On the ultraviolet behaviour of N=8 supergravity amplitudes

Pierre VANHOVE

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Janvier 2010

IHES/P/10/02

On the ultraviolet behaviour of $\mathcal{N} = 8$ supergravity amplitudes*

Pierre Vanhove

IHES, Le Bois-Marie, 35 route de Chartres, F-91440 Bures-sur-Yvette, France
Institut de Physique Théorique, CEA/Saclay, F-91191 Gif-sur-Yvette, France

We discuss the constraints imposed by the extended supersymmetry on the ultraviolet behaviour of $\mathcal{N} = 8$ supergravity.

1. Introduction

There have been recent tremendous progress in the evaluation of multiloop amplitudes in maximally supersymmetric string theory [1–5], in $\mathcal{N} = 8$ supergravity in various dimensions [6–9], and the analysis of the constraints from the extended supersymmetry on possible counter-terms to ultraviolet divergences [10,11]. It has been shown that up to and including four-loop order that the four-graviton amplitudes in $\mathcal{N} = 8$ supergravity in four dimensions are free of ultraviolet divergences [7–9]. One important question is to determine when the first ultraviolet divergence appears in four dimensions. In this text we indicate the various constraints derived from the implementation of maximal supersymmetry.

The mass dimension of the L-loop gravity amplitude in D dimensions is given by

$$[\mathfrak{M}^{(D)}_{n,L}] = \text{mass}^{(D-2)L+2} \tag{1}$$

In $\mathcal{N} = 8$ supergravity half of the supersymmetry are explicitly realized at each loop order and the four-point amplitudes factorize the dimension eight operator

$$\widehat{\mathbf{R}}^4 = \kappa_{(D)}^4 \, K_{A_1\cdots A_4} \tilde{K}_{B_1\cdots B_4} \prod_{i=1}^{4} \zeta_i^{A_i B_i} \tag{2}$$

given by the fourth power of the linearized supercurvature — defined in eq. (7.4.57) of [12] (see as well [13]) where A_i, B_i are the labels of the $\mathcal{N} = 8$ supergraviton multiplet. We have used $\kappa_{(D)}$ for the D-dimensional Newton's constant. In particular the amplitudes between any four states $\phi_1,\dots,\phi_4$ in the massless supergravity multiplet take the form

$$\mathfrak{M}^{(D)}_{4,L}(\phi_1, \dots, \phi_4) = \widehat{\mathbf{R}}^4 \, \widehat{\mathcal{I}}^{(D)}_{4,3}(k_1, \dots, k_4) \tag{3}$$

*IHES/P/10/02, IPHT-T-09/190. Contribution to the proceedings of the Twelfth Marcel Grossmann Meeting, Paris, 12-18 July 2009.

where $\widehat{\mathcal{I}}^{(D)}_{4,L}(k_1,\dots,k_4)$ does not depend on the helicities, and is of superficial mass dimension $(D-2)L-6$. In the case of the four-graviton amplitudes the tensorial factor in (2) will be denoted $\mathcal{R}^4$.

$\triangleright$ This formula indicates that the $\mathcal{R}^4$ operator would appear at $L = 6/(D-2)$ loops, which is $(L,D) = (1,8), (2,5), (3,4), (6,3)$. Non-renormalisation theorems in string theory [3,4] and explicit field theory computations [14] confirm the logarithmic divergence of the one-loop amplitude in $D = 8$, but rule out the other divergences in $D < 8$.

$\triangleright$ Because on-shell the operator $\partial^2\mathcal{R}^4$ vanishes the next operator is the dimension 12 coupling $\partial^4\mathcal{R}^4$ could appear at $(L,D) = (2,7), (5,4)$. The explicit computations in [7] confirm the divergence in $D = 7$. The evaluation of higher-loop amplitudes in field theory [8] and non-renormalisation theorems in string theory [2,3,15] rule out the appearance of this divergences in $D < 7$.

$\triangleright$ The dimension 14 operator $\partial^6\mathcal{R}^4$ could appear at $(L,D) = (2,8), (3,6), (4,5), (6,4)$. The computations in [7–9] confirm the divergences in $(L,D) = (2,8), (3,6)$ and rule out the divergences in $(L,D) = (4,5), (6,4)$.

$\triangleright$ The dimension 16 operator $\partial^8\mathcal{R}^4$ could appear at $(L,D) = (2,9), (7,4)$. The computations in [7] confirm the divergences in $(L,D) = (2,9)$. There is currently no explicit evaluation of the contribution $(L,D) = (7,4)$. We discuss below the constraints from supersymmetry.

