Minimal Basis for Gauge Theory Amplitudes

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Identities based on monodromy for integrations in string theory are used to derive relations between different color ordered tree-level amplitudes in both bosonic and supersymmetric string theory. These relations imply that the color ordered tree-level n-point gauge theory amplitudes can be expanded in a minimal basis of $(n-3)!$ amplitudes. This result holds for any choice of polarizations of the external states and in any number of dimensions.

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INTRODUCTION

The search for a consistent theoretical framework of particle physics has led to remarkable progress in the understanding of fundamental interactions in Nature. String theory provides a very general unified language that naturally incorporates field theories of phenomenological interest and gravity in the low-energy limit. Much can be learned from studying the organizational and computational inspiration it poses [1]. One striking aspect is the link string theory can provide between gravity and gauge theories. Concrete examples of such relationships include the Kawai-Lewellen-Tye [2] relations which connect amplitudes in closed and open string theories. In the low-energy limit this gives a very puzzling and non-trivial map between perturbative amplitudes in gravity and Yang-Mills theory that is far from obvious when viewed at the field theory perspective [3].

In this Letter, we will consider a set of relations among tree-level string theory amplitudes that are implied by their defining integrals. Different color orderings of external legs are connected to specific integration regimes on the string world sheet, but they can be related to each other through monodromy relations. In the field theory limit the phase relations between different integrals induced by these monodromy considerations reduce to a set of equations linking gauge theory amplitudes with different color traces. We first remark that by cyclicity of the trace the number of color ordered amplitudes is reduced from $n!$ to $(n-1)!$ The full set of monodromy relations for the color-ordered amplitudes imply a drastic reduction of the number of independent amplitudes in the n-point case. The number of basis amplitudes is in this way reduced from $(n-1)!$ to $(n-3)!$ Analogously to the Kawai-Lewellen-Tye relations, the detailed understanding of the underlying identities at the gauge theory level poses an interesting challenge. The existence of a minimal number of $(n-3)!$ basis amplitudes in gauge theory, and an associated set of identities, has been conjectured by Bern et al. [4] (see also ref. [5] for the extension to gauge theory with matter) and already checked explicitly to a high number of external legs with different combinations of external states and helicities. The origin of this reduction in basis amplitudes appears in a particularly transparent manner from string theory.

We will here briefly recall how to derive these monodromy-induced relations for string theory amplitudes. The n-point amplitude in open string theory with $U(N)$ gauge group reads

$$\mathcal{A}_n = ig_{\rm YM}^{n-2}\,(2\pi)^D\,\delta^D(k_1+\cdots+k_n) \sum_{(a_1,\dots,a_n)\in S_n/\mathbb{Z}_n} \text{tr}(T^{a_1}\cdots T^{a_n})\,\mathcal{A}(a_1,\cdots,a_n)\,, \tag{1}$$

where D is any number of dimensions obtained by dimensional reduction from 26 dimensions if we consider the bosonic string, or 10 dimensions in the supersymmetric case. In fact our considerations are completely general and without reference to any specific string theory. The color-ordered amplitudes on the disc are given by [1]

$$\mathcal{A}(a_1,\cdots,a_n) = \int\prod_{i=1}^{n} dz_i\,\frac{|z_{ab}\,z_{ac}\,z_{bc}|}{dz_a dz_b dz_c}\prod_{i=1}^{n-1} H(x_{a_{i+1}}-x_{a_i}) \times \prod_{1\le i<j\le n} |x_i-x_j|^{2\alpha' k_i\cdot k_j}\,F_n\,, \tag{2}$$

with $dz_i = dx_i$ and $z_{ij} = x_i - x_j$ for the bosonic case and $dz_i = dx_i d\theta_i$ and $z_{ij} = x_i - x_j + \theta_i\theta_j$ for the supersymmetric case. The ordering of the external legs is enforced by the product of Heaviside functions such that $H(x)=0$ for $x<0$ and $H(x)=1$ for $x\ge 0$. The Möbius $SL(2,\mathbb{R})$ invariance requires one to fix the position of three points. A traditional choice is $x_1=0$, $x_{n-1}=1$ and $x_n=+\infty$, supplemented by the condition $\theta_{n-1}=\theta_n=0$ in the superstring case.

