On Vassiliev invariants of braid groups of the sphere

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Abstract. We construct a universal Vassiliev invariant for braid groups of the sphere and the mapping class groups of the sphere with n punctures. The case of a sphere is different from the classical braid groups or braids of oriented surfaces of genus strictly greater than zero, since Vassiliev invariants in a group without 2-torsion do not distinguish elements of braid group of a sphere.

Contents

1. Introduction

The theory of Vassiliev (or finite type) invariants starts with the works of V. A. Vasiliev [25, 26] though the ideas which lie in the foundations of this theory can be found in the work of M. Gousarov [13]. The basic idea is classical in Mathematics: to introduce a filtration in a complicated fundamental object such that the corresponding associated graded object is simpler and sometimes possible to describe. The construction of spectral sequences have similar features. A lot of progress had been done during the last couple of decades in the theory of Vassiliev invariants of knots, the basic object of the study. Also analogous constructions were done for braids. After the work of T. Stanford [24] S. Papadima [22] constructed a universal Vassiliev invariant ("Kontsevich integral") over $\mathbb{Z}$ for classical braids. A similar construction was done by J. Mostovoy and S. Willerton [21]. For braids on oriented closed surfaces of genus $g \geq 1$ this was done by J. Gonzáles-Meneses and L. Paris [12]. This universal Vassiliev invariant is not multiplicative, and as it was shown by Bellingeri and Funar [2] a multiplicative one in the case of positive genus ($g \geq 1$) does not exist. Here we consider the case of sphere S^2. This case is interesting because the braid groups on the sphere contain torsion and Vassiliev invariants in rational numbers does not give a complete set of invariants [5, 6]. Our universal invariant is not multiplicative. Usually multiplicative universal Vassiliev invariant is constructed over rational numbers, which is not good for the sphere, but on the other hand there is no interdiction of existence of such an invariant over the integers.

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We also study the mapping class group of the sphere with n punctures, this group is closely connected to the braid group of the sphere. It turns out that for these two types of groups as for the braid groups examined before the study of Vassiliev invariants, topological by their nature reduces to purely algebraic constructions and facts.

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2. Braid groups of the sphere and the mapping class groups of the sphere with n punctures

Let M be a topological space and let M^n be the n-fold Cartesian product of M. The *n-th ordered configuration space* $F(M,n)$ is defined by

$$F(M,n) = \{(x_1,\dots,x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}$$

with subspace topology of M^n. The symmetric group Σ_n acts on $F(M,n)$ by permuting coordinates. The orbit space

$$B(M,n) = F(M,n)/\Sigma_n$$

is called the *n-th unordered configuration space*. The *braid group* $B_n(M)$ is defined to be the fundamental group $\pi_1(B(M,n))$. The *pure braid group* $P_n(M)$ is defined to be the fundamental group $\pi_1(F(M,n)$. From the covering $F(M,n) \to F(M,n)/\Sigma_n$, we get a short exact sequence of groups

$$\{1\} \to P_n(M) \to B_n(M) \to \Sigma_n \to \{1\}. \tag{2.1}$$

We will use later the following classical Fadell-Neuwirth Theorem.

Theorem 2.1. [7] *For $n > m$ the coordinate projection (forgetting of $n-m$ coordinates)*

$$\delta_m^{(n)}\colon F(M,n) \to F(M,m),\ (x_1,\dots,x_n) \mapsto (x_1,\dots,x_m)$$

is a fiber bundle with fiber $F(M \setminus Q_m, n-m)$, where Q_m is a set of m distinct points in M.

In this work we consider the classical braids which are braids of the disc: $M = D^2$, and the case of the sphere S^2.

Usually the braid group $Br_n = B_n(D^2)$ is given by the following Artin presentation [1]. It has the generators σ_i, $i = 1,\dots,n-1$, and two types of relations:

$$\begin{cases} \sigma_i\sigma_j &= \sigma_j\,\sigma_i, \quad \text{if } \ |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}\,. \end{cases} \tag{2.2}$$

The generators $a_{i,j}$, $1 \le i < j \le n$ of the pure braid group P_n (of a disc) can be described as elements of the braid group Br_n by the formula:

$$a_{i,j} = \sigma_{j-1}\dots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\dots\sigma_{j-1}^{-1}.$$

The defining relations among $a_{i,j}$, which are called the *Burau relations* ([4], [20]) are as follows:

$$\begin{cases} a_{i,j}a_{k,l} = a_{k,l}a_{i,j} \text{ for } i<j<k<l \text{ and } i<k<l<j, \\ a_{i,j}a_{i,k}a_{j,k} = a_{i,k}a_{j,k}a_{i,j} \text{ for } i<j<k, \\ a_{i,k}a_{j,k}a_{i,j} = a_{j,k}a_{i,j}a_{i,k} \text{ for } i<j<k, \\ a_{i,k}a_{j,k}a_{j,l}a_{j,k}^{-1} = a_{j,k}a_{j,l}a_{j,k}^{-1}a_{i,k} \text{ for } i<j<k<l. \end{cases} \tag{2.3}$$

It was proved by O. Zariski [28] and then rediscovered by E. Fadell and J. Van Buskirk [8] that a presentation of the braid group on a 2-sphere can be given with the generators σ_i, $i = 1, ..., n-1$, the same as for the classical braid group, satisfying the braid relations (2.2) and the following sphere relation:

$$\sigma_1\sigma_2 \dots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \dots \sigma_2\sigma_1 = 1. \tag{2.4}$$

Let Δ be the Garside's fundamental element in the braid group Br_n [9]. It can be expressed in particular by the following word in canonical generators:

$$\Delta = \sigma_1 \dots \sigma_{n-1}\sigma_1 \dots \sigma_{n-2} \dots \sigma_1\sigma_2\sigma_1.$$

If we use Garside's notation $\Pi_t \equiv \sigma_1 \dots \sigma_t$, $1 \le t \le n-1$, then $\Delta \equiv \Pi_{n-1} \dots \Pi_1$.

