Incompressibility of generic orthogonal grassmannians

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Abstract. Given a non-degenerate quadratic form over a field such that its maximal orthogonal grassmannian is 2-incompressible (a condition satisfied for generic quadratic forms of arbitrary dimension), we apply the theory of upper motives to show that all other orthogonal grassmannians of this quadratic form are 2-incompressible. This computes the canonical 2-dimension of any projective homogeneous variety (i.e., orthogonal flag variety) associated to the quadratic form. Moreover, we show that the Chow motives with coefficients in $\mathbb{F}_2$ (and therefore also in any field of characteristic 2, [2]) of those grassmannians are indecomposable. That is quite unexpected, especially after a recent result of [9] on decomposability of the motives of incompressible *twisted grassmannians*.

In this note, we are working with the 2-motives of certain smooth projective varieties associated to quadratic forms over fields of arbitrary characteristic. We refer to [3] for notation and basic results concerning the quadratic forms. By 2-motives, we mean the Grothendieck Chow motives with coefficients in the finite field $\mathbb{F}_2$ as introduced in [3]. We are using the theory of upper motives conceived in [5] and [7].

Let φ be a non-zero non-degenerate quadratic form over a field F (which may have characteristic 2). For any integer r with $0 \le r \le (\dim \varphi)/2$ we write $X_r = X_r(\varphi)$ for the variety of r-dimensional totally isotropic subspaces of φ.

For any r, the variety X_r is smooth and projective. It is geometrically connected if and only if $r \neq (\dim \varphi)/2$. In particular, X_r is connected for any r if $\dim \varphi$ is odd. For even-dimensional φ and $r = (\dim \varphi)/2$, the variety X_r is connected if and only the discriminant of φ is non-trivial.

If the variety X_r is not connected, it has two connected components and they are isomorphic. In particular, the dimension of X_r is always the dimension of any its connected component. Here is a formula for the dimension, where $d := \dim \varphi$:

$$\dim X_r = r(r-1)/2 + r(d-2r).$$

In the case where the quadratic form φ is "generic enough" (the precise condition is formulated in terms of the variety X_r with maximal r), we are going to show (see Theorems 2.1, 3.1, and 4.1) that the 2-motive of X_r is indecomposable, if we are away from the two exceptional cases described below (where the motive evidently decomposes).

Each of the both exceptional cases arises only if the dimension of φ is even and the discriminant of φ is trivial. The first case is the case of $r = (\dim \varphi)/2$, where the variety X_r has two connected components. Our assumption on φ ensures that the 2-motive of each component of X_r is indecomposable.

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The second case is the case of $r = (\dim \varphi)/2 - 1$, where the variety X_r is a rank r projective bundle over a component of X_{r+1}. Therefore, the 2-motive of X_r is a sum of shifts of $r+1$ copies of the 2-motive of a component of X_{r+1}, and the summands of this decomposition are indecomposable.

We recall that a connected smooth projective variety X is called 2-*incompressible*, if its *canonical* 2-*dimension*, as defined in [3, §90], takes its maximal value $\dim X$. This in particular implies that any rational map $X \dashrightarrow X$ is dominant, i.e., that X is *incompressible*.

Any projective homogeneous variety X having indecomposable 2-motive, is 2-incompressible, [5, §2e]. Therefore our indecomposability results imply 2-incompressibility of the corresponding varieties.

Let us point out that our incompressibility results compute the canonical 2-dimension of any projective homogeneous variety (i.e., orthogonal flag variety) associated to a quadratic form which is "generic enough". This is so indeed because for an arbitrary non-degenerate quadratic form φ and an arbitrary sequence of integers $r_1, \dots, r_k$ with $0 < r_1 < \dots < r_k \leq (\dim \varphi)/2$ we have an orthogonal flag variety $X_{r_1,\dots,r_k}$, the variety of flags of totally isotropic subspaces of φ of dimensions $r_1, \dots, r_k$, and the canonical 2-dimension (of a component) of this variety coincides with the canonical 2-dimension of (a component of) X_{r_k}.

