How to take advantage of the blur between the finite and the infinite

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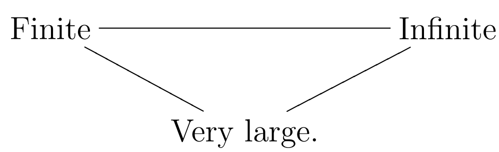
In order to come within the scope of the congress theme, I would like to speak about not the *square of opposition* but the *triangle of contradiction* which one finds when one centres one's interest on the integers and the infinite. It is usual indeed to contrast the finite with the infinite. I claim that there is a grey area in between, where the large numbers are, hence the following triangle:



According to Cantor's and Dedekind's definition, a set is *infinite* if it can be put in bijection with a proper subset. A set is therefore *finite* if it cannot be put in bijection with a proper subset, that is, a subset strictly contained within itself. From this, it follows that $\mathbb{N}$, the set of integers, is infinite. Indeed, as remarked already by Galileo, much to his surprise, the doubling process puts in a one-to-one correspondance the numbers $0, 1, 2, 3, \ldots$ with the even numbers $0, 2, 4, 6, \ldots$ Hence the collection of even numbers is as large as the collection of all numbers. That is, against a well-known axiom of Euclides, the part is as big as the whole!

Mathematical induction

A basic feature of the ordinary numbers is the possibility to *count*, that is to enunciate them in order

$$\text{zero, one, two, three,} \ldots$$

where each number is named following its predecessor, without any omission. What makes this sequence infinite, is that it never stops: if a child can count up to one thousand, he can go further and name one thousand and one.

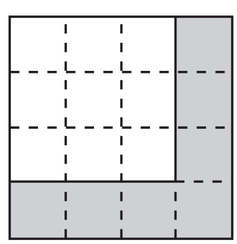
A mathematical principle, expressing this intuitive remark, is the *principle of mathematical* (or complete) *induction*. Euclides knew already that the sum of odd numbers in order is a square, namely

$$\begin{aligned}
&1 = 1^2\,,\\
&1+3 = 2^2\,,\\
&1+3+5 = 3^2\,,\\
&1+3+5+7 = 4^2\,,\\
&\ldots\ldots\ldots\ldots
\end{aligned}$$

This can be checked by direct calculation in any given case, but in order to make it a general rule, we need a systematic procedure. We can use the following sequence of equalities

$$\begin{aligned}
&1^2 = 1\,,\\
&2^2 = 1^2 + 3\,,\\
&3^2 = 2^2 + 5\,,\\
&4^2 = 3^2 + 7\,,\\
&\ldots\ldots\ldots\ldots
\end{aligned}$$

Each of these identities can be visualized using the “gnomon” (Greek word for “tee-square”), as known to Euclides.



$4^2 = 3^2 + 7$

Hence, the calculation of the sum of the first four odd numbers requires four steps, and similarly we need thousands steps for the sum of thousands odd numbers. In the classical age, one would be satisfied in describing explicitely the first few steps, and concluding with "and so forth", when the pattern of each step is sufficiently clear.

To prove the veracity of a property $P(n)$ for every integer n, one uses *mathematical induction*. Historically speaking, one can trace the standard principle of induction back to Fermat and Pascal. How was the reasoning done before them? As in the previous example of the gnomon, one would explain how to go from 0 to 1, then from 1 to 2, then from 2 to 3, and so forth... To know the veracity of $P(17)$, one conducted consequently 17 demonstrations. More exactly, the ellipsis following "and so forth" means that the reader was asked to fill in the blank, with only the indication of how to do it. What is new with Pascal (especially in his work about the *arithmetic triangle*), is that one goes from n to $n+1$ in full generality. This lies on the progress in algebra made by Viète in the previous century. A single demonstration is made now! After checking the starting point, obviously. It is accepted as a logical principle that this sole demonstration replaces all intermediate demonstrations. But does this sole demonstration actually replace 100 000 billion demonstrations such?

The principle of complete induction was reformulated by Peano at the end of the 19^{th} century using his logical formal language:

The property[1] "$\forall\, n \geq 0 \cdot P(n)$" is implied by

$$\text{“}P(0) \text{ and } \forall\, n \geq 1 \cdot (P(n) \Rightarrow P(n+1))\text{”}\,.$$

Peano proceeded then to define the elementary operations (sum, product, exponentiation) on numbers and to prove their elementary properties using this general principle. Poincaré was very impressed by this achievement, and both Peano and Poincaré claimed that it was there the foundation of all existing mathematics.

