a-Maximization in N=1 Supersymmetric Spin(10) Gauge Theories

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abstract

A summary is reported on our previous publications about four-dimensional $\mathcal{N}=1$ supersymmetric $Spin(10)$ gauge theory with chiral superfields in the spinor and vector representations in the non-Abelian Coulomb phase. Carrying out the method of a-maximization, we exlpored decoupling operators in the infared and the renormalization flow of the theory. We also give a brief review on the non-Abelian Coulomb phase of the theory after recalling the unitarity bound and the a-maximization procedure in four-dimensional conformal field theory.

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Chapter 1

Introduction

Four-dimensional $\mathcal{N} = 1$ $Spin(10)$ gauge theory with one chiral superfield in the spinor representation and N_Q chiral superfields in the vector representation has rich and intriguing dynamics. In particular, for no vector representations, supersymmetry is dynamically broken [1, 2]. For $7 \leq N_Q \leq 21$ vectors, the theory is in the non-Abelian Coulomb phase, and has a dual description at the non-trivial fixed point [3, 4]. It also leads at some points in the moduli space to the duality [5] between chiral and vector-like gauge theories, as well as the one discussed in [6]. The analysis has been extended for more spinors in [7].

When a $\mathcal{N} = 1$ supersymmetric theory is in the non-Abelian Coulomb phase, it must be left invariant under conformal symmetry at a non-trivial infrared fixed point, and some exact results can be obtained by $\mathcal{N} = 1$ superconformal symmetry. In particular, the scaling dimension $D(\mathcal{O})$ of a gauge invariant chiral primary operator $\mathcal{O}$ can be determined by its $U(1)_R$ charge $R(\mathcal{O})$ as

$$D(\mathcal{O}) = \frac{3}{2}R(\mathcal{O}).$$

The unitarity of representations of conformal symmetry [8] requires the scaling dimension $D(\mathcal{O})$ of a scalar field $\mathcal{O}$ to satisfy

$$D(\mathcal{O}) \geq 1.$$

However, one sometimes encounters a gauge invariant chiral primary spinless operator $\mathcal{O}$ which appears to satisfy the inequality $R(\mathcal{O}) < 2/3$. It has been discussed that such an operator decouples as a free field from the remaining interacting system, and an accidental $U(1)$ symmetry is enhanced in the infrared to fix the $U(1)_R$ charge of the operator $\mathcal{O}$ to $2/3$ [9, 10, 11].

One can see that one of the examples is $Spin(7)$ gauge theory with $N_f = 7$ spinors Q^i $(i = 1, \cdots, N_f)$ and with no superpotential, where electric-magnetic duality was found for $7 \leq N_f \leq 14$ in [5]. Its dual or magnetic theory is an $SU(N_f - 4)$ gauge theory with N_f antifundamentals $\bar{q}_i$ and a single symmetric tensor s, along with gauge singlets M^{ij}, which can be identified with Q^iQ^j in the electric theory. The superpotential W_{mag} of the magnetic theory is given by

$$W_{\rm mag} = \frac{\tilde{h}}{\tilde{\mu}^2} M^{ij} \bar{q}_i\, s\, \bar{q}_j + \frac{1}{\tilde{\mu}^{N_f - 7}} \det s,$$

where $\tilde{\mu}$ is a dimensionful parameter to give the correct mass dimension to M^{ij}, and the dimensionless parameter $\tilde{h}$ shows up because we assume that the field M^{ij} has the canonical kinetic term.

Since the $U(1)_R$ charge of the spinors Q^i is given by $1 - (5/N_f)$, the gauge invariant operator M^{ij} appears to violate the unitarity bound for $N_f = 7$ and therefore propagates as a free field at the infrared fixed point.

In the magnetic theory, it suggests that the coupling $\tilde{h}$ in the superpotential $W_{\rm mag}$ goes to zero in the infrared. Therefore, at the infrared fixed point, the superpotential of the magnetic theory becomes

$$W_{\rm IR} = \frac{1}{\tilde{\mu}^{N_f - 7}} \det s.$$

Contrary to the electric theory, on the magnetic side, the gauge invariant operator M^{ij} is an elementary field. Therefore, the vanishing of the coupling $\tilde{h}$ implies that M^{ij} may be a free field.

On the contrary, suppose that we start with the superpotential $W_{\rm IR}$ at the infrared fixed point. The fields $\bar{q}_i$ and s are still interacting, and the $U(1)_R$ charge $2 - 2(1 - (5/N_f)) = 10/N_f$ of $\bar{q}_i\, s\, \bar{q}_j$ is greater than $2 - (2/3)$ for $N_f = 7$. Let us introduce an elementary field M_{ij}, which has carries $U(1)_R$ charge $2/3$, as M_{ij} is a free field. Therefore, the interaction $M^{ij} \bar{q}_i\, s\, \bar{q}_j$ is an irrelevant operator in the superpotential at the infrared fixed point, and it is consistent with the implication of the unitarity bound for the operator M^{ij}.

One thus sees that the magnetic description yields a simple explanation of the prescription [11] for a composite operator hitting the unitarity bound, which has been explained by using the auxiliary field method [12]. The prescription [11] and the explanation [12] will be described in section 2.3.

Furthermore, when $\tilde{h}$ vanishes, one no longer has the F-term condition

$$\frac{\tilde{h}}{\tilde{\mu}^2}\, \bar{q}_i\, s\, \bar{q}_j = \frac{\partial W_{\rm mag}}{\partial M^{ij}} = 0,$$

and thus the gauge invariant operators $N_{ij} = \bar{q}_i\, s\, \bar{q}_j$ becomes a non-trivial chiral primary operator at the infrared fixed point. One can easily see that the resulting magnetic theory at the fixed point has a different electric dual from the original electric theory with no superpotential.

In fact, its electric dual is the same as the original electric theory except that it has the non-zero superpotential

$$W_{\rm ele} = \frac{1}{\mu} N_{ij} Q^i Q^j,$$

along with free singlets M^{ij}. Thus, one can conclude that these two electric theories are identical at the infrared fixed point. It also means that the original dual pair of the $Spin(7)$ gauge theory with no superpotential and the magnetic theory with the superpotential $W_{\rm mag}$ flows into another dual pair of the $Spin(7)$ theory with the superpotential $W_{\rm ele}$ and the magnetic dual with vanishing $\tilde{h}$ in the superpotential in the infrared.

We have thus seen that the $U(1)_R$ charge assignments were very important to understand the decoupling operator and the renormalization flow of the theory. However, when a superconformal field theory is also invariant under other global $U(1)$ symmetries besides the $U(1)_R$ symmetry, which linear combination of those $U(1)$ generators yields the superconformal $U(1)_R$ symmetry is dynamically determined. In [13], Intriligator and Wecht has proposed the method of a-maximization to pick up the superconformal $U(1)_R$ among all the linear combinations of the $U(1)$ generators.

In this review article, we will give a summary on our previous publications [14, 15] about the non-Abelian Coulomb phase of four-dimensional $\mathcal{N} = 1$ supersymmetric $Spin(10)$ gauge theories with one and two chiral superfields in the spinor representation and N_Q chiral fields in the vector representation. They are invariant under two global $U(1)$ symmetries, and one needs to apply a-maximization to determine the superconformal $U(1)_R$ symmetry.

In chapter 2, we give a brief review on $\mathcal{N} = 1$ superconformal algebra and its lowest weight representations. We will discuss the unitarity of representations of the conformal algebra, following [16, 17] to give the unitarity bound, although a complete derivation [8] of it will not be given in this article. The relation of chiral primary operators and a chiral ring will be explained. After giving the definition of the conformal anomalies or the central charges c and a in four dimensions and recalling their relation [18, 19] to the 't Hooft anomalies, we will explain the method of a-maximization in some detail.

In chapter 3, we will review the non-Abelian Coulomb phase of the supersymmetric $Spin(10)$ theory with one spinor and $7 \le N_Q \le 21$ vectors [3, 4] in section 3.1 and the one with two spinors and $6 \le N_Q \le 19$ vectors [7] in section 3.2. Their dual descriptions [3, 4, 7] of the theories at the infrared fixed point will briefly be explained. In chapter 4, we will describe our results obtained by the application of a-maximization to the theories.

In chapter 5, we will discuss consistency checks about our results and their implication for the renormalization flow of the theories, as discussed above for the $Spin(7)$ theory. In particular, one will note in the $Spin(10)$ theory that the decoupling meson operator is given by an elementary field on the magnetic side, which also yields a simple explanation for the prescription [11], as in the $Spin(7)$ theory. In addition, we will use the auxiliary field method on the electric side to attempt to describe the decoupling of the composite meson operator. The flow of the electric-magnetic dual pair into another pair will also be seen in the $Spin(10)$ theory.

Chapter 6 will be devoted to summary and outlook.

Chapter 2

$\mathcal{N} = 1$ Superconformal Field Theory

2.1 $\mathcal{N} = 1$ Superconformal Algebra

In this section, we will briefly review basic facts about four-dimensional $\mathcal{N} = 1$ superconformal algebra. They will frequently be used in the rest of this article.

Conformal transformations are defined, upon acting on the spacetime coordinates x^μ, as those preserving the metric $\eta_{\mu\nu}$ up to an overall nowhere-zero function $\Omega^2(x)$. The metric $\eta_{\mu\nu}$ thus transforms under a conformal transformation as

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \;\rightarrow\; ds'^2 = \eta_{\mu\nu}dx'^\mu dx'^\nu = \Omega^2(x)\eta_{\mu\nu}dx^\mu dx^\nu. \tag{2.1}$$

They all are generated by the infinitesimal transformations

$$\begin{array}{lcl} \text{Lorentz transformation} & : & x'^\mu = x^\mu - \omega^{\mu\nu}x_\nu, \\ \text{Dilatation (Dilation)} & : & x'^\mu = (1-\varepsilon)x^\mu, \\ \text{Translation} & : & x'^\mu = x^\mu + a^\mu, \\ \text{Special conformal transformation} & : & x'^\mu = x^\mu - 2b^\nu x_\nu x^\mu + b^\mu x^\nu x_\nu, \end{array} \tag{2.2}$$

and thus the generators of them can be read as

$$\begin{array}{lcl} \text{Lorentz transformation} & : & M_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu), \\ \text{Dilatation (Dilation)} & : & D = ix^\nu\partial_\nu, \\ \text{Translation} & : & P_\mu = -i\partial_\mu, \\ \text{Special conformal transformation} & : & K_\mu = -i(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu), \end{array} \tag{2.3}$$

as a representation on the spacetime coordinates x^μ. They satisfy the commutation relations

$$\begin{aligned} &[M_{\mu\nu},\, M_{\rho\sigma}] = i\left(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}\right), \\ &[M_{\mu\nu},\, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \qquad [M_{\mu\nu},\, K_\rho] = i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \\ &[D,\, P_\mu] = -iP_\mu, \qquad [D,\, K_\mu] = iK_\mu, \qquad [D,\, M_{\mu\nu}] = 0, \\ &[P_\mu,\, P_\nu] = [K_\mu,\, K_\nu] = 0, \qquad [P_\mu,\, K_\nu] = -2i\left(\eta_{\mu\nu}D + M_{\mu\nu}\right). \end{aligned} \tag{2.4}$$

(1) $d = j_1 = j_2 = 0$
(2) $j_1 \neq 0, j_2 \neq 0, d > j_1 + j_2 + 2$
(3) $j_1 j_2 = 0, d > j_1 + j_2 + 1$
(4) $j_1 \neq 0, j_2 \neq 0, d = j_1 + j_2 + 2$
(5) $j_1 j_2 = 0, d = j_1 + j_2 + 1$

Table 2.1: Unitary irreducible representations

They are isomorphic to the Lie algebra of $SO(2,4)$. The generators S_{AB} $(A, B = -1, \cdots, 4)$ of the Lie algebra satisfy

$$[S_{AB},\, S_{CD}] = i\left(\eta_{AC}S_{BD} - \eta_{AD}S_{BC} - \eta_{BC}S_{AD} + \eta_{BD}S_{AC}\right), \tag{2.5}$$

where $\eta_{ab} = \mathrm{diag}(-1,-1,1,1,1,1)$. With the identification

$$S_{\mu\nu} = M_{\mu\nu}, \qquad S_{\mu\,-1} = \frac{1}{2}(P_\mu + K_\mu), \qquad S_{\mu\,4} = \frac{1}{2}(P_\mu - K_\mu), \qquad S_{-1\,4} = D, \tag{2.6}$$

the algebra (2.5) retains the commutation relations (2.4) of the conformal algebra.

Considering a unitary representation of the conformal algebra (2.4) reveals a bound on its conformal dimension d - the eigenvalue of the dilatation operator D. All unitary irreducible representations with non-negative energy has been classified by Mack [8]. It was shown that such a representation always has the lowest weight state, and that all the representations are sorted into five classes, which are labeled by the conformal dimension d and the spins (j_1, j_2) of the Lorentz group $SL(2,\mathbf{C})$ of their lowest weight states, as listed in Table 2.1.

One can see from Table 2.1 that apart from the trivial identity operator in (1), the conformal dimension d of an operator should satisfy

$$d \geq 1 \quad \text{for scalar}, \qquad d \geq \frac{3}{2} \quad \text{for spinor}, \qquad d \geq 3 \quad \text{for vector}, \tag{2.7}$$

depending on the spin (j_1, j_2) of the operator. These bounds on conformal dimensions are imposed by the unitarity of representations and are referred to as the unitarity bound. The inequalities are saturated if and only if their fields are free for the scalar and the spinor case. For the vector case, the field must be gauge invariant. The equality is satisfied for the vector case by a conserving current. We will frequently use the unitarity bound for a scalar field in our analysis in the later chapters.

We therefore will sketch a derivation of the unitary bounds. For a complete proof and detailed discussions[1], see [8, 16, 17]. The maximal compact subalgebra of the $SO(2,4)$ is given by $SO(2) \times SO(4)$, whose generators can be taken as S_{-10} for the $SO(2)$ and S_{ab}

[1] Further restrictions have been discussed very recently in [20, 21]

$(a, b = 1, \cdots, 4)$ for the $SO(4)$. The remaining generators of the $SO(2,4)$ are S_{-1a} and S_{0a}, which are combined to define

$$E_a^{\pm} = (S_{a-1} \mp i S_{a0}) , \tag{2.8}$$

transforming in the representation $(\pm 1, \mathbf{4})$, respectively under the $SO(2) \times SO(4)$ rotations. From the commutation relations (2.5), one can see that these generators $E_a^{\pm}$ satisfy

$$\left[E_a^- ,\, E_b^+\right] = 2\,(\delta_{ab}\, S_{-10} - i\, S_{ab})\,. \tag{2.9}$$

One thus, may regard E_a^+ and E_a^- as a raising operator and a lowering operator, respectively. Since we assume that the generators S_{AB} are all hermitian, E_a^+ and E_a^- are adjoint to each other; $(E_a^{\pm})^{\dagger} = E_a^{\mp}$.

The $SO(4)$ algebra is isomorphic to the Lie algebra of $SU(2) \times SU(2)$, and the generators are given by two copies of $SU(2)$ generators $S_i^{(1)}$, $S_i^{(2)}$ $(i = 1, 2, 3)$ defined by

$$S_i^{(1)} = \frac{1}{2}\left(\frac{1}{2}\epsilon_{ikl4} S_{kl} - S_{i4}\right) , \qquad S_i^{(2)} = \frac{1}{2}\left(\frac{1}{2}\epsilon_{ikl4} S_{kl} + S_{i4}\right) . \tag{2.10}$$

A state of a unitary irreducible representation of the $SO(2) \times SO(4)$ algebra is specified with eigenvalue d of S_{-10} and two pairs of spin quantum numbers (j_1, m_1) and (j_2, m_2), and is denoted as $|d; j_1, m_1; j_2, m_2\rangle$. In particular, for the lowest weight state $|d; j_1, -j_1; j_2, -j_2\rangle$ of the unitary irreducible representation, we will adopt a convention of writing simply $|d, j_1, j_2\rangle$ as shorthand.

A unitary irreducible representation of the $SO(2,4)$ algebra may be given by a set of unitary irreducible representations of the $SO(2) \times SO(4)$ subalgebra. The unitary irreducible representations of the $SO(2) \times SO(4)$ can be specified by the lowest weight state $|d, j_1, j_2\rangle$ of each of them. A unitary irreducible representation of the $SO(2) \times SO(4)$ contained in a unitary irreducible representation of the $SO(2,4)$ algebra may be mapped by the ladder operators $E_a^{\pm}$ into another unitary irreducible representation of the $SO(2) \times SO(4)$ in the same representation of the $SO(2,4)$ algebra.

Among unitary irreducible representations of the $SO(2) \times SO(4)$ in a unitary irreducible representation of the $SO(2,4)$ algebra, let us pick up the unitary irreducible representation of the $SO(2) \times SO(4)$ specified by the lowest weight state $|d, j_1, j_2\rangle$ with the lowest value of d. Since the lowering operators E_a^- carry charge -1 of the $SO(2)$, the lowest weight state $|d, j_1, j_2\rangle$ must be annihilated by the E_a^-, because there are no lowest weight states with lower eigenvalues of S_{-10} than d in the unitary irreducible representation of the $SO(2,4)$.

Then the first non-trivial state one would like to study about its unitarity would be $E_a^+ |d, j_1, j_2\rangle$. The unitarity requires the matrix

$$\begin{aligned} M_{(m_1', m_2', a)\,(m_1, m_2, b)} &= \langle d; j_1, m_1'; j_2, m_2'|\, E_a^- E_b^+ \,|d; j_1, m_1; j_2, m_2\rangle \\ &= \langle d; j_1, m_1'; j_2, m_2'|\, \left[E_a^- ,\, E_b^+\right] |d; j_1, m_1; j_2, m_2\rangle \end{aligned} \tag{2.11}$$

to be positive definite, if $E_a^+ \,|d, j_1, j_2\rangle$ is not vanishing. The commutator gives $2\,(\delta_{ab}\, S_{-10} - i\, S_{ab})$. Following Minwalla's trick [16], let us recall that $-iS_{ab}$ may be rewritten as

$$-iS_{ab} = -\frac{i}{2}\,(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})\, S_{cd} = \frac{1}{2}\,(S_{cd})_{ab}\, S_{cd}$$

and that $((S_{cd})_{ab})$ is a matrix in the representation **4** of S_{cd}. The representation **4** of the $SO(4)$ algebra is the spins $(j_1 = 1/2, j_2 = 1/2)$ of the isomorphic $SU(2) \times SU(2)$ algebra, and the matrices (δ_{ab}) and $((S_{cd})_{ab})$ can be regarded as the matrix elements of the identity operator and the generator S_{cd};

$$\begin{aligned} \delta_{ab} \quad &\to \quad \langle \alpha_1', \alpha_2'|\, \mathbb{1}\, |\alpha_1, \alpha_2\rangle\,, \\ (S_{cd})_{ab} \quad &\to \quad \langle \alpha_1', \alpha_2'|\, S_{cd}\, |\alpha_1, \alpha_2\rangle\,, \end{aligned} \tag{2.12}$$

where the state $|\alpha_1, \alpha_2\rangle$ is a shorthand for $|j_1 = 1/2, \alpha_1; j_2 = 1/2, \alpha_2\rangle$, and the similar shorthand was also used for the bra states. The matrix $M_{(m_1', m_2', a)\,(m_1, m_2, b)}$ is then given by the operator $2\mathbb{1} \otimes D + S_{cd} \otimes S_{cd}$ sandwiched by the bra $\langle \alpha_1', \alpha_2'| \otimes \langle d; j_1, m_1'; j_2, m_2'|$ and the ket $|\alpha_1, \alpha_2\rangle \otimes |d; j_1, m_1; j_2, m_2\rangle$. The tensor product $S_{cd} \otimes S_{cd}$ may be rewritten as

$$S_{cd} \otimes S_{cd} = \frac{1}{2}\left[(S_{cd} \otimes \mathbb{1} + \mathbb{1} \otimes S_{cd})^2 - (S_{cd} \otimes \mathbb{1})^2 - (\mathbb{1} \otimes S_{cd})^2 \right],$$

and one thus finds that the matrix $M_{(m_1', m_2', a)\,(m_1, m_2, b)}$ is given by

$$\langle \alpha_1', \alpha_2', m_1', m_2'|\, 2\left[\mathbb{1} \otimes D + \frac{1}{4}\left[(S_{cd} \otimes \mathbb{1} + \mathbb{1} \otimes S_{cd})^2 - (S_{cd} \otimes \mathbb{1})^2 - (\mathbb{1} \otimes S_{cd})^2 \right] \right] |\alpha_1, \alpha_2, m_1, m_2\rangle$$

