Open Gromov-Witten invariants and superpotentials for semi-Fano toric surfaces

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OPEN GROMOV-WITTEN INVARIANTS AND SUPERPOTENTIALS FOR SEMI-FANO TORIC SURFACES

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Abstract. We compute the open Gromov-Witten invariants for every compact semi-Fano toric surface, i.e. a toric surface X with nef anticanonical bundle. Unlike the Fano case, this involves non-trivial obstructions in the corresponding moduli problem.

As an application, an explicit expression of the superpotential W for the mirror of X is obtained, which in turn gives an explicit ring presentation of the small quantum cohomology of X. We also give a computational verification of the natural ring isomorphism between the small quantum cohomology of X and the Jacobian ring of W.

1. Introduction

In this paper we investigate the SYZ mirror symmetry for compact semi-Fano toric surfaces, that is, toric surfaces with nef anti-canonical bundles, or equivalently, every toric divisor is at most a (-2)-curve.

The celebrated SYZ mirror symmetry was initiated from the work of Strominger-Yau-Zaslow [22]. For a compact toric manifold X, its SYZ mirror is given by a Landau-Ginzburg model which consists of a domain $\check{X} \subset (\mathbb{C}^*)^n$ and a holomorphic function called the superpotential $W : \check{X} \to \mathbb{C}$. To compute the superpotential, the open GW-invariants which count holomorphic disks play a fundamental role.

When the toric manifold is Fano, various aspects of the SYZ mirror symmetry have been investigated, e.g., Cho and Oh [8] classified holomorphic disks with boundary in Lagrangian torus fibers, and computed the superpotential for the mirror. However, in the non-Fano situation, the moduli of holomorphic disks contains bubble configurations and have a nontrivial obstruction theory, which make explicit computations much more difficult. The only known results are the computations of superpotentials of the Hirzebruch surface $\mathbb{F}_2$ by Fukaya, Oh, Ohta and Ono [11] using their big machinery, and $\mathbb{F}_2$ and $\mathbb{F}_3$ by Auroux [2] via wall-crossing. More recently, using a formula relating open and closed GW-invariants proved by the first author [4], the open GW invariants of all torci CY surfaces and certain CY-threefolds, including the canonical bundles of toric Del Pezzo surfaces, were computed in the joint works [17, 18] of the second author with Leung and Wu.

The main result of this paper calculates the superpotential, or equivalently all genus zero open GW-invariants for Maslov index 2 classes, for every semi-Fano toric surface.

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Theorem 1.1. *Let X be a compact semi-Fano toric surface. Let $b \in \pi_2(X, \mathbf{T})$ be a class of disks with Maslov index two bounded by a Lagrangian torus fiber $\mathbf{T}$. Then the genus zero one-pointed open GW-invariant n_b is either one or zero according to whether b is admissible or not.*

As a consequence, the superpotential for the mirror of X is

$$W = \sum_{\substack{b \text{ admissible} \\ b \in \pi_2(X, \mathbf{T})}} Z_b.$$

Here, b is admissible iff $b = \beta + \sum_k s_k D_k$, where

(1) $\beta \in \pi_2(X, \mathbf{T})$ intersects a unique irreducible toric divisor D_0 once;
(2) D_k's are toric divisors which form a chain of (-2)-curves;
(3) Both $s_0 \geq s_1 \geq s_2 \geq \cdots$ and $s_0 \geq s_{-1} \geq s_{-2} \geq \cdots$ are nondecreasing integer sequences with $|s_k - s_{k+1}| = 0$ or 1 for each k, and the last term of each sequence is not greater than one.

For each $b \in \pi_2(X, \mathbf{T})$, Z_b is a holomorphic function on the dual torus bundle of X defined by Equation (2).

The proof of Theorem 1.1 is based on the comparison of open GW-invariants and ordinary GW-invariants in [4] (and its generalization in [17]), and the result on local GW-invariants obtained by Bryan and Leung [3]. The idea is similar to the proof of Theorem 4.2 in [18].

As an application, we give a verification based on direct computations that there is a natural ring isomorphism between the small quantum cohomology $QH^*(X)$ of a semi-Fano toric surface X and the Jacobian ring $Jac(W)$ of its superpotential W.

Corollary 1.2. *Let X be a compact semi-Fano toric surface, and W the superpotential for its mirror. Then there is a natural ring isomorphism*

$$QH^*(X) \cong \mathrm{Jac}(W). \tag{1}$$

In the final stage of the preparation of this paper, a preprint [12] by Fukaya, Oh, Ohta and Ono appeared on the arXiv. They proved that for every compact toric manifold X and $\mathbf{b} \in H_*(X)$,

$$QH^*_{\mathbf{b}}(X) \cong \mathrm{Jac}(W_{\mathbf{b}})$$

where $QH^*_{\mathbf{b}}(X)$ is the big quantum cohomology ring and $W_{\mathbf{b}}$ is the superpotential bulk-deformed by $\mathbf{b}$. Their proof uses their big machinery of Lagrangian Floer theory and does not involve explicit computations of open Gromov-Witten invariants. Corollary 1.2 can be obtained as a special case of their theorem.

By the isomorphism (1), for every semi-Fano toric surface X, our explicit expression for the supepotential W leads to an explicit presentation of the small quantum cohomology ring $QH^*(X)$. Indeed we can obtain more:

Corollary 1.3. *Let X be a compact semi-Fano toric surface and $\mathbf{b} = D + aX$ be a linear combination of toric cycles, where D is a toric divisor and $a \in \mathbb{C}$. Then the bulk-deformed superpotential is*

$$W_{\mathbf{b}} = a + \sum_{\beta \text{ admissible}} \exp(\langle \beta, D \rangle) Z_\beta.$$

Then by using the results of FOOO mentioned above, an explicit ring presentation of $QH^*_{\mathbf{b}}(X)$ is obtained for $\mathbf{b} \in H_2(X) \oplus H_4(X)$.

Remark 1.4. *FOOO* [9, 10, 12] *used Novikov ring instead of* $\mathbb{C}$ *as the coefficient ring, which is more appropriate in general. Throughout this paper we stick to the tradition of using* $\mathbb{C}$ *as the coefficient ring because* W *is a finite sum for toric semi-Fano surfaces. All the statements in this paper remains unchanged if* $\mathbb{C}$ *is replaced by the Novikov ring.*

This paper is organized as follows. Section 2 is a review on toric manifolds and their Landau-Ginzburg mirrors that we need. In Section 3 we compute the open GW-invariants of semi-Fano toric surfaces and prove Theorem 1.1. In Section 4, we outline our computational proof of the isomorphism $QH^*(X) \cong Jac(W)$ and demonstrate the explicit calculations by several examples. Corollary 1.3 is proved in Section 5, and we end by some comments on bulk-deformation by points.

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2. Landau-Ginzburg mirror of toric manifolds

We set up the notations and review some basic facts in toric geometry and mirror symmetry that we need in this paper.

2.1. **A quick review on toric manifolds.**

Let $N \cong \mathbb{Z}^n$ be a lattice of rank n. For simplicity we'll always use the notation $N_R := N \otimes R$ for a $\mathbb{Z}$-module R. Let X_Σ be a compact complex toric n-fold X_Σ defined by a fan Σ supported in $N_\mathbb{R}$. X_Σ admits an action by the complex torus $N_\mathbb{C}/N \cong (\mathbb{C}^\times)^n$, whence its name 'toric manifold'. There is an open orbit in X_Σ on which $N_\mathbb{C}/N$ acts freely, and by abuse of notation we'll also denote this orbit by $N_\mathbb{C}/N \subset X_\Sigma$.

We denote by M the dual lattice of N. Every lattice point $\nu \in M$ gives a nowhere-zero holomorphic function $\exp(\nu\,, \cdot) : N_\mathbb{C}/N \to \mathbb{C}$ which extends to a meromorphic function on X_Σ. Its zero and pole set gives a toric divisor which is linearly equivalent to 0. (A divisor D in X_Σ is toric if D is invariant under the action of $N_\mathbb{C}/N$ on X_Σ.)

If we further equip X_Σ with a toric Kähler form ω, then the action of $N_\mathbb{R}/N$ on X_Σ induces a moment map

$$\mu_0 : X_\Sigma \to M_\mathbb{R},$$

whose image is a polytope $P \subset M_\mathbb{R}$ defined by a system of inequalities

$$(v_i\,, \cdot) \geq c_i,\ i = 1, \ldots, d,$$

where v_i are all primitive generators of rays of Σ, and $c_i \in \mathbb{R}$ are some suitable constants.

P admits a natural stratification by its faces. Each codimension-one face $T_i \subset P$ which is normal to $v_i \in N$ gives an irreducible toric divisor $D_i = \mu_0^{-1}(T_i) \subset X_\Sigma$ for

$i = 1, \dots, d$, and all other toric divisors are generated by $\{D_i\}_{i=1}^d$. For example, the anti-canonical divisor of X_Σ is given by $\sum_{i=1}^d D_i$.

2.2. **Gromov-Witten invariants.** First we recall the definition of the ordinary GW-invariants of a projective manifold.

