Kramers-Wannier Duality for Non-Abelian Lattice Spin Systems and Hecke Surfaces

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Preface

In this paper I discuss some topics which have long interested me. These themes relate with the following subjects:

1. Duality transformations for generalized Potts models.
2. Hecke surfaces and K- regular graphs.

Each of them relates with deep mathematical and physical theories and they have nothing in common at the first sight. However, it become more evident in the last years that a deep internal relations between all these problems exist. Especially interesting and mysterious is the role of Hecke groups in this context. I consider only few examples of these topics.

The paper is mainly expository.Some of the results are based on the papers jointly written with R.Brooks and V.Buchstaber.

I would like to mention especially Robert Brooks, whose untimely death left me without remarkable friend and coauthor.His ideas of spectral characteristics of Laplacians on "typical" Riemann surfaces are currently not enough appreciated and then will be undoubtedly recognized,

Chapter 1

Duality transformations for generalized Potts models.

In this chapter we discuss some old and new results concerning Kramers-Wannier Duality for spin systems with non-abelian symmetry.

Introduction.

In the classical paper of H. Kramers and G.Wannier [1] a special symmetry was discovered, which relates low-temperature and high-temperature phases in the planar Ising model. The corresponding transformation, the Kramers-Wannier (KW) transform, is a special nonlocal substitution of a variable in the partition function. This substitution transforms the partition function W defined by the initial "spin" variables taking values in Z_2 and determined on the vertices of the original lattice L to the partition function $\tilde{W}$ determined on the dual lattice L^* spin variables taking values in Z_2.

Furthermore, we will use the following transformation of Boltzmann factor

$$\beta \to \beta^* = arth\ e^{-2\beta},\ \beta = (kT)^{-1} \tag{0.1}$$

to get the correct form of the dual partition function $\tilde{W}$.

The existence of such transformations is a general property of lattice spin systems that possess a discrete (and not only discrete) group of symmetry. The KW-transform allows the determination, for many physically important systems, of the point of phase transition in cases when the explicit analytical form of a partition function is unknown.

Generalizations of the KW transform in spin systems with different symmetry groups is essential for many problems in statistical physics and field theory. In fact, it is very important to carry out KW transforms for 4-dimensional gauge theories in which corresponding phases are free quarks and quarks confinement. In this case we need to construct KW transforms for non-abelian groups.

The KW-transform for systems with a commutative symmetry group K, particularly Z_n and Z (like the Ising Z_2-model), can be carried out by general methods. In this case the KW-transform is a Fourier transform from a spin system on the lattice L to the spin system on the dual lattice $\tilde{L}$ with spin variables taking values in the group $\hat{K}$, the group of characters of K. This result was obtained by a number of authors, see [2,3,4] and references cited in it. From the mathematical point of view this result is a generalization of the classical Poisson summation formula for the group Z.

We present some results which solve this problem for non-commutative groups. Our lectures based on the papers [5, 25]. For the sake of a volume limit we omit some examples but add the outline of our construction for the compact case. The efficacy of our approach was illustrated by examples of KW transforms for the icosahedron I_5 and dihedral groups D_n [5]. These examples are also interesting for physical applications, for example, to search out the line of phase transitions in quasicrystals with the icosahedral symmetry or discotic liquid crystals with the symmetry D_n.

The main result of our paper is the definition of the generalized KW-transform, based on the mapping of the group algebra $C(G)$ to the space of complex-valued functions on G. The construction of this transformation clarifies its real meaning and offers far-reaching generalization papers [2, 6, 7].

In section 1 we recall, following the paper [2], the construction of the KW-transform for abelian groups. In section 2 we introduce some relevant algebra notions like the group algebra $C(G)$ and the space of regular functions $C[G]$. We also construct the canonical pairing of $C(G)$ with $C[G]$. In section 3 we describe orbits of the adjoint representation and the regular representation of the group G. In the section 4 we carry out the generalized KW-transform for finite groups and in the section 5 apply our general results to special cases of subgroups of the group SO(3).In the section 6 we study the compact case.

In the conclusion we discuss some applications of these results, in particular some connections with quantum groups.

1. KW-duality for abelian systems.

Let us recall the construction of KW-duality for commutative groups. We shall follow the paper [2]. Let us consider a planar square lattice L with unit edge. Let $x = \{x_\mu\} = \{x_1, x_2\}$ (where x_1 and x_2 are integers) represent a vertex, and $e^\alpha_\mu = \{e^1_\mu, e^2_\mu\} = \delta^\alpha_\mu$ basis vectors of L. We will often use the notation $x + \hat{\alpha} \equiv \{x_\mu + e^\alpha_\mu\}$. A double index x, α is convenient for denoting the edge in the lattice which connects the vertices x and $x + \hat{\alpha}$. In what follows we shall also need the dual lattice, $\tilde{L}$

whose vertices are at the centers of the faces of the original lattice L.We denote the coordinates of a vertex of $\tilde{L}$ by $\tilde{x}$:

$$\tilde{x} = \{x_\mu + 1/2e^1_\mu + 1/2e^2_\mu\}.$$

We define spin variables s_x on vertices of L, these take values in some manifold M, which we call the spin space. We confine ourselves to the case of a finite set M.

The simplest Hamiltonian of such a spin system involves only interactions of nearest neighbors

$$\mathcal{H} = \sum_{x,\alpha} H(s_x, s_{x+\hat{\alpha}}), \tag{1.1}$$

where the Hamiltonian $H(s, s')$ is a real function of a pair of points from M, with the properties

$$H(s, s') = H(s', s), \tag{1.2a}$$

$$H(s, s') \geqslant 0 \text{ for arbitrary } s,\ s' \in M,\ H(s, s) = 0. \tag{1.2.b}$$

The Hamiltonian prescribes a structure similar on M to a metric structure (which in the general case is not metric, since we nowhere require that the triangle inequality hold), which we shall call the H structure.

Of particular interest are examples in which the manifold M is a homogeneous space, i.e., there exists a group G of transformations of M which preserves the H structure: $H(gs, gs') = H(s, s')$ for arbitrary $s, s' \in M$. In this case the spin system has global symmetry with group G.