with a shorthand $|\alpha_1, \alpha_2, m_1, m_2\rangle$ for $|\alpha_1, \alpha_2\rangle \otimes |d; j_1, m_1; j_2, m_2\rangle$ and $\langle \alpha_1', \alpha_2', m_1', m_2'|$ for $\langle \alpha_1', \alpha_2'| \otimes \langle d; j_1, m_1'; j_2, m_2'|$. The ket $|\alpha_1, \alpha_2\rangle \otimes |d; j_1, m_1; j_2, m_2\rangle$ is in the representation $(\frac{1}{2}, \frac{1}{2}) \otimes (j_1, j_2)$, which is decomposed into irreducible representations as

$$(j_1 + \frac{1}{2}, j_2 + \frac{1}{2}) \oplus (j_1 + \frac{1}{2}, j_2 - \frac{1}{2}) \oplus (j_1 - \frac{1}{2}, j_2 + \frac{1}{2}) \oplus (j_1 - \frac{1}{2}, j_2 - \frac{1}{2})$$

for $j_1 \geq 1/2$ and $j_2 \geq 1/2$. Using elementary facts of the $SU(2)$ spin operators

$$\frac{1}{4}\,(S_{cd})^2 = \sum_{i=1}^{3} \left[\left(S_i^{(1)}\right)^2 + \left(S_i^{(2)}\right)^2 \right]$$

and

$$\sum_{i=1}^{3} (S_i)^2\, |j, m\rangle = j(j+1)\, |j, m\rangle\,,$$

one can see that, among the operators in the above expression of the matrix $M_{(m_1', m_2', a)\,(m_1, m_2, b)}$, only $(S_{cd} \otimes \mathbb{1} + \mathbb{1} \otimes S_{cd})^2$ depends on the above irreducible representations to give its eigenvalues. The contribution from the other operators yields

$$2\left[d - 2 \times \frac{3}{4} - j_1(j_1+1) - j_2(j_2+1) \right],$$

and the operator $(S_{cd}\otimes \mathbb{1}+\mathbb{1}\otimes S_{cd})^2$ takes the minimal value in $(j_1-\frac{1}{2},j_2-\frac{1}{2})$ for $j_1\geq 1/2$ and $j_2\geq 1/2$ among the four irreducible representations to contribute

$$2\left[(j_1-\frac{1}{2})(j_1+\frac{1}{2})+(j_2-\frac{1}{2})(j_2+\frac{1}{2})\right]$$

to the matrix $M_{(m_1',m_2',a)\,(m_1,m_2,b)}$. One obtains the total contribution

$$2\,(d-j_1-j_2-2)$$

to it. For $j=j_1\geq 1/2$ and $j_2=0$ or $j=j_2\geq 1/2$ and $j_1=0$, the minimal value of $(S_{cd}\otimes \mathbb{1}+\mathbb{1}\otimes S_{cd})^2$ similarly contributes

$$2\left[(j-\frac{1}{2})(j+\frac{1}{2})+\frac{3}{4}\right]$$

to give the total contribution

$$2\left[d+(j-\frac{1}{2})(j+\frac{1}{2})-j(j+1)-\frac{3}{4}\right]=2\,(d-j-1)\,,$$

while it is obvious that the total contribution is $2d$ for a scalar $j_1=j_2=0$.

The requirement that the eigenvalues of the matrix $M_{(m_1',m_2',a)\,(m_1,m_2,b)}$ be non-negative gives the inequalities

$$\begin{aligned}
&d\geq j_1+j_2+2 && (j_1\geq\frac{1}{2},j_2\geq\frac{1}{2}),\\
&d\geq j+1 && (j=j_1\geq\frac{1}{2},j_2=0 \text{ or } j_1=0,j=j_2\geq\frac{1}{2}),\\
&d\geq 0 && (j_1=0,j_2=0).
\end{aligned}\tag{2.13}$$

The equality is satisfied by the trivial state

$$E_a^+\,|d,j_1,j_2\rangle=0.\tag{2.14}$$

Converting the vector index of E_a^+ to a pair of spinor indices (α_1,α_2) of the $SU(2)\times SU(2)$, and also $|d;j_1,m_1;j_2,m_2\rangle$ to $|d;\beta_1,\cdots,\beta_{2j_1};\gamma_1,\cdots,\gamma_{2j_2}\rangle$, one can make the condition (2.14) more precise. In fact, for $j_1\geq\frac{1}{2},j_2\geq\frac{1}{2}$, since the irreducible representation $(j_1-\frac{1}{2},j_2-\frac{1}{2})$ gives the minimal eigenvalue, the condition (2.14) means the contraction of both the indices α_1 and α_2 with the ones of the state to form the $(j_1-\frac{1}{2},j_2-\frac{1}{2})$ appropriately. For the remaining cases but the $j_1=j_2=0$, the contraction is done for either of the indices. There is no contraction for $j_1=j_2=0$, and (2.14) means that the representation with $d=j_1=j_2=0$ must be one-dimensional.

The inequalities (2.13) are the necessary condition for the lowest weight representation to be unitary, but it is also the sufficient condition except for the scalar case $j_1=j_2=0$, which has been proved by Mack [8]. The inequalities (2.13) for those cases are referred to as the unitarity bounds, as mentioned before.

In order to obtain the unitarity bound for $j_1 = j_2 = 0$, among the next non-trivial states, let us take the state $\sum_a E_a^+ E_a^+ |d, j_1 = 0, j_2 = 0\rangle$ to calculate the norm

$$\sum_{a,b} \langle d,0,0| E_a^- E_a^- E_b^+ E_b^+ |d,0,0\rangle = 32 \ \langle d,0,0| \Big[(S_{-10})^2 - S_{-10} \Big] |d,0,0\rangle$$
$$= 32 \left(d^2 - d \right) \ \langle d,0,0 \,| d,0,0\rangle \,. \tag{2.15}$$

The unitarity then requires that

$$d(d-1) \geq 0. \tag{2.16}$$

Therefore, if a state $|d, j_1 = 0, j_2 = 0\rangle$ is not trivial, the unitarity requires that

$$d \geq 1. \tag{2.17}$$

It is the unitarity bound [8] for a scalar field. In particular, the equality $d = 1$ is satisfied if and only if

$$\sum_{a=1}^{4} E_a^+ E_a^+ |d,0,0\rangle = 0. \tag{2.18}$$

So far we have been discussing eigenvalues of the operator S_{-10}, as well as spins (j_1, j_2) of the $SO(4)$ subalgebra, but we would like to relate them to eigenvalue of the dilatation generator $D = S_{-14}$ of the $SO(1,1)$ subgroup and spins of the algebra of the Lorentz group $SO(1,3)$. It can be done by a similarity transformation exchanging the coordinates x^0 and x^4. In fact, the similarity transformation

$$\begin{aligned} iS_{0J} &= e^{\frac{\pi}{2}S_{40}} S_{4J} e^{-\frac{\pi}{2}S_{40}}, \qquad & iS_{4J} &= e^{\frac{\pi}{2}S_{40}} S_{0J} e^{-\frac{\pi}{2}S_{40}}, \\ S_{IJ} &= e^{\frac{\pi}{2}S_{40}} S_{IJ} e^{-\frac{\pi}{2}S_{40}}, \qquad & S_{40} &= e^{\frac{\pi}{2}S_{40}} S_{40} e^{-\frac{\pi}{2}S_{40}}, \quad (I, J = -1, 1, 2, 3) \end{aligned} \tag{2.19}$$

relates the operators D, P_μ and K_μ to S_{-10}, $E_a^\pm$ as

$$\begin{aligned} D &= -ie^{\frac{\pi}{2}S_{40}} S_{-10} e^{-\frac{\pi}{2}S_{40}}, \\ iP_0 &= e^{\frac{\pi}{2}S_{40}} E_4^+ e^{-\frac{\pi}{2}S_{40}}, \qquad & P_i &= e^{\frac{\pi}{2}S_{40}} E_i^+ e^{-\frac{\pi}{2}S_{40}}, \\ iK_0 &= e^{\frac{\pi}{2}S_{40}} E_4^- e^{-\frac{\pi}{2}S_{40}}, \qquad & K_i &= e^{\frac{\pi}{2}S_{40}} E_i^- e^{-\frac{\pi}{2}S_{40}}. \end{aligned} \tag{2.20}$$

Incidentally, the spin operators $J_i^{(1)}$, $J_i^{(2)}$ $(i = 1,2,3)$ of the Lorentz group $SO(1,3)$ are given by

$$J_i^{(1)} = \frac{1}{2}\left(\frac{1}{2}\epsilon_{ikl} M_{kl} - iM_{i0} \right), \qquad J_i^{(2)} = \frac{1}{2}\left(\frac{1}{2}\epsilon_{ikl} M_{kl} + iM_{i0} \right),$$

and they are related to the $SO(4)$ spin operators $S_i^{(1)}$, $S_i^{(2)}$ $(i = 1, 2, 3)$ as

$$J_i^{(1)} = e^{\frac{\pi}{2}S_{40}} S_i^{(1)} e^{-\frac{\pi}{2}S_{40}}, \qquad J_i^{(2)} = e^{\frac{\pi}{2}S_{40}} S_i^{(2)} e^{-\frac{\pi}{2}S_{40}}. \tag{2.21}$$

One then finds that an eigenstate $|d; j_1, m_1; j_2, m_2\rangle$ of the original operators can be mapped to $|d; j_1, m_1; j_2, m_2) = \exp((\pi/2)S_{40})\,|d; j_1, m_1; j_2, m_2\rangle$, which is an eigenstate [2] of D, J_3^L and J_3^R. In particular, one can see that the state $|d; j_1, m_1; j_2, m_2 d, j_1, j_2)$ has Lorentz spins $(j_1, m_1; j_2, m_2)$, and

$$D\,|d; j_1, m_1; j_2, m_2) = -ie^{\frac{\pi}{2}S_{40}} S_{-10}\,|d; j_1, m_1; j_2, m_2\rangle = -id\,|d; j_1, m_1; j_2, m_2)\,. \tag{2.22}$$

It means that under a dilatation $x^\mu \to e^{-a}x^\mu = e^{iaD}x^\mu$, the state $|d; j_1, m_1; j_2, m_2)$ transforms as

$$|d; j_1, m_1; j_2, m_2) \quad \to \quad e^{da}\,|d; j_1, m_1; j_2, m_2)\,, \tag{2.23}$$

which is consistent with the fact that it carries conformal dimension d in the usual sense [3].

Therefore, the conformal dimension d of a scalar state $|d, j_1 = 0, j_2 = 0)$ must satisfy the unitarity bound (2.17). Since the state $\sum_{a=1}^4 E_a^+ E_a^+ |d, 0, 0\rangle$ is transformed by the similarity transformation (2.19) to be $(-P_0P_0 + \sum_{i=1}^3 P_iP_i)\,|d, 0, 0)$, the condition (2.18) for $d = 1$ yields

$$\eta^{\mu\nu} P_\mu P_\nu |d, 0, 0\rangle = 0. \tag{2.24}$$

If one may regard the state $|d, 0, 0)$ as $\phi(0)\,|0, 0, 0)$ with a scalar field $\phi(x)$, it suggests that

$$\partial^\mu \partial_\mu \phi(x) = 0, \tag{2.25}$$

which is the Klein-Gordon equation of a free scalar field. The equality of the unitarity bound (2.17) is thus satisfied if and only if the field is free [4].

Let us proceed to the superconformal algebra. Besides the generators of the conformal algebra, there are the supersymmetry generator Q_α, the superconformal generator $S_{\dot\alpha}$, and the superconformal $U(1)_R$ generator R, which satisfy the commutation relations

$$\begin{aligned}
&\{Q_\alpha, Q^\dagger_{\dot\beta}\} = 2\sigma^\mu{}_{\alpha\dot\beta} P_\mu, \qquad \{S_\alpha, S^\dagger_{\dot\beta}\} = 2\sigma^\mu{}_{\alpha\dot\beta} K_\mu,\\
&[Q_\alpha, K^\mu] = \sigma^\mu{}_{\alpha\dot\beta} S^{\dagger\dot\beta}, \qquad [S_\alpha, P^\mu] = \sigma^\mu{}_{\alpha\dot\beta} Q^{\dagger\dot\beta},\\
&[D, Q_\alpha] = -\frac{i}{2} Q_\alpha, \quad [D, S_\alpha] = +\frac{i}{2} S_\alpha, \quad [R, Q_\alpha] = -Q_\alpha, \quad [R, S_\alpha] = +S_\alpha,\\
&[Q_\alpha, M^{\mu\nu}] = i\sigma^{\mu\nu}{}_\alpha{}^\beta Q_\beta, \qquad [S_\alpha, M^{\mu\nu}] = i\sigma^{\mu\nu}{}_\alpha{}^\beta S_\beta,\\
&\{S_\alpha, Q^\beta\} = -\delta_\alpha{}^\beta (2iD + 3R) - 2i\sigma^{\mu\nu}{}_\alpha{}^\beta M_{\mu\nu}.
\end{aligned} \tag{2.26}$$

[2] In order for the generators D, P_μ, K_μ, $M_{\mu\nu}$ to be hermitian, one needs to take the dual basis to be $(d; j_1, m_1; j_2, m_2| = \langle d; j_1, m_1; j_2, m_2| \exp((\pi/2)S_{40})$, as explained in [17].

[3] Previously, we have referred to eigenvalue of the operator D not iD as conformal dimension less rigorously. But, this is the definition of conformal dimension, which we will use henceforth.

[4] For $(j_1 = 1/2, j_2 = 0)$ or $(j_1 = 0, j_2 = 1/2)$, the condition (2.14) yields a free Dirac equation, while for $(j_1 = 1, j_2 = 0)$ or $(j_1 = 0, j_2 = 1)$, it gives a free Maxwell equation. Finally, for $(j_1 = 1/2, j_2 = 1/2)$, one finds the conservation law of a gauge invariant current.

The superconformal algebra is isomorphic to the Lie algebra of $SU(2,2|1)$. In addition to conformal dimension d and spins $j_{1,2}$, the superconformal $U(1)_R$ charge r - the eigenvalue of R - is used to specify unitary irreducible representations with their lowest weight (d, j_1, j_2, r).

Since the superconformal algebra never closes without the superconformal $U(1)_R$ generator R, any superconformal field theory must be invariant under the superconformal $U(1)_R$ transformation. Note that the superconformal $U(1)_R$ generator R is unambiguously determined as in the right hand side of the commutation relation $\{S_\alpha, Q^\beta\}$. Therefore, even in a case where a superconformal field theory has more than one global $U(1)$ symmetry, the superconformal $U(1)_R$ charge should be single out uniquely.

When a gauge theory is conformal invariant, it resides at the infrared fixed point, where the beta function of the gauge coupling g must be vanish. The NSVZ exact beta function of the coupling constant g of a gauge group G is given in [22] by

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{3T(G) - \sum_i T(r_i)(1-\gamma_i)}{1 - T(G)(g^2/8\pi^2)}, \tag{2.27}$$

where γ_i are the anomalous dimension of a matter fields labeled by i. $T(\rho_i)$ denotes the index of its representation ρ_i, and $T(G)$ is the index of the adjoint representation [5]. Its vanishing at the IR fixed point implies that the anomalous dimensions should satisfy

$$3T(G) - \sum_i T(r_i)(1-\gamma_i) = 0. \tag{2.28}$$

The anomalous dimension γ_i at the infrared fixed point is related to the conformal dimension as $D_i = 1+\gamma_i/2$, and further can be given in terms of the $U(1)_R$ charge by $\gamma_i = 3R_i - 2$, jumping ahead to (2.33). Thus, the equation (2.28) can be rewritten as

$$T(G) + \sum_i T(r_i)(R_i - 1) = 0. \tag{2.29}$$

This is exactly the same as the anomaly free condition of the superconformal $U(1)_R$ symmetry. It guarantees that the superconformal $U(1)_R$ symmetry at a infrared fixed point does not suffer from the anomalies caused by gauge interactions.

In a superconformal field theory, several fields with different Lorentz spins are transformed with each other by superconformal transformations and form a superconformal multiplet. An local operator $\mathcal{O}(x)$ with the lowest conformal dimension in a superconformal multiplet is called a primary operator. It is known[6] that the other operators in the same irreducible superconformal multiplet can be obtained by successively acting the supersymmetry generators Q_α and $Q^\dagger_{\dot\alpha}$ on the primary operator.

Taking account of the commutation relations $[D, Q_\alpha] = -\frac{i}{2}Q_\alpha$ and $[D, S_\alpha] = +\frac{i}{2}S_\alpha$ in (2.26), one may regard the supersymmetry generators $Q_\alpha, Q^\dagger_{\dot\alpha}$ as raising operators to raise

[5] In our convention, $T(G) = N$ for $G = SU(N)$.

[6] For a review of superconformal transformation, see *e.g.*, [23].

conformal dimension, and the superconformal generators $S_\alpha, S^\dagger_{\dot\alpha}$ as lowering operators. In fact, one can confirm this for an operator $\mathcal{O}$ of conformal dimension d by the Jacobi identity

$$\begin{aligned}
[D,[Q_\alpha,\mathcal{O}\}\} &= [[D,Q_\alpha],\mathcal{O}\}+[Q_\alpha,[D,\mathcal{O}\}\} \\
&= -\frac{i}{2}[Q_\alpha,\mathcal{O}\}-id[Q_\alpha,\mathcal{O}\} = -i\left(d+\frac{1}{2}\right)[Q_\alpha,\mathcal{O}\}.
\end{aligned} \tag{2.30}$$

Similarly, the commutation relations $[D,Q^\dagger_{\dot\alpha}] = -\frac{i}{2}Q^\dagger_{\dot\alpha}$ and $[D,S^\dagger_{\dot\alpha}] = +\frac{i}{2}S^\dagger_{\dot\alpha}$ suggest that $Q^\dagger_{\dot\alpha}$ and $S^\dagger_{\dot\alpha}$ are raising and lowering operators, respectively.

As a primary operator has the lowest conformal dimension, it must satisfy the primary condition

$$[S_\alpha,\ \mathcal{O}(0)\} = [S^\dagger_{\dot\alpha},\ \mathcal{O}(0)\} = 0. \tag{2.31}$$

When the primary operator also satisfies the chiral condition

$$[Q^\dagger_{\dot\alpha},\mathcal{O}(0)\} = 0, \tag{2.32}$$

it is referred to as a chiral primary operator.

A detailed study of unitary irreducible representations of the superconformal algebra [24, 25] shows that the inequality

$$D(\mathcal{O}) \geq \frac{3}{2}R(\mathcal{O})$$

must be satisfied for any local scalar operator $\mathcal{O}(x)$, where $D(\mathcal{O})$ and $R(\mathcal{O})$ are the conformal dimension and the $U(1)_R$ charge of the operator $\mathcal{O}$, respectively. It is remarkable that the equality

$$D(\mathcal{O}) = \frac{3}{2}R(\mathcal{O}) \tag{2.33}$$

is satisfied if and only if $\mathcal{O}(x)$ is a chiral primary operator.

Therefore, in particular, if the chiral primary operator carries no spin $(j_1,j_2)=(0,0)$, the unitarity bound (2.17) means that

$$R(\mathcal{O}) \geq \frac{2}{3}. \tag{2.34}$$

In fact, one can obtain the equality (2.33) for a chiral primary operator $\mathcal{O}$ by calculating $[\{S^{\dagger\dot\alpha},Q^\dagger_{\dot\beta}\},\mathcal{O}]$ in two different ways. On one hand, by using the commutation relation of the generators, one finds that

$$\begin{aligned}
[\{S^{\dagger\dot\alpha},Q^\dagger_{\dot\beta}\},\mathcal{O}] &= [-\delta^{\dot\alpha}{}_{\dot\beta}(2iD-3R)-2i\bar\sigma_{\mu\nu}{}^{\dot\alpha}{}_{\dot\beta}M^{\mu\nu},\mathcal{O}] \\
&= -\delta^{\dot\alpha}{}_{\dot\beta}(2d-3r)\mathcal{O}.
\end{aligned} \tag{2.35}$$

On the other hand, by using the Jacobi identity

$$[\{S^{\dagger\dot{\alpha}}, Q^{\dagger}_{\dot{\beta}}\}, \mathcal{O}] = \{[Q^{\dagger}_{\dot{\beta}}, \mathcal{O}], S^{\dagger\dot{\alpha}}\} + \{[S^{\dagger\dot{\alpha}}, \mathcal{O}], Q^{\dagger}_{\dot{\beta}}\}, \tag{2.36}$$

one can see that it vanishes due to the chiral condition and the primary condition. Thus, if $\mathcal{O}(x)$ is a chiral primary operator, the equality (2.33) is satisfied.