All helicity dependence of the external states is contained in the F_n factor. For tachyons, one has

$F_n = 1$ while for n gauge bosons of helicities h_i one arrives at $F_n = \exp - \sum_{i\neq j} \big(\frac{\sqrt{\alpha'}(h_i\cdot k_j)}{(x_i - x_j)} - 2\frac{(h_i\cdot h_j)}{(x_i-x_j)^2}\big)|_{\text{multilinear in } h_i}$ for the bosonic string and at $F_n = \int \prod_{i=1}^n d\eta_i \, \exp - \sum_{i\neq j} \big(\frac{\eta_i \sqrt{\alpha'}(\theta_i-\theta_j)(h_i\cdot k_j) - \eta_i\eta_j(h_i\cdot h_j)}{(x_i-x_j+\theta_i\theta_j)} \big)$ for the superstring case where η_i are anticommuting variables.

THE FOUR-POINT AMPLITUDE

In order to understand the relations that monodromy imposes we will begin with a discussion of the relations that arise at four points [6, 7]. In that case, we can expand the amplitude $\mathcal{A}_4 \sim g^2_{\rm YM}\ {\rm tr}(T^1T^2T^3T^4)\mathcal{A}(1,2,3,4)$ plus permutations.

For simplicity, we phrase the discussion in terms of tachyon amplitudes. With the choice $x_1 = 0$, $x_3 = 1$ and $x_4 = +\infty$, all three different color-ordered amplitudes $\mathcal{A}(i,j,k,l)$ are given by the same integrand $|x_2|^{2\alpha'\, k_1\cdot k_2}|1-x_2|^{2\alpha'\, k_2\cdot k_3}$ but with x_2 integrated over different domains:

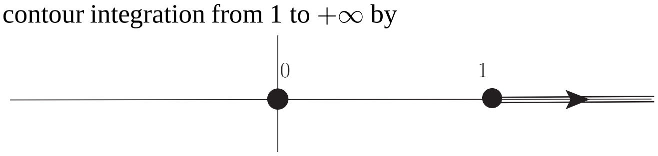
$$\mathcal{A}(1,2,3,4) \;=\; \int_0^1 dx \quad x^{2\alpha'\, k_1\cdot k_2}(1-x)^{2\alpha'\, k_2\cdot k_3}\,, \qquad (3)$$

$$\mathcal{A}(1,3,2,4) \;=\; \int_1^\infty dx \quad x^{2\alpha'\, k_1\cdot k_2}(x-1)^{2\alpha'\, k_2\cdot k_3}\,, \qquad (4)$$

$$\mathcal{A}(2,1,3,4) \;=\; \int_{-\infty}^0 dx\, (-x)^{2\alpha'\, k_1\cdot k_2}(1-x)^{2\alpha'\, k_2\cdot k_3}\,. \qquad (5)$$

By a relabeling of indices, we can derive all the four-point relations shown below from just the first of these integrals. However, the present form is more useful for exploiting the monodromy relations [6, 7] satisfied by the n-point amplitudes.

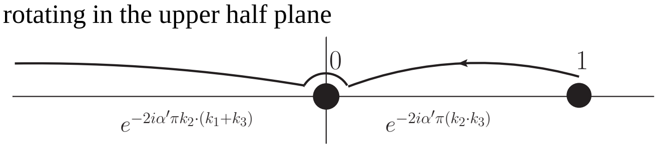
We first consider $\mathcal{A}(1,3,2,4)$, where we can indicate the contour integration from 1 to $+\infty$ by



Assuming that the $\alpha'\, k_i\cdot k_j$ are complex with negative real parts, we can deform the integration region so that instead of integrating between from 1 to $+\infty$ on the real line we integrate either on a contour slightly above or below the real axis. By deforming each of the contours, one can convert the expression into an integration from $-\infty$ to 1. When rotating the contours one needs to include the appropriate phases each time x passes through $y = 0$ or $y = 1$,

$$(x-y)^\alpha = (y-x)^\alpha \times \begin{cases} e^{+i\pi\,\alpha} & \text{for clockwise rotation}\,,\\ e^{-i\pi\,\alpha} & \text{for counterclockwise rotation}\,. \end{cases}$$

The deformation of the integration region can thus be done by rotating in the upper half plane