For the pure braid group on a 2-sphere let us introduce the elements $a_{i,j}$ for all i, j by the formulas:

$$\begin{cases} a_{j,i} = a_{i,j} & \text{for } i < j \le n, \\ a_{i,i} = 1. \end{cases} \tag{2.5}$$

The pure braid group on a 2-sphere has the generators $a_{i,j}$ which satisfy the Burau relations (2.3), the relations (2.5), and the following relations [10]:

$$a_{i,i+1}a_{i,i+2} \dots a_{i,i+n-1} = 1 \quad \text{for all } i \le n,$$

with the convention that indices are considered $\mod n$: $k + n = k$. Note that Δ^2 is a pure braid and can be expressed by the following formula

$$\Delta^2 = (a_{1,2}a_{1,3} \dots a_{1,n})(a_{2,3}a_{2,4} \dots a_{2,n}) \dots (a_{n-1,n}) = \\ (a_{1,2})(a_{1,3}a_{2,3})(a_{1,4}a_{2,4}a_{3,4}) \dots (a_{1,n} \dots\dots a_{n-1,n}).$$

This element of the braid group generates its center.

Another object of our study is the mapping class groups of the sphere with n punctures. The (general) mapping class group is an important object in Topology, Complex Analysis, Algebraic Geometry and other domains. It is a rare situation when the method of Algebraic Topology works perfectly well, the application of the functor of fundamental group completely solves the topological problem: group of isotopy classes of homeomorphisms is described in terms of automorphisms of the fundamental group of the corresponding surface, as the Dehn-Nilsen-Baer theorem states (see [15], for example).

Let $S_{g,b,n}$ be an oriented surface of genus g with b boundary components and we remind that Q_n denotes a set of n punctures (marked points) in the surface. Consider the group $\mathrm{Homeo}(S_{g,b,n})$ of orientation preserving self-homeomorphisms of $S_{g,b,n}$ which fix pointwise the boundary (if $b > 0$) and map the set Q_n into itself.

Let $\mathrm{Homeo}^0(S_{g,b,n})$ be the normal subgroup of self-homeomorphisms of $S_{g,b,n}$ which are isotopic to identity. Then the *mapping class group* $M_{g,b,n}$ is defined as a quotient group

$$M_{g,b,n} = \mathrm{Homeo}(S_{g,b,n}) / \mathrm{Homeo}^0(S_{g,b,n}).$$

These groups are closely connected with braid groups. W. Magnus in [18] interpreted the n-braid group as the mapping class group of an n-punctured disc with the fixed boundary:

$$Br_n \cong M_{0,1,n}.$$

Like braid groups the groups $M_{g,b,n}$ has a natural epimorphism to the symmetric group Σ_n with the kernel called the *pure mapping class group* $PM_{g,b,n}$, so there exists an exact sequence:

$$1 \to PM_{g,b,n} \to M_{g,b,n} \to \Sigma_n \to 1.$$

Geometrically the pure mapping class group $PM_{g,b,n}$ consists of isotopy classes of homeomorphisms that preserve the punctures pointwise.

In the paper we consider the pure mapping class group $PM_{0,0,n}$ of a punctured 2-sphere (so the genus is equal to 0) with no boundary components that we simply denote by PM_n; the same way we denote further $M_{0,0,n}$ simply by M_n.

The group PM_n is closely related to the pure braid group $P_n(S^2)$ on the 2-sphere as well as its non-pure analogue M_n is related with the (total) braid group $B_n(S^2)$ on the 2-sphere.

W. Magnus obtained in [18] (see also [19]) a presentation of the mapping class group M_n for the n-punctured 2-sphere. It has the same generators as $B_n(S^2)$ and a complete set of relations consists of (2.2), (2.4) and the following relation

$$(\sigma_1\sigma_2 \dots \sigma_{n-2}\sigma_{n-1})^n = 1. \tag{2.6}$$

For the generators σ_1, σ_2, $\dots$, σ_{n-2}, σ_{n-1}, subject to the braid relations (2.2) the condition (2.6) is equivalent to the following relation

$$\Delta^2 = 1.$$

Using Theorem 2.1 we have the following morphism of fibrations

$$\begin{tikzcd}
F(D^2 \setminus \{p_1, p_2\}, n-2) \arrow[r, "i"] \arrow[d] & F(D^2, n) \arrow[r, "\delta_2"] \arrow[d] & F(D^2, 2) \arrow[d] \\
F(S^2 \setminus \{p_1, p_2, p_3\}, n-2) \arrow[r, "i"] & F(S^2, n+1) \arrow[r, "\delta_3"] & F(S^2, 3),
\end{tikzcd} \tag{2.7}$$

where the vertical lines are induced by an inclusion of a disc into the sphere. Let us denote by $P_n(S^2_3)$ the pure braid group on n strands of a 2-sphere with three points deleted or equivalently the subgroup of the pure braid group on $n+2$ strands of a disc where (say, the last) two strands are fixed as trivial (unbraided) strands which is also equal to the fundamental group of $F(S^2 \setminus \{p_1, p_2, p_3\}, n)$.