[1] That is, for all integers $n \geq 0$, $P(n)$ is valid.

Infinite descent (or regression)

Looking for an explanation (or for a proof) can lead to unlimited regression:

– Why?

– Because so and so.

– But then why?

– Because...

This was illustrated in a comic vein by Lewis Carroll in the funny logical tale of Achilles and the tortoise. In the mathematical practice, it has been accepted since Euclides that one should stop questioning somewhere, and start from *axioms*, that is unchallenged truths.

Contemporary of Pascal, Fermat made a very ingenious application of such an *infinite descent*. Fermat wanted to prove that a fourth power is never a sum of two fourth powers. He argues by contradiction: if there is an example of the form $a^4 = b^4 + c^2$ with positive integers a, b, c, we can produce out of it another example $a'^4 = b'^4 + c'^2$, smaller in the sense that $a' + b' + c'$ is smaller than $a + b + c$. Nothing prevents us to continue forever. Let us compare this to the ordinary induction:

- start from 0,
- increase by one, n becoming $n + 1$,
- continue forever.

In Fermat's case:

- start from some value of the index $a + b + c$,
- decrease the index by a variable amount,
- continue forever.

Is this possible?

By way of explanation, let us use a metaphor. We are in an elevator, which can only descend. From time to time, it will stop; if the door doesn't open, press the button to restart it. When the elevator reaches the ground floor, it will stop there and the door will open. Common sense tells us that the nightmare will end up, and we will be able to leave at the ground floor. More seriously, it is impossible to have a sequence of *positive integral numbers* which decreases forever

$$a_1 > a_2 > a_3 > \ldots > a_n > a_{n+1} > \ldots$$

With *fractional numbers*, this is possible, as seen from the example

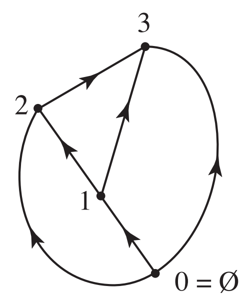
$$1 > \frac{1}{2} > \frac{1}{4} > \frac{1}{8} > \ldots$$

which lies at the heart of Zeno's paradox (Achilles and the tortoise).

Assuming the impossibility of infinite descent, Fermat concludes, by *reductio ad absurdum*, that the original assumption $a^4 = b^4 + c^2$ is impossible. Let us reflect for a while at this proof. Fermat is bold in two ways: first, he imagines an infinite construction in a virtual world, which at the end will be shown not to exist! Second, he introduces a *new logical principle*: the nonexistence of infinite descending sequences. This logical principle is logically equivalent to the following: every nonempty set of positive integers contains a smallest element. Using Cantor's definitions, this states that the positive integers, linearly ordered in the usual way, form a *well-ordered set*. Notice that, logically speaking, this is a *second-order property:* it is not a statement about individual numbers, but about sets (or properties) of numbers. In the proper logical environment, the nonexistence of infinite descent is just another, more sophisticated, version of the principle of complete induction. Notice also that the *reductio ad absurdum* is accomplished, not by exhibiting an explicit contradiction, but by contradicting an abstract logical principle!

Let us add a few remarks. First, the method of infinite descent was resurrected by Mordell in 1922, and by André Weil in his thesis (1928) for arithmetical purposes. At about the same time, Emmy Noether introduced the similar principle of *minimal chain condition* (and the maximal one), in the theory of ideals, and this is now a widely used method in algebra. Second, the introduction of topoi in logic has given way to a deeper understanding of the various logical forms of the principle of complete induction, as well as of Dedekind's definition of a finite set. Finally, the *axiom of foundation* was proposed in set theory by von Neumann: it is not possible to have an infinite sequence of sets $X_0, X_1, X_2, \ldots$ where X_1 is an element of X_0, X_2

is an element of $X_1, \ldots$ This axiom allows to view sets (and in particular numbers) in a hierarchical way starting from the empty set $\emptyset$[2].



An arrow in this diagram corresponds to membership:
the arrow $2 \to 3$ means that 2 is an element of 3!

Jacob's ladder and the grey zone

The principle of complete induction can be depicted as Jacob's ladder: man can ascend with it from the earth to the sky, and angels can descend from the sky to the earth.

