $\mathcal{S}_4(2Nh+p)$. There is a unique 1-dimensional subspace $S(1,1) \subset S_4(2h-1)$ among the $\mathcal{B}_4$-invariant subspaces displayed in Proposition 14.4 corresponding to a non-unitary local field of isospin and conformal dimension $h-1$:

$$\Delta_{h-1} = \frac{(h-1)\,h}{h} = h-1\ . \tag{14.35}$$

It has rational correlation functions; in particular, the 4-point amplitude $f_{h-1}^{(h-1)}\ (\xi,\eta)$ (9.30)–(9.32) is a polynomial [91]. It therefore gives rise to a non-unitary local extension of the $\widehat{su}(2)_k$ current algebra that deserves a further study.

15 Outlook

In conclusion we shall sum up the philosophy underlying these notes – and the ensuing choice of material – and will then list some related topics which appear to be interesting and important but remain outside the scope of the present lectures.

It is natural from both physical and mathematical point of view to associate with any "symmetry" (meaning symmetry group, Lie algebra or a generalization thereof) a family (or "category") of representations equipped with a *tensor product.* The fact that the tensor product of representations is again a representation (of the same symmetry) leads us to the concept of a *coproduct.* The *commutant* of a tensor product representation yields the notion of a *braid group* which reduces to a *permutation group* when the symmetry is described by an ordinary group. If we think of *irreducible representations* as describing *elementary objects* (particles, excitation) then the behaviour under braiding (that exchanges elementary objects) would determine the particle *statistics.* We are thus led to consider the pair *symmetry and statistics* as a whole. The generalization or *deformation* of one requires a similar deformation of the other.

Quantum groups (and their generalizations [23]) are coupled to *braid group statistics* (as already the title of these lectures suggests). Existing attempts at phenomenological applications of "q-symmetries" (viewing q as one more parameter to fit data), that ignore the (necessarily!) accompanying it braid group statistics, are, in our opinion, ill conceived.

The appearance of *monodromy* (a *normal subgroup* of the braid group) is a sign of the presence of multivalued correlation functions which naturally arise in a non-simply connected *configuration space* – that is the case of dimension two. In higher dimensions the *fundamental group* π_1 of configuration space is trivial. Indeed, the deep analysis of Doplicher-Haag-Roberts of the structure of *superselection sectors* in a local relativistic quantum theory (a work spanned over more than 20 years, culminating in [48], and recounted in [87]) demonstrates that the gauge symmetry (of the first kind) of local observables is implemented by a compact group and is thus accompanied by a permutation group (para) statistics (reduced, essentially, to the familiar Bose and Fermi statistics). We, hence, only consider applications of quantum symmetry and braid group statistics to 2-dimensional conformal field theory, cf. [147]. (Our analysis of such "applications" is restricted to the formalism. The relevance, say, of anyonic statistics to the theory of fractional quantum Hall effect is only alluded to.) We have given more room to the (mathematically) intriguing non-abelian QUEA which appear as gauge symmetries of chiral conformal fields. A gauge symmetry, by its definition, does *not* affect observables. Accordingly, it is only manifest after one splits the observable $2D$ fields into *chiral vertex operators*, corresponding to the splitting of $2D$ correlation functions (single valued in the Euclidean domain) into multivalued *conformal blocks.*

Among the big omissions from the present survey the closest in spirit – and thereby particularly regrettable – is the Chern[54]-Simons theory about which we shall just say a few words and give a few references.

The *Chern-Simons theory* is a *topological gauge theory* on a three (space-time) dimensional manifold M. Here by topological we mean that its action does not depend on the metric on M. Let $\boldsymbol{A}$ be a connection one-form with values in a Lie algebra $\mathcal{G}$. (For $\mathcal{G} = u(n)$ – a commonly encountered example – this means that $\boldsymbol{A}$ is an antihermitean $n \times n$ matrix of 1-forms.) The curvature 2-form $\boldsymbol{F}$ is defined, as usual, by

$$\boldsymbol{F} = d\boldsymbol{A} + \boldsymbol{A} \wedge \boldsymbol{A} \,. \tag{15.1}$$

[54]The Chinese American mathematician Shiing-Shen Chern (1911-2004), a leading differential geometer of 20^{th} century, wrote the paper on Chern-Simons forms in 1974 with his student Jim Simons.