The following statement follows from the normal forms of the groups $P_n(S^2)$ and PM_n given in [10] and on the geometrical level it follows from diagram (2.7) and it was expressed in [11]. Note that the groups $P_2(S^2)$ and PM_3 are trivial.

Theorem 2.2. *(i) The pure braid group on a 2-sphere $P_n(S^2)$ for $n \geq 3$ is isomorphic to the direct product of the cyclic group C_2 of order 2 (generated by Δ^2) and PM_n.*

(ii) The pure braid group P_n for $n \geq 2$ is isomorphic to the direct product of the infinite cyclic group C (generated by Δ^2) and PM_{n+1}.

(iii) The groups PM_n and $P_{n-3}(S^2_3)$ are isomorphic for $n \geq 4$.

(iv) There is a commutative diagram of groups and homomorphisms

$$\begin{array}{ccc} P_n \cong & PM_{n+1} \times C \\ \Big\downarrow \rho_p & \Big\downarrow \delta\times\rho \\ P_n(S^2) \cong & PM_n \times C_2, \end{array} \tag{2.8}$$

where ρ_p is the canonical epimorphism $P_n \to P_n(S^2)$, δ is induced by the Fadell-Neuwirth fibration and ρ is the canonical epimorphism of the infinite cyclic group onto the cyclic group of order 2.

The isomorphism of the part (i) of Theorem 2.2 is compatible with the homomorphisms $p_i : P_n(S^2) \to P_{n-1}(S^2)$ and $pm_i : PM_n \to PM_{n-1}$ consisting of deleting one strand or forgetting one point, that means that the following diagram is commutative

$$\begin{array}{ccc} P_n(S^2) \cong & PM_n \times C_2 \\ \Big\downarrow p_i & \Big\downarrow pm_i\times id \\ P_{n-1}(S^2) \cong & PM_{n-1} \times C_2. \end{array}$$

3. Lie algebras from descending central series of groups

For a group G the descending central series

$$G = \Gamma_1 > \Gamma_2 > \dots > \Gamma_i > \Gamma_{i+1} > \dots .$$

is defined by the formula

$$\Gamma_1 = G, \quad \Gamma_{i+1} = [\Gamma_i, G].$$

This series of groups gives rise to the associated graded Lie algebra (over $\mathbb{Z}$) $gr^*_\Gamma(G)$

$$gr^i_\Gamma(G) = \Gamma_i/\Gamma_{i+1}.$$

A presentation of the Lie algebra $gr^*_\Gamma(P_n)$ for the pure braid group was done in the work of T. Kohno [16], and can be described as follows. It is the quotient of the free Lie algebra $L[A_{i,j} | \, 1 \le i < j \le n]$ generated by elements $A_{i,j}$ with $1 \le i < j \le n$ modulo the "infinitesimal braid relations" or "horizontal $4T$ relations" given as follows:

$$\begin{cases} [A_{i,j}, A_{s,t}] = 0, \text{ if } \{i,j\} \cap \{s,t\} = \phi, \\ [A_{i,j}, A_{i,k} + A_{j,k}] = 0, \text{ if } i < j < k, \\ [A_{i,k}, A_{i,j} + A_{j,k}] = 0, \text{ if } i < j < k. \end{cases} \tag{3.1}$$

It is convenient sometimes to have conventions like (2.5). So let us introduce the generators $A_{i,j}$, $1 \le i,j \le n$, not necessary $i < j$, by the formulae

$$\begin{cases} A_{j,i} = A_{i,j} \ \text{ for } \ 1 \le i < j \le n, \\ A_{i,i} = 0 \ \text{ for all } \ 1 \le i \le n. \end{cases}$$

For this set of generators the defining relations (3.1) can be rewritten as follows

$$\begin{cases} A_{i,j} = A_{j,i} \ \text{ for } \ 1 \leq i, j \leq n, \\ A_{i,i} = 0 \ \text{ for } \ 1 \leq i \leq n, \\ [A_{i,j}, A_{s,t}] = 0, \ \text{if } \{i,j\} \cap \{s,t\} = \phi, \\ [A_{i,j}, A_{i,k} + A_{j,k}] = 0 \ \text{ for all different } \ i, j, k. \end{cases} \tag{3.2}$$

Y. Ihara in [14] gave a presentation of the Lie algebra $gr^*_\Gamma(P_n(S^2))$ of the pure braid group of a sphere. It is the quotient of the free Lie algebra $L[B_{i,j}| \, 1 \leq i, j \leq n]$ generated by elements $B_{i,j}$ with $1 \leq i, j \leq n$ modulo the following relations:

$$\begin{cases} B_{i,j} = B_{j,i} \ \text{ for } 1 \leq i, j \leq n, \\ B_{i,i} = 0 \ \text{ for } 1 \leq i \leq n, \\ [B_{i,j}, B_{s,t}] = 0, \ \text{if } \{i,j\} \cap \{s,t\} = \phi, \\ \sum_{j=1}^{n} B_{i,j} = 0, \ \text{for } 1 \leq i \leq n. \end{cases} \tag{3.3}$$

It is also a quotient algebra of the Lie algebra $gr^*_\Gamma(P_n)$: the relations of the last type in (3.2) are the consequences of the third and the forth type relations in (3.3).