Let $\beta \in H_2(X, \mathbb{Z})$ be a 2-cycle in a smooth projective variety X. Let $\overline{M}_{g,k}(X, \beta)$ be the moduli space of stable maps

$$f : (C; x_1, \cdots x_k) \longrightarrow X,$$

where C is a genus g nodal curve with k marked points and $f_*[C] = \beta$. Let $\mathrm{ev}_i : \overline{M}_{g,k}(X, \beta) \to X$ $(i = 1, \dots, k)$ be the evaluation maps $f \mapsto f(x_i)$.

Definition 2.1. *For cohomology classes $\gamma_i \in H^*(X)$, $1 \leq i \leq k$, the GW-invariant of $\{\gamma_1, \cdots, \gamma_k\}$ is*

$$GW_{g,k}^{X,\beta}(\gamma_1, \cdots, \gamma_k) := \int_{[\overline{M}_{g,k}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^k \mathrm{ev}_i^*(\gamma_i).$$

Analogously, we have the open Gromov-Witten invariants defined by FOOO [9], and they are briefly described as follows. Let $X = X_\Sigma$ be a toric manifold defined by a fan Σ. For a Lagrangian torus $\mathbf{T} \subset X$, let $\pi_2(X, \mathbf{T})$ be the group of homotopy classes of maps

$$u : (\Delta, \partial\Delta) \longrightarrow (X, \mathbf{T})$$

where $\Delta := \{z \in \mathbb{C} : |z| \leq 1\}$ denotes the closed unit disk in $\mathbb{C}$. Then $\pi_2(X, \mathbf{T})$ is generated by the basic disk classes $\beta_i \in \pi_2(X, \mathbf{T})$ which correspond to the primitive generators $v_i \in N$ of rays in Σ for $i = 1, \dots, d$. The two most important classical symplectic invariants associated to $\beta \in \pi_2(X, \mathbf{T})$ are its symplectic area $\int_\beta \omega$ and its Maslov index $\mu(\beta)$.

Now for $\beta \in \pi_2(X, \mathbf{T})$, let $\overline{M}_k(\mathbf{T}, \beta)$ be the moduli space of stable maps from a bordered Riemann surface of genus zero with k boundary marked points respecting the cyclic order of the boundary in the class β. Notice that the bordered Riemann surface may have disk or sphere bubbles. It is known that $\overline{M}_k(\mathbf{T}, \beta)$ has expected dimension $n + \mu(\beta) + k - 3$. Let $[\overline{M}_k(\mathbf{T}, \beta)]^{\mathrm{vir}}$ be its virtual fundamental chain constructed in [9]. We let

$$\mathrm{ev}_i : \overline{M}_k(\mathbf{T}, \beta) \longrightarrow \mathbf{T}$$

be the evaluation maps defined by $\mathrm{ev}_i([u; p_0, \dots, p_{k-1}]) = u(p_i)$ for $0 \leq i \leq k - 1$.

Definition 2.2 ([9])**.** *Given a Lagrangian torus $\mathbf{T} \subset X$ and $\beta \in \pi_2(X, \mathbf{T})$, the genus zero one-pointed open GW-invariant n_β is defined as*

$$n_\beta := \int_{[\overline{M}_1(\mathbf{T},\beta)]^{\mathrm{vir}}} \mathrm{ev}_0^*[\mathrm{pt}]$$

where $[\mathrm{pt}] \in H^n(\mathbf{T})$ is the Poincaré dual of the point class of $\mathbf{T}$.

The invariant n_β can be interpreted as the virtual number of holomorphic disks in β whose boundaries pass through a generic point in $\mathbf{T}$. We should mention that the virtual dimension of $\overline{M}_1(\mathbf{T}, \beta)$ is equal to $\dim \mathbf{T} = n$ if and only if $\mu(\beta) = 2$.

Now let's consider the situation where $X = X_\Sigma$ is a semi-Fano (i.e. with nef anti-canonical bundle) and $\mathbf{T} \subset X$ is a regular torus fiber. By the classification result of Cho-Oh [8], a class $\beta \in \pi_2(X, \mathbf{T})$ represented by a stable disk must be of the form $\beta = \beta' + \alpha$, where β' is represented by a holomorphic disk and $\alpha \in H_2(X)$

is represented by a rational curve. The Maslov index of β' is at least two, and the Chern number $\int_\alpha c_1(X)$ of α must be non-negative since X is semi-Fano. This shows that any stable disk bounded by $\mathbf{T}$ has $\mu \geq 2$, which implies that $\overline{M}_1(\mathbf{T}, \beta)$ has no boundary since there is no disk bubbling, and hence the virtual fundamental chain $[\overline{M}_1(\mathbf{T}, \beta)]$ is indeed a cycle. Thus n_β defined above is indeed a symplectic invariant.

2.3. **The LG mirror of toric manifolds.** The mirror of a toric manifold $X = X_\Sigma$ is a Landau-Ginzburg model $(\check{X}, W)$, which is a complex manifold $\check{X}$ equipped with a holomorphic function $W : \check{X} \to \mathbb{C}$ called the superpotential. This superpotential can be written down in terms of Kähler sizes and open Gromov-Witten invariants of X [8, 1, 6]. The following is a brief review of this procedure from the SYZ viewpoint. See [6] for more details.

First of all, we recall that the semi-flat mirror of X is

$$\check{X}_0 := \left\{(\mathbf{T}_r, \nabla) : r \in P^{\text{int}}, \nabla \text{ is a flat } U(1)\text{-connection on } \mathbf{T}_r\right\},$$

where $\mathbf{T}_r \subset X$ denotes the moment-map fiber over r and P^{int} denotes the interior of P. It is well known that $\check{X}_0$ can be equipped with the so-called semi-flat complex structure, making it into a complex manifold [19]. In this toric case, $\check{X}_0$ is simply $P^{\text{int}} \times M_{\mathbb{R}}/M$ equipped with the standard complex structure.

Let Λ^* be the lattice bundle over B_0 whose fiber at $r \in P^{\text{int}}$ is $\Lambda^*_r = \pi_1(\mathbf{T}_r)$. For each $\lambda \in \Lambda^*$, we may consider the following weighted count of stable holomorphic disks:

$$\mathcal{F}(\lambda) := \sum_{\partial\beta = \lambda} n_\beta \exp\left(-\int_\beta \omega\right).$$

This defines a function $\mathcal{F} : \Lambda^* \to \mathbb{R}$. Applying fiberwise Fourier transform on $\mathcal{F}$, we obtain the superpotential

$$W : \check{X}_0 \to \mathbb{C},$$
$$W(\mathbf{T}_r, \nabla) = \sum_{\beta \in \pi_2(X, \mathbf{T}_r)} n_\beta \exp\left(-\int_\beta \omega\right) \mathrm{Hol}_\nabla(\partial\beta),$$

which defines a holomorphic function on $\check{X}_0$. Notice that the above expression can be an infinite series. Nevertheless we'll see that for semi-Fano toric surfaces, this is just a finite sum and hence there is no convergence issues. For $\beta \in \pi_2(X, \mathbf{T}_r)$, we define a function $Z_\beta : \check{X}_0 \to \mathbb{C}$ by

$$Z_\beta(\mathbf{T}_r, \nabla) := \exp\left(-\int_\beta \omega\right) \mathrm{Hol}_\nabla(\partial\beta), \tag{2}$$

so that the superpotential can be written in the form $W = \sum_{\beta \in \pi_2(X, T_r)} n_\beta Z_\beta$.

It is already known by [8] that $n_{\beta_i} = 1$, where β_i, are the basic disk classes for $i = 1, \ldots, d$ corresponding to the primitive generators $v_i \in N$. Moreover, when X is semi-Fano, the moduli space $\overline{M}_1(\mathbf{T}, \beta)$ is non-empty only when $\beta = \beta_i + \alpha$ for some $i = 1, \ldots, d$ and $\alpha \in H_2(X)$ represented by a rational curve of Chern number zero. Thus we may write

$$W = W_0 + \sum_{i=1}^{d} \sum_{\alpha \neq 0, c_1(\alpha) = 0} n_{\beta_i + \alpha} Z_{\beta_i + \alpha},$$

where $W_0 = \sum_{i=1}^{d} Z_{\beta_i}$. In general it is very hard to compute $n_{\beta_i+\alpha}$ starting from the definition. In the following section, we'll give a method to compute these invariants when X is a semi-Fano toric surface.

3. Disk counting and GW-invariants

3.1. **Toric surfaces.** In this subsection we give some basic results on toric surfaces, which will be needed in the proof of Theorem 1.1. These are probably well-known facts among experts.

We start with the well-known formula for self-intersection number of a compact toric divisor. Let $X = X_\Sigma$ be a smooth toric surface defined by a fan Σ in $\mathbb{Z}^2$. Suppose $D \subset X$ is a compact toric prime divisor. Then D corresponds to a ray $\tau \in \Sigma$, so that $\tau = \sigma^- \cap \sigma^+$ for two 2-dimensional cones $\sigma^-, \sigma^+ \in \Sigma$. (See Figure 1).