An example of a topological action density in four space time dimensions is given by the so called "θ-term"[55] the 4-form $\mathrm{tr}(\boldsymbol{F}\wedge\boldsymbol{F})$ $(=\boldsymbol{F}^{\alpha}_{\beta}\wedge F^{\beta}_{\alpha})$, which is a total derivative (when expressed in terms of $\boldsymbol{A}$). The *Chern-Simons form*

$$\omega_3 = \mathrm{tr}\left(\boldsymbol{A}\wedge d\boldsymbol{A} + \frac{2}{3}\,\boldsymbol{A}\wedge\boldsymbol{A}\wedge\boldsymbol{A}\right) \tag{15.2}$$

is defined to satisfy

$$d\,\omega_3 = \mathrm{tr}(\boldsymbol{F}\wedge\boldsymbol{F})\,. \tag{15.3}$$

In order to verify (15.3) (for ω_3 given by (15.2) and F given by (15.1)) one has to use the cyclicity of the trace and the anticommutativity of 1-forms to deduce

$$\mathrm{tr}\,\boldsymbol{A}^{12k}(=\boldsymbol{A}^{\alpha_1}_{\alpha_2}\wedge\boldsymbol{A}^{\alpha_2}_{\alpha_3}\wedge\ldots\wedge A^{\alpha_{2k}}_{\alpha_1}) = 0 \tag{15.4}$$

(cf. Exercise 10.4). (More generally, the Chern-Simons $(2k-1)$-form ω_{2k-1} is defined to satisfy $d\,\omega_{2k-1} = \mathrm{tr}(\boldsymbol{F}^{\wedge k})$. Verify that

$$\omega_5 = \mathrm{tr}\left(\boldsymbol{F}\wedge\boldsymbol{F}\wedge\boldsymbol{A} - \frac{1}{2}\,\boldsymbol{F}\wedge\boldsymbol{A}\wedge\boldsymbol{A}\wedge\boldsymbol{A} + \frac{1}{10}\,\boldsymbol{A}^{\wedge 5}\right) \tag{15.5}$$

satisfies $d\,\omega_5 = \mathrm{tr}\,\boldsymbol{F}^{\wedge 3}$. Note that ω_3 may be also written as $\omega_3 = \mathrm{tr}\,(\boldsymbol{F}\wedge\boldsymbol{A} - \frac{1}{3}\,\boldsymbol{A}\wedge\boldsymbol{A}\wedge\boldsymbol{A})$.

Varying the (conformally invariant!) Chern-Simons action

$$S = \frac{k}{4\pi}\int_M \mathrm{tr}\left(\boldsymbol{A}\wedge d\boldsymbol{A} + \frac{2}{3}\,\boldsymbol{A}^{\wedge 3}\right) \tag{15.6}$$

we find the equation of motion

$$0 = \frac{\delta S}{\delta\boldsymbol{A}} = \frac{k}{2\pi}\,\boldsymbol{F} \tag{15.7}$$

which says that the curvature is zero, or, in other words, the *connection* $\boldsymbol{A}$ *is flat.* Flat connections are determined entirely by *holonomies* around noncontractible cycles. If $\mathcal{K}$ is an oriented knot[56] then one considers the trace of the holonomy of the gauge connection around $\mathcal{K}$ in a given IR R of $U(n)$, which gives the *Wilson loop operator*, the trace of the path-ordered exponent

$$W^{\mathcal{K}}_R(\boldsymbol{A}) = \mathrm{tr}_R\left(P\exp\oint_{\mathcal{K}}\boldsymbol{A}\right)\,. \tag{15.8}$$

Witten [156] discovered that the vacuum expectation value of this operator for $n=2$ reproduces the *Jones polynomial* invariant [96]. (The appearance of topological invariants in QFT has been suggested earlier by Albert Schwarz.) For M a 3-manifold with boundary Σ Witten demonstrates that the Chern-Simons theory on M with a compact Lie group G and action (15.6) gives rise to a WZNW theory on Σ corresponding to the current algebra $\hat{\mathcal{G}}$ of level k. Recently exact results for perturbative Chern-Simons theory with a complex gauge group have been obtained and applied to study different topological invariants [43] (an outgrow of [85]), demonstrating that Chern-Simons theory continues to be a lively subject.

For a review on Chern-Simons theory with applications to topological strings – see [117]. For the quantization of the Hamiltonian Chern-Simons theory and for the representation theory of Chern-Simons observables (not covered in [117]) – see [2] and [4].

[55] The term $\theta\,\mathrm{tr}(F\wedge F)$ is much discussed in connection with the problem of strong CP violation, [139]; for an instance of a subsequent theoretical study – see [157].

[56] For a survey of modern knot theory – see [110] and [128].

A second important topic outside the scope of these lectures is the application of Hopf algebra techniques to QFT renormalization initiated by Dirk Kreimer and further developed by Connes and Kreimer – see for recent reviews [37, 108] (cf. also [33]).

We have not touched upon the study of *quantum homogeneous spaces* and their possible application as candidates for non-commutative space-time manifolds. Here we feel that a more general point of view, not necessarily related to quantum groups is preferable – see [35, 37]. For interesting purely mathematical results in this direction – see [36] and [40].

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