We shall use the following results which were proved in [17].

Theorem 3.1. *(i) The graded Lie algebra $gr^*_\Gamma(PM_n)$ is the quotient of the free Lie algebra $L[B_{i,j}| \, 1 \leq i, j \leq n]$ modulo the following relations:*

$$\begin{cases} B_{i,j} = B_{j,i} \ \textit{for } 1 \leq i, j \leq n, \\ B_{i,i} = 0 \ \ \textit{for } 1 \leq i \leq n, \\ [B_{i,j}, B_{s,t}] = 0, \ \textit{if } \{i,j\} \cap \{s,t\} = \phi, \\ \sum_{j=1}^{n} B_{i,j} = 0, \ \textit{for } 1 \leq i \leq n, \\ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} B_{i,j} = 0. \end{cases} \tag{3.4}$$

(ii) The graded Lie algebra $gr^*_\Gamma(PM_n)$ is the quotient of the free Lie algebra $L[B_{i,j}| \, 1 \leq i, j \leq n-1]$ generated by the elements $B_{i,j}$, $1 \leq i, j \leq n-1$, (smaller number of generators than in (i)) modulo the following relations:

$$\begin{cases} B_{i,j} = B_{j,i} \ \ \textit{for } \ 1 \leq i, j \leq n-1, \\ B_{i,i} = 0 \ \ \textit{for } \ 1 \leq i \leq n-1, \\ [B_{i,j}, B_{s,t}] = 0, \ \ \textit{if } \ \{i,j\} \cap \{s,t\} = \emptyset, \\ [B_{i,j}, B_{i,k} + B_{j,k}] = 0 \ \ \textit{for all different } \ i, j, k, \\ \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} B_{i,j} = 0. \end{cases} \tag{3.5}$$

Corollary 3.1. *A presentation of the Lie algebra $gr^*_\Gamma(P_n(S^2))$ can be given with generators $A_{i,j}$ with $1 \leq i < j \leq n-1$, modulo the following relations:*

$$\begin{cases} [A_{i,j}, A_{s,t}] = 0, \ \textit{if } \ \{i,j\} \cap \{s,t\} = \phi, \\ [A_{i,j}, A_{i,k} + A_{j,k}] = 0 \ \ \textit{for all different } \ i, j, k, \\ 2(\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} A_{i,j}) = 0. \end{cases}$$

So the element $\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} A_{i,j}$ of order 2 generates the central subalgebra in $gr^*_\Gamma(P_n(S^2))$.

4. Universal Vassiliev invariants for M_n and $B_n(S^2)$

We sketch briefly the basic ideas of the theory of Vassiliev invariants for braids. For the classical braids (i.e. of a disc) it can be found, for example in [22].

Let A be an abelian group, then the group V of all maps (non necessary homomorphisms) from $B_n(S^2)$ to A is called the group of *invariants* of $B_n(S^2)$:

$$V = \mathrm{Map}(B_n(S^2), A).$$

If A is a commutative ring then V becomes an A-module.

Let $\mathbb{Z}[B_n(S^2)]$ be the group ring of the group $B_n(S^2)$, then

$$\mathrm{Map}(B_n(S^2), A) = \mathrm{Hom}(\mathbb{Z}[B_n(S^2)], A).$$

where $\mathrm{Hom}(\mathbb{Z}[B_n(S^2)], A)$ is an abelian group of homomorphisms of the group $\mathbb{Z}[B_n(S^2)]$ into the group A.

We can enlarge an invariant $v \in V$ for singular braids using the rule

$$v(\text{singular crossing of } i\text{-th and } i+1 \text{ strands}) = v(\sigma_i) - v(\sigma_i^{-1}).$$

The elements $\sigma_i - \sigma_i^{-1} \in \mathbb{Z}[B_n(S^2)]$, $i = 1, \dots, n-1$, generate an ideal of the ring $\mathbb{Z}[B_n(S^2)]$ which we denote by W; degrees of his ideal define a multiplicative filtration (*Vassiliev filtration*) $W^m = \Phi^m(\mathbb{Z}[B_n(S^2)])$. An invariants $v \in V$ is called *of degree* m if $v(x) = 0$ for all $x \in \Phi^{m+1}(\mathbb{Z}[B_n(S^2)])$. So the group V_m of invariants of degree m is defined as

$$V_m = \mathrm{Hom}(\mathbb{Z}[B_n(S^2)]/\Phi^{m+1}(\mathbb{Z}[B_n(S^2)]),\ A).$$

The advantage of braids is that this filtration can be characterized completely algebraically. Let S be a map from the symmetric group Σ_n:

$$S : \Sigma_n \to B_n(S^2)$$

which is a section of the canonical epimorphism $B_n(S^2) \to \Sigma_n$ (2.1). It is not a homomorphism which does not exist with such a condition. For example, we can set up $S(s_i) = \sigma_i$. Let I be the augmentation ideal of the group ring $\mathbb{Z}[P_n(S^2)]$. The powers of I generate a filtration of the ring $\mathbb{Z}[P_n(S^2)]$ and hence of the ring $\mathbb{Z}[P_n(S^2)] \otimes \mathbb{Z}[\Sigma_n]$.