The relation (2.33) implies a striking fact - the additivity of the conformal dimensions of chiral primary operators. Since the $U(1)_R$ charge of the product of operators O_1 and O_2) is of course the sum of the charge of each operator;

$$R(O_1 O_2) = R(O_1) + R(O_2),$$

due to (2.33), their conformal dimensions must also be additive;

$$D(O_1 O_2) = D(O_1) + D(O_2).$$

It suggests that the product of two chiral primary operators does not cause a singularity.

Since a product of chiral primary operators at a spacetime point is well-defined without any singularities - just a multiplication of them, all chiral primary operators can form a ring. In a supersymmetric gauge theory with a non-trivial infrared fixed point, it is in fact convenient to consider the set of all the chiral operators, which are not necessarily primary, and to introduce a quotient of the set by a equivalence relation, which we will refer to as a chiral ring. An equivalence class of the chiral ring is mapped into a chiral primary operator, as will be seen just below.

A chiral operator $\mathcal{O}$ satisfies the chiral condition

$$[Q^{\dagger}_{\dot{\alpha}}, \mathcal{O}\} = 0. \tag{2.37}$$

Then, we introduce the equivalence relation

$$\mathcal{O} \simeq \mathcal{O} + [Q^{\dagger}_{\dot{\alpha}}, X^{\dot{\alpha}}\}, \tag{2.38}$$

where $X^{\dot{\alpha}}$ is an arbitrary gauge invariant operator which satisfies

$$\left[Q^{\dagger}_{\dot{\beta}}, [Q^{\dagger}_{\dot{\alpha}}, X^{\dot{\alpha}}\}\right\} = 0. \tag{2.39}$$

The chiral ring is the set consisting of all the gauge invariant chiral operators with the identification (2.38).

The term $[Q^{\dagger}_{\dot{\alpha}}, X^{\dot{\alpha}}\}$ in (2.38) is not primary, because its conformal dimension is greater than that of $X^{\dot{\alpha}}$, as $Q^{\dagger}_{\dot{\alpha}}$ carries conformal dimension a half. Since $X^{\dot{\alpha}}$ and $[Q^{\dagger}_{\dot{\alpha}}, X^{\dot{\alpha}}\}$ reside in the same superconformal multiplet, $[Q^{\dagger}_{\dot{\alpha}}, X^{\dot{\alpha}}\}$ is not primary. Conversely, a chiral non-primary operator $\mathcal{O}$ of conformal dimension d and the superconformal $U(1)_R$ charge r can be rewritten in a form $[Q^\dagger_{\dot\alpha}, X^{\dot\alpha}\}$. In fact, from (2.35) and (2.36) together with the chiral condition (2.32), one can show [7] that

$$\mathcal{O} = -\frac{1}{2(2d-3r)}\{Q^\dagger_{\dot\beta}, [S^{\dagger\dot\alpha}, \mathcal{O}]\}, \tag{2.40}$$

where we used the fact that $2d > 3r$ for a non-primary operator. It thus concludes that an equivalence class of the chiral ring corresponds to a chiral primary operator [8].

The condition (2.38) can be rewritten in terms of a chiral superfield Φ as

$$\Phi \simeq \Phi + \bar{\mathcal{D}}^2 Z, \tag{2.41}$$

with $\bar{\mathcal{D}}^2 = \bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}/2$, where the supercovariant derivative $\bar{\mathcal{D}}_{\dot\alpha}$ is defined by using superspace coordinates $(x, \theta, \bar\theta)$ as

$$\bar{\mathcal{D}}_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha \sigma^\mu{}_{\alpha\dot\alpha}\partial_\mu.$$

This indicates that the lowest component of $\bar{\mathcal{D}}^2$ term is chiral but vanishes in the chiral ring.

Since it is generically formidable to consider the chiral ring at a non-trivial infrared fixed point, one often considers the chiral ring at the ultraviolet cutoff, which is referred to as a classical chiral ring. In the classical chiral ring, one assumes that a relation among gauge invariant operators is determined by the classical equations of motion. One can see that the equations of motion

$$\bar{\mathcal{D}}^2(\Phi^\dagger e^V) + \frac{\partial W(\Phi)}{\partial\Phi} = 0, \tag{2.42}$$

of the chiral superfield Φ yields the F-term condition

$$\frac{\partial W(\Phi)}{\partial\Phi} \simeq 0, \tag{2.43}$$

as a defining equation of the classical chiral ring [9]. However, the classical defining equation (2.43) may be modified quantum mechanically. This deformed chiral ring is called a quantum chiral ring, which corresponds to the set of the chiral primary operators, as discussed previously.

[7] Here, we assumed that $\mathcal{O}$ is a Lorentz scalar for simplicity.

[8] The correspondence can also be seen by explicitly constructing a representation of the superconformal algebra on field operators. For a review, see *e.g.*, [23].

[9] Among gauge invariant operators including gaugino superfield $W_\alpha \propto \bar{\mathcal{D}}^2[e^{-V}D_\alpha e^V]$, there are other relations [26, 27] determined by

$$W^A_\alpha (T^A)^a{}_b\,\phi^b \propto \bar{\mathcal{D}}^2\left[e^{-V}D_\alpha(e^V\phi)\right]^a \sim 0, \tag{2.44}$$

where a is an index of the gauge group and T^A is the generator in the representation R.

2.2 Central Charges in Four-Dimensional Conformal Field Theories

In two-dimensional conformal field theories (CFTs), the central charge c of the Virasoro algebra plays an important role. In a curved background, it is related to the trace anomaly as

$$\langle T^{\mu}{}_{\mu} \rangle = -\frac{c}{12} R, \tag{2.45}$$

where R is the scalar curvature. In a four-dimensional conformal field theory, one also uses the trace anomaly to define the analogs of the two-dimensional central charge c.

In four dimensions, the Weyl tensor $C_{\mu\nu\rho\sigma}$ is given in terms of the Riemann tensor $R_{\mu\nu\rho\sigma}$ as

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\sigma}R_{\nu\rho} - g_{\mu\rho}R_{\nu\sigma} - g_{\nu\sigma}R_{\mu\rho} + g_{\nu\rho}R_{\mu\sigma}\right) + \frac{1}{3}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})R. \tag{2.46}$$

Using the dual $\tilde{R}^{\mu\nu\rho\sigma} = \varepsilon^{\mu\nu\alpha\beta}\varepsilon^{\rho\sigma\gamma\delta}R_{\alpha\beta\gamma\delta}/4$ of the Riemann tensor $R_{\mu\nu\rho\sigma}$, the Euler density is defined by $(1/16\pi^2)R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$. The trace anomaly in a four-dimensional curved background is then calculated to be

$$\langle T^{\mu}{}_{\mu} \rangle = \frac{c}{16\pi^2} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}. \tag{2.47}$$

The coefficients a and c are supposed to play a similar role to the two-dimensional c, and are thus called the central charges of the four-dimensional CFT.

In a four-dimensional superconformal field theory, since one has the energy-momentum tensor $T_{\mu\nu}$ and the $U(1)_R$ current J_R^{μ}, the 't Hooft anomaly coefficients $\mathrm{Tr}R^3$ and $\mathrm{Tr}R$ can be defined as the coefficients of the divergence of the three-point functions $\langle J_R^{\mu} J_R^{\nu} J_R^{\rho} \rangle$ and $\langle J_R^{\lambda} T^{\mu\nu} T^{\rho\sigma} \rangle$, respectively. Interestingly, the central charges c and a are related to these coefficients $\mathrm{Tr}R^3$ and $\mathrm{Tr}R$ [18, 19] via

$$a = \frac{3}{32}\left(3\mathrm{Tr}R^3 - \mathrm{Tr}R\right), \tag{2.48}$$

$$c = \frac{1}{32}\left(9\mathrm{Tr}R^3 - 5\mathrm{Tr}R\right). \tag{2.49}$$

When an asymptotically free gauge theory becomes strongly interacting in the infrared, generically at a non-trivial infrared fixed point, it is difficult to calculate these coefficients $\mathrm{Tr}R^3$ and $\mathrm{Tr}R$, let alone the central charges c and a. However, using the 't Hooft anomaly matching condition, the coefficients $\mathrm{Tr}R^3$ and $\mathrm{Tr}R$ in the infrared can be obtained by calculating them in terms of elementary fields at high energies. In fact, one can find that

$$\mathrm{Tr}R^3 = \sum_i (R_i)^3, \qquad \mathrm{Tr}R = \sum_i R_i, \tag{2.50}$$

where R_i is the superconformal $U(1)_R$ charge of the elementary chiral fermion ψ_i.

In two-dimensional conformal field theory, there is Zamolodchikov's c-theorem [28]. Let us recall the c-theorem by considering an renormalization group (RG) flow connecting a ultraviolet (UV) fixed point to an infrared (IR) fixed point. One has a two-dimensional conformal field theory with the central charge $c_{\rm UV}$ at the UV fixed point and also the one with the central charge $c_{\rm IR}$ at the IR fixed point. The RG flow is described by the coupling constants $g^i(t)$ at the energy scale $e^{-t}\Lambda$, and its gradient is given by their beta functions $\beta^i(g)$.

Zamolodchikov's c-theorem then states that there exists a function $c(g^i(t))$ connecting $c_{\rm UV}$ with $c_{\rm IR}$, which monotonically decreases throughout the RG flow;

$$c(g^i_{\rm UV}) = c_{\rm UV}, \quad c(g^i_{\rm IR}) = c_{\rm IR}, \quad \frac{d}{dt}c(t) \equiv -\beta^i(g)\frac{\partial}{\partial g^i}c(g) \leq 0, \tag{2.51}$$

where $g^i_{\rm UV}$ and $g^i_{\rm IR}$ are the values of the couplings at the UV and IR fixed point, respectively. The inequality is saturated only at the fixed points. The existence of such a function $c(g)$ insures that the central charge at the IR fixed point is always smaller than that at the UV fixed point;

$$c_{\rm IR} < c_{\rm UV}. \tag{2.52}$$

The c-theorem is consistent with the interpretation that the central charge measures the number of the degrees of freedom of a CFT. Through the RG flow, massive modes are integrated out and the number of the degrees of freedom decreases.

An extension of Zamolodchikov's c-theorem to four-dimensional CFTs has been proposed in [29]. In a four-dimensional conformal theory, as the counterpart of the two-dimensional central charge, either of the anomaly coefficients a and c in (2.47) may be chosen. The anomaly coefficient c is known to violate the inequality $c_{\rm IR} < c_{\rm UV}$ in some examples. On the other hand, it is known that $a_{\rm IR} < a_{\rm UV}$ is satisfied in a large class of four-dimensional field theories. Therefore, the anomaly coefficient a was expected to satisfy a four-dimensional analog of the c-theorem, and the conjecture is called the "a-theorem conjecture". There has been much evidence found for the conjecture so far.

However, strikingly, a counter-example was found by [30] recently [10] to rule out the a-theorem conjecture. However, since it is known that the a-theorem conjecture is satisfied in a quite large class of four-dimensional field theories, it is conceivable that a variant of the a-theorem conjecture may be satisfied in four-dimensional conformal field theory [11].

2.3 a-Maximization

In section 2.1, we have explained that the conformal dimension of a chiral primary operator is exactly determined by its $U(1)_R$ charge as in (2.33). Therefore, it is very important to

[10] Only one counter example is known at present.

[11] For a recent overview of the subject and a-maximization, see [31].

know the superconformal $U(1)_R$ charges of all the chiral primary operators in a four-dimensional superconformal field theory.

Let us suppose that in a four-dimensional supersymmetric gauge theory, there is only one global $U(1)$ symmetry rotating the gaugino in the ultraviolet, and that the gauge theory flows into a non-trivial fixed point in the infrared. If no additional global $U(1)$ symmetry is enhanced in the infrared, the global $U(1)$ symmetry itself yields the superconformal $U(1)_R$ symmetry at the infrared fixed point. Therefore, from the $U(1)$ charge assignment of the elementary fields, one can read the superconformal $U(1)_R$ charges of gauge invariant operators.

However, besides the $U(1)$ symmetry, if there are additional global $U(1)$ symmetries in the ultraviolet, one cannot a prior determine which linear combination of the global $U(1)$ symmetries gives the superconformal $U(1)_R$ symmetry at the infrared fixed point, even though no $U(1)$ symmetry enhancement occurs. Intriligator and Wecht have given a prescription in [13] to pick up the superconformal $U(1)_R$ symmetry among all the linear combinations of the global $U(1)$ symmetries. In this section, we will explain their prescription *i.e.*, a-maximization.

As mentioned above, let us suppose that the supersymmetric gauge theory with more than one anomaly-free $U(1)$ symmetry flows into its non-trivial infrared fixed point. Let us denote the $U(1)$ transformation rotating the gaugino as $U(1)_\lambda$. One then can choose the rest of the $U(1)$ transformations so that the gaugino are left invariant under them. Let us label them by I and denote them as $U(1)_I$. At the infrared fixed point, if there occurs no additional $U(1)$ symmetry enhancement, the superconformal $U(1)_R$ symmetry should be a linear combination of the $U(1)$ symmetries. In particular, the $U(1)_R$ charge $R_{\mathcal{O}}$ of an operator $\mathcal{O}$ may be given by the flavor $U(1)_I$ charges $F_{\mathcal{O}I}$ and the $U(1)_\lambda$ charge $\Lambda_{\mathcal{O}}$ of $\mathcal{O}$ as

$$R_{\mathcal{O}}(x) = \Lambda_{\mathcal{O}} + \sum_I x^I F_{\mathcal{O}I}, \tag{2.53}$$

where x^I taking a real value determines which linear combination of the $U(1)$ symmetries yields the superconformal $U(1)_R$ symmetry.

The superconformal $U(1)_R$ symmetry is distinguished from the other linear combinations of the global $U(1)$ symmetries by the fact that its $U(1)_R$ current is in the same superconformal multiplet as the energy-momentum tensor. The fact has another consequence that the 't Hooft anomaly coefficient $\mathrm{Tr} F_I R(x) R(x)$ of the three-point function with the $U(1)_I$ current inserted at one vertex and the $U(1)_R$ current at each of the two remaining vertices is related to the one $\mathrm{Tr} F_I$ with the same $U(1)_I$ current at one vertex and the energy-momentum tensor at each of the remaining two vertices. More precisely, one has

$$9\, \mathrm{Tr} F_I R(x) R(x) = \mathrm{Tr} F_I. \tag{2.54}$$

Let us briefly explain the relation (2.54). See [13] for more detail. Generically, even in a non-supersymmetric field theory, coupling each current j_i^μ ($i = 1, 2$) of two global $U(1)$ symmetries to a background gauge field A_μ^i with the field strength $F_{\mu\nu}^i$ in a background metric $g_{\mu\nu}$, causes the non-conservation of the current j_i^μ due to their 't Hooft anomalies; for example,

$$\partial_\mu j_1^\mu = \frac{k_{111}}{48\pi^2} F^{1\mu\nu} \tilde{F}^1_{\mu\nu} + \frac{k_{112}}{16\pi^2} F^{1\mu\nu} \tilde{F}^2_{\mu\nu} + \frac{k_{122}}{16\pi^2} F^{2\mu\nu} \tilde{F}^2_{\mu\nu} + \frac{k_1}{384\pi^2} R^{\mu\nu\rho\sigma} \tilde{R}_{\mu\nu\rho\sigma}, \tag{2.55}$$

where k_{111}, k_{112}, k_{122}, and k_1 are the 't Hooft anomaly coefficients. Let us return to our supersymmetric theory. Taking the current j_1^μ to be one of the $U(1)$ symmetries other than $U(1)_R$, turning off the background A_μ^1, and regarding the background A_μ^2 as that coupled to the $U(1)_R$ current, one can see that

$$\partial_\mu j_I^\mu = \frac{1}{16\pi^2} \left[\mathrm{Tr} F_I R(x) R(x) \right] F^{\mu\nu} \tilde{F}_{\mu\nu} + \frac{1}{384\pi^2} \left[\mathrm{Tr} F_I \right] R^{\mu\nu\rho\sigma} \tilde{R}_{\mu\nu\rho\sigma}, \tag{2.56}$$

where we denoted j_1^μ as j_I^μ, and $F_{\mu\nu}^2$ was rewritten as $F_{\mu\nu}$.

In the superfield formalism, the background field strength $F_{\mu\nu}$ coupled to the $U(1)_R$ current and the background curvature $R_{\mu\nu\rho\sigma}$ forms the superWeyl tensor $\mathcal{W}_{\alpha\beta\gamma}$ as components. The current j_I^μ is also given by a component of a current superfield J_I as

$$j_I^\mu = \sigma^\mu{}_{\alpha\dot{\beta}} \left[\nabla^\alpha, \, \overline{\nabla}^{\dot{\beta}} \right] J_I|_{\theta=0} \, . \tag{2.57}$$

In this background, it was discussed in [13] that the current superfield J_I satisfies

$$\overline{D}^2 J_I = \frac{1}{384\pi^2} \left[\mathrm{Tr} F_I \right] \frac{1}{2} \mathcal{W}_{\alpha\beta\gamma} \mathcal{W}^{\alpha\beta\gamma}. \tag{2.58}$$

Then, one can deduce that

$$\partial_\mu j_I^\mu = -\frac{i}{4} \left[\nabla^2, \, \overline{\nabla}^2 \right] J_I|_{\theta=0} = \frac{1}{384\pi^2} \left[\mathrm{Tr} F_I \right] \left[\frac{8}{3} F^{\mu\nu} \tilde{F}_{\mu\nu} + R^{\mu\nu\rho\sigma} \tilde{R}_{\mu\nu\rho\sigma} \right]. \tag{2.59}$$

It can be compared to (2.56), which proves (2.54).

Instead of the non-trivial background metric and gauge field coupled to the $U(1)_R$ current, turning on background gauge fields A_μ^I with the field strength $F_{\mu\nu}^I$ coupled to the rest of the global $U(1)$ currents, the non-conservation of the $U(1)_R$ current j_R^μ is found to be

$$\partial_\mu j_R^\mu = \frac{1}{16\pi^2} \sum_{I,J} \left[\mathrm{Tr} R(x) F_I F_J \right] F^{I\mu\nu} \tilde{F}^J_{\mu\nu}.$$

It has been discussed in [18] that the 't Hooft anomaly coefficient $\mathrm{Tr} R F_I F_J$ is proportional to a positive definite matrix τ^{IJ} appearing in the two-point function

$$\langle j_\mu^I(x) j_\nu^J(0) \rangle = \frac{1}{16\pi^4} \tau^{IJ} \left(\eta_{\mu\nu} \partial^\rho \partial_\rho - \partial_\mu \partial_\nu \right) \left(\frac{1}{x^4} \right)$$

of the $U(1)_I$ current j^I_μ and the $U(1)_J$ current j^J_μ. More precisely, one has

$$\mathrm{Tr}R(x)F_IF_J = -\frac{1}{3}\tau^{IJ}.$$

Regarding $\mathrm{Tr}RF_IF_J$ as a matrix with indices I and J, one can see that it is negative definite; Schematically,

$$\mathrm{Tr}R(x)F_IF_J < 0. \tag{2.60}$$

As we have seen so thus, the linear combination of the global $U(1)$ symmetries giving the superconformal $U(1)_R$ symmetry should satisfy the conditions (2.54) and (2.60). On the contrary, if one regards the equation (2.53) as parametrizing all the combinations of the $U(1)$ symmetries with the parameters x^I, instead of the definite values x^I, the solution x^I to the equations (2.54) and (2.60) may be a candidate for giving the superconformal $U(1)_R$ symmetry. Let us call this parametrized $U(1)_R$ charge $R(x)$ with x^I, the trial $U(1)_R$ charge. Substituting the trial $U(1)_R$ charge $R(x)$ into the central charge a in (2.48) formally, one obtains

$$a(x) = \frac{3}{32}\left[3\,\mathrm{Tr}R(x)^3 - \mathrm{Tr}R(x)\right], \tag{2.61}$$

which was called a trial a-function in [13]. It does not only give the actual value of the central charge a with the value x^I giving the superconformal $U(1)_R$ symmetry at the infrared fixed point, and it also yields a concise method to express the conditions (2.54) and (2.60) as

$$\frac{\partial}{\partial x^I}\,a(x) = 0, \qquad \frac{\partial^2}{\partial x^I\,\partial x^J}\,a(x) < 0, \tag{2.62}$$

for all I, J. It means that the candidate x^I giving the $U(1)_R$ symmetry must be a local maximum of the trial a-function $a(x)$. The a-maximization procedure is thus to find local maxima x^I of the trial a-function $a(x)$ to determine the linear combination of the $U(1)$ symmetries giving the superconformal $U(1)_R$ symmetry at an infrared fixed point.