Because the original amplitude is real, the real part of this contour integral expresses the original amplitude

$$\begin{aligned}\mathcal{A}(1,3,2,4) = -\Re\mathrm{e}\Big(&e^{-2i\alpha'\pi\, k_2\cdot k_3}\,\mathcal{A}(1,2,3,4)\\ &+ e^{-2i\alpha'\pi\, k_2\cdot(k_1+k_3)}\,\mathcal{A}(2,1,3,4)\Big)\,,\end{aligned} \qquad (6)$$

where the minus sign arises from the reversed orientation of the contour. The imaginary part vanishes:

$$\begin{aligned}0 = \Im\mathrm{m}\Big(&e^{-2i\alpha'\pi\, k_2\cdot k_3}\,\mathcal{A}(1,2,3,4)\\ &+ e^{-2i\alpha'\pi\, k_2\cdot(k_1+k_3)}\,\mathcal{A}(2,1,3,4)\Big)\,.\end{aligned} \qquad (7)$$

This system of equations implies that all amplitudes can be related to $\mathcal{A}(1,2,3,4)$:

$$\begin{aligned}\mathcal{A}(1,3,2,4) &= \frac{\sin(2\alpha'\pi\, k_1\cdot k_2)}{\sin(2\alpha'\pi\, k_2\cdot k_4)}\,\mathcal{A}(1,2,3,4)\,,\\ \mathcal{A}(2,1,3,4) &= \frac{\sin(2\alpha'\pi\, k_2\cdot k_3)}{\sin(2\alpha'\pi\, k_2\cdot k_4)}\,\mathcal{A}(1,2,3,4)\,,\end{aligned} \qquad (8)$$

where we have used momentum conservation and the on-shell condition, here, for a tachyon, $\alpha' k^2 = -1$. For other external states of higher spin with the inclusion of the appropriate F_n factor, the integrals change in order to restore the identities (including sign factors for the fermionic statistics of half-integer spins). These relations are valid for all four-point amplitudes in bosonic and supersymmetric string theory, as can immediately be checked using the explicit expressions for such string amplitudes.

Taking the limit $\alpha' \to 0$, we get the following relations between field theory amplitudes:

$$\begin{aligned}A(1,3,2,4) &= \frac{k_1\cdot k_2}{k_2\cdot k_4}\,A(1,2,3,4)\,,\\ A(2,1,3,4) &= \frac{k_2\cdot k_3}{k_2\cdot k_4}\,A(1,2,3,4)\,.\end{aligned} \qquad (9)$$

These identities agree with those of ref. [4].

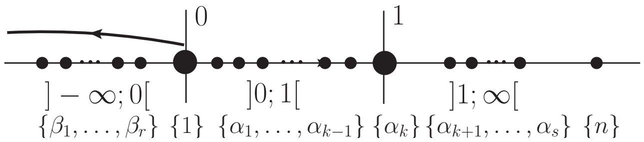
THE N-POINT AMPLITUDE

We now turn to the general n-point case. By generalizing the four-point case we will prove that any color-ordered n-point amplitude can be expressed in terms of a minimal basis of $(n-3)!$ amplitudes $\mathcal{B}$. In the field theory limit these relations reduce to the new amplitude relations conjectured in ref. [4].

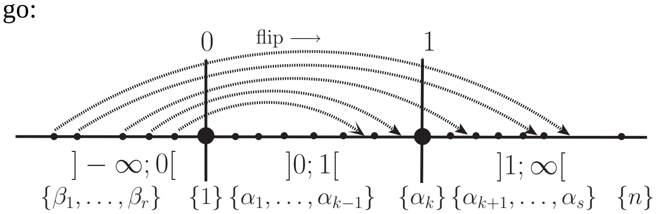
First, we show to how to reduce the number of independent amplitudes from $(n-1)!$ to $(n-2)!$ In this way we derive a string theory generalization of the so-called Kleiss-Kuijf relations in field theory [8, 9]. Indeed, in the limit $\alpha' \to 0$, our relations reduce to those, providing an immediate and alternative proof of them.