$\triangleright$ The dimension 18 operator $\partial^{10}\mathcal{R}^4$ could appear at $(L,D) = (2,10), (4,6), (8,4)$. Explicit computations confirm the divergence in $(L,D) = (2,10), (4,6)$. There is currently no explicit evaluation of the contribution $(L,D) = (8,4)$. We discuss below the constraints from supersymmetry.

$\triangleright$ The dimension 20 operator $\partial^{12}\mathcal{R}^4$ could appear at $(L,D) = (2,11), (3,8), (6,5), (9,4)$. Explicit computations confirm the divergences in $(L,D) = (2,11), (3,8)$. There is currently no direct evaluation of the $(L,D) = (6,5), (9,4)$ contributions. In fact whatever is the form of the five-loop four-graviton amplitude (i.e. factorizing the operator $\partial^8\mathcal{R}^4$ or $\partial^{10}\mathcal{R}^4$) there will be a logarithmic divergence at $L = 6$ associated with $\partial^{12}\mathcal{R}^4$ in $D = 5$. If $\mathcal{N} = 8$ supergravity has an ultraviolet divergence in $D = 4$ there will always be a nine-loop divergence. Actually, supersymmetry cannot rule out this divergence in $D = 4$ [3,4]. *The nine-loop four-graviton amplitude in four dimensions is ultraviolet finite if and only if $\mathcal{N} = 8$ supergravity is ultraviolet finite in four dimensions.*

If one parametrizes the superficial power counting of the ultraviolet behaviour of the amplitude as

$$[\mathfrak{M}^{(D)}_{4,L}] = \Lambda^{(D-2)L-6-2\beta_L}\,\partial^{2\beta_L}\,\hat{\mathcal{R}}^4 \tag{4}$$

the critical dimension for ultraviolet divergences in the four-graviton amplitude is given by

$$D \geq 2 + \frac{6+2\beta_L}{L} \tag{5}$$

Up to an including four-loop the supersymmetry constraints [2,8,14] implies that $\beta_L = L$.

When only the simple large-λ regulator of the pure spinor string formalism is used one can argue [2,4] that $\beta_L = L$ for $L \leq 6$ and $\beta_L = 6$ for $L \geq 6$ leading to the critical dimension for UV divergence

$$\begin{aligned} D &\geq 4 + \frac{6}{L}; \qquad \text{for } L \leq 6 \\ D &\geq 2 + \frac{18}{L}; \qquad \text{for } L \geq 6 \end{aligned} \tag{6}$$

which implies a nine-loop divergence in four dimensions [4]. From genus five possible divergences from the tip of the pure spinor cone that would requiere the use of the small-λ complicated regulator [1,16] can restrict $\beta_L = 4$ for $L \geq 4$

$$\begin{aligned} D &\geq 4 + \frac{6}{L}; \qquad \text{for } L \leq 4 \\ D &\geq 2 + \frac{14}{L}; \qquad \text{for } L \geq 4 \end{aligned} \tag{7}$$

leading to seven-loop divergence in four dimensions.

2. D-term and F-term in extended supergravity

The issue of correctly identifying the ultraviolet behaviour of a supersymmetric theory is equivalent to the understanding of which operators are true F-terms satisfying non-renormalisation theorems, and which operators are D-terms receiving quantum corrections to all orders [17].

By partial integration over the superspace variables it is possible to rewrite D-term as "fake" F-term, and detecting the true D-term nature of an operator can be non-obvious. In four-point open string amplitudes the true D-term nature of the $\partial^2 \mathrm{tr} F^4$ interaction became manifest by the appearance of inverse derivatives after integrating over the string theory moduli [5]. In the four-graviton amplitudes it was confirmed in [5] that no such inverse derivative factors arise up to and including genus four implying that the operators $\mathcal{R}^4$, $\partial^4\mathcal{R}^4$, and $\partial^6\mathcal{R}^4$ are F-terms satisfying non-renormalisation theorems [2,3] .

In the pure spinor formalism for maximally supersymmetric closed string theory, D-terms arise explicitly as soon as the small-λ regulator from the tip of the pure spinor cone enters in the evaluation of the amplitude [1,16]. In the four-graviton amplitude arises the folllowing full superspace integral of the dimension one superfield $W_{\alpha\beta} = F_{\alpha\beta} + \cdots + \theta^\gamma\theta^\delta\, R_{\alpha\beta\gamma\delta} + \cdots$

$$\int d^{32}\theta\, W^2 = D^{12}\mathcal{R}^4 + \text{susy completion} \tag{8}$$

which is a D-term. From genus five order singularities from the tip of the pure spinor cone can modify the naive zero mode counting so that the amplitude contributes to the $\partial^8\mathcal{R}^4$ interactions instead of $\partial^{10}\,\mathcal{R}^4$.