Important special cases are systems on groups. For these the spin manifold coincides with a group G: $s_i = g_i \in G$, and the Hamiltonian is invariant under left and right translations:

$$H(hg, hg') = H(gh, g'h) = H(g, g') \text{ for arbitraty } h \in G \tag{1.3}$$

The general H function of the system on the group can therefore be put in the form

$$H(g_1, g_2) = H(g_1 g_2^{-1}) = \sum_p h(p)\chi_p(g_1 g_2^{-1}), \tag{1.4}$$

where $\chi_p(g)$ are the characters of the p-th irreducible representations of the group G, and the constants $h(p)$ are chosen so that H has the properties (1.2) and are otherwise arbitrary.

The partition function of the general spin system with the Hamiltonian (1.1) is

$$Z = \sum_{s_x \in M} \prod_{x,\alpha} W(s_x, s_{x+\alpha}), \tag{1.5}$$

where

$$W(s, s') = exp\{-H(s, s')\}. \tag{1.6}$$

According to Eq.(1.2) the function W has the properties

$$W(s, s') = W(s', s),\ 0 \leqslant W(s, s') \leqslant 1,\ W(s, s) = 1 \tag{1.7}$$

For the system on a group we have also

$$W(g_1, g_2) = W(g_1 g_2^{-1}),\ W(g^{-1}) = W(g) \tag{1.8}$$

For a spin system on a group G the sum over states (1.5) can be put in the following equivalent form:

$$Z = \sum_{g_{x,\alpha} \in G} \prod_{x,\alpha} W(g_{x,\alpha}) \prod_{\hat{x}} \delta(Q_{\tilde{x}}, I), \tag{1.9}$$

where the summation variables $g_{x,\alpha}$ are defined on the edges of the lattice

$$Q_{\tilde{x}} = g_{x,1}\ g_{x+\hat{1},2} g^{-1}_{x+\hat{2},1} g^{-1}_{x,2}, \tag{1.10}$$

and the δ-function is defined by the formula

$$\delta(g, I) = \begin{cases} 1, & \text{if } g = I, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, the general solution of the connection equation $Q_{\tilde{x}} = I$ is

$$g_{x,\alpha} = g_x g^{-1}_{x+\hat{\alpha}}$$

and this brings us back to Eq.(1.5).

Systems on commutative groups are a special case, in which the δ-function in Eq.(1.9) can be factorized in the following way:

$$\delta(Q_{\tilde{x}}, I) = \sum_p \chi_p(Q_{\tilde{x}}) = \sum_p \chi_p(g_{x,1})\chi_p(g_{x+\hat{1},2})\chi^{-1}_p(g_{x+\hat{2},1})\chi^{-1}_p(g_{x,2}). \tag{1.11}$$

This sort of factorization is of decisive importance and allow for a unified presentation of the KW transform for all commutative groups.

We note that for a commutative group G all irreducible representations are one-dimensional and their characters χ_p form a commutative group $\hat{G}$ (the character group) with a group multiplication defined in accordance with the tensor product of representations. By definition

$$\chi_{p_1 p_2}(g) = \chi_{p_1}(g)\chi_{p_2}(g),\ \chi_{p^{-1}}(g) = \chi^{-1}_p(g),$$

and the unit element of $\tilde{G}$ corresponds to the identity representation of G. Accordingly, the summation in Eq.(1.11) can be regarded as a summation over the elements of the dual group $\hat{G}$.

Substituting the expansion (1.11) in Eq.(1.9), an obvious regrouping of factors yields

$$Z = \sum_{s_{x,\alpha}\in G} \prod_{x,\alpha} W(g_{x,\alpha}) \prod_{\tilde{x}} \sum_{p_{\tilde{x}}} \chi_{p_{\tilde{x}}}(g_{x,1})\chi_{p_{\tilde{x}}}(g_{x+\hat{1},2})\chi_{p^{-1}_{\tilde{x}}}(g_{x+\hat{2},1})\chi_{p^{-1}_{\tilde{x}}}(g_{x,2}) =$$

$$= \sum_{p_{\tilde{x}}\in G} \prod_{\tilde{x},\alpha} \tilde{W}(p_{\tilde{x}} p^{-1}_{\tilde{x}+\hat{\alpha}}), \tag{1.12}$$

$$\tilde{W}(p_{\tilde{x}} p^{-1}_{\tilde{x}+\hat{\alpha}}) = \sum_{g\in G} W(g)\chi_{p_{\tilde{x}}}(g)\chi_{p^{-1}_{\tilde{x}+\alpha}}(g) = \sum_{g\in G} W(g)\chi_{p_{\tilde{x}} p^{-1}_{\tilde{x}+\hat{\alpha}}}(g). \tag{1.13}$$

The expression (1.12) defines a new, dual, spin system on the dual group $\hat{G}$ with a new Hamiltonian $\tilde{H}$, which is defined the formula

$$exp\{-\tilde{H}(p)\} = \tilde{W}(p) \tag{1.14}$$

The result can be formulated in the following way.

Proposition 1.1. A spin system on a commutative group G with a Hamiltonian $H(g)(g \in G)$ is equivalent to a spin system on the character group $\hat{G}$ (and on the dual lattice) with the Hamiltonian $\tilde{H}(p)(p \in \hat{G})$ given by the Fourier transform

$$exp\{-\tilde{H}(p)\} = \sum_{g\in G} exp\{-H(g)\}\chi_p(g). \tag{1.15}$$

This is a Kramers-Wannier transform. In contradistinction to the "order variables" g_x the name "disorder variables" can be given to the dual spins $p_{\tilde{x}}$.

2. Algebraic constructions.[1]

A) The group algebra $C(G)$ of G.

Let G be a finite group of order n with elements $\{g_1 = e, ..., g_n\}$.