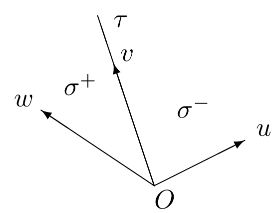

Figure 1. Cones corresponding to a compact divisor.

Let τ be generated by $v \in \mathbb{Z}^2$, σ^- be generated by u, v and σ^+ be generated by v, w such that u, v, w are placed in a counterclockwise fashion. Then the self-intersection of D is given by

$$D^2 = -\begin{vmatrix} u_1 & w_1 \\ u_2 & w_2 \end{vmatrix}$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Proposition 3.1. *Let $D = \cup_{i=1}^{l} D_i$ be a connected union of compact toric prime divisors with $D_i^2 = -2$, and τ_i be the ray corresponding to D_i. Suppose $\sigma_i \in \Sigma$ are 2-dimensional cones so that $\tau_i = \sigma_{i-1} \cap \sigma_i$. Then the cone $\cup_{i=0}^{n} \sigma_i$ is strictly convex.*

Proof. Suppose τ_i is generated by $v_i \in \mathbb{Z}^2$. Without loss of generality, we can assume v_i are labeled in a counterclockwise order as vectors in $\mathbb{R}^2$. We further let σ_0 be generated by v_0, v_1; and σ_n be generated by v_n, v_{n+1}.

Let

$$v_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix}.$$

Since D_i is a (-2)-curve, we have

$$\begin{vmatrix} a_{i-1} & a_{i+1} \\ b_{i-1} & b_{i+1} \end{vmatrix} = 2.$$

In other words, the area of the triangle spanned by v_{i-1} and v_{i+1} is 1.

On the other hand, let A be the triangle spanned by vectors v_{i-1} and v_i; and let B be the triangle spanned by v_i and v_{i+1}. Since X is smooth, the areas of A and B are $\frac{1}{2}$. Now because the sum of areas of A and B is 1, which is equal to the area

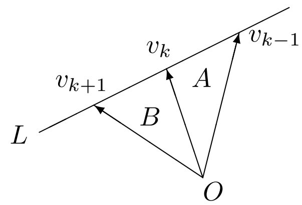

FIGURE 2. -2 toric divisors.

of the triangle spanned by v_{i-1} and v_{i+1}, we know the heads of the vectors v_{i-1}, v_i and v_{i+1} are on the same line L. Moreover,

$$v_i = \frac{1}{2}(v_{i-1} + v_{i+1}).$$

Now since the heads of all vectors v_i are on the same line, the cone $\cup_{i=0}^{n}\sigma_i$ must be strictly convex. □

3.2. **Proof of Theorem 1.1.** In this subsection, we give a proof of Theorem 1.1.

Let X be a compact semi-Fano toric surface. Let $D_1, \cdots, D_d$ be the toric prime divisors of X. Let $\mathbf{T}$ be a Lagrangian torus fiber and let $\beta_i \in \pi_2(X, \mathbf{T})$ be the relative homotopy class of a Maslov index 2 disk such that $\beta_i \cdot D_j = \delta_{ij}$.

Given any $b \in \pi_2(X, \mathbf{T})$ of Maslov index two. Recall that $\overline{M}_1(\mathbf{T}, b)$ is the moduli space of stable maps from bordered Riemann surfaces of genus zero with one boundary marked point to X in the class b. It is known that $\overline{M}_1(\mathbf{T}, b)$ is empty unless $b = \beta_i$, or $b = \beta_i + \alpha$ for some $i \in \{1, \ldots, d\}$ and $\alpha \in H_2(X, \mathbb{Z})$ with $c_1(\alpha) = 0$. Moreover, such an α must be of the form $\alpha = \sum s_k D_k$ where all D_k have self-intersection -2.

Our goal is to calculate the open GW-invariant n_b for all classes b. To state the result, we need the following definitions.

Definition 3.2. *Let $m_1, m_2 \in \mathbb{Z}$. We call a sequence $\{s_k : m_1 \leq k \leq m_2\}$ admissible with center l if each s_k is a positive integer, and*

(1) $s_i \leq s_{i+1} \leq s_i + 1$ when $i < l$;
(2) $s_i \geq s_{i+1} \geq s_i - 1$ when $i \geq l$;
(3) $s_{m_1}, s_{m_2} \leq 1$.

For any toric prime divisor D_i with self-intersection -2, we have a maximal chain $D_i^{\max}$ of compact toric (-2)-divisors containing D_i. Given a sequence $\{s_k\}$, we have an induced sequence $\{\tilde{s}_k\}$ with respect to D_i, defined as $\tilde{s}_j = s_j$ if $D_j \subset D_i^{\max}$ and $s_j = 0$ otherwise.

Definition 3.3. *Let $b = \beta_i + \alpha$ with $\alpha = \sum s_k D_k$. We say b is admissible if $D_i^2 = -2$ and the sequence $\{s_k\}$ is identical to its induced sequence with respect to D_i, and $\{s_k\}$ is admissible with center i.*

To prove Theorem 1.1, we recall the local GW-invariants of a configuration of $\mathbf{P}^1$'s which was obtained by Jim Bryan and Conan Leung in [3].

Let $L(n)$ be a genus 0 nodal curve consisting of a linear chain of $2n+1$ smooth components $L_{-n}, \cdots, L_n$ with an additional smooth component L_* meeting L_0. So we have $L_n \cap L_m = \emptyset$ unless $|n-m| = 1$ and $L_* \cap L_n = \emptyset$ unless $n = 0$. It was shown in [3] that $L(n)$ can be embedded into a smooth surface S so that all L_i are

(-2)-curves and L_* is a (-1)-curve, where S can be taken as a certain blowup of $\mathbf{P}^2$ along points.

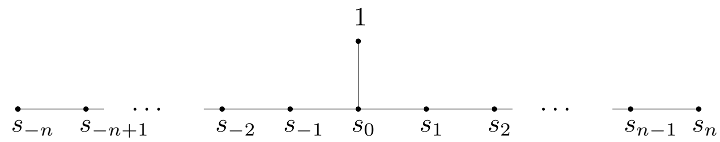

FIGURE 3. The graph of $L(n)$.

The local GW-invariants of $L(n)$ is well-defined, at least for curve classes

$$L_* + \sum_{k=-n}^{n} s_k L_k, \quad s_k \geq 0.$$

Theorem 3.4. [3] *The genus zero local GW-invariants $N(s_k)$ of $L(n)$ for classes $L_* + \sum_{k=-n}^{n} s_k L_k$ is given by*

$$N(s_k) = \begin{cases} 1 & \text{if } \{s_k\} \text{ is admissible with center } 0. \\ 0 & \text{otherwise.} \end{cases}$$

We remark that here admissible with center 0 is an equivalent term for 1-admissible used in [3].

We come to prove our main result Theorem 1.1.

Proof of Theorem 1.1. Given a semi-Fano toric surface X defined by a fan Σ, we would like to compute the open GW-invariant n_b for $b \in \pi_2(X, \mathbf{T})$. First of all, by [8, 9], n_b is non-zero only when $b = \beta_i + \alpha$ for some i and $\alpha \in H_2(X, \mathbb{Z})$ represented by rational curves with $c_1(\alpha) = 0$. It is already known that $n_b = 1$ when $\alpha = 0$, so it suffices to consider $\alpha \neq 0$.

Suppose $n_{\beta_i + \alpha} \neq 0$ and $\alpha \neq 0$. Then D_i must have self-intersection -2, and α must be of the form $\alpha = \sum_{k \in I} s_k D_k$, where I is the index set containing all the natural numbers k such that $D_k \subset D_i^{\max}$, and $s_i \neq 0$. We want to show that the sequence $\{s_k\}$ is admissible, and in such cases $n_b = 1$.

This is done by equating the open GW-invariant n_b to an ordinary GW-invariant of yet another toric manifold Y, which is a toric modification of X. The modification is done as follows. Let v_i be the primitive generator of the ray of Σ corresponding to D_i, and let Σ_1 be the refinement of Σ by adding the ray generated by $v_\infty := -v_i$ (and then completing it into a convex fan). In general the corresponding toric variety X_{Σ_1} may not be smooth. If this is the case, then we take a toric desingularization Y of X_{Σ_1} by adding rays which are adjacent to v_∞. By abuse of notations we still denote the divisors in Y corresponding to v_l's by D_l, and $\alpha = \sum_{k \in I} s_k D_k$ is regarded as a homology class in Y. We remark that the above procedure does nothing if the ray generated by v_∞ is already in Σ.

Notice that in Σ, the ray generated by v_∞ cannot be adjacent to those generated by v_k for $k \in I$ (I is the index set introduced above) by using the fact that D_k's have self-intersection (-2). Then the newly added rays are not adjacent to any v_k for $k \in I$, and thus each $D_k \subset Y$ for $k \in I$ still has self-intersection number (-2). Let $f \in H_2(Y)$ be the fiber class, that is, $f = \beta_i + \beta_\infty$, where β_∞ is the disk class corresponding to v_∞.