Our starting point will be the most general amplitude, given in term of an integral with three fixed points, one at 0: $x_1 = 0$, one at 1: $x_{\alpha_k} = 1$, and one at $+\infty$: $x_n = +\infty$. There can then be r points $\{\beta_1,\ldots,\beta_r\}$ in the interval $]-\infty, 0[$,

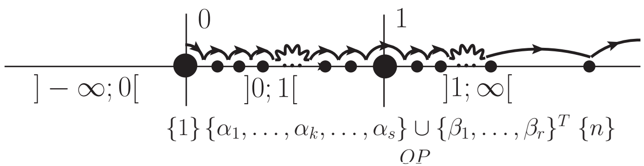
$k-1$ points $\{\alpha_1,\ldots,\alpha_{k-1}\}$ in the interval $]0,1[$ and $s-k$ points $\{\alpha_{k+1},\ldots,\alpha_s\}$ in the interval $]1,+\infty[$. Both r and k are arbitrary, and of course $s = n-r-2$. (We use the notation $]a,b] = \{x|a < x \leq b\}$.) We first focus on the integrations of the $\{\beta_1,\ldots,\beta_r\}$ variables in the amplitude $\mathcal{A}(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n)$, illustrated in the figure below:



By analytic continuation of the integration region $]-\infty,0[$ we now flip the β_i-integrations into the region $]0,+\infty[$ in one go:



We thus have an identity that relates the original integral with integrations in the domain $]-\infty,0[$ with a sum of integrations in the complementary region $]0,+\infty[$.



Taking the real parts of this n-point equation we arrive at the following amplitude relation:

$$\mathcal{A}_n(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n) \;=\; (-1)^r \times \qquad (10)$$
$$\Re\mathrm{e}\Big[\prod_{1\leq i<j\leq r} e^{2i\pi\alpha'(k_{\beta_i}\cdot k_{\beta_j})} \sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}} \prod_{i=1}^{r}\prod_{j=1}^{s} e^{(\alpha_i,\beta_j)}\mathcal{A}_n(1,\sigma,n)\Big],$$

with $e^{(\alpha,\beta)} \equiv e^{2i\pi\alpha'(k_\alpha\cdot k_\beta)}$ if $x_\beta > x_\alpha$ and 1 otherwise. The $(-1)^r$ arises because the flip is reversing the r integrations over the β_i-variables. In (10) the sum runs over the ordered set of permutations that preserves the order within each set. These new relations between string theory amplitudes are generalizations of the field theory Kleiss-Kuijf relations,

$$A_n(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n) = (-1)^r \sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}} A_n(1,\sigma,n)\,, \qquad (11)$$

to which they reduce when $\alpha' \to 0$ since all phases become unity in that limit. The string theory relations (10) reduce the set of independent amplitudes from $(n-1)!$ to $(n-2)!$ in detail by eliminating all amplitudes with legs in the integration interval $]-\infty,0[$ in favor of those with legs in the interval $]0,+\infty[$, with the two extreme ends fixed. However, we have not yet used all the information contained in these n-point monodromy relations.

Because the amplitudes $\mathcal{A}_n(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n)$ are real, the imaginary parts of the n-point relations give

$$0 = \Im\mathrm{m}\Big[\prod_{1\leq i<j\leq r} e^{2i\pi\alpha'(k_{\beta_i}\cdot k_{\beta_j})} \sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}} \prod_{i=1}^{r}\prod_{j=1}^{s} e^{(\alpha_i,\beta_j)}\mathcal{A}_n(1,\sigma,n)\Big]. \qquad (12)$$

By systematically using these relations, we can connect all amplitudes which have points in the region $]1,+\infty[$ with amplitudes which have points only in the region $[0,1]$ (and one leg fixed at infinity).

Our proof will be by explicit construction. A generic amplitude (with $x_{\alpha_k}=1$) $\mathcal{A}_n(1,\alpha_1,\ldots,\alpha_k,\gamma_1,\ldots,\gamma_{n-2-k},n)$ can be uniquely eliminated by considering the string theory relations (10) for the amplitude $\mathcal{A}_n(\gamma_{n-2-k},\ldots,\gamma_1,1,\alpha_1,\ldots,\alpha_k,n)$. This way we have an explicit expression for $\mathcal{A}_n(1,\alpha_1,\ldots,\alpha_k,\gamma_1,\ldots,\gamma_{n-2-k},n)$ in terms of amplitudes with at least one γ_i among the set $\{\alpha_1,\ldots,\alpha_k\}$ and now with at most $n-3-k$ elements between $]1,+\infty[$. Proceeding iteratively on the number of elements in $\{\gamma\}$ starting with $n-2-k$ elements, we can apply (12) to express all amplitudes having points in the interval $]1,+\infty[$ in terms of $(n-3)!$ amplitudes restricted to the interval $[0,1]$.