The motivic indecomposability of the varieties X_r contrasts with a recent result of M. Zhykhovich [9] saying that for any prime p, any central division F-algebra D of degree p^n for some n, and any r with $0 < r < n$ (and $r \neq 1$ if $p = 2$), the p-motive of the variety of the right ideals of reduced dimension p^r in D (this variety is known to be p-incompressible and is a twisted form of the grassmannian of p^r-dimensional subspaces in a p^n-dimensional vector space) is decomposable.

The paper is organized as follows. In §1 we recall the necessary aspects of the theory of upper motives. In the next sections we establish our main result: in §2 for odd-dimensional forms (Theorem 2.1), in §3 for even-dimensional forms of trivial discriminant (Theorem 3.1), and finally in §4 for even-dimensional forms of non-trivial discriminant (Theorem 4.1).

1. Upper motives

In this section, φ is an arbitrary non-degenerate quadratic form over F. We are considering the corresponding varieties X_r with $0 \leq r \leq (\dim \varphi)/2$.

We recall that the following Krull-Schmidt principle holds: for any finite separable field extension K/F, any summand of the 2-motive of the F-variety $(X_r)_K$ decomposes and in a unique way in a finite direct sum of indecomposable motives, see [1] and/or [7, Corollary 2.2].

Below we are assuming that $K = F$ if the dimension of φ is odd or if the dimension of φ is even and the discriminant of φ is trivial. If the dimension of φ is even and the discriminant of φ is non-trivial, then $K = F$ or K is the discriminant field extension of F (which is always separable and quadratic). Note that in any case the connected components of the variety $(X_r)_K$ are isomorphic.

Let X be a connected component of $(X_r)_K$. We consider X as an F-variety. The 0-codimensional Chow group of the 2-motive of X is an $\mathbb{F}_2$-vector space of dimension

1. Therefore the complete decomposition of the 2-motive of X contains precisely one summand summand $U((X_r)_K)$ with the same property: the 0-codimensional Chow group of $U((X_r)_K)$ is an $\mathbb{F}_2$-vector space of dimension 1. The motive $U((X_r)_K)$ is determined by φ, r and K uniquely up to an isomorphism. We call it the *upper motive* of $(X_r)_K$.

Here are the main results of [5] and [7] applied to the varieties X_r:

Theorem 1.1. *Any indecomposable summand of the 2-motive of X_r is isomorphic to a shift of the motive $U((X_s)_K)$ for some K as above and for some $s \geq r$.*

Some additional information can be derived from the proofs given in [5] and [7]:

Lemma 1.2. *Let φ and r be such that X_r is geometrically connected. Let L/F be a field extension such that the field extension $L(X_r)/F(X_r)$ is purely transcendental. Let M be an indecomposable summand of the 2-motive of X_r. Let $U((X_s)_{KL})(j)$ be an indecomposable summand of M_L with the smallest shifting number j (with some s and some K). Then $M \simeq U((X_s)_K)(j)$ (with the same s, K, and j).*

Proof. The case of $L = F(X_r)$ is actually proved in the proofs of [5, Theorem 3.5] and [7, Theorem 1.1]. It implies the general case because the extension $L(X_r)/F(X_r)$ is purely transcendental. □

Remark 1.3. Lemma 1.2 will provide us with a doubled amount of information if we take into account the duality, [3, §65]. Namely, if M is a summand of the 2-motive of X_r, then $M^*(\dim X_r)$ is also a summand of the 2-motive of X_r. Therefore, being interested in understanding the summand M, we may apply Lemma 1.2 not only to M itself, but also to $M^*(\dim X_r)$.

We finish this section by discussing isomorphism criteria for the motives $U(X_r)_K$ easily derived from the general isomorphism criterion for upper motives, [5, Corollary 2.15]. For odd-dimensional φ, we have $U(X_r) \simeq U(X_s)$ for some $r < s$ if and only if the Witt index of the quadratic form $\varphi_{F(X_r)}$ is $\geq s$ (informally speaking, this means that the *splitting pattern* of φ "makes a jump" at least from $r-1$ to s). For φ of even dimension $2n$ and of trivial discriminant, we always have $U(X_{n-1}) \simeq U(X_n)$; for $r < s < n$, we have $U(X_r) \simeq U(X_s)$ once again if and only if the Witt index of the quadratic form $\varphi_{F(X_r)}$ is $\geq s$. Finally, for $2n$-dimensional φ of non-trivial discriminant, the isomorphism criterion for $U(X_r)$ and $U(X_s)$ with $r < s < n$ is the same, but we never have $U(X_r) \simeq U(X_n)$ with $r \neq n$ nor we have $U(X_r) \simeq U((X_s)_K)$ with $r, s \neq n$. On the contrary, we have $U(X_n) \simeq U((X_n)_K)$, where K is the discriminant extension. As to the criterion of isomorphism for the upper motives of the F-varieties $(X_r)_K$ and $(X_s)_K$, it coincides with the criterion for the K-varieties $(X_r)_K$ and $(X_s)_K$ related to the quadratic form φ_K of trivial discriminant, the case discussed above already.