In an asymptotically free gauge theory, the trial a-function can be calculated [12] in terms of the trial $U(1)_R$ charges of elementary spinor fields in the ultraviolet to be

$$a_0(x) = a_G + \sum_i \left[3\left(R_i(x)-1\right)^3 - \left(R_i(x)-1\right)\right] \tag{2.63}$$

with the trial $U(1)_R$ charge $R_i(x)$ of the chiral superfield Φ_i whose spinor component is one of the elementary spinor fields, where the sum runs over all the elementary matter fields, and the contribution a_G comes from the gauginos and is by definition independent of x^I. If

[12] In this paper, we are not interested in the overall normalization of the a-function and will thus omit it henceforth.

no accidental $U(1)$ symmetry is not enhanced in the infrared, thanks to 't Hooft anomaly matching, one can use this trial a-function in (2.63) as a trial a-function in the infrared to find the $U(1)_R$ symmetry at a infrared fixed point.

However, when the trial $U(1)_R$ charge of a gauge invariant chiral primary operator $\mathcal{O}$ violates the inequality (2.34) at a point of x^I, if the point was a local maximum of the trial a-function $a_0(x)$ in (2.63), one would encounter an inconsistency with the unitarity of the theory; the inequality (2.34) for any gauge invariant operators should be satisfied in a unitary theory, as explained in the previous sections. Therefore, it suggests that the trial a-function $a_0(x)$ calculated in the ultraviolet cannot be used at the point of x^I, where the trial $U(1)_R$ charge of any gauge invariant chiral primary operator violates the unitarity bound (2.34), to identify the superconformal $U(1)_R$ symmetry.

For such a point of x^I, it has been proposed in [11] to replace the trial a-function $a_0(x)$ by

$$a(x) = a_0(x) + \sum_i \left[-a_{\mathcal{O}_i}\left(R_{\mathcal{O}_i}(x)\right) + a_{\mathcal{O}_i}\left(\frac{2}{3}\right)\right], \tag{2.64}$$

with the sum running over all the gauge invariant chiral primary operators whose trial $U(1)_R$ charge $R_{\mathcal{O}_i}(x)$ violates the unitarity bound (2.34), where $a_{\mathcal{O}}(R)$ is a function of a parameter R;

$$a_{\mathcal{O}}\left(R\right) = d_{\mathcal{O}}\left[3\left(R-1\right)^3 - \left(R-1\right)\right], \tag{2.65}$$

with $d_{\mathcal{O}}$ the number of the components of $\mathcal{O}$. When the trial $U(1)_R$ charge of a gauge invariant operator violates the unitarity bound (2.34), what is really happening is that the operator become free at the infrared fixed point. Therefore, it has superconformal $U(1)_R$ charge $2/3$. Therefore, the improvement in (2.64) may be interpreted as subtracting the individual contribution $a_{\mathcal{O}_i}\left(R_{\mathcal{O}_i}(x)\right)$ of wrongly interacting $\mathcal{O}_i$ and adding the contribution $a_{\mathcal{O}_i}\left(2/3\right)$ of free $\mathcal{O}_i$ to the trial a-function $a_0(x)$.

Since the function $a_{\mathcal{O}}(R)$ for any operator $\mathcal{O}$ has a critical point at $R = 2/3$; $a'_{\mathcal{O}}(R = 2/3) = 0$, at a point of x^I where an operator $\mathcal{O}$ has trial $U(1)_R$ charge just $2/3$, the trial a-function $a(x)$ and all its first derivatives have the same value as its improvement $a(x) - a_{\mathcal{O}}\left(R_{\mathcal{O}}(x)\right) + a_{\mathcal{O}}\left(2/3\right)$ and its corresponding first derivatives, respectively.

Within a region with the same content of gauge invariant operators apparently violating the unitarity bound (2.34) in the whole parameter space $\{x^I\}$, one uses a single trial a-function. Let us call the trial a-function in such a region the local trial a-function. Since the whole parameter space $\{x^I\}$ is covered with such regions, all the local trial a-functions are combined into a global trial a-function, which is a continuous function of x^I. Since a local trial a-function is in fact a polynomial of degree 3 in x^Is, one can find at most a single local maximum, but in another region, one could obtain another local maximum. It may suggest that one could find more than one local maximum over the whole parameter space to lose definitive results on which linear combination of the $U(1)$ symmetries is the superconformal $U(1)_R$ symmetry which we search for. However, we will see that it is not the case for the theory which we will deal with in this article [13].

An explanation was given to carry out the prescription (2.64) in [12] by introducing a Lagrange multiplier field L and a free field M. Let us suppose to turn on the superpotential

$$W = L\mathcal{O} + hLM, \tag{2.66}$$

with a coupling constant h. As far as the coupling h is non-zero, by shifting $M \to M - \mathcal{O}/h$ and by integrating out M and L, one can return to the original theory.

In the new system, the trial a-function is given by

$$\tilde{a}(x) = a(x) + a_L(R_L(x)) + a_M(R_M(x)), \tag{2.67}$$

where $a(x)$ is the trial a-function of the original theory. Since the $U(1)_R$ charge $R_L(x)$ of L is given by $R_L(x) = 2 - R_{\mathcal{O}}(x)$, one can see that

$$a_L(R_L(x)) = a_L(2 - R_{\mathcal{O}}(x)) = -a_L(R_{\mathcal{O}}(x)) = -a_{\mathcal{O}}(R_{\mathcal{O}}(x)). \tag{2.68}$$

With a non-zero h, it cancels the contribution $a_M(R_M(x))$ in $\tilde{a}(x)$ to give the original $a(x)$.

Let us take the coupling h to zero, and assume that the resulting theory has a non-trivial infrared fixed point. The remaining term in the superpotential gives the F-term condition $\mathcal{O} = 0$. It means that the operator $\mathcal{O}$ vanishes in the original interacting system, and that the field M is propagating freely. Therefore, it describes the same low-energy physics as when the operator $\mathcal{O}$ hits the unitarity bound. In fact, when the superconformal $U(1)_R$ charge $R(\mathcal{O})$ of $\mathcal{O}$ is less than $2/3$, since the superpotential has $U(1)_R$ charge two, the Lagrange multiplier L has $U(1)_R$ charge $2 - R(\mathcal{O})$. The free field M has $U(1)_R$ charge $2/3$, and the $U(1)_R$ charge of the operator LM is more than two. Therefore, LM is an irrelevant operator as a perturbation about the infrared fixed point. This is a consistent result. One then finds the trial a-function

$$\tilde{a}(x) = a(x) - a_{\mathcal{O}}(R_{\mathcal{O}}(x)) + a_M(R_M = \frac{2}{3}), \tag{2.69}$$

which reproduces the prescription (2.64).

In [14], an extension of this argument has been discussed by using the auxiliary field method in the $Spin(10)$ gauge theory, where the discussion was armed with the electric-magnetic duality, as will be explained in detail in section 5.

[13] In the case where more than one local maximum of a global trial a-function are found, one could invoke a diagnostic conjectured by Intriligator [32]. The weak version of the diagnostic states that the correct infrared phase is the one with the larger value of the conformal anomaly a. He used it and the strong version to predict whether the phase under consideration is infrared free or interactingly conformal.

Chapter 3

$Spin(10)$ Theories and their Electric-Magnetic Duals

We will explain the models which we will carry out the a-maximization procedure to study its physics at non-trivial infrared fixed point in more detail. Therefore, we will only focus on the non-Abelian Coulomb phase of them. For the other phases, see [3, 4, 33].

3.1 The Theory with One Spinor

In this section, we will briefly explain the non-Abelian Coulomb phase of four-dimensional $\mathcal{N} = 1$ supersymmetric $Spin(10)$ gauge theory with a single chiral superfield Ψ in the spinor representation and N_Q chiral superfields Q^i ($i = 1, \cdots, N_Q$) in the vector representation. First, we will not turn on any superpotentials, but in the next subsection, we will discuss electric-magnetic duality of the theory with a superpotential.

The model is in the non-Abelian Coulomb phase for $7 \leq N_Q \leq 21$, where it has a non-trivial infrared fixed point [3, 4]. It was discussed in [3, 4] that the dual description is also available at the infrared fixed point.

This theory has the global symmetries $SU(N_Q) \times U(1)_F \times U(1)_\lambda$, which are not broken by any anomalies. Under the global symmetries, the matter fields transform as in Table 3.1. Here, we chose a basis of the generators of the $U(1)$ groups; under the $U(1)_F$ transformation, the gaugino W_α are not rotated, while under $U(1)_\lambda$ transformation, it has charge one. There is also an anomalous $U(1)_A$ symmetry. The charge of it for each field is also given in Table 3.1.

The gauge invariant generators of the classical chiral ring of this theory are given by

$$\begin{aligned}
&M^{ij} = Q^{ai}Q^{aj}, \\
&Y^i = \Psi^T C\Gamma^a \Psi Q^{ai}, \\
&B^{i_1\cdots i_5} = \Psi^T C\Gamma^{a_1\cdots a_5}\Psi Q^{a_1 i_1}\cdots Q^{a_5 i_5}, \\
&E^{i_1\cdots i_9} = \Psi^T C\Gamma^{a_1\cdots a_9}\Psi Q^{a_1 i_1}\cdots Q^{a_9 i_9},
\end{aligned}$$

	$Spin(10)$	$SU(N_Q)$ $i, j, \cdots$	$U(1)_F$	$U(1)_\lambda$	$U(1)_A$
Q^i	**10**	$\square$	-2	1	1
Ψ	**16**	**1**	N_Q	-3	0
W_α	Adjoint	**1**	0	1	0
Λ^{22-N_Q}	**1**	**1**	0	0	$2N_Q$

Table 3.1: The matter contents of the electric theory and their charges.

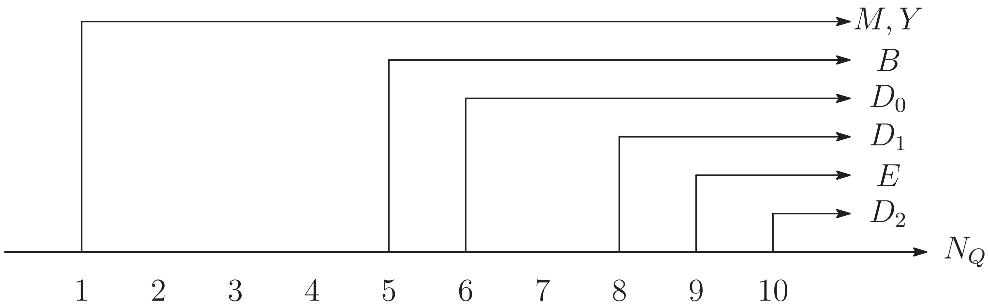

Figure 3.1: The number N_Q of the vectors Q^i where the gauge invariant operators exist.

$$\begin{aligned}
&{D_0}^{i_1 \cdots i_6} = \varepsilon^{a_1 \cdots a_{10}} Q^{a_1 i_1} \cdots Q^{a_6 i_6} {W_\alpha}^{a_7 a_8} W^{\alpha a_9 a_{10}}, \\
&{D_1}_\alpha^{i_1 \cdots i_8} = \varepsilon^{a_1 \cdots a_{10}} Q^{a_1 i_1} \cdots Q^{a_8 i_8} {W_\alpha}^{a_9 a_{10}}, \\
&{D_2}^{i_1 \cdots i_{10}} = \varepsilon^{a_1 \cdots a_{10}} Q^{a_1 i_1} \cdots Q^{a_{10} i_{10}}, \\
&S = \mathrm{Tr} W_\alpha W^\alpha.
\end{aligned} \tag{3.1}$$

Here, a and $a_1, a_2, \cdots$ are $Spin(10)$ gauge indices. The matrix C is the charge conjugation matrix, and $\Gamma^{a_1 \cdots a_n}$ is defined as an antisymmetrized product of $Spin(10)$ gamma matrices as

$$\Gamma^{a_1 \cdots a_n} = \frac{1}{n!} \Gamma^{[a_1} \cdots \Gamma^{a_n]}.$$

Thus, a spinor bi-linear $\Psi^T C \Gamma^{a_1 \cdots a_n} \Psi$ is an antisymmetric tensor of rank n. Taking account of the number of the antisymmetrized indices of the $SU(N_Q)$ global symmetry, one can see that which operators exist depends on N_Q. The values of N_Q where each operator exists are illustrated in Figure 3.1.

For $7 \leq N_Q \leq 21$, as mentioned above, the dual description of the original theory is available, and we will call it the magnetic theory, while the original theory will be called the electric theory. The magnetic theory is given by $SU(N_Q - 5)$ gauge theory with N_Q antifundamentals $\bar{q}_i$, a single fundamental q, a symmetric tensor s and singlets M^{ij} and Y^i. It has the superpotential

$$W_{\mathrm{mag}} = \frac{\tilde{h}}{\tilde{\mu}^2} M^{ij} \bar{q}_i \, s \, \bar{q}_j + \frac{\tilde{h}'}{\tilde{\mu}^2} Y^i \, q \bar{q}_i + \frac{1}{\tilde{\mu}^{N_Q - 8}} \det s. \tag{3.2}$$

	$SU(N_Q-5)$ $a,b,\cdots$	$SU(N_Q)$ $i,j,\cdots$	$U(1)_F$	$U(1)_\lambda$
$\bar{q}_{ai}$	$\bar{\square}$	$\bar{\square}$	2	$-\frac{1}{N_Q-5}$
q^a	$\square$	$\mathbf{1}$	$-2N_Q$	$\frac{7N_Q-34}{N_Q-5}$
s^{ab}	$\square\square$	$\mathbf{1}$	0	$\frac{2}{N_Q-5}$
M^{ij}	$\mathbf{1}$	$\square\square$	-4	2
Y^i	$\mathbf{1}$	$\square$	$2N_Q-2$	-5
w_α	Adjoint	$\mathbf{1}$	0	1

Table 3.2: The matter contents of the magnetic theory.

Only for $N_Q = 7$, one has the additional term

$$\frac{\tilde{h}''}{\tilde{\mu}^{15}}\epsilon_{i_1\cdots i_7}\epsilon_{j_1\cdots j_7}M^{i_1j_1}\cdots M^{i_6j_6}Y^{i_7}Y^{j_7} \tag{3.3}$$

in the above superpotential $W_{\rm mag}$. As discussed in [3, 4], we need this extra term to obtain the superpotential for $N_Q = 6$ by giving mass to a vector, although its origin is not identified clearly.

This magnetic theory is asymptotically free for $N_Q > 7$. For $N_Q = 7$, since the term $\det s$ in the superpotential becomes a mass term of the symmetric tensor s, it decouples in the infrared, where the gauge coupling becomes asymptotically free. Since the electric theory is asymptotically free for $N_Q \leq 21$, the region $7 \leq N_Q \leq 21$ is believed to be a conformal window, where a non-trivial infrared fixed point exists.

This theory has the same quantum global symmetries $SU(N_Q)\times U(1)_F\times U(1)_\lambda$ as the electric theory. The charges of the elementary fields in the magnetic theory under the quantum global symmetry $U(1)\times U(1)_\lambda$ is determined by the superpotential $W_{\rm mag}$. Under the global symmetries, the charges of the matters are summarized in Table 3.2, with the gaugino w_α. The basis of the global $U(1)$ symmetries is chosen to be the same as in the electric theory. The 't Hooft anomaly matching conditions are satisfied by the elementary fields in the magnetic theory, which is one of the strongest evidences of the duality [3, 4].

The classical chiral ring of the magnetic theory is generated by the elementary singlets M^{ij} and Y^i along with the composite operators

$$\begin{aligned}
&(*B)_{i_1\cdots i_{N_Q-5}} \sim \varepsilon^{a_1\cdots a_{N_Q-5}}\bar{q}_{a_1i_1}\cdots\bar{q}_{a_{N_Q-5}i_{N_Q-5}},\\
&(*D_1)_{\alpha i_1\cdots i_{N_Q-8}} \sim \varepsilon_{a_1\cdots a_{N_Q-5}}(s\bar{q}_{i_1})^{a_1}\cdots(s\bar{q}_{i_{N_Q-8}})^{a_{N_Q-8}}(sw_\alpha)^{a_{N_Q-7}a_{N_Q-6}}q^{a_{N_Q-5}},\\
&(*D_2)_{i_1\cdots i_{N_Q-10}} \sim \varepsilon_{a_1\cdots a_{N_Q-5}}(s\bar{q}_{i_1})^{a_1}\cdots(s\bar{q}_{i_{N_Q-10}})^{a_{N_Q-10}}\\
&\qquad\qquad\qquad\times(sw_\alpha)^{a_{N_Q-9}a_{N_Q-8}}(sw^\alpha)^{a_{N_Q-7}a_{N_Q-6}}q^{a_{N_Q-5}},\\
&(*E)_{i_1\cdots i_{N_Q-9}} \sim \varepsilon_{a_1\cdots a_{N_Q-5}}(s\bar{q}_{i_1})^{a_1}\cdots(s\bar{q}_{i_{N_Q-9}})^{a_{N_Q-9}}\\
&\qquad\qquad\qquad\times(sw_\alpha)^{a_{N_Q-8}a_{N_Q-7}}(sw_\alpha)^{a_{N_Q-6}a_{N_Q-5}},\\
&S = {\rm Tr}w^\alpha w_\alpha,
\end{aligned} \tag{3.4}$$

Gauge Invariant Operators $\mathcal{O}$	$U(1)_F$	$U(1)_\lambda$
$M \sim Q^2$	-4	2
$Y \sim Q\Psi^2$	$2N_Q - 2$	-5
$B \sim Q^5\Psi^2 \sim \bar{q}^{N_Q-5}$	$2N_Q - 10$	-1
$E \sim Q^9\Psi^2 \sim (s\bar{q})^{N_Q-9}(sw)^2$	$2N_Q - 18$	3
$D_n \sim Q^{6+2n}W^{2-n} \sim (s\bar{q})^{N_Q-6-2n}(sw)^n q$	$-4n - 12$	$n + 8$

Table 3.3: The charges of the gauge invariant operators with respect to the $U(1) \times U(1)_\lambda$ symmetry.

where the operation $*$ on the gauge invariant operators denotes the Hodge duality with respect to the flavor $SU(N_Q)$ symmetry. The other conceivable gauge invariant chiral superfields such as $N_{ij} = \bar{q}_i\, s\, \bar{q}_j$, $\det s$, $q\bar{q}_i$ are redundant, due to the F-term condition from the superpotential W_{mag}.

The mapping of the gauge invariant operators between the electric theory and the magnetic theory is shown in Table 3.3. The same symbols are used for the corresponding operators in (3.1) and (3.4). We can check that the corresponding operators have the same quantum numbers by using Table 3.1 and Table 3.2. It is interesting to note that the classical moduli parameters D_2 and E in the electric theory are given by the gauge invariant operators containing the dual gaugino superfield w_α.

However, the electric gauge invariant operator D_0 in (3.1) does not have its counterpart in the classical chiral ring of the magnetic theory. This discrepancy seems to be a serious problem. Indeed, the "quantum" chiral ring of both the theories must be identical as long as they are dual. Since at present we do not know the precise description of the quantum chiral rings, we cannot decide whether the discrepancy actually exists quantum-mechanically. Therefore, there are no convincing reasons to believe that all the other non-trivial checks discussed in [3, 4] are only accidental. Thus, in this paper, we assume that the classical chiral rings are deformed by the quantum effects to be identical quantum-mechanically. However, it is still unclear whether D_0 is in the quantum chiral ring or not [1]. For our analysis, this discrepancy causes a problem which prevents us from obtaining the complete trial a-function defined globally, as will be discussed in the next chapter. However, it will turns out that the local maximum of the trial a-function, which will be found in the next chapter, is not affected by this issue.