That is, there is no gap between the earth and the sky. Nevertheless, it seems natural to admit that there is a horizon on both ends, that there are altitudes we will not dare to climb to, and that from the sky, it will be difficult to imagine how to go down to the ground. There is room for a grey zone, or a cloudy level, impenetrable to the eyesight from both sides. This is particularly clear in ethical problems, where the distinction between white and black is not that sharp. To define the precise boundary between life and death, or between the living and the non-living in biology, is foolish.

But in the surest of all sciences, mathematics, the boundary is also not completely sharp between truth and falsity. There are many historical examples where a revision was necessary: Euclides was considered the perfect model, until a revision was felt necessary in the 18$^{\text{th}}$ century (Legendre) and finally accomplished (after Pasch, Veronese and Peano) by D. Hilbert in his famous book "Grundlagen der Geometrie". For instance, one of the first constructions in Euclides' book I is of an equilateral triangle. This is accomplished by drawing circles and looking at their intersection, but nothing in

[2] 0 is the empty set $\emptyset$, 1 is $\{0\}$, 2 is $\{0,1\}$, *etc.*

Euclides' axiomatics guarantees (what is clear intuitively) that circles should meet in this construction.

A more difficult situation arose at the beginning of calculus. The notion of a curve is not defined in a way which enables to guarantee the existence of a tangent; the length of a circle, the area within a curve, were not properly defined notions. Galois himself, as a high school student, pretended to prove the existence of the derivative of an "arbitrary" function, a few years before the counter-examples of Liouville. The shaky foundations of calculus by Leibniz or Newton, using infinitesimals or fluxions, were accepted with some embarrassment until the revision by Cauchy, Weierstrass,... in the 19^{th} century. But this didn't hamper a fruitful development in the hands of Newton, Euler, Lagrange and many others during the 18^{th} century.

In modern times, after the invention of set theory by Cantor, a major revision of mathematics occurred, first suggested by Hilbert, and put in textbook form in the treatise of Bourbaki. Every mathematical notion is now reduced to a specific set-theoretic construction. The notion of set (or class), and the activity of classifying being so fundamental, premathematical in their essence, there can be no doubt about the meaning of a set, and about the operations on sets. Nevertheless, to discard annoying paradoxes, it was felt necessary to rely on an axiomatic basis which is far from being intuitive. Also, the mathematical practice, in its daily use of categories, plays with dangerous notions like the category of all sets, worst the category of all categories. This can be cured by resorting to *large cardinals*, but the lack of evidence for the existence of these objects, as well as some logical difficulties (non-provability, doubts about consistency) leave the "working mathematician" in a state of uneasiness.

To borrow an allegory from Hermann Weyl, the mathematical activity takes place in a playground well lit by the sun of axiomatics. But it is surrounded by bushes, muddy and full of traps – another grey zone. It is dangerous to wander there, but knowledge progresses by curiosity: take your chances! I don't believe in absolute rigour, but I'm a pragmatic. Every good mathematical invention will survive, albeit perhaps in a different ideological environment.

Mastering the numbers

Mathematicians are creators of virtual and ideal realities. These realities make sense only when they are able to incarnate in a project, not of society but of civilization. Mathematics and its progress are part of civilization and heritage of civilization. The first mathematical tools were forged by the calculation of areas and lengths, also from astronomy seen as a science of usage to elaborate calendars. Navigation requires knowledge of the sky and the celestial motions. Lévi-Strauss, among other ethnologists, maintains that the beginning of social organization started with the invention of incest. By that, he means that humans lived before in hordes, in clans. Relations between males and females were not codified. From the moment incest was invented, that is the rule of exogamy, a spouse must be sought in another clan, so rules of negociation must be invented, so rules of cooperation between clans must be laid down. Gradually, as society develops, the economic needs create needs in storage, accounting, trade, currencies, and all this means a good handling of numbers and arithmetic. My thesis is that if mathematics creates ideal objects, these objects are motivated by civilization needs and mathematicians give back to society what they have created. This can be seen across a generation. I maintain that in my childhood, negative numbers were not perceived by the ordinary citizen, whereas today, it is clear that negative numbers are part of common heritage in our countries. Lazare Carnot, a great statesman and a great mathematician, wrote in one of his deepest books a whole diatribe against negative numbers.