By Theorem 1.1 in [4] and its generalization in [17], we have the following equality between open and closed GW-invariants

$$n_b = GW_{0,1}^{Y,\alpha+f}([\mathrm{pt}]).$$

The proof is done by equating the open moduli $\overline{M}_1^{\mathrm{ev}=p}(X,b)$ with the closed moduli $\overline{M}_1^{\mathrm{ev}=p}(Y,f+\alpha)$ together with their Kuranishi structures. We refer the details to [4, 17].

Next we identify $GW_{0,1}^{Y,\alpha+f}([\mathrm{pt}])$ with the local GW-invariant of a configuration of $\mathbf{P}^1$'s.

Let $\tilde{Y}$ be the blowup of Y at a generic point p. Then, by the result of Hu [15] and Gathmann [13], which relates GW-invariants of blowups along points, we know that the GW-invariant of Y for a class γ with one point constraint is equal to that of $\tilde{Y}$ for the strict transform of γ without this point constraint. More precisely, we have

$$GW_{0,1}^{Y,\alpha+f}([\mathrm{pt}]) = GW_{0,0}^{\tilde{Y},\alpha+f'},$$

where f' is the strict transform f, which is the class of a (-1)-curve.

Because $\alpha = \sum s_k D_k$, with all D_k have self-intersection -2, it is easy to see that every curve in $\alpha + f'$ is a tree of $\mathbf{P}^1$'s, with the same configuration as $L(n)$, up to a relabeling of its index. Therefore, $GW_{0,0}^{\tilde{Y},\alpha+f'}$ is exactly the local GW-invariant of $L(n)$. Theorem 1.1 now follows from Theorem 3.4. $\square$

Theorem 1.1 allows us to explicitly compute the superpotential for the mirror of any compact semi-Fano toric surface. Since these surfaces are completely classified (there are totally 16 such surfaces, including the Fano ones), we can compute all the superpotentials; a list of the results is given in the appendix.

4. Small quantum cohomology and Jacobian ring

For a toric Fano manifold X, the map

$$\psi : QH^*(X) \to Jac(W),\ D_i \mapsto Z_{\beta_i},$$

gives a canonical ring isomorphism between the small quantum cohomology $QH^*(X)$ of X and the Jacobian ring $Jac(W)$ of the superpotential W [6, 9]. Recall that the Jacobian ring of W is defined as

$$Jac(W) = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]/\langle \partial_1 W, \ldots, \partial_n W\rangle,$$

where ∂_j denotes $z_j \frac{\partial}{\partial z_j}$ and $n = \dim X$. In the non-Fano case, it is expected that we still have an isomorphism $QH^*(X) \cong Jac(W)$,[1] but the map $\psi : QH^*(X) \to Jac(W)$ needs to be modified by quantum corrections.

In the following, we briefly recall the definition of the corrected map following Fukaya, Oh, Ohta and Ono [9, 10]. As before, X is a compact toric manifold and $\mathbf{T}$ is a Lagrangian torus fiber. Consider the moduli space $\overline{M}_{k,l}(\mathbf{T},\beta)$ of stable maps from genus 0 bordered Riemann surfaces to (X,L) with k boundary marked points and l interior marked point in the class β. We have evaluation maps

$$ev^{\mathrm{int}} : \overline{M}_{k,l}(\mathbf{T},\beta) \to X^l,\ [u;p_0,p_1,\ldots,p_{k-1};z_1,\ldots,z_l] \mapsto (u(z_1),\ldots,u(z_l)),$$

[1]This is now proved in the recent work [12] of Fukaya, Oh, Ohta and Ono (as a special case of their main result).

and

$$ev_i : \overline{M}_{k,1}(\mathbf{T}, \beta) \to \mathbf{T},\ [u; p_0, p_1, \ldots, p_{k-1}; z] \mapsto u(p_i),$$

$i = 0, 1, \ldots, k-1$, at the interior and boundary marked points respectively.

Let $V_1, \ldots, V_l \subset X$ be toric subvarieties. Consider the fiber product

$$\overline{M}_{1,l}(\mathbf{T}, \beta; V_1, \ldots, V_l) := \overline{M}_{1,l}(\mathbf{T}, \beta)_{ev^{\mathrm{int}}} \times_{X^l} \left(\prod_{j=1}^{l} V_j \right).$$

More precisely, $\overline{M}_{1,l}(\mathbf{T}, \beta; V_1, \ldots, V_l)$ is the set of all elements

$$([u; p_0; z_1, \ldots, z_l], x_1, \ldots, x_l) \in \overline{M}_{1,l}(\mathbf{T}, \beta) \times \prod_{j=1}^{l} V_j$$

such that $u(z_1, \ldots, z_l) = (x_1, \ldots, x_l)$. The virtual dimension of $\overline{M}_{1,l}(\mathbf{T}, \beta; V_1, \ldots, V_l)$ is $n + \mu(\beta) + 2l - 2 - \sum_{j=1}^{l} \mathrm{codim}_{\mathbb{R}}(V_j)$.

Definition 4.1 ([10, 11])**.** *The genus zero open GW-invariant $n(\beta; V_1, \ldots, V_l)$ is defined as*

$$n(\beta; V_1, \ldots, V_l) = \int_{[\overline{M}_{1,l}(\mathbf{T}, \beta; V_1, \ldots, V_l)]^{\mathrm{vir}}} ev_0^*[\mathrm{pt}].$$

It is non-zero only when the virtual dimension matches, that is, $\mu(\beta) = 2 - 2l + \sum_{j=1}^{l} \mathit{codim}_{\mathbb{R}}(V_j)$.

By Lemma 6.8 in [10], the number $n(\beta; V_1, \ldots, V_l) \in \mathbb{Q}$ is independent of the auxiliary perturbation data used to define $[\overline{M}_{1,1}(\mathbf{T}, \beta; V)]^{\mathrm{vir}}$. Definition 2.2 is the special case when $l = 0$.

Choose an additive basis $\{T_i = \mathrm{PD}[V_i]\}$ of $H^*(X, \mathbb{C})$ represented by the Poincaré duals of fundamental classes of toric subvarieties $V_i \subset X$.

Definition 4.2 ([10, 11])**.** *Define an additive map $\psi : QH^*(X) \to Jac(W)$ by setting*

$$\psi(T_i) = \sum_{\beta : \mu(\beta) = \mathit{codim}_{\mathbb{R}}(V_i)} n(\beta; V_i) Z_\beta,$$

and extending linearly.

Remark 4.3. *Fukaya, Oh, Ohta and Ono* [10] *also study the so-called potential function with bulk of a toric manifold X, by incorporating deformations of Floer cohomology by cycles on the ambient space X. (In contrast, the superpotential, or what Fukaya, Oh, Ohta and Ono called the potential function, W just encodes deformations of Floer cohomology by the cycles on L.) In the recent preprint* [12]*, they proved that the Jacobian ring of the potential function with bulk is canonically isomorphic to the big quantum cohomology ring of X. The map $\psi : QH^*(X) \to Jac(W)$ we discuss here is a special case of this isomorphism, when the bulk deformation is set to zero. We'll also discuss the potential function with bulk in Section 5.*

Now, for the toric prime divisors $D_1, \ldots, D_d$, the map ψ is given by

$$D_i \mapsto \sum_{\beta : \mu(\beta) = 2} n(\beta; D_i) Z_\beta.$$

A special case of Lemma 9.2 in [10] gives the following analogue of the divisor equation for open GW-invariants.

Proposition 4.4 ([10])**.** *If D is a toric divisor, then we have the following equality*

$$n(\beta; D) = (D \cdot \beta) n_\beta.$$

Combining with our Theorem 1.1, we can compute the map $\psi : QH^*(X) \to Jac(W)$ on toric divisors for any compact semi-Fano toric surface. As an application, we outline a proof of Corollary 1.2 in the following.

To begin with, recall that the cohomology ring $H^*(X, \mathbb{C})$ of a compact toric manifold X is generated by the divisor classes $D_1, \ldots, D_d \in H^2(X, \mathbb{C})$. Moreover, a presentation of $H^*(X, \mathbb{C})$ is given by

$$H^*(X, \mathbb{C}) = \mathbb{C}[D_1, \ldots, D_d]/(\mathcal{L} + \mathcal{SR}),$$

where $\mathcal{L}$ is the ideal generated by linear equivalences among divisors and $\mathcal{SR}$ is the Stanley-Reisner ideal generated by primitive relations.