Definition 1. The group algebra $C(G)$ of G is n-dimensional algebra over the complex field $\mathbf{C}$ with basis $\{g_1 = e, ...g_n\}$. A general element $u = c(g) \in C(G)$ is

$$u = \sum \alpha_i g_i \qquad (2.1).$$

The product of two elements (convolution) $u, v \in C(G)$ is defined as

$$uv = \left(\sum_{i=1}^{n} \alpha_i g_i\right)\left(\sum_{i=1}^{n} \beta_j g_j\right) = \sum_{1}^{k} (\gamma_k g_k), \ \gamma_k = \sum_{g_i g_j = g_k} \alpha_i \beta_j \qquad (2.2)$$

B) The ring of functions $C[G]$ on G.

Definition 2. $C[G]$ is a linear space of all complex-valued functions on G and the product is defined pointwise:

$$\left(f_1 \cdot f_2\right)(g) = f_1(g) f_2(g) \qquad (2.3)$$

C) let us determine the canonical pairing $< \cdot, \cdot >$ of these two spaces

$$C(G) \otimes C[G] \to \mathbf{C},$$

$$\text{if } u \in C(G), \text{ and } f \in C[G] \text{ then}$$

$$u \otimes f \to < u, f >= \sum \alpha_i f(g_i) \qquad (2.4)$$

We choose as a basis in $C[G]$ functions such that $< g_i, g^j >= \delta_i^j$ here δ_i^j is the Kronecker symbol.

This pairing enables us to identify $C(G)$ and $C[G]$ as vector spaces.

3. Canonical actions of the Group G.

We now define two canonical representations, the adjoint representation on $C(G)$ and the regular representation on $C[G]$.

A) $T(g) : C(G)$

The adjoint representation is defined on the basis consisting of elements of G by

$$g : \ g_i \to g g_i g^{-1} \qquad (3.1)$$

The adjoint representation ad G decomposes in the direct sum of irreducible representations and split $C(G)$ in the sum of subspaces invariant under the adjoint action.

Each irreducible subspace H_i relates with the orbit of ad G (3.1). The number of H_i is equal to m, the number of elements in the space $C(G)/[C(G), C(G)]$, here $[C(G), C(G)]$ denotes the commutant of $C(G)$.

B) $\tilde{T}(g) : C[(G]$

[1] For further details of exploiting algebraic constructions one can consult the books [8,9].

Let us define the canonical representation $\tilde{T}$ in the space $C[G]$ as the (right) regular representation as:

$$T(g) : f \Rightarrow T(g) : f(g_k) = f(g_k g),\ g \in G,\ f(g) \in C[G]. \tag{3.2}$$

It is well known that, in the decomposition of the regular representation into irreducible ones all irreducible representations appear with multiplicity equal to the dimension of the representation.

$$\tilde{T} = \sum d_k V_k$$

where V_k is the irreducible representation of degree k and d_k is the degree (dimension) of V_k (multiplicity of irreducible representation).

Proposition 3.1. The number m of irreducible representations $\tilde{T}$ is equal to the number of orbits of T.

C) The canonical scalar product in the space $C[G]$ is

$$< f_1, f_2 >= 1/n \sum_{k=1}^{n} f_1(g_k)\bar{f}_2(g_k),\ \ f_1, f_2 \in C[G] \tag{3.4}$$

The characters $\chi_i(g)$ of the irreducible representation of G form the set of orthogonal functions with respect to the scalar product (3.4).

Now we construct the basis in the space $C[G]$. Let us choose the character $\chi_k(g)$ and act on $\chi_k(g)$ by the group G with the help of the right regular representation:

$$R_{g_l}\chi_k(g),\ l = 1, ..., n \tag{3.5}$$

We obtain the space V_k with dim $V_k = |\chi_k(g)|^2$. As a result we get the factorization of $C[G]$:

$$C[G] = \sum_{k \in \mathfrak{M}_G} V_k,\ \mathfrak{M}_G = \{k = 1, .., m_G\},$$

where m_G is the number of irreducible representations of G.

Orthonormalizing the set of functions (3.5) we obtain the basis in the space V_k. Since V_k are pairwise orthogonal, applying this procedure to all characters χ_k we obtain the desired basis in $C[G]$.

Definition 3. We shall call the dual space $\hat{G}$ to G the basis in $C[G]$ which we construct in the section C.

Motivations for such definition ensue from the case of a commutative group K. The characters of K are one-dimensional and the action of G on characters is simply the multiplication on the scalar, the eigenvalue of the operator R_g. The derived basis is the same as the set of elements of the group $\hat{K}$.

4. The KW-transform for finite groups.

Let us consider the adjoint representation ad G of G, on the space $C(G)$, induced by

$$g : g_k \rightarrow g g_k g^{-1}$$

Let us denote by g_k^G the orbit relative to the adjoint action for $g_k \in G$, and by $\delta_k \in C[G]$ it characteristic function:

$$\delta_k(g_s) = \begin{cases} 1, & \text{if } g_s \in g_k^G, \\ 0, & \text{otherwise.} \end{cases}$$

Let m_G be the number of conjugacy classes relative to the adjoint action of G. Let us choose representations of the classes

$$g_1, ..., g_{k_j}.$$

Lemma 4.1. A linear map

$$W : C(G) \to \mathbf{C}$$

satisfies the condition

$$W(g_k) = W(g_l g_k g_l^{-1}) \tag{4.1}$$

for every $g_l \in G$, off

$$W = \sum_{j=1}^{m} \gamma_j \delta_{k_j} \in C[G] = Hom(C(G), \mathbf{C}),$$

i.e.

$$W(g_s) = \sum \gamma_j \delta_{k_j}(g_s) \tag{4.2}$$

We obtain a general form of the adjoint invariant linear mapping, if we choose as $\gamma = (\gamma_1, ..., \gamma_m)$, the vector of free parameters.

Now we shall find the form of a general linear mapping:

$$\hat{W} : C[G] \to \mathbf{C}$$

determined by the characters $\chi^i(G)$.

The set of characters $\chi^1, ..., \chi^m$ of the irreducible representation of G form the orthonormalized basis (relative to the scalar product (3.4)) in $C[G]$. Here and further χ^1 is the character of the trivial one-dimensional representation.