Proposition 4.1. *There is an isomorphism of abelian groups with filtration*

$$\mathbb{Z}[P_n(S^2)] \otimes \mathbb{Z}[\Sigma_n] \cong \mathbb{Z}[B_n(S^2)],$$

which is induced by the canonical inclusion of the pure braids and the map S, the ring $\mathbb{Z}[B_n(S^2)]$ is equipped with Vassiliev filtration.

□

The same constructions can be done for the mapping class group M_n and the analogue of Proposition 4.1 is true.

Let c be the generator of the infinite cyclic group C and let $\mathbb{Z}[C]$ be the group ring of C. We denote by C_2 the cyclic group of the order 2 with the generator a, $\mathbb{Z}[C_2]$ is the group ring of C_2 and we define the homomorphism

$$\rho : \mathbb{Z}[C] \to \mathbb{Z}[C_2],$$

by the formula

$$\rho(c) = a.$$

Proposition 4.2. *There are isomorphisms of rings*

$$\mathbb{Z}[P_n] \cong \mathbb{Z}[PM_{n+1}] \otimes \mathbb{Z}[C],$$

$$\mathbb{Z}[P_n(S^2)] \cong \mathbb{Z}[PM_n] \otimes \mathbb{Z}[C_2],$$

which can be included into the following commutative diagram

$$\begin{tikzcd} \mathbb{Z}[P_n] \cong \arrow[d, "\rho_p"] & \mathbb{Z}[PM_{n+1}] \otimes \mathbb{Z}[C] \arrow[d, "\delta\otimes\rho"] \\ \mathbb{Z}[P_n(S^2)] \cong & \mathbb{Z}[PM_n] \otimes \mathbb{Z}[C_2], \end{tikzcd}$$

where the morphisms ρ_p and δ in the diagram are induced by the corresponding morphisms of the diagram (2.8).

Proof. It follows from Theorem 2.2. □

Proposition 4.3. *The intersections of Vassiliev filtration for $\mathbb{Z}[B_n(S^2)]$ and $\mathbb{Z}[M_n]$ are trivial*

$$\bigcap_{m\geq 0} \Phi^m(\mathbb{Z}[B_n(S^2)]) = 0, \qquad \bigcap_{m\geq 0} \Phi^m(\mathbb{Z}[M_n]) = 0.$$

The groups $\Phi^m(\mathbb{Z}[M_n])/\Phi^{m+1}(\mathbb{Z}[M_n])$ are torsion free. The group PM_n is residually torsion free nilpotent, the group $P_n(S^2)$ is residually nilpotent.

Proof. Proposition 4.1 reduces the question to the I-adic filtration of $\mathbb{Z}[PM_n]$ and $\mathbb{Z}[P_n(S^2)]$. In the case of $\mathbb{Z}[PM_n]$ the situation is the same as for the pure braid group of a disc, PM_n is a subgroup of P_{n-1} (cf Theorem 2.2) and hence it is residually torsion free nilpotent. For the group ring $\mathbb{Z}[P_n(S^2)]$ because of its structure as the tensor product (Proposition 4.2) we have

$$I^m(\mathbb{Z}[P_n(S^2)]) \cong \mathrm{Im}(\oplus_{k=0}^m I^{m-k}(\mathbb{Z}[PM_n]) \otimes I^k(\mathbb{Z}[C_2]) \to \mathbb{Z}[P_n(S^2)]),$$

where Im denotes as usual the image of the map given in parentheses. This group ring can be also presented as the following direct sum

$$\mathbb{Z}[P_n(S^2)] \cong \mathbb{Z}[PM_n] \oplus \mathbb{Z}[PM_n]\{(a-1)\} \tag{4.1}$$

where by a we denoted the generator of cyclic group C_2 of order 2. For an element $z \in \mathbb{Z}[PM_n]$ let

$$z = \sum_i n_i b_i.$$

be a decomposition of z with respect to the base of the free abelian group $\mathbb{Z}[PM_n]$. Let the greatest common divisor of the set of numbers $\{n_i\}$ be equal to $2^k n$ where n is coprime with 2. Let $v(z)$ be the order function (see [3, chapter III, § 2.2], for example) of the I-adic filtration on $\mathbb{Z}[PM_n]$ then the order function $v_2(z)$ of the I-adic filtration on the second summand in the right hand part of formula (4.1) is given as follows

$$v_2(z(a-1)) = v(z) + k + 1.$$

The intersection $\bigcap_{k=0}^{\infty} I^k(\mathbb{Z}[C_2])$ is trivial, the group $\mathbb{Z}[PM_n]$ is residually torsion free nilpotent, so

$$\bigcap_{k=0}^{\infty} I^k(\mathbb{Z}[P_n(S^2)]) = 0.$$

We remind that integral dimension subgroups (for the group $P_n(S^2)$) are defined by the formula

$$D_m(P_n(S^2)) = P_n(S^2) \cap (1 + I^m),$$

where the intersection is taken in $\mathbb{Z}[P_n(S^2)]$. Hence the intersection of $D_m(P_n(S^2)$ is trivial the group $P_n(S^2)$ is residually nilpotent (of course not residually torsion free nilpotent); the last fact can be also seen directly. □