By a result of Siebert and Tian [21], when X is semi-Fano, the small quantum cohomology $QH^*(X)$ is also generated by the divisor classes $D_1, \ldots, D_d$ and a presentation of $QH^*(X)$ is given by replacing each relation in $\mathcal{SR}$ by its quantum counterpart, i.e. denoting the quantum Stanley-Reisner ideal by $\mathcal{SR}_Q$, then we have

$$QH^*(X) = \mathbb{C}[D_1, \ldots, D_d]/(\mathcal{L} + \mathcal{SR}_Q).$$

Consider the case when $X = X_\Sigma$ is a semi-Fani toric surface. We also assume that X is not $\mathbb{P}^2$. Then any primitive collection is of the form $\mathfrak{P} = \{v_i, v_j\}$ so that v_i, v_j do not generate a cone in Σ. To compute $\mathcal{SR}_Q$, we need to calculate $D_i * D_j$, where $*$ denotes the small quantum product. Choose dual bases $\{D_m\}$, $\{D^m\}$ of $H^2(X)$, both represented by toric divisors. Then, by the divisor equation and a straightforward manipulation, we have

$$\begin{aligned} D_i * D_j &= \sum_{\alpha : c_1(\alpha) = 2} (D_i \cdot \alpha)(D_j \cdot \alpha) GW_{0,1}^{X,\alpha}([\mathrm{pt}]) q^\alpha \\ &\quad + \sum_m \left(\sum_{\alpha : c_1(\alpha) = 1} (D_i \cdot \alpha)(D_j \cdot \alpha)(D^m \cdot \alpha) GW_{0,0}^{X,\alpha} q^\alpha \right) D_m. \end{aligned}$$

The GW-invariants $GW_{0,1}^{X,\alpha}([\mathrm{pt}]), GW_{0,0}^{X,\alpha}$ can be computed using the results of Bryan-Leung [3] as follows. To compute $GW_{0,1}^{X,\alpha}([\mathrm{pt}])$, note that we have $c_1(\alpha) = 2$ so that $\alpha^2 = 0$. Such an α must be of the form $\alpha' + f$ where α' is represented by a chain of (-2)-toric prime divisors and f is a fiber class. We are therefore in exactly the same situation as in the proof of Theorem 1.1. Hence, $GW_{0,1}^{X,\alpha}([\mathrm{pt}])$ can be computed as before.

As for $GW_{0,0}^{X,\alpha}$, we have $c_1(\alpha) = 1$, so that α is represented by a chain $\sum_{k=-p}^{q} s_k D_{i_k}$ of toric prime divisors such that $D_{i_k}^2 = -2$ for all $k \neq 0$, $D_{i_0}^2 = -1$ and $s_0 = 1$. The results of Bryan and Leung also apply in this situation: namely, the GW-invariant $GW_{0,0}^{X,\alpha} = 1$ if both the chains $\sum_{k=-p}^{0} s_k D_{i_k}$ and $\sum_{k=0}^{q} s_k D_{i_k}$ are admissible with center 0 and $GW_{0,0}^{X,\alpha} = 0$ otherwise.

Let us give an example to illustrate the explicit computations.

Example. Let Σ be the fan whose rays are generated by

$$v_1 = (1,0), v_2 = (0,1), v_3 = (-1,-1), v_4 = (0,-1), v_5 = (1,-1), v_6 = (2,-1).$$

This determines a toric surface X. We equip X with a toric Kähler form such that the polytope P is given by

$$\begin{aligned} P &= \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0, 0 \leq x_2 \leq t_1 + t_3 + 2t_4, x_1 + x_2 \leq t_1 + t_2 + 2t_3 + 3t_4, \\ & \qquad t_1 + t_4 + x_1 - x_2 \geq 0, t_1 + 2x_1 - x_2 \geq 0\}, \end{aligned}$$

where $t_i > 0$ are the Kähler parameters.

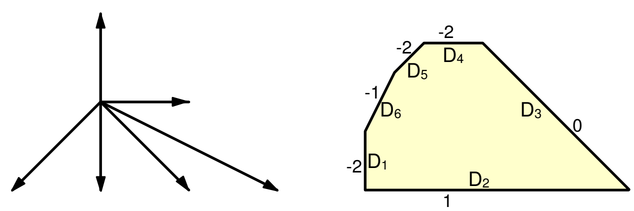

Figure 4. The fan Σ and the polytope P defining X. The numbers beside the divisors indicate their self-intersection numbers.

The linear equivalences among divisors are generated by the following two relations

$$\begin{aligned} D_1 - D_3 + D_5 + 2D_6 = 0, \\ D_2 - D_3 - D_4 - D_5 - D_6 = 0. \end{aligned}$$

Hence, $H^2(X)$ is of rank 4. We choose the dual bases $\{D^m\}$ and $\{D_m\}$ to be $\{D_1, D_4, D_5, D_6\}$ and $\{D_2, D_3, D_4 + 2D_3, D_1 + 2D_2\}$ respectively.

We can now start to compute the primitive relations. For example, we want to compute $D_2 * D_4$. We need to look for all curve classes with $c_1 = 1, 2$ which intersect both D_2 and D_4 non-trivially. There are two such classes with $c_1 = 2$: the classes represented by D_3 and $D_3 + D_4$, and also two with $c_1 = 1$: the classes represented by $D_1 + D_5 + D_6$ and $D_1 + D_4 + D_5 + D_6$. Since all these configurations are admissible, the corresponding GW-invariants are all equal to one, by the above discussion. Hence, we get

$$\begin{aligned} D_2 * D_4 &= q_1 q_3 q_4^2 - q_1 q_2 q_3 q_4^2 + q_1 q_3 q_4(-D_2 + D_3 - (D_4 + 2D_3) + (D_1 + 2D_2)) \\ & \quad - q_1 q_2 q_3 q_4(-D_2 - D_3 + (D_1 + 2D_2)) \\ &= q_1 q_3 q_4^2 - q_1 q_2 q_3 q_4^2 + q_1 q_3 q_4 (D_1 + D_5 + D_6) \\ & \quad - q_1 q_2 q_3 q_4 (D_1 + D_4 + D_5 + D_6), \end{aligned}$$

where we have used linear equivalences to get the second equality. Similarly, we can compute all other primitive relations.

Having computed all the primitive relations, we can go on to show the following

Lemma 4.5. *The map*

$$\psi : \mathbb{C}[D_1, \ldots, D_d] \to \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}], \ D_i \mapsto \sum_{\beta : \mu(\beta) = 2} n(\beta; D_i) Z_\beta$$

defines a ring homomorphism $\psi : QH^*(X) \to Jac(W)$.

Sketch of proof. First of all, we show that the ideal $\mathcal{L}$ of linear equivalences is mapped to the ideal $\langle \partial_1 W, \ldots, \partial_n W\rangle$ by ψ. Linear equivalences are generated by the relations $\sum_{i=1}^d v_i^j D_i = 0$, $j = 1, 2$, where we write $v_i = (v_i^1, v_i^2)$ in coordinates. By Proposition 4.4, we have

$$\begin{aligned}
\psi(D_i) &= \sum_{k=1}^{d} \sum_{\alpha: c_1(\alpha)=0} n(\beta_k + \alpha; D_i) Z_{\beta_k+\alpha} \\
&= \sum_{k=1}^{d} \sum_{\alpha: c_1(\alpha)=0} (D_i \cdot (\beta_k + \alpha)) n(\beta_k + \alpha) Z_{\beta_k+\alpha}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\psi\left(\sum_{i=1}^{d} v_i^j D_i\right) &= \sum_{i=1}^{d} v_i^j \left(\sum_{k=1}^{d} \sum_{\alpha: c_1(\alpha)=0} (D_i \cdot (\beta_k + \alpha)) n(\beta_k + \alpha) Z_{\beta_k+\alpha}\right) \\
&= \sum_{k=1}^{d} \sum_{\alpha: c_1(\alpha)=0} \left(\sum_{i=1}^{d} v_i^j (\delta_{ik} + D_i \cdot \alpha)\right) n(\beta_k + \alpha; D_i) Z_{\beta_k+\alpha} \\
&= \sum_{k=1}^{d} \sum_{\alpha: c_1(\alpha)=0} v_k^j n(\beta_k + \alpha; D_i) Z_{\beta_k+\alpha} \\
&= \partial_j W.
\end{aligned}$$

Next, we need to show that each primitive relation is mapped by ψ to a relation in the ideal $\langle \partial_1 W, \ldots, \partial_n W\rangle$. This can be done by explicit computations. Again, we illustrate this by an example.