2. Odd-dimensional quadratic forms

Let F be a field, n an integer ≥ 0, φ a non-degenerate $(2n+1)$-dimensional quadratic form over F. We assume that the following equivalent conditions hold:

(1) the variety X_n is 2-incompressible;
(2) the 2-motive of the variety X_n is indecomposable;

(3) the J-invariant of the quadratic form φ (as defined in [3, §88]) takes its maximal value:
$$J(\varphi) = \{1, 2, \dots, n\}.$$

The equivalence (1) ⇔ (3) is proved in [3, §90]. The equivalence (2) ⇔ (3) follows from [8]. Also, the equivalence (1) ⇔ (2) is a consequence of Theorem 1.1 and [4, Theorem 5.1].

The above conditions imply that the *splitting pattern* of φ "has no jumps". This means that the upper motives $U(X_0), U(X_1), \dots, U(X_n)$ are pairwise non-isomorphic.

The conditions (1–3) are satisfied if the degree of any closed point on X_n is divisible by 2^n. The condition on the closed points is satisfied if the even Clifford algebra of φ is a division algebra. Finally, the condition on the even Clifford algebra is satisfied if $F = k(t_0, \dots, t_{2n})$, where k is a field and $t_0, \dots, t_{2n}$ are variables, and $\varphi = \langle t_0\rangle \perp [t_1, t_2] \perp \dots \perp [t_{2n-1}, t_{2n}]$ (a sort of *generic* $(2n + 1)$-dimensional quadratic form).

Theorem 2.1. *Let φ be a non-degenerate $(2n + 1)$-dimensional quadratic form over a field F such that the variety X_n is 2-incompressible. Then for any r with $0 \le r \le n$, the 2-motive of the variety X_r is indecomposable. In particular, all X_r are 2-incompressible.*

Proof. We induct on n. The induction base is the trivial case of $n = 0$. Now we assume that $n \ge 1$.

We do a descending induction on r. The induction base is the case of $r = n$ which is served by our assumption on φ. Now we assume that $r < n$. Since the case of $r = 0$ is trivial, we may assume that $r > 0$.

Let $L := F(X_1)$. We have $\varphi_L \simeq \mathbb{H} \perp \psi$, where ψ is a quadratic form over L of dimension $2(n-1)+1$ and $\mathbb{H}$ is the hyperbolic plane. According to [3, §88], the assumption on the J-invariant (assumption (3)) still holds for ψ.

For any s with $0 \le s \le n-1$, we write Y_s for the variety $X_s(\psi)$. By [6, Theorem 15.8], the 2-motive of the L-variety $(X_r)_L$ decomposes in a sum of three summands:
$$M(X_r)_L \simeq M(Y_{r-1}) \oplus M(Y_r)(i) \oplus M(Y_{r-1})(j),$$
where $i := (\dim X_r - \dim Y_r)/2$ and $j := \dim X_r - \dim Y_{r-1}$. By the induction hypothesis, each of these three summands is indecomposable. It follows (taking into account the duality like in Remark 1.3) that if the motive of X_r (over F) is decomposable, then it has a summand M with $M_L \simeq M(Y_r)(i) = U(Y_r)(i)$. Note that $U(Y_r) \simeq U((X_{r+1})_L)$. By Lemma 1.2, $M \simeq U(X_{r+1})(i)$, that is, $U(X_{r+1})_L \simeq M(Y_r)$, where $U(X_{r+1})$ is the upper motive of the variety X_{r+1}. By the induction hypothesis, the 2-motive of X_{r+1} is indecomposable. In particular, $U(X_{r+1}) = M(X_{r+1})$. Therefore we have an isomorphism $M(X_{r+1})_L \simeq M(Y_r)$ and, in particular, $\dim X_{r+1} = \dim Y_r$. However $\dim X_{r+1} - \dim Y_r = 2n - r - 1 > n - 1 \ge 0$. □

3. Even-dimensional quadratic forms of trivial discriminant

Let F be a field, n an integer ≥ 1, φ a non-degenerate $(2n)$-dimensional quadratic form over F of trivial discriminant. In this case the variety X_n has two (isomorphic) connected components, and we write X'_n for a component of the variety X_n.