The filtered algebra $\mathcal{P}_n$ is defined as the universal enveloping algebra of the Lie algebra $gr^*_\Gamma(P_n)$ for the standard pure braid group

$$\mathcal{P}_n = U(gr^*_\Gamma(P_n)).$$

Its completion $\widehat{\mathcal{P}_n}$ is the target of the universal Vassiliev invariant for the pure braids [22]

$$\mu : \mathbb{Z}[P_n] \to \widehat{\mathcal{P}_n}.$$

Let $\mathcal{PM}_n$ be the universal enveloping algebra of the Lie algebra $gr^*_\Gamma(PM_n)$; so as an associative algebra it has the generators which are in one-to-one correspondence with the generators $B_{i,j}$ of $gr^*_\Gamma(PM_n)$, say it will be $x_{i,j}$, $1 \leq i, j \leq n$, which satisfy the associative form of relations (3.5). Also we denote by $\mathcal{P}_n(S^2)$ the universal enveloping algebra of the Lie algebra $gr^*_\Gamma(P_n(S^2))$.

As usual one can define a Hausdorff filtration (intersection is zero) on $\mathcal{PM}_n$ and on $\mathcal{P}_n(S^2)$ by giving a degree 1 to each generator $x_{i,j}$. The canonical epimorphism of groups $\rho_p : P_n \to P_n(S^2)$ induces an epimorphism of filtered algebras

$$\rho_a : \mathcal{P}_n \to \mathcal{P}_n(S^2)$$

We denote by $\widehat{\mathcal{PM}_n}$ the completion of $\mathcal{PM}_n$ with respect to the topology, defined by this filtration. The same way $\widehat{\mathcal{P}_n}(S^2)$ is the completion of $\mathcal{P}_n(S^2)$. The algebra $\widehat{\mathcal{PM}_n}$ can be also described as an algebra of noncommutative power series of $x_{i,j}$ factorized by the closed ideal generated by the left hand sides of relations (3.5).

Let $\widehat{\mathcal{A}}$ be an associative algebra with unit such that as an abelian group it is isomorphic to the direct sum of integers and 2-adic numbers $\mathbb{Z} \oplus \mathbb{Z}_2$. We denote the generator of the first summand by 1 and the generator of the second summand by x. The multiplication in $\widehat{\mathcal{A}}$ is given by the rule

$$x^2 = -2x.$$

This algebra is filtered as follows $\Phi^0 = \widehat{\mathcal{A}}$, $\Phi^1 = \mathbb{Z}_2$, Φ^m is generated by $2^m x$, for $m = 2, 3 \ldots$.

We define the homomorphisms

$$\alpha : \mathbb{Z}[C_2)] \to \widehat{\mathcal{A}},$$

$$\chi : \mathbb{Z}[C] \to \mathbb{Z}[[y]],$$

$$\beta : \mathbb{Z}[[y]] \to \widehat{\mathcal{A}}$$

by the formulae

$$\alpha(a) = 1 + x, \quad \chi(c) = 1 + y, \quad \beta(y) = x.$$

Proposition 4.4. *The homomorphisms of rings α and χ respect the filtration and induce a multiplicative isomorphism at the associated graded level. They fit in the following commutative* *diagram of homomorphisms of rings.*

$$\begin{tikzcd}
\mathbb{Z}[C] \arrow[r, "\rho"] \arrow[d, "\chi"] & \mathbb{Z}[C_2] \arrow[d, "\alpha"] \\
\mathbb{Z}[[y]] \arrow[r, "\beta"] & \widehat{\mathcal{A}}.
\end{tikzcd}$$

Proof. It suffices to verify that

$$(a-1)^2 = a^2 - 2a + 1 = 2 - 2a = -2(a-1).$$

□

Proposition 4.5. *There are isomorphisms of filtered rings*

$$\widehat{\mathcal{P}}_n \cong \widehat{\mathcal{PM}}_{n+1} \widehat{\otimes} \mathbb{Z}[[y]],$$

$$\widehat{\mathcal{P}}_n(S^2) \cong \widehat{\mathcal{PM}}_n \widehat{\otimes} \widehat{\mathcal{A}},$$

which can be included into the following commutative diagram of filtered ring homomorphisms

$$\begin{tikzcd}
\widehat{\mathcal{P}}_n \cong \arrow[d, "\widehat{\rho}_a"] & \widehat{\mathcal{PM}}_{n+1} \widehat{\otimes} \mathbb{Z}[[y]] \arrow[d, "\widehat{\delta} \widehat{\otimes} \beta"] \\
\widehat{\mathcal{P}}_n(S^2) \cong & \widehat{\mathcal{PM}}_n \widehat{\otimes} \widehat{\mathcal{A}},
\end{tikzcd}$$

where the morphisms $\widehat{\rho_a}$ and $\widehat{\delta}$ in the diagram are induced by the corresponding morphisms of the diagram (2.8).