We get $\hat{W} = \sum \hat{\gamma}_j \chi^j$ as

$$\hat{W}(\psi) = \sum_{j=1}^{m} \hat{\gamma}_j < \chi^j, \psi > \tag{4.3}$$

since characters of representations by lemma 4.1 are ad-invariant functions, we introduce the matrix $\Gamma = \gamma_j^l$ using the expansion

$$\chi^l = \sum_{j=1}^{m} \gamma_j^l \delta_{k_j} \tag{4.4}$$

Let us denote by $g^0, ..., g^{m-1}$ the orthonormalized basis in the algebra $C[G]$, dual to the basis $g_0, .., g_{m-1}$ in the group algebra $C(G)$, i.e.

$$< g^i, g_j >= \delta^i_j.$$

Let D be the duality map:

$$D : C(G) \to C[G], \; D(g_k) = g^k. \tag{4.5}$$

Theorem 4.1. If we pose

$$\gamma_j = \sum_{l=1}^{m} \gamma_j^l \hat{\gamma}_l, \; j = 1, ..., m$$

then by the canonical duality D the linear map

$$W : C(G) \to \mathbf{C}, \; W(g) = \sum_{j=1}^{m} \gamma_j \delta_{k_j}(g)$$

pass to the linear map

$$\hat{W} : C[G] \to \mathbf{C},\ \hat{W}(\psi) = \sum_{j=1}^{m} \hat{\gamma}_j < \chi^j, \psi >$$

and maps W and $\hat{W}$ themselves will be determined by the same function, more precisely

$$W(g_s) = \sum_{j=1}^{m} \gamma_j \delta_{k_j}(g_s) = n \sum_{j=1}^{m} \hat{\gamma}_j \chi^j(g_s) = n\hat{W}(g^s) \tag{4.6}$$

Proof. For any g_s we have

$$W(g_s) = \sum_{j=1}^{m} \gamma_j \delta_{k_j}(g_s) = \sum_{j=1}^{m} \sum_{l=1}^{m} \gamma_j^l \hat{\gamma}_l \delta_{k_j}(g_s) = \sum_{l=1}^{m} \hat{\gamma}_l (\sum_{j=1}^{m} \gamma_j^l \delta_{k_j})(g_s) =$$

$$= \sum_{l=1}^{m} \hat{\gamma}_l \chi^l(g_s) = n \sum_{l=1}^{m} \hat{\gamma}_l < \chi^l, g^s >= n\hat{W}(g^s)$$

Definition 4. We shall call the transform

$$W(g_s) = \sum \gamma_j \delta_{k_j}(g_s) \to \hat{W}(g^s) = 1/n \sum_l \gamma_l \chi^l(g^s)$$

$$\text{where: } \gamma_j = \sum_{l=1}^{m} \gamma_j^l \hat{\gamma}_l \tag{4.7}$$

the Kramers-Wannier transform for finite groups.

In the next section we consider several examples which confirm the coincidence of our approach with former one in the known cases and enables us to find explicit K-W transforms in some earlier unknown cases.See also for other examples [5].

5. Examples.

A) Commutative case $G = Z_n$

Let us consider first the special case $G = Z_3 = \{1, g, g^2\}$. In this case $\delta_j = \delta(g - g^{j-1}),\ j = 1, 2, 3$. Then

$$\begin{aligned} \chi^1 &= \delta_1 + \ \delta_2 \ + \delta_3 \\ \chi^2 &= \delta_1 + \ z\delta_2 \ + z^2\delta_3 \\ \chi^3 &= \delta_1 + z^2\delta_2 \ + z\delta_3, \text{ as } z^4 = z \end{aligned} \tag{5.1}$$

where $z = exp 2\pi i/3$, and $\chi^k(g^j) = z^{(k-1)j}$, $(k = 1, 2, 3)$ are the characters of one-dimensional representations. Hence

$$\Gamma = (\gamma_j^l) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & z & z^2 \\ 1 & z^2 & z \end{pmatrix} \tag{5.2}$$

and we get $\hat{\gamma} = \Gamma^{-1}\gamma$.

If we choose $\gamma_1 = 1,\ \gamma_2 = \gamma_3 = \gamma$, we obtain

$$\hat{\gamma}_1 = \frac{1 + 2\gamma}{3};\ \hat{\gamma}_2 = \hat{\gamma}_3 = \frac{1 - \gamma}{3}, \tag{5.3}$$

$$\text{and hence } \hat{\gamma}_2/\hat{\gamma}_1 = \frac{1-\gamma}{1+2\gamma} \tag{5.4}$$

For the general case of the group Z_n we have to replace the formula (5.1) for characters $\chi^1, ..., \chi^n$ to

$$\begin{aligned} &\chi^1 = \delta_1 + \ \delta_2 \ + ... + \delta_n \\ &\chi^2 = \delta_1 + \ z\delta_2 \ + z^{n-1}\delta_n \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &\chi^n_l = \delta_1 + z^{(n-1)}\delta_2 \ + z\delta_n \end{aligned} \tag{5.5}$$

and for $\Gamma = (\gamma^l_j)$ we get

$$\Gamma = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & z & \cdots & z^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & z^{n-1} & \cdots & z \end{pmatrix} \tag{5.6}$$

In the special case of choosing parameters γ_j:

$$\gamma_1 = 1, \gamma_2, ..., \gamma_n = \gamma$$

we obtain

$$\frac{\hat{\gamma}_j}{\hat{\gamma}_1} = \frac{1-\gamma}{1+(n-1)\gamma} \tag{5.7}$$

These formulas coincide with the similar one in the paper [2].

B) The group S_3.