Explicitly, the five-point case gives

$$\begin{aligned}
\mathcal{S}_{k_2,k_5}\,\mathcal{A}(2,1,3,4,5) &= \mathcal{S}_{k_2,k_3+k_4}\,\mathcal{A}(1,2,3,4,5) + \mathcal{S}_{k_2,k_4}\,\mathcal{A}(1,3,2,4,5)\,,\\
\mathcal{S}_{k_3,k_5}\,\mathcal{A}(1,2,4,3,5) &= \mathcal{S}_{k_3,k_1+k_2}\,\mathcal{A}(1,2,3,4,5) + \mathcal{S}_{k_1,k_3}\,\mathcal{A}(1,3,2,4,5)\,,\\
\mathcal{S}_{k_2,k_5}\mathcal{S}_{k_1,k_4}\,\mathcal{A}(2,3,1,4,5) &= -\mathcal{S}_{k_1,k_2}\mathcal{S}_{k_3,k_4}\,\mathcal{A}(1,2,3,4,5) - \mathcal{S}_{k_2,k_4}\mathcal{S}_{k_1,k_3+k_4}\,\mathcal{A}(1,3,2,4,5)\,,\\
\mathcal{S}_{k_3,k_5}\mathcal{S}_{k_1,k_4}\,\mathcal{A}(1,4,2,3,5) &= -\mathcal{S}_{k_1,k_2}\mathcal{S}_{k_3,k_4}\,\mathcal{A}(1,2,3,4,5) - \mathcal{S}_{k_1,k_3}\mathcal{S}_{k_4,k_1+k_2}\,\mathcal{A}(1,3,2,4,5)\,,\\
\mathcal{S}_{k_1,k_4}\mathcal{S}_{k_2,k_5}\mathcal{S}_{k_3,k_5}\,\mathcal{A}(2,1,4,3,5) &= \Big(\mathcal{S}_{k_2,k_3+k_4}\mathcal{S}_{k_3,k_1+k_2}\mathcal{S}_{k_1,k_4} - \mathcal{S}_{k_2,k_3}\mathcal{S}_{k_1,k_2}\mathcal{S}_{k_3,k_4}\Big)\,\mathcal{A}(1,2,3,4,5)\\
&\quad + \mathcal{S}_{k_1,k_3}\mathcal{S}_{k_2,k_4}\mathcal{S}_{k_5,k_2+k_3}\,\mathcal{A}(1,3,2,4,5)\,,
\end{aligned} \qquad (13)$$

where we have introduced the notation $\mathcal{S}_{p,q} \equiv \sin(2\alpha'\pi\, p\cdot q)$. Analogous equations are obtained by the exchange of labels $2 \leftrightarrow 3$. It is immediate to verify these relations from the explicit form of tree amplitudes in string theory amplitudes in string theory given in [10–12]. In the field theory limit they reduce to the relations discussed in ref. [4].

GRAVITY AMPLITUDES

We finally turn towards implications of these results for gravity amplitudes. The n-point closed string amplitudes can be represented as a left/right product of color-ordered open string amplitudes through the Kawai-Lewellen-Tye relations [2]. Using the result of the previous section, we can expand each open string amplitude of this sum in the basis of open string amplitudes $(\mathcal{B}^I, \tilde{\mathcal{B}}^J)$:

$$\mathcal{M}_n = \alpha' \left(\frac{\kappa}{\alpha'}\right)^{n-2} \sum_{1\leq I,J\leq (n-3)!} \mathcal{G}_{IJ}(\{k_i\})\, \mathcal{B}^I \tilde{\mathcal{B}}^J\,. \quad (14)$$

The holomorphic factorization of the amplitude into left and right open string amplitudes introduces $n-3$ extra phase factors [2] of the type discussed above and the entries of the matrix $\mathcal{G}$ are rational functions of degree $n-3$ in the quantities $\sin(2\pi\alpha'\, p\cdot q)$. Since the matrix is symmetric this provides a left/right symmetric expression for the the gravity amplitudes in terms of the color ordered gauge theory amplitudes.