Proof. The statements follow from the facts about the direct product of the Lie algebras similar to Theorem 2.2. □

The maps

$$\kappa : \mathbb{Z}[PM_n] \to \widehat{\mathcal{PM}}_n$$

and

$$K_M : \mathbb{Z}[M_n] \to \widehat{\mathcal{M}}_n$$

can be defined following the same steps as the definition of the universal Vassiliev invariant in [22]. However it is more simple to use the universal invariant from [22, Theorem 1.1] and define κ as the following composition

$$\mathbb{Z}[PM_{n+1}] \to \mathbb{Z}[P_n] \xrightarrow{\mu} \widehat{\mathcal{P}}_n \to \widehat{\mathcal{PM}}_{n+1},$$

where the first map is the canonical inclusion and the last one is the canonical projection. We can also reason inversely: at first construct κ, then define the map $\widehat{\kappa \otimes \chi}$ as the composition

$$\mathbb{Z}[PM_{n+1}] \otimes \mathbb{Z}[C] \xrightarrow{\kappa \otimes \chi} \mathcal{PM}_{n+1} \otimes \mathbb{Z}[[y]] \to \mathcal{PM}_{n+1} \widehat{\otimes} \mathbb{Z}[[y]],$$

where the last map is the completion, and then define μ using the following diagram

$$\begin{tikzcd}
\mathbb{Z}[P_n] \arrow[d, "\mu"] & \cong \mathbb{Z}[PM_{n+1}] \otimes \mathbb{Z}[C] \arrow[d, "\widehat{\kappa \otimes \chi}"] \\
\widehat{\mathcal{P}}_n \cong & \widehat{\mathcal{PM}}_{n+1} \widehat{\otimes} \mathbb{Z}[[y]].
\end{tikzcd} \tag{4.2}$$

The map μ defined by (4.2) is a universal Vassiliev invariant for the classical braids, though it may not coincide with the map constructed in [22] which is not unique.

Theorem 4.1. *The map*

$$\kappa : \mathbb{Z}[PM_n] \to \widehat{\mathcal{PM}}_n$$

respects the filtration and induces a multiplicative isomorphism at the associated graded level.

Proof. Two proofs can be done. The first one is to follow the steps of the proof of the same fact for the classical pure braid groups [22]; this is possible because the group PM_n has the similar structure as P_n: it is an iterated semidirect product of free groups [10]. In particular the algebra $\mathcal{PM}_n$ has no torsion and the Quillen map [23]

$$\mathcal{PM}_n \to gr_I^* \mathbb{Z}[PM_n] \tag{4.3}$$

to the graded object associated to the I-adic filtration of the ring $\mathbb{Z}[PM_n]$ becomes an isomorphism with $gr^*(\kappa)$ as its inverse. Another proof is to apply the fact that P_n is the product of PM_n with the infinite cyclic group generated by the center of P_n, and then use Proposition 4.2 and diagram (4.2). □

We define the map

$$\lambda : \mathbb{Z}[P_n(S^2)] \to \widehat{\mathcal{P}_n}(S^2)$$

using the following diagram

$$\begin{tikzcd}
\mathbb{Z}[PM_n] \otimes \mathbb{Z}[C_2] \arrow[d, "\widehat{\kappa \otimes \alpha}"] & \cong \mathbb{Z}[P_n(S^2)] \arrow[d, "\lambda"] \\
\widehat{\mathcal{PM}}_n \widehat{\otimes} \widehat{\mathcal{A}} & \cong \widehat{\mathcal{P}_n}(S^2).
\end{tikzcd}$$

Theorem 4.2. *The map*

$$\lambda : \mathbb{Z}[P_n(S^2)] \to \widehat{\mathcal{P}_n}(S^2)$$

respects the filtration, induces a multiplicative isomorphism at the associated graded level and fits in the following diagram of filtered rings

$$\begin{tikzcd}
\mathbb{Z}[P_n] \arrow[r, "\rho_p"] \arrow[d, "\mu"] & \mathbb{Z}[P_n(S^2)] \arrow[d, "\lambda"] \\
\widehat{\mathcal{P}}_n \arrow[r, "\widehat{\rho_a}"] & \widehat{\mathcal{P}_n}(S^2).
\end{tikzcd} \tag{4.4}$$

Proof. The maps κ and α respect the filtration and induce a multiplicative isomorphism at the associated graded level, so this is true for $\widehat{\kappa \otimes \alpha}$. We continue diagram (4.2)

$$\begin{tikzcd}
\mathbb{Z}[P_n] \cong \mathbb{Z}[PM_{n+1}] \otimes \mathbb{Z}[C] \arrow[r, "\delta \otimes \rho"] \arrow[d, "\mu"] \arrow[d, shift left=8ex, "\widehat{\kappa \otimes \chi}"] & \mathbb{Z}[PM_n] \otimes \mathbb{Z}[C_2] \cong \mathbb{Z}[P_n(S^2)] \arrow[d, "\widehat{\kappa \otimes \alpha}"] \arrow[d, shift left=8ex, "\lambda"] \\
\widehat{\mathcal{P}}_n \cong \widehat{\mathcal{PM}}_{n+1} \widehat{\otimes} \mathbb{Z}[[y]] \arrow[r, "\widehat{\delta} \widehat{\otimes} \beta"] & \widehat{\mathcal{PM}}_n \widehat{\otimes} \widehat{\mathcal{A}} \cong \widehat{\mathcal{P}}_n(S^2).
\end{tikzcd}$$