We assume that the following equivalent conditions hold:

(1) the variety X'_n is 2-incompressible;
(2) the 2-motive of the variety X'_n is indecomposable;
(3) the J-invariant of the quadratic form φ takes its maximal value:

$$J(\varphi) = \{1, 2, \dots, n-1\}.$$

The above conditions imply that the upper motives

$$U(X_0), U(X_1), \dots, U(X_{n-2}), U(X_{n-1})$$

are pairwise non-isomorphic. We recall that $U(X_{n-1}) \simeq U(X_n)$.

The conditions (1–3) are satisfied if the degree of any closed point on X_n is divisible by 2^{n-1}. The condition on the closed points is satisfied if the even Clifford algebra of φ is the direct product of two copies of a division algebra. Finally, the condition on the even Clifford algebra is satisfied if F is the discriminant quadratic extension over $k(t_1, \dots, t_{2n})$ of the quadratic form $[t_1, t_2]\perp \dots \perp[t_{2n-1}, t_{2n}]$, where k is a field and $t_1, \dots, t_{2n}$ are variables, and $\varphi = ([t_1, t_2]\perp \dots \perp[t_{2n-1}, t_{2n}])_F$ (a sort of *generic* $(2n)$-dimensional quadratic form of trivial discriminant).

Theorem 3.1. *Let φ be a non-degenerate $(2n)$-dimensional quadratic form over a field F such that the discriminant of φ is trivial and a component of the variety X_n is 2-incompressible. Then for any r with $0 \le r \le n-2$, the 2-motive of the variety X_r is indecomposable. In particular, X_r is 2-incompressible for such r.*

Proof. We induct on n. The induction base is the trivial case of $n = 1$. Now we assume that $n \ge 2$.

We do a descending induction on $r \le n-2$. Since the case of $r = 0$ is trivial, we may assume that $r > 0$ (and, in particular, $n \ge 3$).

Let $L := F(X_1)$. We have $\varphi_L \simeq \mathbb{H}\perp\psi$, where ψ is a quadratic form over L of dimension $2(n-1)$. The discriminant of ψ is trivial. According to [3, §88], the assumption on the J-invariant holds for ψ.

For any s with $0 \le s \le n-1$, we write Y_s for the variety $X_s(\psi)$. By [6, Theorem 15.8], the 2-motive of the L-variety $(X_r)_L$ decomposes in a sum of three summands:

$$M(X_r)_L \simeq M(Y_{r-1}) \oplus M(Y_r)(i) \oplus M(Y_{r-1})(j),$$

where, as before, $i := (\dim X_r - \dim Y_r)/2$ and $j := \dim X_r - \dim Y_{r-1}$. By the induction hypothesis, the motive $M(Y_{r-1})$ is indecomposable. However the motive $M(Y_r)$ is indecomposable (if and) only if $r \ne n-2$. Let us treat the case of $r = n-2$ first.

In this case the complete decomposition of $M(Y_r)$ looks as follows:

$$M(Y_{n-2}) \simeq M(Y'_{n-1}) \oplus M(Y'_{n-1})(1) \oplus \dots \oplus M(Y'_{n-1})(n-2),$$

where Y'_{n-1} is a component of the variety Y_{n-1}. Therefore $U(X_n) = M(X'_n)$ is not a shift of a summand of $M(X_{n-2})$. Indeed, otherwise

$$M(X'_n)_L \simeq M(Y'_{n-1}) \oplus M(Y'_{n-1})(n-1)$$

would be a shift of a summand of $M(X_{n-2})_L$ but it is not because

$$M(Y_{n-2}) = U(Y_{n-2}) \not\simeq U(Y_n) = M(Y'_n).$$

It follows that the motive of X_{n-2} is indecomposable. This is the base case of our descending induction on r. Below we assume that $r < n-2$.