As a direct application of our procedure, we can rewrite the Kawai-Lewellen-Tye relations at four-point level as

$$\mathcal{M}_4 = \frac{\kappa^2}{\alpha'}\, \frac{\mathcal{S}_{k_1,k_2}\mathcal{S}_{k_1,k_4}}{\mathcal{S}_{k_1,k_3}}\, |\mathcal{A}_4(1,2,3,4)|^2\,. \quad (15)$$

Similarly, the five point closed string amplitude takes the symmetric form

$$\begin{aligned}\mathcal{M}_5 \;=\; \frac{\kappa^3}{\alpha'^2}\Big[&\mathcal{G}_{11}\,|\mathcal{A}_5(1,2,3,4,5)|^2 + \mathcal{G}_{22}|\mathcal{A}_5(1,3,2,4,5)|^2\\ &+\mathcal{G}_{12}\big(\mathcal{A}_5(1,2,3,4,5)\tilde{\mathcal{A}}_5(1,3,2,4,5) \\ &+\mathcal{A}_5(1,3,2,4,5)\tilde{\mathcal{A}}_5(1,2,3,4,5)\big)\Big],\end{aligned} \quad (16)$$

where

$$\mathcal{S}_{k_2,k_5}\mathcal{S}_{k_3,k_5}\mathcal{S}_{k_1,k_4}\,\mathcal{G}_{11} \;=\; \mathcal{S}_{k_1,k_2}\mathcal{S}_{k_3,k_4}\Big(\mathcal{S}_{k_2,k_3+k_4}\mathcal{S}_{k_3,k_1+k_2}\mathcal{S}_{k_1,k_4} - \mathcal{S}_{k_2,k_3}\mathcal{S}_{k_1,k_2}\mathcal{S}_{k_3,k_4}\Big)\,, \quad (17)$$

$$\mathcal{S}_{k_3,k_5}\mathcal{S}_{k_2,k_5}\mathcal{S}_{k_1,k_4}\,\mathcal{G}_{22} \;=\; \mathcal{S}_{k_1,k_3}\mathcal{S}_{k_2,k_4}\Big(\mathcal{S}_{k_3,k_2+k_4}\mathcal{S}_{k_2,k_1+k_3}\mathcal{S}_{k_1,k_4} - \mathcal{S}_{k_2,k_3}\mathcal{S}_{k_1,k_3}\mathcal{S}_{k_2,k_4}\Big)\,, \quad (18)$$

$$\mathcal{S}_{k_2,k_5}\mathcal{S}_{k_3,k_5}\mathcal{S}_{k_1,k_4}\,\mathcal{G}_{12} \;=\; \mathcal{S}_{k_1,k_2}\mathcal{S}_{k_1,k_3}\mathcal{S}_{k_2,k_4}\mathcal{S}_{k_3,k_4}\,\mathcal{S}_{k_5,k_2+k_3}\,. \quad (19)$$

In the limit $\alpha'\to 0$ the $\mathcal{S}_{p,q}$ are replaced by the scalar products $2\pi\,\alpha'\,(p\cdot q)$. They lead to an expression for the field theory gravity amplitude that reproduces the results of [4]. It is now clear how this symmetric form can be proven for any number of external states.

CONCLUSION

To conclude, we have derived a new series of amplitude identities based on monodromy for integrations in string theory, providing relations between different color-ordered amplitudes in either bosonic and supersymmetric string theory. As a first step, we have derived the string theory generalization of Kleiss-Kuijf relations, thus providing a new and very simple proof of these relations also in the field theory limit. Our main result is the proof that there is a minimal basis of only $(n-3)!$ amplitudes in which all other amplitudes can be expanded. This follows from fixing three of the n external legs at 0, 1 and $+\infty$ using the $SL(2,\mathbb{R})$ invariance of the amplitudes, and forcing the remaining $n-3$ coordinates to lie in the interval $[0,1]$. Because the monodromy relations hold for all polarization configurations and any smaller number of dimensions by a trivial dimensional reduction, it follows immediately that they hold for any choice of external legs corresponding to the full $\mathcal{N}=1$, $D=10$ supermultiplet and dimensional reductions thereof. The field theory limit of these relations generalize and prove for any number of external legs the new amplitude relations recently conjectured by Bern et al. [4] in gauge theory. The string theory monodromy identities for the Kawai-Lewellen-Tye relationship between closed and open string amplitudes give highly symmetric forms for tree-level amplitudes between any external states in the $\mathcal{N}=8$, $D=4$ supermultiplet. This and other related issues will be discussed in detail elsewhere.

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