[1] In [3], it was discussed that the gauge invariant operator D_0 in the electric theory correspond to the operator $(s\bar{q})^{N_Q-6}q$ in the magnetic theory. Indeed, they have the same charges of all the global symmetries. However, this operator can be rewritten as $(q\bar{q}) \cdot B \cdot M$ by using F term condition in the magnetic theory. Furthermore, the operator $q\bar{q}$ is redundant from the F-term condition. Thus, the candidate D_0 vanishes in the classical chiral ring of the magnetic theory. If the correspondence stated in [3] is still correct, the classical chiral ring must be modified quantum-mechanically, *i.e.*, D_0 in the electric theory vanishes or $(s\bar{q})^{N_Q-6}q$ in the magnetic theory appears as a non-trivial generator in the quantum chiral ring.

3.1.1 Turning on a Superpotential

Let us turn to the electric theory with the superpotential

$$W_{\text{ele}} = \frac{1}{\mu^2} N_{ij} Q^i Q^j.$$

by introducing the extra singlet N_{ij} into the theory. In the next chapter, we will see that the previous electric theory flows into this theory in the infrared for $7 \leq N_Q \leq 9$.

From the F-term condition

$$\frac{\partial}{\partial N_{ij}} W_{\text{ele}} = \frac{1}{\mu^2} Q^i Q^j = 0,$$

one can see that the moduli parameter $M^{ij} = Q^i Q^j$ are eliminated, and instead that the new moduli parameter N_{ij} shows up.

In order to obtain the dual description of this electric theory, one needs to get rid of the gauge singlet operator M^{ij} in the previous magnetic theory, and then one obtains the superpotential W_{mag} without its first term $M^{ij} \bar{q}_i\, s\, \bar{q}_j$ due to the absence of M^{ij} in this case. The F-term condition from the modified magnetic superpotential does not impose any constraints on the gauge invariant operator $N_{ij} = \bar{q}_i\, s\, \bar{q}_j$, which was redundant in the original theory. Although the use of N_{ij} seems the abuse of the notation, the two on the both sides are in the same representation

$$N_{ij} : (\,\square\!\square\,, 4, 0)$$

of the global symmetries $SU(N_Q) \times U(1)_F \times U(1)_\lambda$ and can thus be identified.

We will see in the following chapter that the field N_{ij} plays an important role, when the gauge invariant operator M^{ij} hits the unitarity bound in the original $Spin(10)$ theory. Note that all the gauge invariant operators in the previous dual pair are retained except for M^{ij} in this dual pair.

3.2 The Theory with Two Spinors

In this section, we will add one more spinor to the electric theory in the previous section. More precisely, we will discuss four-dimensional $\mathcal{N} = 1$ supersymmetric $Spin(10)$ gauge theory with two chiral superfields Ψ_I ($I = 1, 2$) in the spinor representation, and N_Q chiral superfields Q^i ($i = 1, \cdots, N_Q$) in the vector representation. We will also turn on no superpotentials. The remarkable difference from the theory with the single spinor is that its dual magnetic theory has two gauge groups, as will be explained in detail later. This theory is believed to be in the non-Abelian Coulomb phase for $6 \leq N_Q \leq 19$, where the electric-magnetic duality is available [7]. The quantum global symmetries are $SU(N_Q) \times SU(2) \times U(1)_F \times U(1)_\lambda$. Under the $U(1)_F$ transformation, the gaugino is not

	$Spin(10)$	$SU(N_Q)$ $i, j, \cdots$	$SU(2)$ $I, J, \cdots$	$U(1)_F$	$U(1)_\lambda$
Q^i	$\mathbf{10}$	$\square$	$\mathbf{1}$	-4	1
Ψ^I	$\mathbf{16}$	$\mathbf{1}$	$\mathbf{2}$	N_Q	-1

Table 3.4: The matter contents of the electric theory.

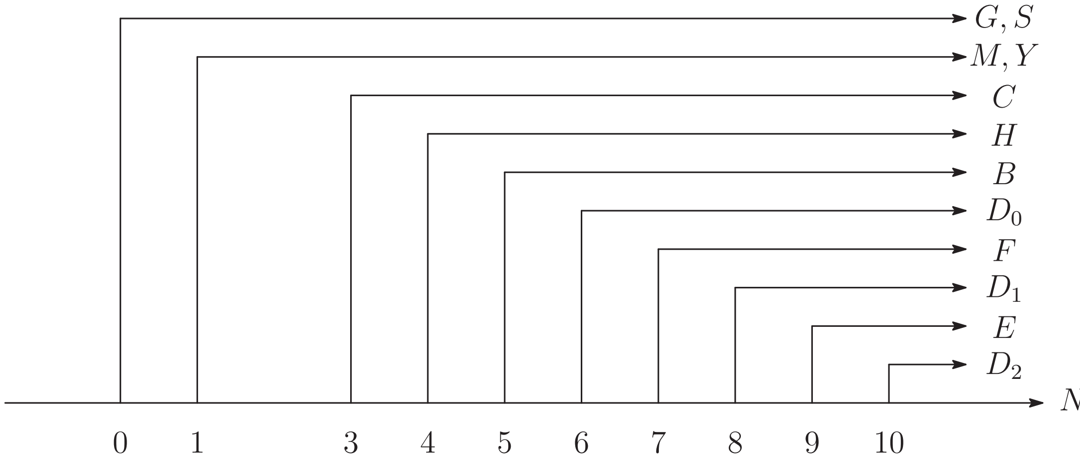

Figure 3.2: The number N_Q of the vectors Q^i where the gauge invariant operators exist.

rotated, while it has charge one under the $U(1)_\lambda$ transformation. The charges of the matters are listed in Table 3.4.

The gauge invariant generators of the classical chiral ring of this theory are given by

$$\begin{aligned}
&M^{ij} = Q^{ai}Q^{aj},\\
&Y_X^i = \Psi_I^T C(\sigma_2\sigma_X)^{IJ}\Gamma^a\Psi_J Q^{ai},\\
&C^{i_1\cdots i_3} = \Psi_I^T C(\sigma_2)^{IJ}\Gamma^{a_1\cdots a_3}\Psi_J Q^{a_1 i_1}\cdots Q^{a_3 i_3},\\
&B_X^{i_1\cdots i_5} = \Psi_I^T C(\sigma_2\sigma_X)^{IJ}\Gamma^{a_1\cdots a_5}\Psi_J Q^{a_1 i_1}\cdots Q^{a_5 i_5},\\
&F^{i_1\cdots i_7} = \Psi_I^T C(\sigma_2)^{IJ}\Gamma^{a_1\cdots a_7}\Psi_J Q^{a_1 i_1}\cdots Q^{a_7 i_7},\\
&E_X^{i_1\cdots i_9} = \Psi_I^T C(\sigma_2\sigma_X)^{IJ}\Gamma^{a_1\cdots a_9}\Psi_J Q^{a_1 i_1}\cdots Q^{a_9 i_9},\\
&G = \Psi_I^T C(\sigma_2\sigma_X)^{IJ}\Gamma^a\Psi_J\Psi_K^T C(\sigma_2\sigma_X)^{KL}\Gamma^a\Psi_L,\\
&H^{i_1\cdots i_4} = \Psi_I^T C(\sigma_2\sigma_X)^{IJ}\Gamma^{a_1\cdots a_5}\Psi_J\Psi_K^T C(\sigma_2\sigma_X)^{KL}\Gamma^{a_1}\Psi_L Q^{a_2 i_2}\cdots Q^{a_5 i_5},\\
&{D_0}^{i_1\cdots i_6} = \varepsilon^{a_1\cdots a_{10}}Q^{a_1 i_1}\cdots Q^{a_6 i_6}{W_\alpha}^{a_7 a_8}W^{\alpha a_9 a_{10}},\\
&{D_1}_\alpha^{i_1\cdots i_8} = \varepsilon^{a_1\cdots a_{10}}Q^{a_1 i_1}\cdots Q^{a_8 i_8}{W_\alpha}^{a_9 a_{10}},\\
&{D_2}^{i_1\cdots i_{10}} = \varepsilon^{a_1\cdots a_{10}}Q^{a_1 i_1}\cdots Q^{a_{10} i_{10}},\\
&S = \operatorname{Tr} W^\alpha W_\alpha,
\end{aligned}\tag{3.5}$$

where the gauge indices a and $a_1, a_2, \cdots$ and the charge conjugation matrix C are the same as in the one spinor case. The matrices σ_X $(X = 1, 2, 3)$ are the Pauli matrices for the flavor group of the spinors. Figure 3.2 displays what gauge invariant operators are available at each value of N_Q.

	$SU(N_Q-3)$ $a, b, \cdots$	$Sp(1)$ $\alpha, \beta, \cdots$	$SU(N_Q)$ $i, j, \cdots$	$SU(2)$ $I, J, \cdots$	$U(1)_F$	$U(1)_\lambda$
$\bar{q}_{ai}$	$\bar{\square}$	$\mathbf{1}$	$\bar{\square}$	$\mathbf{1}$	$2\frac{N_Q-6}{N_Q-3}$	$\frac{1}{N_Q-3}$
$\bar{q}'{}_a{}^{\alpha I}$	$\bar{\square}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$-\frac{2N_Q}{N_Q-3}$	$\frac{N_Q-2}{N_Q-3}$
q_X^a	$\square$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	$-2N_Q\frac{N_Q-4}{N_Q-3}$	$\frac{3N_Q-10}{N_Q-3}$
s^{ab}	$\square\square$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{4N_Q}{N_Q-3}$	$-\frac{2}{N_Q-3}$
$t^{\alpha I}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$2N_Q$	-2
M^{ij}	$\mathbf{1}$	$\mathbf{1}$	$\square\square$	$\mathbf{1}$	-8	2
Y_X^i	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\mathbf{3}$	$2N_Q-4$	-1

Table 3.5: The matter contents of the magnetic theory.

The magnetic theory is given by an $SU(N_Q-3)\times Sp(1)$ gauge theory with matter fields given by Table 3.5, and Its superpotential is given [7] by

$$W_{\text{mag}} = M^{ij}\bar{q}_{ai}s^{ab}\bar{q}_{bj} + Y_X^i\bar{q}_{ai}q_X^a + \varepsilon_{\alpha\beta}\varepsilon_{IJ}\bar{q}'{}_a{}^{\alpha I}s^{ab}\bar{q}'{}_b{}^{\beta J} + \varepsilon_{\alpha\beta}(\sigma_X\sigma_2)_{IJ}\bar{q}'{}_a{}^{\alpha I}q_X^a t^{\beta J}. \tag{3.6}$$

One can check that it has the same quantum global symmetries as the electric theory. The one-loop beta functions show that the $SU(N_Q-3)$ gauge coupling constant is asymptotically free for $N_Q \geq 7$, while the $Sp(1)$ gauge coupling constant is asymptotically free for $N_Q \leq 7$. Thus, either of them is asymptotically free for arbitrary N_Q. Therefore, the dual pair does not have the free magnetic phase.

The magnetic theory has the counterpart of all the gauge invariant operators of the electric theory. They are the fundamental singlets M^{ij} and Y_X^i and the composites

$$\begin{aligned}
&(*C)_{i_1\cdots i_{N_Q-3}} \sim \varepsilon^{a_1\cdots a_{N_Q-3}}\bar{q}_{a_1i_1}\cdots\bar{q}_{a_{N_Q-3}i_{N_Q-3}},\\
&(*B)_{Xi_1\cdots i_{N_Q-5}} \sim \varepsilon^{a_1\cdots a_{N_Q-3}}\varepsilon_{\alpha\beta}\bar{q}_{a_1i_1}\cdots\bar{q}_{a_{N_Q-5}i_{N_Q-5}}\bar{q}'{}^{\alpha I}_{a_{N_Q-4}}(\sigma_2\sigma_X)_{IJ}\bar{q}'{}^{\beta J}_{a_{N_Q-3}},\\
&(*F)_{i_1\cdots i_{N_Q-7}} \sim \varepsilon^{a_1\cdots a_{N_Q-3}}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\bar{q}_{a_1i_1}\cdots\bar{q}_{a_{N_Q-7}i_{N_Q-7}}\\
&\qquad\qquad \times\bar{q}'{}^{\alpha I}_{a_{N_Q-6}}(\sigma_2\sigma_X)_{IJ}\bar{q}'{}^{\beta J}_{a_{N_Q-5}}\bar{q}'{}^{\gamma K}_{a_{N_Q-4}}(\sigma_2\sigma_X)_{KL}\bar{q}'{}^{\delta L}_{a_{N_Q-3}},\\
&(*E)_{Xi_1\cdots i_{N_Q-9}} \sim \varepsilon_{a_1\cdots a_{N_Q-3}}\varepsilon_{XYZ}(s\bar{q}_{i_1})^{a_1}\cdots(s\bar{q}_{i_{N_Q-9}})^{a_{N_Q-9}}\\
&\qquad\qquad \times(sw_\alpha)^{a_{N_Q-8}a_{N_Q-7}}(sw_\alpha)^{a_{N_Q-6}a_{N_Q-5}}q_Y^{N_Q-4}q_Z^{N_Q-3},\\
&G \sim \varepsilon_{\alpha\beta}t^{\alpha I}(\sigma_2)_{IJ}t^{\beta J},\\
&(*H)_{i_1\cdots i_{N_Q-4}} \sim \varepsilon_{IJ}\varepsilon^{a_1\cdots a_{N_Q-3}}\varepsilon_{\alpha\beta}\bar{q}_{a_1i_1}\cdots\bar{q}_{a_{N_Q-4}i_{N_Q-4}}\bar{q}'{}^{\alpha I}_{a_{N_Q-3}}t^{\beta J},\\
&(*D_0)_{i_1\cdots i_{N_Q-6}} \sim \varepsilon_{XYZ}\varepsilon_{a_1\cdots a_{N_Q-3}}(s\bar{q}_{i_1})^{a_1}\cdots(s\bar{q}_{i_{N_Q-6}})^{a_{N_Q-6}}q_X^{a_{N_Q-5}}q_Y^{a_{N_Q-4}}q_Z^{a_{N_Q-3}},\\
&(*D_1)_{\alpha i_1\cdots i_{N_Q-8}} \sim \varepsilon_{a_1\cdots a_{N_Q-3}}\varepsilon_{XYZ}(s\bar{q}_{i_1})^{a_1}\cdots(s\bar{q}_{i_{N_Q-8}})^{a_{N_Q-8}}\\
&\qquad\qquad \times(sw^\alpha)^{a_{N_Q-7}a_{N_Q-6}}q_X^{a_{N_Q-5}}q_Y^{a_{N_Q-4}}q_Z^{a_{N_Q-3}},\\
&(*D_2)_{i_1\cdots i_{N_Q-10}} \sim \varepsilon_{a_1\cdots a_{N_Q-3}}\varepsilon_{XYZ}(s\bar{q}_{i_1})^{a_1}\cdots(s\bar{q}_{i_{N_Q-10}})^{a_{N_Q-10}}\\
&\qquad\qquad \times(sw_\alpha)^{a_{N_Q-9}a_{N_Q-8}}(sw^\alpha)^{a_{N_Q-7}a_{N_Q-6}}q_X^{a_{N_Q-5}}q_Y^{a_{N_Q-4}}q_Z^{a_{N_Q-3}},\\
&S \sim \operatorname{Tr} w_\alpha w^\alpha, \quad S' \sim \operatorname{Tr}\tilde{w}_\alpha\tilde{w}^\alpha,
\end{aligned} \tag{3.7}$$

Gauge Invariant Operators $\mathcal{O}$	$U(1)_F$	$U(1)_\lambda$
$M{\sim}Q^2$	-8	2
$Y{\sim}Q\Psi^2$	$2N_Q-4$	-1
$C{\sim}\,Q^3\Psi^2{\sim}\,\bar{q}^{N_Q-3}$	$2N_Q-12$	1
$B{\sim}\,Q^5\Psi^2{\sim}\,\bar{q}^{N_Q-5}\bar{q}'^2$	$2N_Q-20$	3
$F{\sim}\,Q^7\Psi^2{\sim}\,\bar{q}^{N_Q-7}\bar{q}'^4$	$2N_Q-28$	5
$E{\sim}Q^9\Psi^2{\sim}\,(s\bar{q})^{N_Q-9}(sw)^2q^2$	$2N_Q-36$	7
$G{\sim}\,\Psi^4{\sim}\,t^2$	$4N_Q$	-4
$H{\sim}\,Q^4\Psi^2{\sim}\,\bar{q}^{N_Q-4}\bar{q}'t$	$4N_Q-16$	0
$D_n{\sim}Q^{6+2n}W^{2-n}{\sim}\,(s\bar{q})^{N_Q-6-2n}(sw)^nq^3$	$-8n-24$	$n+8$

Table 3.6: The charges of the gauge invariant operators with respect to the $U(1)\times U(1)_\lambda$ symmetry.

where w_α and $\tilde{w}_\alpha$ are the gaugino superfields of the $SU(N_Q-3)$ and $Sp(1)$ gauge interactions, respectively,[2] and the operation $*$ is the Hodge duality for the flavor $SU(N_Q)$ indices. We can check that each of the operators has the same quantum numbers as that of the electric theory, as shown in Table 3.6.

However, there exist more gauge invariant generators in the magnetic theory than in the electric one, as we pointed out in [15]. They are given by

$$\begin{aligned}
&U_0 = \det s,\\
&U_{1XY} = \varepsilon_{a_1\cdots a_{N_Q-3}}\varepsilon_{b_1\cdots b_{N_Q-3}}s^{a_1b_1}\cdots s^{a_{N_Q-4}b_{N_Q-4}}q_X^{a_{N_Q-3}}q_Y^{b_{N_Q-3}},\\
&U_{2XY} = \varepsilon_{XX_1X_2}\varepsilon_{YY_1Y_2}\varepsilon_{a_1\cdots a_{N_Q-3}}\varepsilon_{b_1\cdots b_{N_Q-3}}\\
&\qquad\quad\times s^{a_1b_1}\cdots s^{a_{N_Q-5}b_{N_Q-5}}q_{X_1}^{a_{N_Q}-4}q_{X_2}^{a_{N_Q}-3}q_{Y_1}^{b_{N_Q}-4}q_{Y_2}^{b_{N_Q}-3},\\
&U_3 = \varepsilon_{X_1X_2X_3}\varepsilon_{Y_1Y_2Y_3}\varepsilon_{a_1\cdots a_{N_Q-3}}\varepsilon_{b_1\cdots b_{N_Q-3}}\\
&\qquad\quad\times s^{a_1b_1}\cdots s^{a_{N_Q-6}b_{N_Q-6}}q_{X_1}^{a_{N_Q}-5}q_{X_2}^{a_{N_Q}-4}q_{X_3}^{a_{N_Q}-3}q_{Y_1}^{b_{N_Q}-5}q_{Y_2}^{b_{N_Q}-4}q_{Y_3}^{b_{N_Q}-3},\\
&(*E_0)_{Xi_1\cdots i_{N_Q-5}} = \varepsilon_{XYZ}\varepsilon_{a_1\cdots a_{N_Q-3}}(s\bar{q}_{i_1})^{a_1}\cdots(s\bar{q}_{i_{N_Q-5}})^{a_{N_Q-5}}q_Y^{a_{N_Q-4}}q_Z^{a_{N_Q-3}},\\
&(*E_1)_{\alpha Xi_1\cdots i_{N_Q-7}} = \varepsilon_{a_1\cdots a_{N_Q-3}}\varepsilon_{XYZ}(s\bar{q}_{i_1})^{a_1}\cdots(s\bar{q}_{i_{N_Q-7}})^{a_{N_Q-7}}\\
&\qquad\qquad\qquad\qquad\times(sw^\alpha)^{a_{N_Q-6}a_{N_Q-5}}q_Y^{a_{N_Q-4}}q_Z^{a_{N_Q-3}},\\
&(*I_0)_{Xi_1\cdots i_{N_Q-4}} = \varepsilon_{a_1\cdots a_{N_Q-3}}(s\bar{q}_{i_1})^{a_1}\cdots(s\bar{q}_{i_{N_Q-4}})^{a_{N_Q-4}}q_X^{a_{N_Q-3}},\\
&(*I_1)_{\alpha Xi_1\cdots i_{N_Q-6}} = \varepsilon_{a_1\cdots a_{N_Q-3}}(s\bar{q}_{i_1})^{a_1}\cdots(s\bar{q}_{i_{N_Q-6}})^{a_{N_Q-6}}(sw^\alpha)^{a_{N_Q-5}a_{N_Q-4}}q_X^{a_{N_Q-3}},\\
&(*I_2)_{Xi_1\cdots i_{N_Q-8}} = \varepsilon_{a_1\cdots a_{N_Q-3}}(s\bar{q}_{i_1})^{a_1}\cdots(s\bar{q}_{i_{N_Q-8}})^{a_{N_Q-8}}\\
&\qquad\qquad\qquad\qquad\times(sw_\alpha)^{a_{N_Q-7}a_{N_Q-6}}(sw^\alpha)^{a_{N_Q-5}a_{N_Q-4}}q_X^{a_{N_Q-3}},\\
&(*J_1)_{\alpha i_1\cdots i_{N_Q-5}} = \varepsilon_{a_1\cdots a_{N_Q-3}}(s\bar{q}_{i_1})^{a_1}\cdots(s\bar{q}_{i_{N_Q-5}})^{a_{N_Q-5}}(sw^\alpha)^{a_{N_Q-4}a_{N_Q-3}},\\
&(*J_2)_{i_1\cdots i_{N_Q-7}} = \varepsilon_{a_1\cdots a_{N_Q-3}}(s\bar{q}_{i_1})^{a_1}\cdots(s\bar{q}_{i_{N_Q-7}})^{a_{N_Q-7}}\\
&\qquad\qquad\qquad\qquad\times(sw_\alpha)^{a_{N_Q-6}a_{N_Q-5}}(sw^\alpha)^{a_{N_Q-4}a_{N_Q-3}}.
\end{aligned}\tag{3.8}$$

[2]The index α on the gaugino superfields w_α and $\tilde{w}_\alpha$ is that of Lorentz spinors, which would not cause any confusion.