This is the first non-trivial example of non-abelian groups which was studied in [2]. Following our general approach we split the group S_3 in 3 classes of conjugacy elements or 3 orbits:

$$S_3 = \{\Omega_1 = \{e\},\ \Omega_2 = \{a, a^2\},\ \Omega_3 = \{b, ab, a^2b\}\}$$

The characteristic functions are:

$$\delta_1 = \delta(\Omega_1) = \delta(g-e),\ \delta_2 = \delta(\Omega_2),\ \delta_3 = \delta(\Omega_3).$$

Following our general procedure (see 4.4) and using

$$\begin{aligned} &\chi^1 = \delta_1 + \ \delta_2 \ + \delta_3 \\ &\chi^2 = \delta_1 + \ \delta_2 \ - \delta_3 \\ &\chi^3 = 2\delta_1 - \delta_2 \end{aligned}$$

we get the matrix

$$\Gamma = (\gamma^l_j) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

and hence $\hat{\gamma} = \Gamma^{-1}\gamma$

$$\begin{aligned} &\hat{\gamma}_1 = \frac{1}{6}(\gamma_1 + \ 2\gamma_2 \ + 3\gamma_3) \\ &\hat{\gamma}_2 = \frac{1}{6}(\gamma_1 + \ 2\gamma_2 \ - 3\gamma_3) \\ &\hat{\gamma}_3 = \frac{1}{3}(\gamma_1 - \gamma_2) \end{aligned} \tag{5.8}$$

with the following relation:

$$\hat{\gamma}_1 + \hat{\gamma}_2 + 2\hat{\gamma}_3 = \gamma_1$$

If we choose the free parameters $\gamma_1, \gamma_2, \gamma_3$ as $1, \gamma_2, \gamma_3$ we obtain two independent parameters $\hat{\eta}_1, \hat{\eta}_2$

$$\hat{\eta}_1 = \frac{\hat{\gamma}_2}{\hat{\gamma}_1} = \frac{1 + 2\gamma_2 - 3\gamma_3}{1 + 2\gamma_2 + 3\gamma_3}, \ \hat{\eta}_2 = \frac{\hat{\gamma}_3}{\hat{\gamma}_1} = \frac{2(1 - \gamma_2)}{1 + 2\gamma_2 + 3\gamma_3} \tag{5.9}$$

which coincide with the formula (5.7) in the paper [2].

Remark 1. Let us mention the missing of factor 2 in the nominator of $\hat{\eta}_2$ in (5.7) in the paper [2].

6. The KW transform for compact groups.

In this section we give an outline of construction of KW transform for compact "gauge" groups. The corresponding construction can be carried out parallel to the finite case. However, it is substantially more complicated. As formerly, we restrict to the case of a square lattice $L \subset R^2$. All necessary materials regarding the theory of representations of compact groups can be find in [10, 11]

A) Let G be a compact connected group. G is isomorphic to $A \times G_1$, where A is a compact abelian group (torus T^n) and G_1 is a semisimple compact group. In the case of an abelian group A the KW transform can be carried out by the general method of section 2. Therefore, in what follows we restrict to the case of a semisimple compact group G.

B) Group algebra $C(G)$. The natural analog of a group algebra for finite group will be some functional space endowed with the product operation as a convolution. It is possible to choose as such space $L^1(G, dg)$, the space of summable functions, or $L^2(G, dg)$, or a subspace $H(G, dg)$ of continuous functions on G.It is more convenient to consider a completion of these spaces by norm:

$$||f|| = sup_T ||T(f)||,$$

where T runs over all unitary representations of the group G. The algebra $C^*(G)$ is called the C^*-algebra of G.

C) $C[G]$. $C[G]$ is a linear functional space (e.g. $L^1(G, dg), L^2(G, dg), ...$) and the product is defined pointwise :

$$(f_1 \cdot f_2)(g) = f_1(g)\, f_2(g)\,.$$

There is a well known theorem of I. Gelfand and D. Raikov asserting that for any locally compact group there exist irreducible unitary representations and the system of such representations is complete.

To construct an analog of a basis in $C(G)$ for a compact case we need some generalization of Schur-Frobenius theorem (see C in Sec. 4). In our case we use the theorem of Peter-Weyl [10].

Theorem Peter-Weyl.

The set of linear combinations of matrix elements of irreducible representation is dense in the space $H(G, dg), L^2(G, dg)$.

The orthogonal relations for matrix elements of a unitary representations can be proved in the same way as for finite groups.

D) The basis in $C(G)$. To construct a relevant basis in the space $C(G)$ we use the construction of irreducible representations by the orbit method. Let us recall that a coadjoint orbit of a group G is an orbit in the space g^* dual to the Lie algebra g of G. If we have an adjoint representation T of G we can determine the coadjoint representation T^* of G which acts in the space g^*. We call such a representation as a coadjoint representation.

Proposition. For a compact semisimple group G a coadjoint representation of G is equivalent to the adjoint representation.

This is evident, since exists Cartan-Killing Ad invariant form on g. For any compact group G there exists only finite number of co(adjoint) orbits Ω_i.The stabilizers of elements $x \in g$ form a finite number k of conjugate classes of subgroups of G. Let $G_i\,(1 \leq i \leq k)$ be a representative of these classes. Then any adjoint orbit is isomorphic to the coset space $\Omega_i = G/G_i$. So we can choose a basis $\delta_j(\Omega_i)$ in $C(G)$ $\delta_j(\Omega_i)$. To complete our proof we use the following statement of Gelfand and Naimark [10]. We omit some technical conditions.

Theorem (Gelfand - Naimark). There exists a one-to-one correspondence between representations of a group algebra $C(G)$ and unitary representation of the group G. So as in the finite case we determine the KW transform for a compact group as

$$\mathcal{D}:\ C(G) \to C[G]\,. \tag{6.1}$$

Conclusion.

Our approach to the KW-transform has important applications. We briefly discuss some of them, intending to return to these problems in the forthcoming publications.

A. KW-transforms and Quantum groups.

We refer reader to [12, 13] for all notations and following references in the theory of Hopf algebras and Quantum groups.

Let us consider the algebra $C[G]$. If we endow $C[G]$ by the operation of coproduct $\triangle : C[G] \to C[G] \otimes C[G]$ induced by the multiplication in the group G, the algebra $C[G]$ becomes Hopf algebra. Using natural dual to $C[G]$ the algebra $C(G)$, we are able to construct another Hopf algebra, (quantum) double $D(G) = C[G] \otimes C(G)$ [12]. Since transformations W and $\hat{W}$ acts as $W : C(G) \to \mathbf{C}$ and $\hat{W} : C[G] \to \mathbf{C}$, i.e. $W \in C[G] = Hom(C(G), \mathbf{C})$ and $\hat{W} \in C[G] = Hom(C(G), \mathbf{C})$ that is $W \otimes \hat{W} \in D(G)$. The KW-transform yields to explicit solutions of Yang-Baxter equations related with the quantum group D(G).