Consider X in the previous example. By Theorem 1.1, we can compute the superpotential explicitly. The result is given by

$$\begin{aligned}
W &= (1 + q_1) z_1 + z_2 + \frac{q_1 q_2 q_3^2 q_4^3}{z_1 z_2} + (1 + q_2 + q_2 q_3) \frac{q_1 q_3 q_4^2}{z_2} \\
&\quad + (1 + q_3 + q_2 q_3) \frac{q_1 q_4 z_1}{z_2} + \frac{q_1 z_1^2}{z_2},
\end{aligned}$$

where $q_l = \exp(-t_l)$, $l = 1, \ldots, 4$. We can also compute the images of the divisors D_i under ψ:

$$\begin{aligned}
\psi(D_1) &= (1 - q_1) z_1, \\
\psi(D_2) &= z_2 + q_1 z_1, \\
\psi(D_3) &= \frac{q_1 q_2 q_3^2 q_4^3}{z_1 z_2} + (q_2 + q_2 q_3) \frac{q_1 q_3 q_4^2}{z_2} + \frac{q_1 q_2 q_3 q_4 z_1}{z_2}, \\
\psi(D_4) &= (1 - q_2)\left(\frac{q_1 q_3 q_4^2}{z_2} + \frac{q_1 q_3 q_4 z_1}{z_2}\right), \\
\psi(D_5) &= (1 - q_3)\left(\frac{q_1 q_4 z_1}{z_2} + \frac{q_1 q_2 q_3 q_4^2}{z_2}\right), \\
\psi(D_6) &= \frac{q_1 z_1^2}{z_2} + q_1 z_1 + (q_3 + q_2 q_3) \frac{q_1 q_4 z_1}{z_2} + \frac{q_1 q_2 q_3^2 q_4^2}{z_2}.
\end{aligned}$$

Using what we have computed before,

$$\begin{aligned} D_2 * D_4 &= q_1q_3q_4^2 - q_1q_2q_3q_4^2 + q_1q_3q_4(D_1+D_5+D_6) \\ &\quad -q_1q_2q_3q_4(D_1+D_4+D_5+D_6) \\ &= q_1q_3q_4[(1-q_2)(q_4+D_1+D_5+D_6)-q_2D_4]. \end{aligned}$$

This is mapped by ψ to

$$\begin{aligned} & q_1q_3q_4[(1-q_2)(q_4+z_1+\frac{q_1z_1^2}{z_2}+(1+q_2q_3)\frac{q_1q_4z_1}{z_2}+\frac{q_1q_2q_3q_4^2}{z_2}) \\ & \quad -q_2(1-q_2)(\frac{q_1q_3q_4^2}{z_2}+q_3\frac{q_1q_4z_1}{z_2})] \\ = \; & q_1q_3q_4(1-q_2)(q_4+z_1+\frac{q_1z_1^2}{z_2}+\frac{q_1q_4z_1}{z_2}), \end{aligned}$$

which is exactly $\psi(D_2)\cdot\psi(D_4)$.

Similarly, we can show that $\psi(\mathcal{SR}_Q)=\{0\}\subset Jac(W)$. Hence, ψ defines a ring homomorphism $\psi: QH^*(X)\to Jac(W)$. $\square$

Corollary 1.2 now follows from the following lemma.

Lemma 4.6. *For generic choices of the Kähler parameters q_l, $\psi: QH^*(X)\to Jac(W)$ is a bijective map.*

Sketch of proof. Having computed the superpotential W and the images of the divisors D_i under ψ, we can check surjectivity of ψ in a straightforward way. For instance, for the surface X in the previous example, we have

$$z_1=\psi((1-q_1)^{-1}D_1), z_2=\psi(D_2-q_1(1-q_1)^{-1}D_1),$$
$$z_2^{-1}=\psi([q_1q_3q_4^2(1-q_2)(1-q_2q_3)]^{-1}D_4-[q_1q_4^2(1-q_3)(1-q_2q_3)]^{-1}D_5).$$

Also, since we have the relation $\partial_1 W=0$ which gives

$$z_1^{-1}=(q_1q_2q_3^2q_4^3)^{-1}[(1+q_1)z_1z_2+(1+q_3+q_2q_3)q_1q_4z_1+2q_1z_1^2],$$

and ψ is a homomorphism, z_1^{-1} also lies in the image of ψ. The surjectivity of ψ for all other examples can be checked in this way.

On the other hand, by Proposition 3.7 and Lemma 3.9 in Iritani [16] (which were proved by using Kouchnirenko's results), we have $\dim H^*(X)=\dim Jac(W)$ for generic choices of the Kähler parameters q_l. Hence, $\psi: QH^*(X)\to Jac(W)$ is bijective. $\square$

5. The big quantum cohomology

5.1. **The potential with bulk.** For a Lagrangian torus fiber $\mathbf{T}$ in a compact toric manifold X and $\mathbf{b}\in\mathcal{A}$, where $\mathcal{A}:=\mathbb{C}\langle$toric invariant cycles$\rangle$, FOOO [10] defined the potential with bulk $W_{\mathbf{b}}$ as

$$W_{\mathbf{b}} := \sum_{\substack{\beta\in\pi_2(X,\mathbf{T})\\ l\geq 0}} \frac{1}{l!} n_l(\beta;\underbrace{\mathbf{b},\ldots,\mathbf{b}}_{l})Z_\beta$$

where the open Gromov-Witten invariants $n(\beta;V_1,\ldots,V_l)$ (see Definition 4.1) extend multilinearly to give a function $n_l:\pi_2(X,\mathbf{T})\times\mathcal{A}^{\otimes l}\to\mathbb{C}$. In a recent preprint [12] they proved that

$$QH^*_{\mathbf{b}}(X)\cong \mathrm{Jac}(W_{\mathbf{b}}).$$

Thus an explicit expression of $W_{\mathbf{b}}$ would give an explicit presentation of the big quantum cohomology ring $QH^*_{\mathbf{b}}(X)$.

In the previous section, we have given an explicit expression of $W_{\mathbf{b}}$ when $\mathbf{b} = 0$ for a semi-Fano toric surface X. We consider its potential with bulk in this section. For the purpose of computing $QH^*_{\mathbf{b}}(X)$, it is enough to consider $\mathbf{b} = aX + D + cp$, where D is a toric divisor, p is the intersection point of two toric prime divisors (say D_1 and D_2), and $a, c \in \mathbb{C}$.

Proposition 5.1 (Restatement of Corollary 1.3)**.** *Let X be a semi-Fano toric surface, and $\mathbf{b} = aX + D + cp$ as described above. Then*

$$W_{\mathbf{b}} = a + \sum_{\beta \neq 0} \exp(\langle \beta, D \rangle) \left(\sum_{k=0}^{\infty} \frac{c^k}{k!} n_k(\beta; p, \ldots, p) \right) Z_\beta.$$

In particular, when $c = 0$,

$$W_{\mathbf{b}} = a + \sum_{\beta \ \textit{admissible}} \exp(\langle \beta, D \rangle) Z_\beta.$$

Proof. When $\beta \neq 0$,

$$n_k(\beta; [X], \gamma_1, \ldots, \gamma_{k-1}) = 0$$

for all $k \geq 1$ and $\gamma_1, \ldots, \gamma_{k-1} \in H_*(X)$ due to dimension reason. Thus

$$\begin{aligned} W_{\mathbf{b}} &:= \sum_{\substack{\beta \in \pi_2(X, \mathbf{T}) \\ l \geq 0}} \frac{1}{l!} n_l(\beta; \mathbf{b}, \ldots, \mathbf{b}) Z_\beta \\ &= \sum_{l \geq 0} \frac{1}{l!} n_l(0; \mathbf{b}, \ldots, \mathbf{b}) + \sum_{\substack{\beta \neq 0 \\ l \geq 0}} \frac{1}{l!} n_l(\beta; D + cp, \ldots, D + cp) Z_\beta. \end{aligned}$$

Moreover, $n_1(0; X) = 1$ ($\overline{M}_{1,1}(\mathbf{T}, 0; X)$ contains the constant map only) and $n_1(0; p) = n_1(0; D) = 0$ (the corresponding moduli spaces are empty). Also by dimension counting, $n_l(0; \gamma_1, \ldots, \gamma_l) = 0$ for all $l \neq 1$. Thus the first term is

$$\sum_{l \geq 0} \frac{1}{l!} n_l(0; \mathbf{b}, \ldots, \mathbf{b}) = a.$$

Using the divisor equation for open Gromov-Witten invariants ([10]; see Proposition 4.4), the second term is

$$\begin{aligned} \sum_{\substack{\beta \neq 0 \\ l \geq 0}} \frac{1}{l!} n_l(\beta; D + cp, \ldots, D + cp) Z_\beta &= \sum_{\substack{\beta \neq 0 \\ l \geq 0}} \frac{1}{l!} \sum_{k=0}^{l} \mathrm{C}^l_k c^k n_l(\beta; \underbrace{D, \ldots, D}_{l-k}, \underbrace{p, \ldots, p}_{k}) Z_\beta \\ &= \sum_{\substack{\beta \neq 0 \\ l \geq 0}} \frac{1}{l!} \sum_{k=0}^{l} \mathrm{C}^l_k c^k (\langle \beta, D \rangle)^{l-k} n_k(\beta; p, \ldots, p) Z_\beta \\ &= \sum_{\substack{\beta \neq 0 \\ j,k \geq 0}} \frac{c^k}{j! k!} (\langle \beta, D \rangle)^j n_k(\beta; p, \ldots, p) Z_\beta \\ &= \sum_{\beta \neq 0} \exp(\langle \beta, D \rangle) \left(\sum_{k=0}^{\infty} \frac{c^k}{k!} n_k(\beta; p, \ldots, p) \right) Z_\beta. \end{aligned}$$

When $c = 0$,

$$W_{\mathbf{b}} = a + \sum_{\beta \neq 0} \exp(\langle \beta, D \rangle) n_\beta Z_\beta.$$

By Theorem 1.1, $n_\beta = 1$ when β is admissible, and 0 otherwise. Thus

$$W_{\mathbf{b}} = a + \sum_{\beta \text{ admissible}} \exp(\langle \beta, D \rangle) Z_\beta.$$

□

5.2. **Speculations and discussions.** In Proposition 5.1, $n_l(\beta; p, \ldots, p)$ $(l \geq 1)$ has not been computed. In the following we give an informal discussion concerning these invariants.