Up to this moment, we have not succeeded to prove after some algebra that these operators are included in the classical chiral ring of the electric theory. We even do not know that it should be. Similarly to the one spinor case in the previous section, we will assume that the classical chiral ring is deformed by the quantum effects and that the quantum chiral rings of both the theories are identical to each other.

We can check that the 't Hooft anomaly matching conditions are satisfied by the magnetic theory.

Chapter 4

The $Spin(10)$ Theories via a-Maximization

In this chapter, we will carry out the method of a-maximization to determine the superconformal $U(1)_R$ charges of all the gauge invariant chiral primary operators in the $Spin(10)$ gauge theories at the infrared fixed point.

In the models, at different trial superconformal $U(1)_R$ charge assignments, different gauge invariant chiral primary operators hit the unitarity bounds. Therefore, one needs to follow the prescription (2.64) to construct the trial a-function globally over all the $U(1)_R$ charge assignments. This will be done for the one spinor case in section 4.1 and for the the two spinor case in section 4.2. There will turn out to exist a local maximum of the trial a-function for each flavor number N_Q in the two case. The local maximum will be confirmed to be the same as in the magnetic description for all the cases.

In particular, among all the cases, the cases where gauge invariant chiral primary operators indeed hit the unitarity bounds, are interesting. In fact, it will turns out that they are elementary fields in the magnetic theory, and one does not need the prescription (2.64) to find the identical local maximum to the one in the electric theory. Therefore, the magnetic description yields another support for the proposal (2.64).

For the interesting cases, furthermore in the next chapter, we will find that the electric theory with no superpotential is identical to the one with a superpotential at the infrared fixed point. The dual pair of the former is thus identical to that of the latter in the infrared. The auxiliary field method in the electric theory offers a satisfying description of the renormalization flow of the dual pairs, which is consistent with the picture in the magnetic theory. In particular, the auxiliary field method gives a clear description of the emergence of new massless degrees of freedom in the electric theory.

Although these results are not affected, there are however a few subtleties, due to the mismatch of the classical chiral rings between the dual pair, and due to the lack of our knowledge about the a-maximization procedure applied to gauge an invariant operator in a non-trivial representation of the Lorentz group, as will be discussed later. Therefore, we haven't confirmed that the local maximum was also the unique local maximum of the global trial a-function.

4.1 The One Spinor Case

In this section, we will use a-maximization to identify the superconformal $U(1)_R$ symmetry of the $Spin(10)$ theory with a single spinor and N_Q vectors for $7 \leq N_Q \leq 21$ at the non-trivial infrared fixed point. We will find that there is a local maximum of the global trial a-function, which is consistent with the conjectured presence of the non-trivial infrared fixed point for $7 \leq N_Q \leq 21$. Furthermore, as we will see below, at the local maximum, the meson M^{ij} hits the unitarity bound for $N_Q = 7, 8, 9$, while no gauge invariant primary operators hit the unitarity bounds for $10 \leq N_Q \leq 21$.

In order to construct the trial a-function in this model, assuming no accidental $U(1)$ symmetry [1] enhanced in the infrared, one can see that a trial superconformal $U(1)_R$ symmetry is given by a linear combination of $U(1)_F$ and $U(1)_\lambda$ in Table 3.1 as

$$U(1)_R = xU(1)_F + U(1)_\lambda \tag{4.1}$$

with a real number x. Thus, the $U(1)_R$ charges of the matter fields can be expressed as

$$R(Q) = -2x + 1, \qquad R(\Psi) = N_Q x - 3. \tag{4.2}$$

We will determine the value of x by using a-maximization to identify the superconformal $U(1)_R$ symmetry at the infrared fixed point. For convenience, we will use a parameter $R \equiv R(Q)$ instead of x throughout this section.

At a particular value of x, if there are no gauge invariant operators hitting the unitarity bounds, we can give the trial a-function in terms of the elementary fields as

$$a_0(R) = 90 + 16F\left[R\left(\Psi\right)\right] + 10N_Q F\left[R\left(Q\right)\right], \tag{4.3}$$

where the function $F(x)$ was defined by $F(x) = 3(x-1)^3 - (x-1)$. The first term on the right hand side of 4.3 comes from the contribution of the gaugino, which are forty-five Weyl spinors of charge one with respect to the $U(1)_R$ symmetry, thus giving $45 \times [3R(\lambda)^3 - R(\lambda)] = 90$.

When some of the gauge invariant chiral primary operators hit the unitarity bounds, they decouple from the remaining system and become free fields of the $U(1)_R$ charge $2/3$. Therefore, following the prescription (2.64) explained in section 2.3, one needs to improve the trial a-function $a_0(R)$ as

$$a(R) = 90 + 16F\left[R\left(\Psi\right)\right] + 10N_Q F\left[R\left(Q\right)\right] - \sum_i \left[F[R(\mathcal{O}_i)] - F_0\right], \tag{4.4}$$

[1] More precisely, we assume no accidental $U(1)$ symmetry enhancement which does not accompany any gauge invariant operators hitting the unitarity bounds.

where $\mathcal{O}_i$ are the gauge invariant operators hitting the unitarity bounds. However, at the values of x with the same set of the operators hitting the unitarity bounds, one can use the same trial a-function (4.4), and, as illustrated for $N_Q = 7$ in Table 3.3 [2], one only have to divide all real values of R into several regions, where one can use one local trial a-function (4.4). Indeed, the unitarity bound of each gauge invariant chiral primary operator yields the condition on R as

$$\begin{aligned}
&R(M) = 2R \geq \tfrac{2}{3} && \Rightarrow && R \geq \tfrac{1}{3}, \\
&R(Y) = N_Q - 6 - (N_Q - 1)R \geq \tfrac{2}{3} && \Rightarrow && R \leq \tfrac{1}{N_Q - 1}\left(N_Q - \tfrac{20}{3}\right), \\
&R(B) = N_Q - 6 - (N_Q - 5)R \geq \tfrac{2}{3} && \Rightarrow && R \leq \tfrac{1}{N_Q - 5}\left(N_Q - \tfrac{20}{3}\right), \\
&R(E) = (N_Q - 6) - (N_Q - 9)R \geq \tfrac{2}{3} && \Rightarrow && R \leq \tfrac{1}{N_Q - 9}\left(N_Q - \tfrac{20}{3}\right), \\
&R(D_0) = 6R + 2 \geq \tfrac{2}{3} && \Rightarrow && R \geq -\tfrac{2}{9}, \\
&R(D_1) = 8R + 1 \geq 1 && \Rightarrow && R \geq 0, \\
&R(D_2) = 10R \geq \tfrac{2}{3} && \Rightarrow && R \geq \tfrac{1}{15},
\end{aligned} \tag{4.5}$$

and, combining these conditions, one may divide all real values of R into several region where one local trial a-function (4.4) can be defined, as sketched for $N_Q = 7$ in Figure 4.1. Combining all the local trial a-functions, one thus obtain the global trial a-function defined over all real values of x or equivalently R.

There is a subtle point, as mentioned above, about what the chiral primary operators are quantum-mechanically at the non-trivial IR fixed point. The gauge invariant chiral superfields M, Y, B, D_2, and E parametrize the classical moduli space of the electric theory. If we assume that the quantum moduli space is the same as the classical one, which is believed to be the case for the conformal window of SQCD, these operators should be chiral primary operators. However, it is not clear whether D_0 is chiral primary or not, as discussed previously. In the region $R < -2/9$ where D_0 hits the unitarity bound, which local trial a-function (4.4) should be used depends on whether D_0 is chiral primary or not in the infrared. Therefore, we will try a-maximization for both cases to find a local maximum in the region $R < -2/9$. However, one can find there is no local maximum in the region for the both cases.

For $N_Q \geq 8$, the gauge invariant operator D_1 is available, as in Figure 3.1, and it is in the spinor representation of the Lorentz group. If the operator D_1 is a chiral primary operator, in the region where it hits the unitarity bound, we cannot construct a local trial a-function, because we at present do not know how to extend the a-maximization procedure to an operator like D_1 in a non-trivial representation of the Lorentz group. Therefore, assuming that the operator D_1 is not a chiral primary operator in the infrared, we will proceed to construct a global trial a-function, and we will see below that the solution to

[2] The unitarity bound for a spin one-half field is given [8] by $D \geq \frac{3}{2}$, which gives the bound for $U(1)_R$ charge; $R \geq 1$.

	Hitting Operators	Hitting Regions
I	M, (D_0)	$R \leq -\frac{2}{9}$
II	M	$-\frac{2}{9} \leq R \leq \frac{1}{18}$
III	M, Y	$\frac{1}{18} \leq R \leq \frac{1}{6}$
IV	M, Y, B	$\frac{1}{6} \leq R \leq \frac{1}{3}$
V	Y, B	$R \geq \frac{1}{3}$

Table 4.1: The four regions of the $U(1)_R$ charge R for $N_Q = 7$

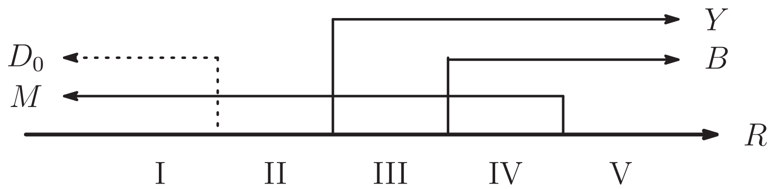

Figure 4.1: A sketch of operators hitting the unitarity bounds for the theory with 7 vectors. Each of the regions from I to V are separated at $R(Q) = -2/9$, $1/18$, $1/6$, $1/3$, respectively. The arrows show the regions where the corresponding operators hit the unitarity bounds.

the a-maximization condition (2.62) is found in the other region, where D_1 does not hit the bound. Therefore, the local maximum remains valid, even when the operator D_1 is indeed chiral primary in the infrared. However, we never exclude the possibility that there is another local maximum in the region where the operator D_1 hits the unitarity bound, if the operator D_1 is chiral primary in the infrared. If this is the case, it would be interesting to determine which the local maximum gives the superconformal $U(1)_R$ symmetry at the infrared fixed point. In addition, there is no such a problem for $N_Q = 7$.

We will demonstrate the a-maximization procedure for the case of $N_Q = 7$ vectors Q^i, and then will report our results on the other values of N_Q. Before proceeding, let us make a comment on the structure of the divided regions of R. When one looks at the operators hitting the unitarity bounds from a large negative value of R to a large positive value, the order of the operators hitting the unitarity bound could change, depending on the number N_Q. It turns out from the unitarity bounds (4.5) that for the cases $N_Q \geq 10$, the order of the operators hitting the bound is the same, This greatly facilitates our study for the cases $N_Q \geq 10$ and allows us to give our results in a uniform way. However, one needs to consider each case for $N_Q = 7, 8, 9$. One also finds that, for the cases $N_Q \geq 10$, there is a region where none of the gauge invariant operators hit the unitarity bounds, but no such regions for the cases $N_Q = 7, 8, 9$.

In the case of $N_Q = 7$ vectors, there are five regions dividing the parameter space of R, as can be seen from Table 4.1 and as illustrated in Figure 4.1. In each region, one finds the local trial a-function $a(R)$ (4.4) as

$$a(R) = \begin{cases} ??? & \left(I : R \leq -\frac{2}{9}\right), \\ a_0(R) + f_M(R) & \left(II : -\frac{2}{9} \leq R \leq \frac{1}{18}\right), \\ a_0(R) + f_M(R) + f_Y(R) & \left(III : \frac{1}{18} \leq R \leq \frac{1}{6}\right), \\ a_0(R) + f_M(R) + f_Y(R) + f_B(R) & \left(IV : \frac{1}{6} \leq R \leq \frac{1}{3}\right), \\ a_0(R) + f_Y(R) + f_B(R) & \left(V : R \geq \frac{1}{3}\right), \end{cases} \tag{4.6}$$

where $f_{\mathcal{O}}(R)$ are defined by

$$\begin{aligned} f_M(R) &= -\frac{N_Q(N_Q+1)}{2}\left[3\left(R(M)-1\right)^3 - \left(R(M)-1\right)\right] + \frac{N_Q(N_Q+1)}{2}\cdot\frac{2}{9}, \\ f_Y(R) &= -N_Q\left[3\left(R(Y)-1\right)^3 - \left(R(Y)-1\right)\right] + N_Q\cdot\frac{2}{9}, \\ f_B(R) &= -\frac{N_Q!}{(N_Q-5)!5!}\left[3\left(R(B)-1\right)^3 - \left(R(B)-1\right)\right] + \frac{N_Q!}{(N_Q-5)!5!}\cdot\frac{2}{9}, \end{aligned} \tag{4.7}$$

with $N_Q = 7$.

For $R \leq -2/9$, where D_0 hits the unitarity bound, as mentioned above, we will try to implement the a-maximization procedure for both of the cases whether D_0 is chiral primary or not with the two functions

$$a(R) = a_0(R) + f_M(R) + f_{D_0}(R), \qquad \text{if } D_0 \text{ is chiral primary}, \tag{4.8}$$

$$a(R) = a_0(R) + f_M(R), \qquad \text{if } D_0 \text{ is not chiral primary}, \tag{4.9}$$

where

$$f_{D_0}(R) = -\frac{N_Q!}{(N_Q-6)!6!}\left[3\left(R(D_0)-1\right)^3 - \left(R(D_0)-1\right)\right] + \frac{N_Q!}{(N_Q-6)!6!}\cdot\frac{2}{9}, \tag{4.10}$$

with $N_Q = 7$. However, one can easily see that both the functions in (4.9) have no local maximum in this range $R \leq -2/9$.

The global trial function $a(R)$ is illustrated in Figure 4.2. Although it is locally a polynomial of degree three in R for each of the five regions, it gives two local minima as a whole. As can be seen from Figure 4.2, there is a unique local maximum, where only the mesons M^{ij} are free and the $U(1)_R$ charge gives $R = 1/30$ in the region II. It is the local maximum

$$\begin{aligned} R(Q) &= \frac{3N_Q^2 - 21N_Q - 12 + 2\sqrt{-(N_Q-6)(N_Q^2-29N_Q+73)}}{3(N_Q+3)(N_Q-1)}, \\ R(\Psi) &= \frac{9N_Q^2 - 33N_Q + 54 + 2N_Q\sqrt{-(N_Q-6)(N_Q^2-29N_Q+73)}}{6(N_Q+3)(N_Q-1)} \end{aligned} \tag{4.11}$$

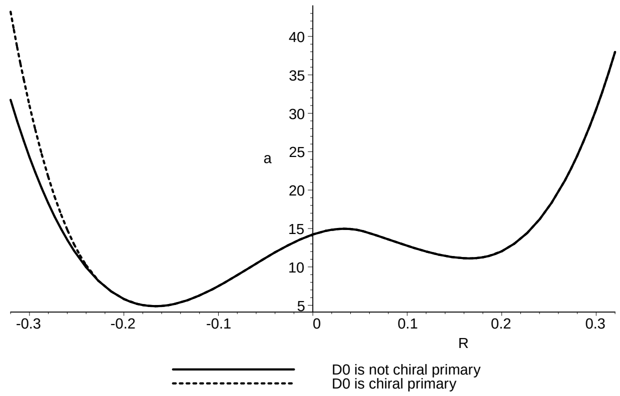

Figure 4.2: The global trial a-function $a(R)$ for $N_Q = 7$. Dotted line corresponds to the case where D_0 is chiral primary while solid line corresponds to the case where D_0 is not chiral primary.

of the function $a_0(R) + f_M(R)$ for $N_Q = 7$.

Strictly speaking, the $U(1)_R$ symmetry should be expressed as $U(1)_R = U(1)_\lambda + xU(1)_F + yU(1)_M$ instead of (4.1) at the local maximum because $U(1)_M$ symmetry which transforms only M^{ij} appears at the IR fixed point. Here, y is determined so that the $U(1)_R$ charge of M^{ij} becomes 2/3. However, the $U(1)_R$ charges of the other gauge invariant operators, except for that of the operator M^{ij}, can be expressed as the sum of those of the component fields given by (4.11), because they have no charges under the $U(1)_M$ symmetry.

For $N_Q = 8, 9$, as can be seen from Table 4.2, there are also five regions on the line of R, as in Figure 4.3. As is different from the case of $N_Q = 7$, there is no region where the three gauge invariant operators M, Y, and B hit the unitarity bounds at the same time, but a new region V, where only the operator Y^i hits the unitarity bound, appears. Only for $N_Q = 9$, the operator E is available, but it does not violate the unitarity bound over all the values of R. If the spinor exotics D_1 are chiral primary in the infrared, our results for the regions I and II would be incomplete. The global trial a-function $a(R)$ is similar to the one for $N_Q = 7$ and have, in the region III, a single local maximum given by (4.11) with $N_Q = 8, 9$ substituted for each case, where also only the meson M^{ij} is hitting the unitarity bound to be free in the infrared. This result does not depend on whether D_0 is chiral primary or not. The local maximum would be retained even after taking account of the exotics D_1.

For $10 \leq N_Q \leq 21$, it is remarkable that there exists a region V with no gauge invariant operators hitting the unitarity bounds, as shown in Figure 4.4. The parameter space of R is divided into seven regions, as can be seen from Table 4.3. The regions I and II could be incomplete due to the exotics D_1. The global trial a-function $a(R)$ has a profile similar to

	Hitting Operators	Hitting Regions
I	M, (D_0)	$R \leq -\frac{2}{9}$
II+III	M	$-\frac{2}{9} \leq R \leq \frac{1}{N_Q-1}(N_Q - \frac{20}{3})$
VI	M, Y	$\frac{1}{N_Q-1}(N_Q - \frac{20}{3}) \leq R \leq \frac{1}{3}$
V	Y	$\frac{1}{3} \leq R \leq \frac{1}{N_Q-5}(N_Q - \frac{20}{3})$
VI	Y, B	$R \geq \frac{1}{N_Q-5}(N_Q - \frac{20}{3})$

Table 4.2: The four regions of the $U(1)_R$ charge R for $N_Q = 8, 9$.

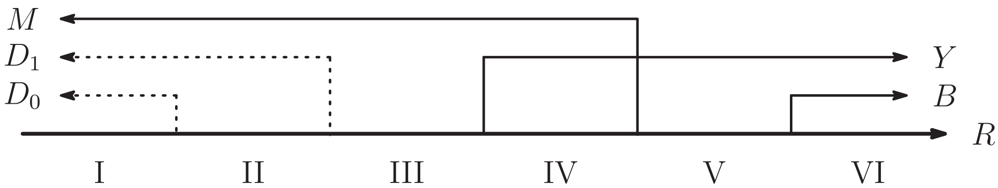

Figure 4.3: The operators hitting the unitarity bounds for the theory with $N_Q = 8$ and 9 vectors. The arrows show the regions where the corresponding operators hit the unitarity bounds.

the one in Figure 4.2. One finds the unique local maximum at

$$R(Q) = \frac{3N_Q^2 - 24N_Q - 15 + \sqrt{2885 - N_Q^2}}{3(N_Q^2 - 5)},$$
$$R(\Psi) = \frac{6N_Q^2 + 90 - N_Q\sqrt{2885 - N_Q^2}}{6(N_Q^2 - 5)} \tag{4.12}$$

in the region V, where no operator hits the unitarity bound. The local maximum also remains valid even after taking account of the unitarity bound of D_1.