This observation leads to very explicit formulas in the structure theory of quantum groups and quantum spin systems.

And last but not least.

B. In our lecture we consider spin systems with a global non-abelian symmetry. It is natural to ask about generalizing proposed technique to systems with a local (gauge) symmetry. The study of such systems including Ising and Potts chiral models, abelian and non-abelian gauge fields is very important for Quantum Field Theory and the Theory of Phase Transitions.

Chapter 2
Hecke surfaces and K- regular graphs.

1. The Basic Construction

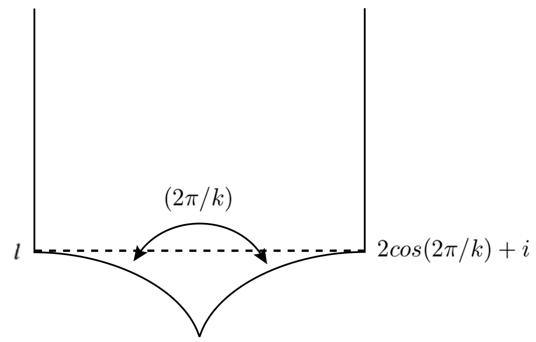
Let Γ be a finite k-regular graph.

Definition 1.1. An orientation $\mathcal{O}$ on Γ is an assignment for each vertex $v \in \Gamma$, of a cyclic ordering of the edges emanating from v.

A graph $(\Gamma, \mathcal{O})$ with orientation is often referred to in the literature as a *fatgraph*.
Generalizing the construction of [15] for the case $k = 3$, we will associate to the oriented graph, $(\Gamma, \mathcal{O})$ a pair of Riemann surfaces, and $S^O(\Gamma, \mathcal{O})$ and $S^C(\Gamma, \mathcal{O})$. $S^O(\Gamma, \mathcal{O})$ will be a finite-area Riemann surface, and $S^C(\Gamma, \mathcal{O})$ will denote its conformal compactification. As in [15], the idea is that the spectral geometry of the non-compact surface $S^O(\Gamma, \mathcal{O})$ is controlled (up to geometric constants) by the spectral geometry of the oriented graph $(\Gamma, \mathcal{O})$, which may then be studied combinatorially. The spectral geometry of the closed surface $S^C(\Gamma, \mathcal{O})$ will be close to the spectral geometry of the open surface $S^O(\Gamma, \mathcal{O})$, provided that $S^O(\Gamma, \mathcal{O})$ satisfies *a large cusps condition*, which will be explained below.
A central part of the construction is the following.

Definition 1.2. For given k, the Hecke group $\mathbf{H}_k$ is the discrete subgroup of $PSL(2, \mathbb{R})$ generated by the matrices

$$A_k = \begin{pmatrix} 1 & 2\cos(\pi/k) \\ 0 & 1 \end{pmatrix} \qquad B_k = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$



A fundamental domain F_k for H_k is given by the region shown in Figure 1, where ρ_0 is the intersection in the upper half plane of the circles of radius 1 centered at 0 and $2\cos(\pi/k)$. Noticing that i is the fixed point of B_k, we see that ρ_0 is the fixed point of

$$A_k B_k = \begin{pmatrix} 2\cos(\pi/k) & -1 \\ 1 & 0 \end{pmatrix},$$

and hence

$$p_0 = \cos(\pi/k) + i\sin(\pi/k).$$

The corresponding circles meet at p_0 with angle $2\pi/k$.

The fact that A_k and B_k generate a discrete group can be read off from the Poincaré Polygon Theorem, the fact that A_k preserves the horocycle $y = 1$, and the fact that A_kB_k is a rotation through angle $2\pi/k$ about p_0 and sends i to $2\cos(\pi/k) + i$.

In the particular case $k = 3$, we have $2\cos(\pi/k) = 1$, and we have the well-known generators and fundamental domain for $PSL(2, \mathbb{Z})$.

2. Hecke Surfaces

For each k, let $\mathcal{H}$ denote the collection of surfaces

$$\mathcal{H}_k = \{S : S = S^C(\Gamma, \mathcal{O}) \text{ for some k-regular } (\Gamma, \mathcal{O})\}.$$

Note that $\mathcal{H}_k$ is precisely the set $\mathcal{B}$ of *Belyi surfaces*, for which several characterizations are known, see [16, 20, 19].

Theorem 2.1. *For each k*

$$\mathcal{H}_k = \mathcal{B}.$$

It follows, for instance, that for any Riemann surface S and for any ε, there is a k-regular $(\Gamma, \mathcal{O})$ such that $S^C(\Gamma, \mathcal{O})$ is ε-close to S (for any reasonable metric on the moduli space of surfaces).

The point here is that the description of S as $S = S^C(\Gamma_k, \mathcal{O}_k)$ for some k may be very complicated, while for another k', the graph $\Gamma_{k'}, \mathcal{O}_{k'}$ might be quite simple.

See the proof related to the graph theory in [29].

3. Riemann surfaces $S^O(\Gamma)$ and$S^C(\Gamma)$

In this section we describe how to read off some geometric properties of the surfaces $S^O(\Gamma)$ and $S^C(\Gamma)$ from the combinatorics of the graph $\Gamma(G, \mathcal{O})$.

Definition 1.3, A left-hand -turn path (LHT) on $\Gamma(G, \mathcal{O})$. is a closed path on Γ such that, at each vertex, the path turns left in the orientation O.

Traveling on a path on Γ which always turn left describes a path on $S^C(\Gamma, \mathcal{O})$ which travels around a cusp . Let l= $l(\Gamma(G), \mathcal{O})$ to be the number of disjoint LHT paths , then the topology of $S^O(\Gamma, \mathcal{O})$ is describable in terms of l and the number of vertices $2n$. The graph Γ divides $S^O(\Gamma, \mathcal{O})$ into l regions , each bordered by a LHT path and containing one cusp in interior. $(\Gamma, \mathcal{O})$ Using the Euler characterstic formula : $\chi(S^O(\Gamma, \mathcal{O}) = 2\text{n} - l\text{n} + l = 2 - 2g$. So the genus g($S^O(\Gamma, \mathcal{O})$) is given by $g = 1 + (n - l)/2$ and the number of cusps is l.