If this happens then the interactions $\partial^8\,\mathcal{R}^4$ will be a D-term receiving contribution to all loop order in string theory and the critical dimensions for ultraviolet divergences will be given by (7) implying a seven-loop divergence in the four-graviton amplitude of $\mathcal{N}=8$ supergravity in $D=4$. This behaviour of the genus five amplitude is compatible with the presence of genus five contribution in the $\partial^4\,\mathcal{R}^4$ interaction coupling derived in [18] from M-theory. This gives support to the bound (7) which implies that the first divergence in the four-graviton amplitude in four dimensions will be at seven loops. A candidate for an on-shell superspace expression for this D-term is the volume of superspace

$$\delta\mathcal{S}_{ct}\sim\kappa_{(4)}^{12}\int d^4x\int d^{32}\theta\,|E| \tag{9}$$

where $|E|$ is the determinant of the superfield vielbein Which would be the seven loop counter-term in four dimensions (see [11] for some comments about this). Integrating over the fermionic variables this would lead to

$$\delta\mathcal{S}_{ct}\sim\kappa_{(4)}^{12}\int d^4x\,\sqrt{-g^{(4)}}\,(D^8\mathcal{R}^4+\text{susy completion})\ . \tag{10}$$

References

1. N. Berkovits, JHEP **0510**, 089 (2005) [arXiv:hep-th/0509120].
   N. Berkovits and N. Nekrasov, JHEP **0612**, 029 (2006) [arXiv:hep-th/0609012].
   Y. Aisaka and N. Berkovits, JHEP **0907**, 062 (2009) [arXiv:0903.3443 [hep-th]].
2. N. Berkovits, Phys. Rev. Lett. **98**, 211601 (2007) [arXiv:hep-th/0609006].
3. M. B. Green, J. G. Russo and P. Vanhove, JHEP **0702**, 099 (2007) [arXiv:hep-th/0610299].
4. M. B. Green, J. G. Russo and P. Vanhove, Phys. Rev. Lett. **98**, 131602 (2007) [arXiv:hep-th/0611273].
5. N. Berkovits, M. B. Green, J. G. Russo and P. Vanhove, JHEP **0911** (2009) 063 [arXiv:0908.1923 [hep-th]].
6. Z. Bern, L. J. Dixon and R. Roiban, Phys. Lett. B **644**, 265 (2007) [arXiv:hep-th/0611086].
7. Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein and J. S. Rozowsky, Nucl. Phys. B **530**, 401 (1998) [arXiv:hep-th/9802162].
8. Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, D. A. Kosower and R. Roiban, Phys. Rev. Lett. **98**, 161303 (2007) [arXiv:hep-th/0702112].
   Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, Phys. Rev. D **78**, 105019 (2008) [arXiv:0808.4112 [hep-th]].
9. Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, Phys. Rev. Lett. **103**, 081301 (2009) [arXiv:0905.2326 [hep-th]].
10. P. S. Howe and K. S. Stelle, Phys. Lett. B **554**, 190 (2003) [arXiv:hep-th/0211279].
11. G. Bossard, P. S. Howe and K. S. Stelle, Gen. Rel. Grav. **41**, 919 (2009) [arXiv:0901.4661 [hep-th]].
    G. Bossard, P. S. Howe and K. S. Stelle, Phys. Lett. B **682** (2009) 137 [arXiv:0908.3883 [hep-th]].
12. M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology," *Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics)*

13. M. B. Green, J. G. Russo and P. Vanhove, JHEP **0802**, 020 (2008) [arXiv:0801.0322 [hep-th]].
14. M. B. Green, J. H. Schwarz and L. Brink, Nucl. Phys. B **198**, 474 (1982).
15. M. B. Green, H. h. Kwon and P. Vanhove, Phys. Rev. D **61**, 104010 (2000) [arXiv:hep-th/9910055].
16. P. A. Grassi and P. Vanhove, JHEP **0905**, 089 (2009) [arXiv:0903.3903 [hep-th]].
17. M. T. Grisaru, W. Siegel and M. Roček, "Improved Methods for Supergraphs," Nucl. Phys. B **159** (1979) 429.
18. M. B. Green, J. G. Russo and P. Vanhove, JHEP **0807**, 126 (2008) [arXiv:0807.0389 [hep-th]].