So far, we have determined the superconformal $U(1)_R$ charges of the gauge invariant chiral primary operators in the electric theory. One might wonder whether the same results

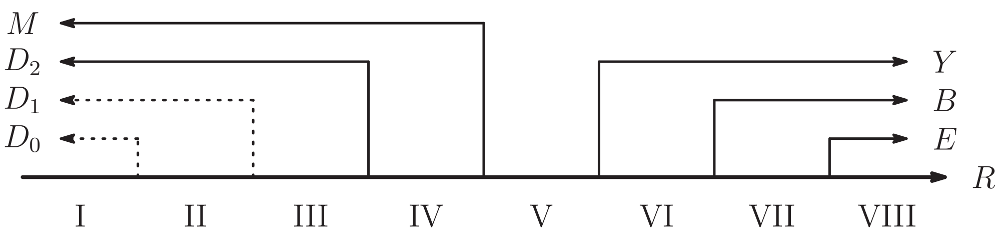

Figure 4.4: The operators hitting the unitarity bounds for the theory with $10 \leq N_Q \leq 21$ vectors. The arrows show the regions where the corresponding operators hit the unitarity bounds.

	Hitting Operators	Hitting Regions
I	M, D_2, (D_0)	$R \leq -\frac{2}{9}$
II+III	M, D_2	$-\frac{2}{9} \leq R \leq \frac{1}{15}$
IV	M	$\frac{1}{15} \leq R \leq \frac{1}{3}$
V	no operators	$\frac{1}{3} \leq R \leq \frac{1}{N_Q-1}(N_Q - \frac{20}{3})$
VI	Y	$\frac{1}{N_Q-1}(N_Q - \frac{20}{3}) \leq R \leq \frac{1}{N_Q-5}(N_Q - \frac{20}{3})$
VII	Y, B	$\frac{1}{N_Q-5}(N_Q - \frac{20}{3}) \leq R \leq \frac{1}{N_Q-9}(N_Q - \frac{20}{3})$
VIII	Y, B, E	$R \geq \frac{1}{N_Q-9}(N_Q - \frac{20}{3})$

Table 4.3: The six regions on the line of the $U(1)_R$ charge R for $10 \leq N_Q \leq 21$.

could be obtained in the magnetic theory. Actually, this is automatically guaranteed by the 't Hooft anomaly matching condition [34]. Since the magnetic theory saturates the anomalies of all the global symmetries of the electric theory [3, 4], the a-function in the magnetic theory is identical to the one in the electric theory, even when the gauge invariant operators hit the unitarity bounds as long as the hitting operators are the same. By using (4.2) and Table 3.2, the $U(1)_R$ charges of the elementary fields of the magnetic theory can also be determined from the $U(1)_R$ charges of Q or Ψ determined above.

The $U(1)_R$ charges of the elementary fields of the electric theory and those of the magnetic theory are plotted in Figure 4.5 and 4.6. They indicate that the $U(1)_R$ charge of each elementary field is close to $2/3$ in the electric theory for large N_Q and in the magnetic theory for small N_Q. One therefore may regard that the electric theory and the magnetic theory are weakly interacting for large and small N_Q, respectively, which is consistent with the conventional expectation.

In the one spinor case, we have found the unique local maximum of the a-function for $10 \leq N_Q \leq 21$, where there are no gauge invariant chiral primary operators hitting the unitarity bounds. On the other hand, for $7 \leq N_Q \leq 9$, one also found the unique local maximum of the a-function, but at the local maximum, one finds that the gauge invariant operators M^{ij} are free fields at the infrared fixed point. Note that the existence of the local maximum is consistent with the conjecture [3, 4] that this theory is in the non-Abelian Coulomb phase for $7 \leq N_Q \leq 21$.

4.2 The Two Spinor Case

In this section, we will briefly give our results about the a-maximization procedure in the $Spin(10)$ theory with two spinors and N_Q vectors for $6 \leq N_Q \leq 19$. Since our analysis of this case is quite similar to that in the previous section, we will not repeat a detailed

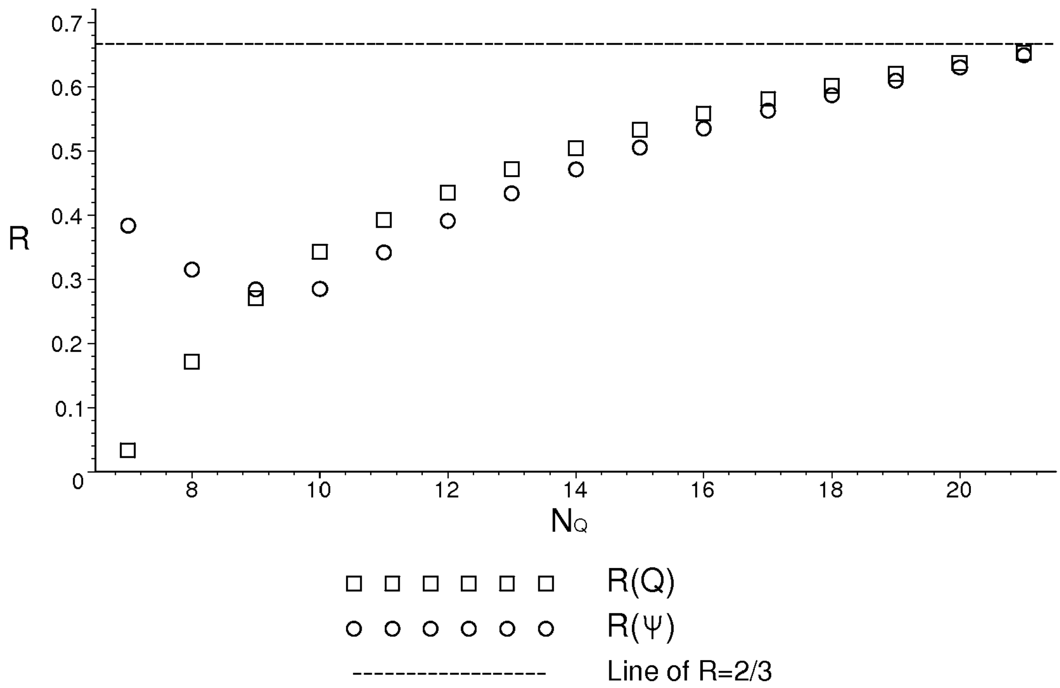

Figure 4.5: $U(1)_R$ charges of vectors Q and spinors Ψ in the electric theory.

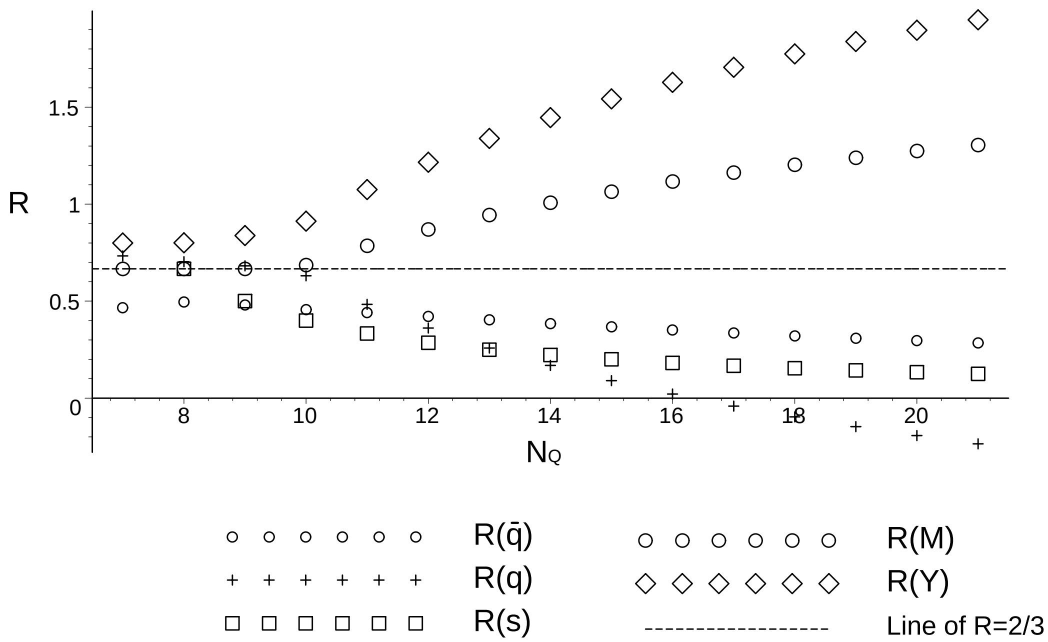

Figure 4.6: $U(1)_R$ charges of antifundamentals $\bar{q}$, a fundamental q, a symmetric tensor s, and singlets M and Y in the magnetic theory.

explanation about it.

However, before proceeding, let us make a comment on two points, which is distinct from the one spinor case.

First, the magnetic theory has two gauge groups. From the one-loop beta functions of the two gauge couplings, one can see that the $SU(N_Q-3)$ gauge coupling is asymptotically free for $N_Q \geq 7$, while the $Sp(1)$ coupling is asymptotically free for $N_Q \leq 7$, perturbatively. Except for $N_Q = 7$, since there is no flavor number N_Q where both of the gauge coupling constants are asymptotically free at the one-loop level, it might happen that either of the gauge interactions could be free at the infrared fixed point [3]. However, assuming below that both of the gauge interactions are not free at the infrared fixed point, a-maximization will be carried out in the magnetic theory.

Second, as was discussed in the previous chapter, the classical chiral ring of the electric theory is not identical to the one of the magnetic theory. Therefore, at some values of the trial $U(1)_R$ charges, the set of the gauge invariant operators hitting the unitarity bounds in the electric theory is different from the one in the magnetic theory, which prevents us from finding the unique and correct trial a-function. This problem is parallel to the problem concerning with the operator D_0 in the one spinor case. However, there are more extra operators as in (3.8) compared to the one spinor case, and, depending on whether each of them is chiral primary or not, there are many possibilities to consider, if we will use the same strategy as in the one spinor case. It is formidable for us to carry out the method of a-maximization for each of all the possibilities. Therefore, we will pick up two of them; the case that all the extra operators in (3.8) are not chiral primary - the classical chiral ring of the electric theory - and the other case that they are all chiral primary - the classical chiral ring of the magnetic theory. Thus, we will carry out the a-maximization procedure for the electric theory and the magnetic theory with their independent classical chiral primary operators. Although we will implement the method of a-maximizationwith the different global trial a-function in the electric theory from the one in the magnetic theory, it will turn out that both the global trial a-functions have the identical local maximum, which is consistent with the duality conjecture [7].

4.2.1 On the Electric Side

Let us begin with the electric theory. Similarly to the one spinor case, the trial $U(1)_R$ charges of the matter fields may be given by

$$R(Q) = -4x + 1, \qquad R(\Psi) = N_Q x - 1, \tag{4.13}$$

with the trial $U(1)_R$ symmetry given by a linear combination of $U(1)_F$ and $U(1)_\lambda$ in Table 3.4 as

$$U(1)_R = xU(1)_F + U(1)_\lambda \tag{4.14}$$

[3] The possibility will be discussed in detail in Chapter 6.

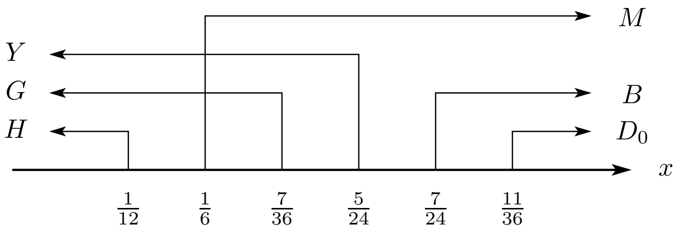

Figure 4.7: The ranges of x where each operator hits the unitarity bound for $N_Q = 6$.

with some real number x, assuming that there are no accidental global $U(1)$ symmetries [4] in the infrared.

For $N_Q = 6$, as can be seen from Figure 3.2, all the gauge invariant operators are M, Y, C, B, G, H, D_0, and S in (3.5). The $U(1)_R$ charges of the gauge invariant operators can be written in terms of x as [5]

$$\begin{aligned} &R(M) = -8x + 2, \quad R(Y) = 8x - 1, \quad R(C) = 1, \quad R(B) = -8x + 3, \\ &R(G) = 24x - 4, \quad R(H) = 8x, \quad R(D_0) = -24x + 8 \end{aligned} \tag{4.15}$$

as can be seen from Table 3.6. The unitarity bound conditions of them divides all the values of x into seven regions, as in Figure 4.7 [6].

The global trial a-function is given by

$$a(x) = \begin{cases} a_0(x) + f_Y(x) + f_G(x) + f_H(x), & \left(x \le \frac{1}{12}\right) \\ a_0(x) + f_Y(x) + f_G(x), & \left(\frac{1}{12} \le x \le \frac{1}{6}\right) \\ a_0(x) + f_M(x) + f_Y(x) + f_G(x), & \left(\frac{1}{6} \le x \le \frac{7}{36}\right) \\ a_0(x) + f_M(x) + f_Y(x), & \left(\frac{7}{36} \le x \le \frac{5}{24}\right) \\ a_0(x) + f_M(x), & \left(\frac{5}{24} \le x \le \frac{7}{24}\right) \\ a_0(x) + f_M(x) + f_B(x), & \left(\frac{7}{24} \le x \le \frac{11}{36}\right) \\ a_0(x) + f_M(x) + f_B(x) + f_{D_0}(R), & \left(\frac{11}{36} \le x\right) \end{cases} \tag{4.16}$$

where $a_0(x)$ is the local trial a-function with no operators hitting the unitarity bounds, and the function $f_{\mathcal{O}}$ is similarly defined to the one in (4.7). The function (4.16) has a unique local maximum at

$$x = \frac{18N_Q + 6 - \sqrt{-4N_Q^3 + 143N_Q^2 - 928N_Q + 1824}}{6(N_Q^2 + 8N_Q - 12)}, \tag{4.17}$$

[4] See the footnote 1 in this chapter.

[5] Since the glueball S of the $U(1)_R$ charge 2 never hits the unitarity bound, we will not take account of S.

[6] Since the operator C does not hit the unitarity bound for any value of x, it does not appear in the figure.

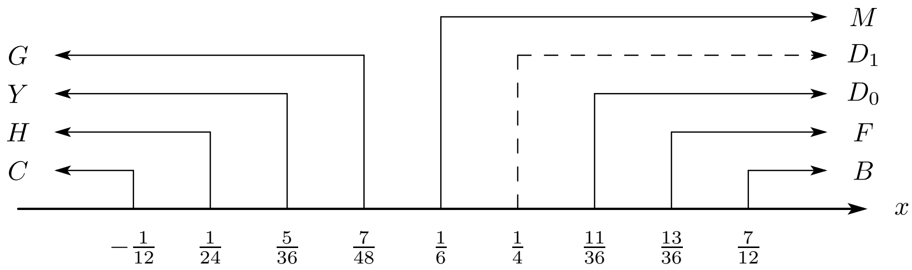

Figure 4.8: The ranges of x where each operator hits the unitarity bound for $N_Q = 8$.

with $N_Q = 6$, where only the operator M^{ij} hits the unitarity bound and is free at the infrared fixed point.

For $N_Q = 7$, one can also find that only M^{ij} hits the unitarity bound at the local maximum (4.17).

For $N_Q = 8$, all the values of x are divided as Figure 4.8 [7]. The global trial a-function has a unique local maximum at

$$x = \frac{12N_Q - \sqrt{2900 - N_Q^2}}{6(N_Q^2 - 20)}, \tag{4.18}$$

where no operators hit the unitarity bounds.

For $9 \leq N_Q \leq 19$, a local maximum is found at (4.18), where no gauge invariant operators hit the unitary bounds.

4.2.2 On the Magnetic Side

Let us turn to the magnetic theory. For $N_Q = 6$, the gauge invariant operators are U_0, U_1, U_2, E_0, I_0, I_1, and J_1 in (3.8), which exist only in the magnetic theory, besides M, Y, C, B, G, H, D_0, and S in (3.7), with their trial $U(1)_R$ charges given by (4.15) and by

$$R(U_0) = 24x - 2, \quad R(U_1) = 4, \quad R(U_2) = -24x + 10, \quad R(E_0) = -8x + 5,$$
$$R(I_0) = 8x + 2, \quad R(I_1) = 3, \quad R(J_1) = 16x. \tag{4.19}$$

Their unitarity bounds are illustrated in Figure 4.9 [8].

For the region $1/9 \leq x \leq 7/18$, where neither of the operators which exist only in the magnetic theory hits the unitarity bound, the global trial a-function should be the same

[7] In this case, the subtlety arises in the region $x \leq 1/4$ due to the lack of our knowledge of a-maximization for Lorentz spinor operators like $D_{1\alpha}$. The unitarity bound for a gauge invariant Lorentz spinor is $R(\mathcal{O}) \geq 1$ [8]. Our strategy for the issue is exactly the same as in the one spinor case.

[8] Since the operators C, U_1 and I_1 do not hit the unitarity bounds for all the values of x, they do not appear in Figure 4.9. The bold arrows correspond to the operators which exist only in the magnetic theory. The dotted arrows correspond to the Lorentz spinor operators, which we ignore as in the previous subsection.

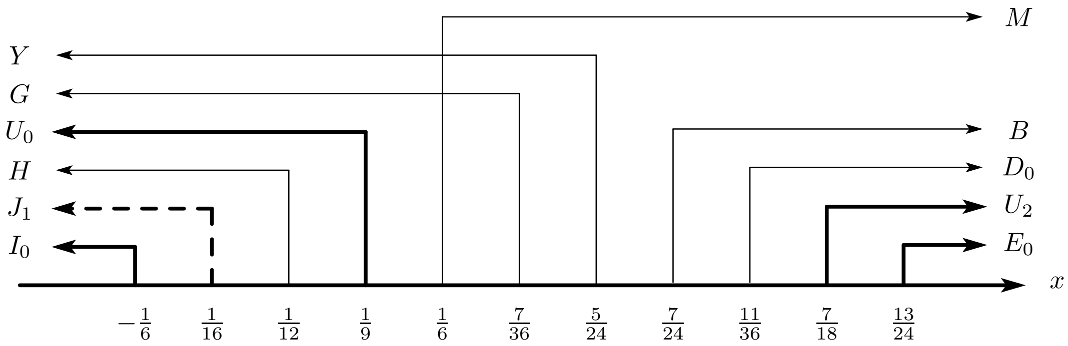

Figure 4.9: The ranges of x where each operator hits the unitarity bound for $N_Q = 6$ in the magnetic theory.

as the one in the electric theory, the latter of which has a local maximum in the range. Therefore, the global trial a-function in magnetic theory has at least one local maximum at the same value of x. One can also show that it has no local maximum outside the region $1/9 \leq x \leq 7/18$.

For $7 \leq N_Q \leq 19$, this is also the case. In the magnetic theory, we obtain the same local maximum as in the electric theory, and there is no other local maximum of the global trial a-function.

Chapter 5

Discussions

We have seen so far that the meson $M^{ij} = Q^i Q^j$ has no interactions for $7 \le N_Q \le 9$ in the one spinor case and for $6 \le N_Q \le 7$ in the two spinor case at the infrared fixed point. In this chapter, by using the electric-magnetic duality [3, 4, 7], we will give more elaborate discussions about what actually happens in the infrared when the meson becomes free.

The meson operator $Q^i Q^j$ in the electric theory corresponds to the elementary singlet M^{ij} in the magnetic theory. For $7 \le N_Q \le 9$ in the one spinor case, the singlet M^{ij} becomes free at the infrared fixed point. Therefore, the coupling constant of the interaction term $M^{ij} \bar{q}_i \, s \, \bar{q}_j$ in the magnetic superpotential (3.2) must vanish at the point. It means that the interaction term should be irrelevant at the infrared fixed point. Since we now know the exact superconformal $U(1)_R$ charges of the chiral primary operators at the same point, and thus the exact conformal dimensions of them, we can precisely determine whether the interaction term $M^{ij} \bar{q}_i \, s \, \bar{q}_j$ is irrelevant or not at the fixed point.