Remark 3.1. The topology of $S^O(\Gamma, \mathcal{O})$is heavily dependent on the choice of orientation Ω.

Example 3.1. [15] The usual orientation on the 3-regular graph which is the 1-skeleton of cube contains six LHT paths, giving the associate surface of sphere with six punctures, while a choice on this can have either two,four or six LHT paths, so that the the associated surface can have genus $0, 1, 2$.

Example 3.2. Platonic solids Let π_k be the k-th Platonic graph of [24]. It is the k-regular graph defined by

$$\{(a,b) \in \mathbb{Z}/k \times \mathbb{Z}/k, a, b \quad \text{relatively prime to } k\}/(a,b) \sim (-a,-b).$$

Two vertices (a,b) and (c,d) are joined by an edge provided that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm 1 (\text{mod } k).$$

An orientation $\mathcal{O}$ on π_k may be defined as follows: at the vertex (a,b), let $< (a,b),(c,d) >$ be an edge. We choose the sign of $c,d)$ so that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv 1 (\text{mod } k).$$

Then the next edge in the cyclic order at (a,b) is $< (a,b),(c-a),(d-b) >$.

With this orientation all *LHT* paths are of length 3, by virtue of the sequence

$$< (a,b),(c,d) > \to < (c-a, d-b), (-a,-b) > \to < (c,d), (a-c, b-d) > .$$

sequence

$$< (a,b),(c,d) > \to < (c-a, d-b), (-a,-b) > \to < (c,d), (a-c, b-d) > .$$

The surface $S^C(\pi_k, \mathcal{O})$ is the *Platonic surface* P_k, which is the compactification of the modular surface

$$S_k = \mathbb{H}^2/\Gamma_k, \qquad \Gamma_k = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\text{mod } k) \right\}.$$

Example 3.3. Let $(\Gamma, \mathcal{O})$ be an oriented k-regular graph, all of whose *LHT* paths have length l. Then we may define the dual oriented graph $(\Gamma^*, \mathcal{O}^*)$ as an l-regular graph, all of whose *LHT* paths are of length k, as follows: the vertices of Γ^* are the *LHT* paths of Γ. The edges $\{e\}$ correspond to the edges e of Γ, and $\{e\}$ joins the two (not necessarily distinct) *LHT* paths to which e belongs.

The orientation $\mathcal{O}^*$ on Γ is given as follows: given a *LHT* path γ and an edge e on Γ, the next element in the cyclic ordering at $\{\gamma\}$ is $\{e'\}$, where e' is the edge following e along the path γ.

Theorem 3.1.

$$S^C(\Gamma, \mathcal{O}) = S^C(\Gamma^*, \mathcal{O}^*).$$

This duality concerns with two types of compactification of surface by horocycles and to add the points of absolute by geodesics going to cusps.

The point here is that it may be that $(\Gamma, \mathcal{O})$ is difficult to analyze, but $(\Gamma^*, \mathcal{O}^*)$ may be relatively easy to understand. For instance, bounding the Cheeger constant and first eigenvalue of the dual Platonic graphs π_k^* uniformly from below is equivalent to Selberg's Theorem [24] up to constant, but the Cheeger constant and first eigenvalue of π_k^* may be calculated in an elementary manner [24].

4. Large Cusps

The geometry of the cusps can also be read off from $\Gamma(G, \mathcal{O})$. In [15] R.Brooks suggested the following construction.

Definition 4.1. Let S^O be a finite area Riemann surface. S^O has cusps of length $\gg L$ (shortly large cusps conditions) if there is a system c_i of closed horocycles such that :
i) Each horocycle has length at least L,
ii) Each cusp is contained in the interior of one of the c_i ,
iii)The interior of the c_i are disjoint.

The importance of this conditions follows from the theorem that asserts:
When S^O satisfies the large cusps conditions, the spectral geometry of S^O and S^C are close. See the exact statement in [16]

The theory as described in [24] is qualitative, but was made quantitative in [26]. We give an outline of the proof. If each cusp has a horocycle of length at least 2π, than you can close off the cusp with a metric of negative curvature by changing the metric conformally inside the cusp. The number 2π arises as necessary condition for this by Gauss-Bonnet theorem. D.Mangoubi [26] shows it is sufficient. The corresponding number for k-regular graphs would be the first integer m such that $2m\cos(\pi/k) > 2\pi$. So the limiting behavior as k is going to infinity is $m = 4$ and for $k = 3$ (modular group), $m = 7$. In particular, Mangoubi calculate how long the cusps must be to guarantee that S^C carries a metric of negative curvature. He shows that this will be the case provided that the cusps have length $\geq 2\pi$.

We remark that the large cusps condition *does not imply* that all the closed paths on the graph Γ are short. It is a condition only on the *LHT* paths. Thus, the oriented graph $(\Gamma, \mathcal{O})$ may have plenty of short geodesics, while still having cusps of length $\geq L$ for some large L.

Of course, it is not always convenient to change the metric within closed horocycles. For instance, the Platonic graphs π_k have *LHT* paths all of length 3, and so do not have large cusps. In [26] it is shown by example that one cannot weaken the large cusps condition by, for instance, replacing horocycles with a general condition such as large geodesic curvature and convexity. However, in special cases we may still modify the metric on $S^O(\Gamma, \mathcal{O})$ in a canonical way to obtain the desired results. Here is an example geared to handle the Platonic graphs:

Theorem 4.1. *There exists a k_0 and a number d_0 with the following property: let $(\Gamma, \mathcal{O})$ be a k-regular graph, for $k \geq k_0$, such that all the LHT paths have length equal to 3. Then there exist neighborhoods of the cusps and of the vertices of $S^O(\Gamma, \mathcal{O})$ and $S^C(\Gamma, \mathcal{O})$, depending only on k_0, such that outside of these neighborhoods, the metrics ds^2_C and ds^2_O satisfy*

$$\frac{1}{d_0} ds^2_O \leq ds^2_C \leq d_0 ds^2_O.$$

The notation is meant to emphasize that we do not have $d_0 \to 0$ as $k \to \infty$.