One of the issues involved in computing these invariants is the presence of 'ghost bubbles' in the moduli space $\overline{M}_{1,l}(\mathbf{T}, \beta; p, \ldots, p)$ (see Figure 5) when p is chosen to be a toric fixed point. On the other hand, if we consider $p_1, \ldots, p_l \in X$ in generic position, which is the approach taken by M. Gross [14] where he used tropical geometry to define the superpotential with bulk, the moduli space $\overline{M}_{1,l}(\mathbf{T}, \beta; p_1, \ldots, p_l)$ does not involve disk bubbling (when β has the suitable Maslov index $\mu(\beta) = 2 - 2l + \sum_{j=1}^{l} \mathrm{codim}_{\mathbb{R}}(V_j)$ so that the moduli has expected dimension $n = \dim \mathbf{T}$), and also ghost bubbles are not present. The invariant $n_l(\beta; p_1, \ldots, p_l)$ can still be defined, and it is easier to compute.

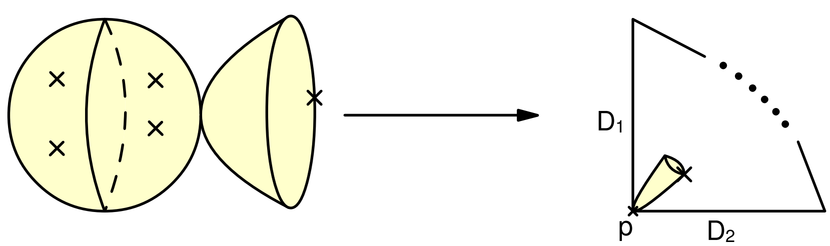

FIGURE 5. Ghost bubbles in $\overline{M}_{1,4}(\mathbf{T}, \beta; p, p, p, p)$. The whole sphere bubble is contracted to the toric fixed point p. The disk class is taken such that $\overline{M}_{1,4}(\mathbf{T}, \beta; p, p, p, p)$ has expected dimension 2. However the actual dimension is bigger than 2 since the interior marked points are free to move in the bubble.

This motivates us to consider $p' \in D_1$ which is *not* fixed by the torus action, and define the invariant $n_l(\beta; p', \ldots, p')$ by taking a generic perturbation of the l points around p'.

Example 5.2 (The Hirzebruch surface $\mathbf{F}_2$). *Consider The Hirzebruch surface* $\mathbf{F}_2$ *whose polytope picture is shown in Figure 6. If we take the above approach, then* $n_l(\beta; p', \ldots, p')$ *equals to* 1 *when* $\beta = l\beta_1 + \beta_i$ *for* $i = 2, 3, 4$ *or* $\beta = l\beta_1 + \beta_4 + D_4$,

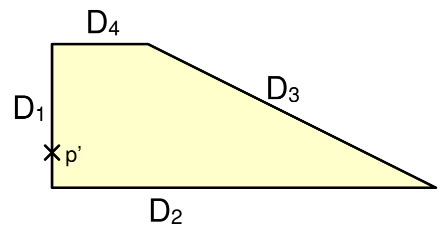

FIGURE 6. The polytope of the Hirzebruch surface $\mathbf{F}_2$.

and 0 *otherwise. Then for* $\mathbf{b} = a[X] + D + cp$*,*

$$\begin{aligned} W_{\mathbf{b}} &= a + \sum_{\beta \neq 0} \exp(\langle \beta, D \rangle) \left(\sum_{k=0}^{\infty} \frac{c^k}{k!} n_k(\beta; p, \ldots, p) \right) Z_\beta \\ &= a + \exp(\langle \beta_1, D \rangle) Z_{\beta_1} + \sum_{i=2}^{4} \exp(c\, \mathbf{e}^{\langle \beta_1, D \rangle} Z_{\beta_1}) \exp(\langle \beta_i, D \rangle) Z_{\beta_i} \\ &\quad + \exp(c\, \mathbf{e}^{\langle \beta_1, D \rangle} Z_{\beta_1}) \exp(\langle \beta_4 + D_4, D \rangle) q_4 Z_{\beta_4}. \end{aligned}$$

The above consideration is tentative, and we are still investigating whether this idea is in the right direction.

APPENDIX A. A LIST OF THE SUPERPOTENTIALS FOR THE MIRRORS OF ALL SEMI-FANO TORIC SURFACES

Using the fact that any smooth compact toric surface is a blowup of either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_m$ ($m \geq 0$) at torus fixed points, it is easy to see that there are finitely many isomorphism classes of semi-Fano toric surfaces. In fact, all except $\mathbb{F}_2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are blowups of $\mathbb{P}^2$; there are 16 of such surfaces, five of which are Fano (namely, $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and the blowup of $\mathbb{P}^2$ at 1, 2 or 3 points).

By using Theorem 1.1, we can compute the superpotentials for the mirrors of all these semi-Fano toric surfaces explicitly. In this appendix, we shall give a list of the superpotentials for the 11 semi-Fano but non-Fano toric surfaces. We enumerate them as $X_1, \ldots, X_{11}$, and each surface is specified by the primitive generators $\rho(\Sigma)$ of rays of its fan and the defining inequalities of its polytope. Also, in the following tables, the t_l's are positive numbers and $q_l = \exp(-t_l)$ (Kähler parameters).

	$\rho(\Sigma)$	polytope P	superpotential W
X_1	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,-2)$ $v_4=(0,-1)$	$x_1\geq 0$ $x_2\geq 0$ $2t_1+t_2-x_1-2x_2\geq 0$ $t_1-x_2\geq 0$	$z_1+z_2+\frac{q_1^2q_2}{z_1z_2^2}+(1+q_2)\frac{q_1}{z_2}$
X_2	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,-1)$ $v_4=(0,-1)$ $v_5=(1,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_1+t_2+2t_3-x_1-x_2\geq 0$ $t_1+t_3-x_2\geq 0$ $t_1+x_1-x_2\geq 0$	$z_1+z_2+\frac{q_1q_2q_3^2}{z_1z_2}+(1+q_2)\frac{q_1q_3}{z_2}+\frac{q_1z_1}{z_2}$
X_3	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,-1)$ $v_4=(0,-1)$ $v_5=(1,-1)$ $v_6=(2,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_1+t_2+2t_3+3t_4-x_1-x_2\geq 0$ $t_1+t_3+2t_4-x_2\geq 0$ $t_1+t_4+x_1-x_2\geq 0$ $t_1+2x_1-x_2\geq 0$	$(1+q_1)z_1+z_2+\frac{q_1q_2q_3^2q_4^3}{z_1z_2}+(1+q_2+q_2q_3)\frac{q_1q_3q_4^2}{z_2}+(1+q_3+q_2q_3)\frac{q_1q_4z_1}{z_2}+\frac{q_1z_1^2}{z_2}$
X_4	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,0)$ $v_4=(0,-1)$ $v_5=(1,-1)$ $v_6=(2,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_2+t_3+t_4-x_1\geq 0$ $t_1+t_3+2t_4-x_2\geq 0$ $t_1+t_4+x_1-x_2\geq 0$ $t_1+2x_1-x_2\geq 0$	$(1+q_1)z_1+z_2+\frac{q_2q_3q_4}{z_1}+\frac{q_1q_3q_4^2}{z_2}+(1+q_3)\frac{q_1q_4z_1}{z_2}+\frac{q_1z_1^2}{z_2}$
X_5	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,0)$ $v_4=(-1,-1)$ $v_5=(0,-1)$ $v_6=(1,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_2+t_3+t_4-x_1\geq 0$ $t_1+t_3+2t_4-x_1-x_2\geq 0$ $t_1+t_4-x_2\geq 0$ $t_1+x_1-x_2\geq 0$	$z_1+z_2+\frac{q_2q_3q_4}{z_1}+\frac{q_1q_3q_4^2}{z_1z_2}+(1+q_3)\frac{q_1q_4}{z_2}+\frac{q_1z_1}{z_2}$
X_6	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,0)$ $v_4=(-1,-1)$ $v_5=(0,-1)$ $v_6=(1,-1)$ $v_7=(2,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_2+t_3+t_4+t_5-x_1\geq 0$ $t_1+t_3+2t_4+3t_5-x_1-x_2\geq 0$ $t_1+t_4+2t_5-x_2\geq 0$ $t_1+t_5+x_1-x_2\geq 0$ $t_1+2x_1-x_2\geq 0$	$(1+q_1)z_1+z_2+\frac{q_2q_3q_4q_5}{z_1}+\frac{q_1q_3q_4^2q_5^3}{z_1z_2}+(1+q_3+q_3q_4)\frac{q_1q_4q_5^2}{z_2}+(1+q_4+q_3q_4)\frac{q_1q_5z_1}{z_2}+\frac{q_1z_1^2}{z_2}$
X_7	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,1)$ $v_4=(-1,0)$ $v_5=(0,-1)$ $v_6=(1,-1)$ $v_7=(2,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_2+t_3-t_1-t_5-x_1+x_2\geq 0$ $t_3+t_4+t_5-x_1\geq 0$ $t_1+t_4+2t_5-x_2\geq 0$ $t_1+t_5+x_1-x_2\geq 0$ $t_1+2x_1-x_2\geq 0$	$(1+q_1)z_1+z_2+\frac{q_2q_3z_2}{q_1q_5z_1}+\frac{q_3q_4q_5}{z_1}+\frac{q_1q_4q_5^2}{z_2}+(1+q_4)\frac{q_1q_5z_1}{z_2}+\frac{q_1z_1^2}{z_2}$