In fact, taking account of the charge assignments in Table 3.2, one can see that the $U(1)_R$ charge of $\bar{q}_i \, s \, \bar{q}_j$ is $4x$, and at the infrared fixed point, $R(\bar{q}_i \, s \, \bar{q}_j) > 4/3$. Since the free meson operator M^{ij} has the $U(1)_R$ charge $2/3$, the $U(1)_R$ charge of the interaction term $M^{ij} \bar{q}_i \, s \, \bar{q}_j$ is greater than 2. Therefore, the interaction term is irrelevant at the infrared fixed point. This is consistent with the result that the meson M^{ij} decouple from the remaining interacting system to be free in the infrared.

To the $6 \le N_Q \le 7$ case with the two spinors, the same argument can be applied to find that the interaction terms of the meson M^{ij} in the magnetic superpotential is irrelevant at the infrared fixed point.

Furthermore, let us consider another implication of the irrelevant interaction term. Since the equation of motion gives

$$\frac{\partial}{\partial M^{ij}} W_{\rm mag} = \frac{\tilde{h}}{\tilde{\mu}^2} N_{ij} = 0,$$

where $N_{ij} = \bar{q}_i \, s \, \bar{q}_j$, if its coupling constant $\tilde{h}$ were not zero, the gauge invariant operators N_{ij} would be redundant. This is indeed the case for $10 \le N_Q \le 21$ with one spinor and for

$8 \leq N_Q \leq 19$ with two spinors. However, for $7 \leq N_Q \leq 9$ with one spinor and $6 \leq N_Q \leq 7$ with two spinors, since $\tilde{h}$ goes to zero[1], the operators N_{ij} do not have to be redundant. Therefore, N_{ij} should be a new generator of the chiral ring in the magnetic theory.

Furthermore, the magnetic theory with vanishing $\tilde{h}$ in the superpotential is dual to the same $Spin(10)$ theory but with the superpotential

$$W_{\rm ele} = N_{ij} Q^i Q^j,$$

with the gauge singlets N_{ij} and the free singlets M^{ij}, which was explained in section 3.1.1 for the theory with one spinor but it is also the case for the theory with two spinors, though we haven't previously mentioned about the latter case. The singlets N_{ij} can be identified with $\bar{q}_i\, s\, \bar{q}_j$. Therefore, the magnetic theory of the original dual pair flows into the magnetic one of another dual pair at the infrared fixed point. It suggests that the original electric theory flows into the electric theory with the superpotential $W_{\rm ele}$ as illustrated in Figure 5.1.

In the electric theory with the superpotential $W_{\rm ele}$, we can carry out the a-maximization procedure in a similar way to what we have done in the previous sections. The values of the trial $U(1)_R$ charge where no gauge invariant operators hit the unitarity bounds in this theory is identical to the values where only the operators M^{ij} hit the unitarity bound in the original electric theory. In the region of the trial $U(1)_R$ charge, the local trial a-function can be calculated in terms of the fundamental fields in the ultraviolet in the former theory to give

$$a_0(R) + \frac{N_Q(N_Q+1)}{2} F[R(N)] + \frac{N_Q(N_Q+1)}{2} F_0,$$

where $a_0(R)$ is given in (4.3), $F(x)$ is defined as $F(x) \equiv 3(x-1)^3 - (x-1)$, and F_0 is the contribution from the free singlets M^{ij}. Since the function $F(x)$ satisfies the relation

$$F(x) + F(2-x) = 0, \tag{5.1}$$

one notices that $F[R(N)] = -F[R(QQ)]$ and that the above a-function is the same as the one in the identical region in the original electric theory. Since the latter a-function are constructed via the prescription of [11], one finds that it is consistent with the electric-magnetic duality.

The origin of the singlet field N_{ij} can also be captured in the original electric theory by using the auxiliary field method. In the original theory, let us introduce the auxiliary fields M^{ij} and the Lagrange multipliers N_{ij} to turn on the superpotential

$$W = N_{ij}\left(Q^i Q^j - h\, M^{ij}\right), \tag{5.2}$$

with the parameter h. It does not change the original theory at all, as far as h is non-zero. The equations of motion give the constraints

$$Q^i Q^j = h\, M^{ij}, \qquad h N_{ij} = 0. \tag{5.3}$$

[1]For $N_Q = 7$, the coupling constant $\tilde{h}''$ of the additional interaction term (3.3) also goes to zero.

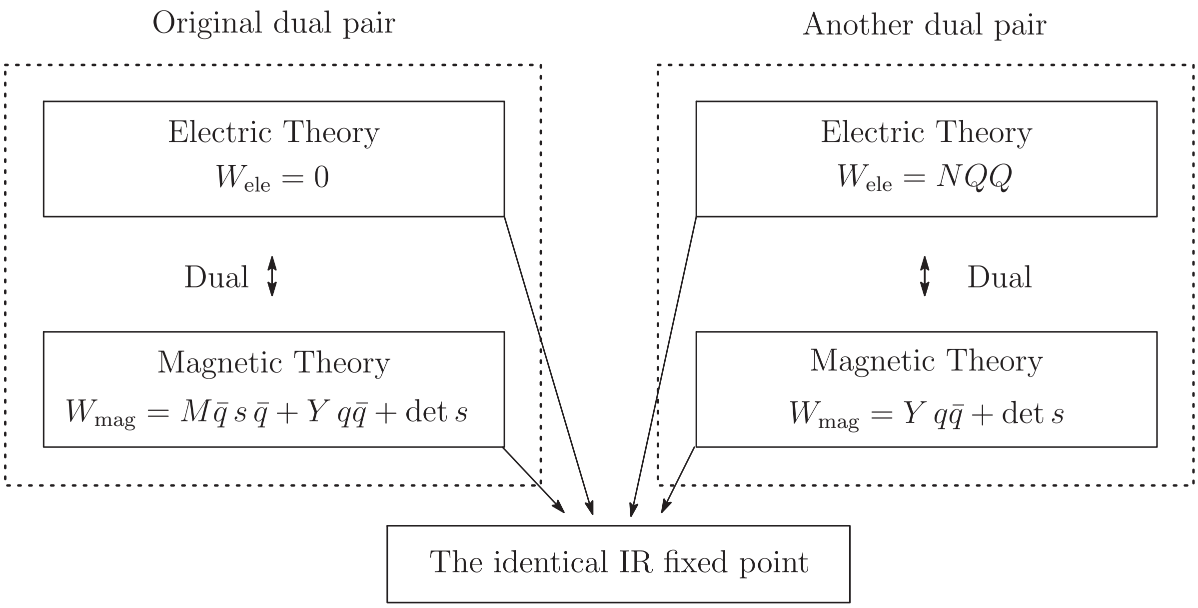

Figure 5.1: Two dual pairs flow to the identical IR fixed point.

Substituting them into (5.2), one can return to the original theory.

One can conceive that when the meson operator hits the unitarity bound, the parameter h goes to zero in the infrared, due to the consistency with the result that the singlet M^{ij} becomes free in the magnetic theory. In this case, the first equation of motion in (5.3) gives $Q^iQ^j = 0$ while the second one gives the trivial identity $0 = 0$. It is consistent with the result that the composites Q^iQ^j decouple from the interacting system in the original theory while the chiral primary operator N_{ij} is gained. Here, the decoupled free meson operators correspond to M^{ij}, which are not related with vectors Q^i of the interacting system any more. Furthermore, when h goes to zero, one obtains the superpotential $W_{\rm ele}$ of the other electric theory introduced in subsection 3.1.1. It means that the original electric theory flows into the other electric theory with $W_{\rm ele}$ and thus is consistent with the magnetic picture.

One may raise a question whether the auxiliary field method affects our results via a-maximization in the last section, because we introduced the auxiliary fields M^{ij} and the Lagrange multipliers N_{ij} charged under $U(1) \times U(1)_R$. This is however not the case, since as has been discussed in [12], the massive fields M^{ij} and N_{ij} do not contribute to the a-function, due to (5.1). But, once the singlet M^{ij} hits the unitarity bound, an accidental $U(1)_M$ symmetry appears to fix the $U(1)_R$ charge of M^{ij} to $2/3$. On the other hand, the singlets N_{ij} are still interacting with the vectors Q^i in the superpotential, and their $U(1)_R$ charge remains unchanged and contributes as $F[R(N_{ij})] = F[2 - R(Q^iQ^j)] = -F[2R(Q)]$ to the a-function;

$$F[R(M)] + F[R(N)] \quad \Rightarrow \quad F(2/3) + F[2 - R(Q^iQ^j)] = -F[2R(Q)] + F_0.$$

One can thus see that it gives the identical procedure to what we have done when the meson M^{ij} hits the unitarity bound. This discussion gives a strong support for the prescription (2.64) in section 2.3.

Chapter 6

Summary and Outlook

In this article, by using the electric-magnetic duality and a-maximization to study four-dimensional supersymmetric $\mathcal{N} = 1$ $Spin(10)$ gauge theories with chiral superfields in the vector and the spinor representations at the superconformal infrared fixed point, we have discussed their low-energy physics. In particular, a-maximization allowed us to understand it in more detail, compared to the previous results [3, 4, 7] on the theories.

In the one spinor case, among $7 \leq N_Q \leq 21$ in the non-Abelian Coulomb phase, for $7 \leq N_Q \leq 9$, only the meson operator hits the unitarity bound to be free in the infrared. At the other flavor number N_Q, no gauge invariant operators hit the unitarity bound.

In the two spinor case, the results are quite parallel to that in the one spinor case. Among $6 \leq N_Q \leq 19$ in the non-Abelian Coulomb phase, for $N_Q = 6, 7$, only the meson operator also hits the unitarity bound to be free in the infrared. At the other flavor number N_Q, no gauge invariant operators hit the unitarity bound.

In both the cases, the local maximum we found was confirmed to be identical in both of the electric theory and the magnetic theory.

We have also discussed the physical implication of the decoupling meson operator - the renormalization flows of two electric-magnetic dual pairs into a single nontrivial infrared fixed point - by the three steps; calculating the conformal dimension of the interaction term of the meson in the magnetic superpotential, finding another electric-magnetic dual pair, and using the auxiliary field method.

In our analysis, two subtle points prevents us from completing the a-maximization procedure for the $Spin(10)$ gauge theories, as discussed in detail. One of them is the dismatch of the classical chiral rings of the electric-magnetic dual pairs. It means that a gauge invariant operator does not have their counterpart in the dual description. Therefore, at the value of the trial $U(1)_R$ charge where the operator hits the unitarity bound, the local trial a-function differs from the one in the dual theory. Thus, the a-maximization procedure might give different results in the electric theory from the one in the magnetic theory. Fortunately, this was not the case for our theories. However, in order to implement our a-maximization procedure completely, we need to understand the chiral ring of the dual theories more or less quantum-mechanically. It would also help to establish the electric-magnetic duality itself of the $Spin(10)$ gauge theories completely,

As for the other subtle point, we need to know how to extend the method of a-maximization for Lorentz spinor operator, and also for an operator in any non-trivial representation of the Lorentz group. In our cases, the operator $D_{1\alpha}$ is such a operator. Again, fortunately, the local maximum of the trial a-function is found to be outside the region where the operator $D_{1\alpha}$ hits the unitarity bound. But, it does not necessarily mean that there is no local maximum inside the region. Therefore, it would be interesting to know the extension of a-maximization for the operator $D_{1\alpha}$.

Besides the two subtleties, during the a-maximization procedure in the two spinor case, we have assumed in the magnetic theory that the gauge coupling constants of the magnetic gauge groups $SU(N_Q-3)$ and $Sp(1)$ both have non-zero values at the infrared fixed point. However, from the one-loop beta functions of the two gauge couplings, one can see that $SU(N_Q-3)$ is asymptotically free for $N_Q \geq 7$ and $Sp(1)$ is asymptotically free for $N_Q \leq 7$, perturbatively. Therefore, either of the gauge interactions could be free at the infrared fixed point. Thus, if the perturbation of both the interactions were reliable even in the infrared, the gauge coupling constant of $SU(N_Q-3)$ would vanish for $N_Q = 6$ and that of $Sp(1)$ would vanish for $8 \leq N_Q \leq 19$ at the fixed point. We will argue just below that the method of a-maximization could also have been used to know whether this is the case or not. Although our results suggest that this is not the case, it may offer another enjoyable application of a-maximization [35, 36].

Let us suppose that, for $N_Q = 6$, the coupling constant g_{SU} of the $SU(N_Q-3)$ gauge interaction goes to zero in the infrared. The NVSZ beta function [22] of the gauge coupling g_{SU} is given by

$$\beta_{SU}(g_{SU}, g_{Sp}) = -\frac{g_{SU}^3}{16\pi^2}\frac{3(N_Q-3) - \sum_i T(\rho_i)(1-\gamma_i(g_{SU}, g_{Sp}))}{1-(N_Q-3)(g_{SU}^2/8\pi^2)}, \tag{6.1}$$

where γ_i is the anomalous dimension of the matter field labeled by i and $T(\rho_i)$ denotes the usual index of its representation ρ_i. Under our assumption, in the infrared the beta function can be expanded in powers of the gauge coupling g_{SU} as

$$\beta_{SU}(g_{SU}, g_{Sp}) = \beta_0(g_{Sp})g_{SU}^3 + \beta_1(g_{Sp})g_{SU}^5 + \cdots, \tag{6.2}$$

where

$$\beta_0(g_{Sp}) = -\frac{1}{16\pi^2}\left[3(N_Q-3) - \sum_i T(\rho_i)(1-\gamma_i(g_{SU}=0, g_{Sp}))\right]. \tag{6.3}$$

Therefore, in order to reach the infrared fixed point $(g_{SU}^* = 0, g_{Sp}^*)$, the beta function coefficient $\beta_0(g_{Sp}^*)$ must be positive - the infrared fixed point $(g_{SU}^* = 0, g_{Sp}^*)$ must be an attractive point of the renormalization group flow.

It is generically difficult to calculate the beta function coefficient $\beta_0(g^*_{Sp})$, especially when the gauge coupling g^*_{Sp} cannot be treated perturbatively. However, the anomalous dimensions $\gamma_i(g^*_{SU} = 0, g^*_{Sp})$ are related to the superconformal $U(1)_R$ charges via $\gamma_i(g^*_{SU} = 0, g^*_{Sp}) = 3R_i - 2$ as in (2.33). In order to obtain the superconformal $U(1)_R$ charges, one may set the gauge coupling g_{SU} to zero at the ultraviolet cutoff and then carry out a-maximization. If one can identify the infrared fixed point $(g^*_{SU} = 0, g^*_{Sp})$ as one of the local maxima of the global trial a-function, and if no gauge invariant *composite* operator hits the unitarity bound, one can determine the $U(1)_R$ charge R_i of the elementary field, and thus the coefficient $\beta_0(g^*_{Sp})$.

In order to determine the coefficient $\beta_0(g^*_{Sp})$, let us begin with the magnetic theory for $N_Q = 6$ with $g_{SU} = 0$ - the $Sp(1)$ gauge theory. All the $Sp(1)$ gauge invariant chiral primary operators are M, Y, $\bar{q}$, q, s, and the composites

$$\varepsilon_{\alpha\beta}\bar{q}'_a{}^{\alpha I}\bar{q}'_b{}^{\beta J}, \quad \varepsilon_{\alpha\beta}\bar{q}'_a{}^{\alpha I}t^{\beta J}, \quad \varepsilon_{\alpha\beta}t^{\alpha I}t^{\beta J}. \tag{6.4}$$

Since we do not impose the anomaly free condition coming from the $SU(N_Q - 3)$ gauge interaction for the global $U(1)$ symmetries, one has one extra global $U(1)$ symmetry, and thus the trial a-function depends on two parameters. It seems formidable to obtain the global trial a-function in the whole two-dimensional parameter space. But, we found at least one local maximum in the range where M, Y, and $\bar{q}$ hit the unitarity bounds. At the local maximum, the $U(1)_R$ charge of each field can numerically be read as follows:

$\bar{q}$	$\bar{q}'$	q	s	t	M	Y
2/3	0.4858	0.9716	1.028	0.5426	2/3	2/3

Substituting the $U(1)_R$ charges in the above list into the coefficient

$$\beta_0(g^*_{Sp}) = -\frac{3}{16\pi^2}\left[(N_Q - 3) - \sum_i T(\rho_i)(R_i - 1)\right], \tag{6.5}$$

since one can see that $[(N_Q - 3) - \sum_i T(\rho_i)(R_i - 1)] \simeq 1$, the coefficient $\beta_0(g^*_{Sp})$ is negative - the infrared fixed point into which the theory with $g_{SU} \neq 0$ at the cutoff never flows in the infrared.

If there was no more local maximum of the global trial a-function, the above argument would prove our assumption about the gauge couplings. Therefore, it would be interesting to carry out the a-maximization procedure completely in this system to confirm the assumption.

In the case where the gauge coupling g_{Sp} goes to zero in the infrared instead, a similar discussion can be made for $8 \leq N_Q \leq 19$. The $SU(N_Q - 3)$ gauge invariant operators are fundamental fields t and the composites

$$\begin{aligned}
A^{\alpha(IJK)} &= q^{aX}(\sigma_X\sigma_2)^{(IJ}\bar{q}'_a{}^{|\alpha|K)}, \\
(P_1)^{\alpha}{}_{i_1\cdots i_{N_Q-4}} &= \varepsilon^{a_1\cdots a_{N_Q-3}}\bar{q}'_{a_1}{}^{\alpha I}\bar{q}_{a_2 i_1}\cdots\bar{q}_{a_{N_Q-3}i_{N_Q-4}}, \\
(P_3)^{\alpha}{}_{i_1\cdots i_{N_Q-6}} &= \varepsilon^{a_1\cdots a_{N_Q-3}}\bar{q}'_{a_1}{}^{\alpha_1 I_1}\cdots\bar{q}'_{a_3}{}^{\alpha_3 I_3}\bar{q}_{a_4 i_1}\cdots\bar{q}_{a_{N_Q-3}i_{N_Q-6}},
\end{aligned} \tag{6.6}$$

N_Q	8	9	10	11	12	13	14	15	16	17	18	19
$\bar{q}$	0.516	0.482	0.462	0.440	0.417	0.396	0.376	0.357	0.339	0.323	0.308	0.295
$\bar{q}'$	0.8	0.833	0.857	0.876	0.891	0.902	0.911	0.919	0.925	0.931	0.935	0.939
q	0.623	0.555	0.462	0.395	0.341	0.297	0.262	0.233	0.208	0.187	0.169	0.154
s	0.4	0.333	0.285	0.248	0.219	0.196	0.178	0.162	0.150	0.139	0.129	0.121
t	2/3	2/3	0.680	0.729	0.769	0.801	0.827	0.849	0.867	0.882	0.895	0.907
M	2/3	0.704	0.791	0.873	0.946	1.012	1.071	1.124	1.171	1.214	1.254	1.289
Y	0.861	0.963	1.076	1.166	1.242	1.307	1.362	1.411	1.453	1.490	1.522	1.551

Table 6.1: The $U(1)_R$ charges under the assumption that $Sp(1)$ gauge factor is IR free.

as well as the operators which are same as $SU(N_Q-3)\times Sp(1)$ gauge invariant operators except for H and G, which can be expressed in this case as the product of the $SU(N_Q-3)$ gauge invariant operators.

We also found the local maximum of the trial a-function; for $N_Q=8$, in the the range where M and t hit the unitarity bounds, for $N_Q=9$, where t hits the unitarity bound, and for $10\le N_Q\le 19$, no operator hits the unitarity bound. The $U(1)_R$ charges of the fields are given in Table 6.1. Using the $U(1)_R$ charges, one can see that the NSVZ beta function becomes negative for all $8\le N_Q\le 19$. It implies that the system does not flow in the infrared into the point which the above found local maximum suggests. Since the analysis of a-maximizationin this case also is far from complete, it would be interesting to be done thoroughly.

In our analysis, the trial a-function has a unique maximum under the assumptions mentioned above. Within a region with the same content of decoupling gauge invariant operators in the whole parameter space, one can find at most a single local maximum, but in another region, one could obtain another local maximum, where one should find the different content of interacting gauge invariant operators. It may suggest that one could find more than one local maximum over the whole parameter space to lose definitive results on which linear combination of the $U(1)$ symmetries is the superconformal $U(1)_R$ symmetry. The weak version of the diagnostic in the paper [32] could however be a way out of this problem. It says, "*the correct IR phase is the one with the larger value of the conformal anomaly* a". It would thus be very interesting to find models with more than one local maximum of the function $a(x)$ and to study the renormalization group flow in such models.

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