Its outer frame is diagram (4.4). $\square$

The symmetric group Σ_n acts on the algebras $\widehat{\mathcal{PM}}_n$ and $\widehat{\mathcal{P}}_n(S^2)$ by the action on the indices of $x_{i,j}$:

$$\sigma(x_{i,j}) = x_{\sigma(i),\sigma(j)}, \quad \sigma \in \Sigma_n.$$

This action preserves the defining relations (3.5) and (3.3). We define the following filtered algebras as the semidirect products with respect to the given action:

$$\widehat{\mathcal{M}}_n = \widehat{\mathcal{PM}}_n \rtimes \mathbb{Z}[\Sigma_n], \tag{4.5}$$

$$\widehat{\mathcal{B}}_n(S^2) = \widehat{\mathcal{P}}_n(S^2) \rtimes \mathbb{Z}[\Sigma_n]. \tag{4.6}$$

According to the Markov normal form for $B_n(S^2)$ proved by R. Gillet and J. Van Buskirk in [10] every element b of $B(S^2)$ can be written uniquely in the form

$$b = qS(p),$$

where $q \in P_n(S^2)$ and p is the permutation defined by the braid b. We define the map

$$K : \mathbb{Z}[B_n(S^2)] \to \widehat{\mathcal{B}}_n(S^2)$$

by the formula

$$K(b) = \lambda(q) \otimes p. \tag{4.7}$$

Theorem 4.3. *The homomorphisms of abelian groups*

$$K_M : \mathbb{Z}[M_n] \to \widehat{\mathcal{M}}_n,$$

$$K : \mathbb{Z}[B_n(S^2)] \to \widehat{\mathcal{B}}_n(S^2)$$

are injections, they respect the filtration, induce a multiplicative isomorphism at the associated graded level and fit in the following diagram of filtered rings

$$\begin{tikzcd}
\mathbb{Z}[P_n(S^2)] \arrow[r, "\rho_p"] \arrow[d, "\mu"] & \mathbb{Z}[M_n] \arrow[d, "\lambda"] \\
\widehat{\mathcal{P}(S^2)}_n \arrow[r, "\widehat{\rho}_a"] & \widehat{\mathcal{M}}_n.
\end{tikzcd}$$

Proof. It follows from Proposition 4.1, Theorem 4.2 and definitions (4.5 – 4.6), (4.7). The part about injectivity follows from Proposition 4.3. $\square$

Corollary 4.1. *The groups $\mathbb{Z}[M_n]/\Phi^m(\mathbb{Z}[M_n])$ and $\widehat{\mathcal{M}_n}/\Phi^m(\widehat{\mathcal{M}_n})$ are isomorphic and are torsion free. There are also isomorphisms of abelian groups*

$$\mathbb{Z}[B_n(S^2)]/\Phi^m(\mathbb{Z}[B_n(S^2)]) \cong \widehat{\mathcal{B}_n}(S^2)/\Phi^m(\widehat{\mathcal{B}_n}(S^2)) \cong \bigoplus_{k=0}^{m-1}(\widehat{\mathcal{M}_n}/\Phi^{m-k}(\widehat{\mathcal{M}_n})) \otimes \mathbb{Z}/2^k.$$

□

Corollary 4.2. *Any Vassiliev invariant of $B_n(S^2)$ to an abelian group without 2-torsion does not distinguish every couple of different elements of $B_n(S^2)$. Vassiliev invariants to a group which has an element of order 2 distinguish any couple of different elements in $B_n(S^2)$.*

In [5, 6] M. Eisermann gave the example of a couple of elements in $B_n(S^2)$ which are not distinguished by Vassiliev invariants in rational numbers. Corollary 4.2 explains this situation.

5. Example

The pure braid group $P_4(S^2)$ of a 2-sphere is isomorphic to the direct product of the cyclic group of order 2 (generated by Δ^2) and the pure braid group on one strand of a 2-sphere with three points deleted, it is the fundamental group of disc with two points deleted, that is a free group F_2 on two generators. Its associated graded Lie algebra is a direct sum of central $\mathbb{Z}/2$ and the free Lie algebra on two generators. The pure mapping class group $PM_{0,4}$ is isomorphic to a free group on two generators. According to Theorem 3.1 its associated graded Lie algebra is the free Lie algebra on two generators. The universal Vassiliev invariant for $PM_{0,4}$ is nothing but Magnus expansion $\mathbb{Z}[F_2] \xrightarrow{\mu_e} \mathbb{Z}\langle\langle x_1, x_2\rangle\rangle$ and the universal invariant for $P_4(S^2)$ is

$$\mathbb{Z}[F_2] \otimes \mathbb{Z}[C_2] \overset{\widehat{\mu_e \otimes \alpha}}{\longrightarrow} \mathbb{Z}\langle\langle x_1, x_2\rangle\rangle \widehat{\otimes} \widehat{\mathcal{A}}.$$

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