	$\rho(\Sigma)$	polytope P	superpotential W
X_8	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,0)$ $v_4=(-2,-1)$ $v_5=(-1,-1)$ $v_6=(0,-1)$ $v_7=(1,-1)$ $v_8=(2,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_2+t_3+t_4+t_5+t_6-x_1\geq 0$ $t_1+t_3+2t_4+3t_5+4t_6-2x_1-x_2\geq 0$ $t_1+t_4+2t_5+3t_6-x_1-x_2\geq 0$ $t_1+t_5+2t_6-x_2\geq 0$ $t_1+t_6+x_1-x_2\geq 0$ $t_1+2x_1-x_2\geq 0$	$(1+q_1)z_1+z_2+(1+\frac{q_1q_5q_6^2}{q_2^2q_3})\frac{q_2q_3q_4q_5q_6}{z_1}+\frac{q_1q_3q_4^2q_5^3q_6^4}{z_1^2z_2}+(1+q_3+q_3q_4+q_3q_4q_5)\frac{q_1q_4q_5^2q_6^3}{z_1z_2}+(1+q_4+q_3q_4+q_4q_5+q_3q_4q_5+q_3q_4^2q_5)\frac{q_1q_5q_6^2}{z_2}+(1+q_5+q_4q_5+q_3q_4q_5)\frac{q_1q_6z_1}{z_2}+\frac{q_1z_1^2}{z_2}$
X_9	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,1)$ $v_4=(-1,0)$ $v_5=(-1,-1)$ $v_6=(0,-1)$ $v_7=(1,-1)$ $v_8=(2,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_2+2t_3+t_4-t_1-t_6-x_1+x_2\geq 0$ $t_3+t_4+t_5+t_6-x_1\geq 0$ $t_1+t_4+2t_5+3t_6-x_1-x_2\geq 0$ $t_1+t_5+2t_6-x_2\geq 0$ $t_1+t_6+x_1-x_2\geq 0$ $t_1+2x_1-x_2\geq 0$	$(1+q_1)z_1+z_2+\frac{q_2q_3^2q_4z_2}{q_1q_6z_1}+(1+q_2)\frac{q_3q_4q_5q_6}{z_1}+\frac{q_1q_4q_5^2q_6^3}{z_1z_2}+(1+q_4+q_4q_5)\frac{q_1q_5q_6^2}{z_2}+(1+q_5+q_4q_5)\frac{q_1q_6z_1}{z_2}+\frac{q_1z_1^2}{z_2}$
X_{10}	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,1)$ $v_4=(-2,1)$ $v_5=(-1,0)$ $v_6=(0,-1)$ $v_7=(1,-1)$ $v_8=(2,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_2+t_3+t_4-t_1-t_6-x_1+x_2\geq 0$ $2t_4+t_5-t_1-t_3-2x_1+x_2\geq 0$ $t_4+t_5+t_6-x_1\geq 0$ $t_1+t_5+2t_6-x_2\geq 0$ $t_1+t_6+x_1-x_2\geq 0$ $t_1+2x_1-x_2\geq 0$	$(1+q_1)z_1+z_2+(1+\frac{q_1q_5q_6^2}{q_2^2q_3})\frac{q_2q_3q_4z_2}{q_1q_6z_1}+\frac{q_4^2q_5z_2}{q_1q_3z_1^2}+(1+q_3)\frac{q_4q_5q_6}{z_1}+\frac{q_1q_5q_6^2}{z_2}+(1+q_5)\frac{q_1q_6z_1}{z_2}+\frac{q_1z_1^2}{z_2}$
X_{11}	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,2)$ $v_4=(-1,1)$ $v_5=(-1,0)$ $v_6=(-1,-1)$ $v_7=(0,-1)$ $v_8=(1,-1)$ $v_9=(2,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_2+2t_3+3t_4+t_5-2t_1-t_6-3t_7-x_1+2x_2\geq 0$ $t_3+2t_4+t_5-t_1-t_7-x_1+x_2\geq 0$ $t_4+t_5+t_6+t_7-x_1\geq 0$ $t_1+t_5+2t_6+3t_7-x_1-x_2\geq 0$ $t_1+t_6+2t_7-x_2\geq 0$ $t_1+t_7+x_1-x_2\geq 0$ $t_1+2x_1-x_2\geq 0$	$(1+q_1+\frac{q_2q_3^2q_4^3q_5}{q_1q_6q_7^3})z_1+(1+\frac{q_2q_3^2q_4^3q_5}{q_1^2q_6q_7^3}+\frac{q_2q_3^2q_4^3q_5}{q_1q_6q_7^3})z_2+\frac{q_2q_3^2q_4^3q_5z_2^2}{q_1^2q_6q_7z_1}+(1+q_2+q_2q_3)\frac{q_3q_4^2q_5z_2}{q_1q_7z_1}+(1+q_3+q_2q_3)\frac{q_4q_5q_6q_7}{z_1}+\frac{q_1q_5q_6^2q_7^3}{z_1z_2}+(1+q_5+q_5q_6)\frac{q_1q_6q_7^2}{z_2}+(1+q_6+q_5q_6)\frac{q_1q_7z_1}{z_2}+\frac{q_1z_1^2}{z_2}$

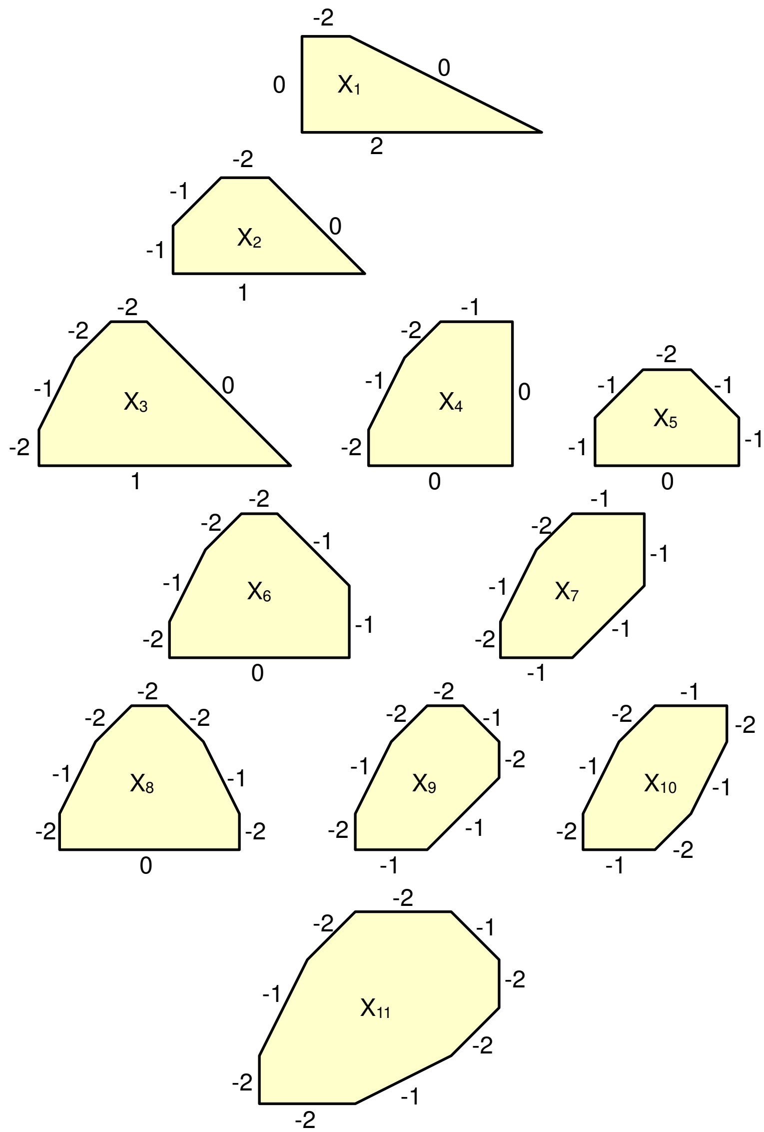

FIGURE 7. Polytopes defining the semi-Fano but non-Fano toric surfaces. The numbers indicate the self-intersection numbers of the toric divisors.

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