5. Geodesics on Graphs and surfaces

. The geodesics of $S^O(\Gamma, \mathcal{O}))$ is possible to describe in terms of $(\Gamma, \mathcal{O})$. Let $\mathcal{L} =$ and $\mathcal{R} =$ A closed path P of length k on the graph may be described by starting at a midpoint of an edge, and then giving a sequence $(w_1, .., w_n)$, where each w_i is either l or r, signifying a left or right turn at the upcoming vertex. Let $M_p = W_1 \cdots W_k$, where $W_j = \mathcal{L} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ if $w_j = l$ and $W_j = \mathcal{R} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ if $w_j = r$.

The closed path P on Γ is then homotopic to a closed geodesic $\gamma(P)$ on $S^O(\Gamma, \mathcal{O}))$ whose length $\gamma(P)$ is given by $2\cosh(\text{length}(\gamma(P))/2) = \text{tr}(M_p)$.

The length $\gamma(P)$ depends strongly on $\mathcal{O}$. For instance, if the path P contains only left hand turns then length $\gamma(P)$ =0. If the path$P$ of length r consists of alternating left and right hand turns, then length$(\gamma(P))$ =$r\log(\frac{3+\sqrt{5}}{2})$.

Remark 5.1. I would like to point out some similarity between the computation of length of geodesics in terms of graphs and construction of von Neumann factors with special values of index. V.Jones proved the following remarkable result [**?**].

Theorem 5.1. *For each integer n.there exist a pair of factor $R_0 \subset R$ and subfactor R_0 of type II_1 with index $[R : R_0]$ equal to $4cos^2(\pi/n)$*

For example, there exists factor with $[R : R_0] = (\frac{3+\sqrt{5}}{2})$.

This similarity is not an accident.It is related to construction of hyperfinite type factors using Bratelli diagrams. On the other hand it is connected with the problem of quantization of moduli space of Riemann surfaces based on the decomposition of surfaces via k-valent graphs. We will discuss the details in a separate publication.

6. The Chromatic Polynomials and generalized Potts models

In this section I consider some relations of Hecke groups with Potts models. I content to classical Potts model in the planar case.I refer for details to the book of R.Baxter [17]

A. Z_n Potts model Let L be a two-dimensional lattice. With each site i we associate a "spin" σ_i which takes n values. Two adjacent spins σ_i and σ_j interact with the energy $-J\delta(\sigma_i, \sigma_j))$ where $\delta(,)$ is the usual Dirac $\delta(,)$ -function. The total energy is

$$E = -J \sum_{(i,j)} \delta(\sigma_i, \sigma_j) \tag{1}$$

where the summation is over all edges(i, j) of L. The partition function is

$$Z_n = \sum_{\sigma} \exp\{K \sum_{(i,j)} \delta(\sigma_i, \sigma_j)\} \tag{2}$$

Here the summation is over all values of spin $\sigma(_i)$.

Remark 6.1. The Potts model is possible to determine on any graph L.

In 1969 P. W. Kasteleyn and C. M. Fortuin have found that Z_n Potts model can be expressed as a dichromatic polynomial ,known in the graph theory (H.Whitney, T.Tutte). We set v=$exp(K)-1$. Consider a typical graph G containing l bonds and c connected components(including isolated sites). Let e be the number of edges of the graph L. Then the summand in (1) is the sum of two terms 1 and $v\delta(i, j)$. So the product can be expanded as the sum of 2^e terms . Each of these 2^e terms can be associated with a bond-graph on L. Then the corresponding term in the expansion contains factor v^l. Summing over independent spins and over all components we obtain the contribution of these terms $n^c v^l$.So the partition function Z_n be the same as in (2). The summation is over all graphs G drawn on L.

The expression (2) is called a dichromatic polynomial or Whitney-Tutte polynomial.

In the anti ferromagnetic case $K = -\infty$ and $v = -1$. $Z_n = \sum q^C (-1)^l = P_n(q)$ reduces to chromatic polynomial.

It is clear that $P_n(q)$ determines the number of ways of coloring the sites of L with q colors ,no two adjacent sites having the same color. So $P_n(q)$ is the polynomial in q, which coincides with partition function Z_n (2) with $v = -1$.

Remark 6.2. It is important to mention that the expression (2) is determined for any complex numbers q, not necessary integers. There is a beautiful conjecture concerning the behavior of zeros of chromatic polynomials.

Beraha Conjecture 6.1. Let us consider a chromatic polynomial $P_n(q)$ for arbitrary large planar graph. Then the real zeros of $P_n(q)$ cluster round limit points. These limit points are so called"Beraha numbers" $q = [2\cos(\pi/k)]^2$, $k = 2, 3..$

This conjecture is still unproved. There is an interesting approach using quantum groups [23]. I would like to outline another approach using Hecke graphs.In this case it is necessary to consider the Caley graph generating by Hecke groups. The partition function of Potts model determined on this graph reduces to the chromatic polynomials with desire properties.

Remark 6.3. We mention at the end that the famous problem of four colors on a planar graph is exactly equivalent to the property that $P_n(4)$ is always equal zero.

7. Conclusions

Our approach to generalized Kramers-Wannier (KW) duality is very natural in the spirit of quantum groups. From this point of view is interesting to study the so called McKay correspondence which attached to any finite group K of $SU(2)$ a certain graph which coincides with affine extensions of Dynkin diagrams . Recently these results were extended by I.Dolgachev to the cocompact discrete subgroups γ of $SU(1,1)$ [28]. It is interesting problem to consider McKay correspondence in the case of Hecke groups.

Remark 7.1. The last which I would like only to mention is the relation of Hecke groups with the two-dimensional quantum field theory. These groups appeared as the monodromy representations of some colored braid groups and determined the correlation functions in Z_3 and parafermionic Potts models [14, 21, 27].

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