Now each of the three summands of the above decomposition of $M(X_r)_L$ is indecomposable. It follows that if the motive of X_r (over F) is decomposable, then it has a summand M with $M_L \simeq M(Y_r)(i)$. By Lemma 1.2, $M \simeq U(X_{r+1})(i)$, that is, $U(X_{r+1})_L \simeq M(Y_r)$. By the induction hypothesis, the 2-motive of X_{r+1} is indecomposable. In particular, $U(X_{r+1}) = M(X_{r+1})$. Therefore we have an isomorphism $M(X_{r+1})_L \simeq M(Y_r)$ and, in particular, $\dim X_{r+1} = \dim Y_r$. However $\dim X_{r+1} - \dim Y_r = 2n - r - 2 > n \geq 3$. □

4. Even-dimensional quadratic forms of non-trivial discriminant

Let F be a field, n an integer ≥ 1, φ a non-degenerate $(2n)$-dimensional quadratic form over F of non-trivial discriminant.

We assume that the following equivalent conditions hold:

(1) the variety X_n is 2-incompressible;
(2) the 2-motive of the variety X_n is indecomposable;
(3) the J-invariant of the quadratic form φ takes its maximal value:

$$J(\varphi) = \{0, 1, 2, \ldots, n-1\}.$$

The above conditions are satisfied if the degree of any closed point on X_n is divisible by 2^n. The condition on the closed points is satisfied if the even Clifford algebra of φ is a division algebra (whose center is the discriminant quadratic extension). Finally, the condition on the even Clifford algebra is satisfied if $F = k(t_1, \ldots, t_{2n})$, where k is a field and $t_1, \ldots, t_{2n}$ are variables, and $\varphi = [t_1, t_2] \perp \ldots \perp [t_{2n-1}, t_{2n}]$ (a sort of *generic* $(2n)$-dimensional quadratic form).

Theorem 4.1. *Let φ be a non-degenerate $(2n)$-dimensional quadratic form over a field F such that the discriminant of φ is non-trivial and the (therefore connected) variety X_n is 2-incompressible. For any r with $0 \leq r \leq n$, the 2-motive of the variety X_r is indecomposable. In particular, all X_r are 2-incompressible.*

Proof. We induct on n. The induction base is the trivial case of $n = 1$. Now we assume that $n \geq 2$.

We do a descending induction on $r \leq n$. The induction base $r = n$ holds by our assumption on φ. Below we are assuming that $r < n$. Since the case of $r = 0$ is trivial, we may assume that $r > 0$.

Let $L := F(X_1)$. We have $\varphi_L \simeq \mathbb{H} \perp \psi$, where ψ is a quadratic form over L of dimension $2(n-1)$. Since F is algebraically closed in L, the discriminant of ψ is non-trivial. Moreover, according to [3, §88], the assumption on the J-invariant holds for ψ.

For any s with $0 \leq s \leq n-1$, we write Y_s for the variety $X_s(\psi)$. By [6, Theorem 15.8], the 2-motive of the L-variety $(X_r)_L$ decomposes in a sum of three summands:

$$M(X_r)_L \simeq M(Y_{r-1}) \oplus M(Y_r)(i) \oplus M(Y_{r-1})(j),$$

where $i := (\dim X_r - \dim Y_r)/2$ and $j := \dim X_r - \dim Y_{r-1}$. By the induction hypothesis, each of the three summands of this decomposition is indecomposable. It follows that if the motive of X_r (over F) is decomposable, then it has a summand M with $M_L \simeq M(Y_r)(i)$. By Lemma 1.2, $M \simeq U(X_{r+1})(i)$, that is, $U(X_{r+1})_L \simeq M(Y_r)$. By the induction hypothesis, the 2-motive of X_{r+1} is indecomposable. In particular, $U(X_{r+1}) = M(X_{r+1})$.

Therefore we have an isomorphism $M(X_{r+1})_L \simeq M(Y_r)$ and, in particular, $\dim X_{r+1} = \dim Y_r$. However $\dim X_{r+1} - \dim Y_r = 2n - r - 2 > n - 2 \geq 0$. □

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