On the side of physics, there has been since the early 20$^{\text{th}}$ century this extraordinary development which goes in the two directions as defined by Pascal, the two infinite, the infinitely large and the infinitely small[3]. The infinitely small is the resurrection of atomistic ideas in the early 20$^{\text{th}}$ century, but with a change of status which allows atoms to move from virtual, intellectual reality to something accessible to the outside by measurements. Einstein's first major scientific contribution is to propose a method for determining Lodschmit-Avogadro's number, that is roughly the number of atoms in 1g of hydrogen or 12g of carbon. Now, until Einstein, one had no idea how to measure this number. With the study of Brownian motion and other effects, Einstein provides a scale here: in the mathematical model of phenom-

[3]Pascal was one of the first to handle the infinitely small as a mathematical tool. He is one of the inventors of infinitesimal calculus, integral calculus more than differential.

ena appears explicitly this number of atoms per gram and in a measurable way. Here we discover with some amazement that in 1g of hydrogen there are 6×10^{23} atoms. It means 600 000 billion billion, it defies the imagination. At the same time, developments in astronomy and astrophysics have gone to another scale. It is only in the mid 19^{th} century that one has a first idea of the distance to the nearest star. Bessel was probably the first to give the first determinations. Following the measurement of the parallax effect[4], we find that the nearest star is about 3 light-years away. At the same period, the speed of light is measured (300 000 km/s). The power of 10 notation is introduced. All the development of science shows that the universe is becoming larger and larger, as Copernicus had known it and as Giordano Bruno proclaimed it at the peril of his life.

Visible and invisible numbers

Here is another way to see that there is a gap between the finite and the infinite. There are the visible and the invisible numbers. The visible numbers are the numbers which can be brought into the field of attentive, clear consciousness. They are sufficiently individualized to be non-confusing. For instance, one can distinguish in a single glance if there are 8 ou 10 objects on a table, but not between 20 and 22 for instance. But the boundaries of the invisible are constantly pushed back. The trick was to invent the compact representation of numbers, with the decimal system. It dates from five centuries in Europe. In many civilizations, numbers beyond 10 000 are not mastered. Now, we are used to handling much larger quantities, especially by using the power of ten notation, and the corresponding prefixes kilo, mega, giga, tera, peta,. . .

When I speak of the visible and invisible numbers, the 20^{th} century represents the momentary end of appropriation of very very large numbers by nonspecialists. That is the test of a scientific notion. It is the day when everyone captures it. The goal of sciences is to be disseminated as widely as possible. There has been all along in recent times an increasing power of numbers. There is still a horizon of invisible numbers, those for which we do not have a convenient mode of representation or intuition, and there are

[4]Something known theoretically to Copernicus, but too small to be observable before Bessel. This impossibility was the first hint of the real size of the universe.

the non-visible which are captured gradually with scientific progress and its dissemination.

All of this is my same concern. How does one capture scientific notions which are initially in the domain of the invisible? Today the best clocks measure time intervals of 10^{-17} second and distances are measured in billions of light-years, so roughly the travel of light in 10^{15} seconds. We have about 30 orders of magnitude. It is the great novelty of the 20$^{\text{th}}$ century physics. The paradox or the miracle of modern physics is that the same laws in physics govern large numbers and small numbers.

Here is the logical perspective which I defend. When you look at a painters' representation of mathematicians in the 16$^{\text{th}}$ century, they are usually represented with a compass or a ruler, a globe,... Mathematicians have tools. Mathematics does not do without tools. Beside this, contemporary logic has developed with formalization by using tools also. Formalization is the possibility to encode a reasoning unassailably. All the construction rules accepted in logic implicitly use a principle of recurrence. The repetition is used at 1$^{\text{st}}$ level, 2$^{\text{nd}}$ level, and so forth. There are programs which are made with loops. With formalization, there is the idea of materializing or incarnating the reasoning in the form of a symbolical sequence. The horizon is that every mathematical reasoning must be able to be encoded with extremely precise rules and without derogation. The thing is that, with the powers play, we get to numbers which exceed capacities[5]. In each construction stands the time factor. This introduces something new which is little taken into account in current expositions of logic. You will be told that every mathematical reasoning can be encoded with a formula, a sequence of symbols. But it is still necessary to write the sequence! We have wanted to ensure the accuracy of mathematical reasoning by mechanizing it, we have wanted to dismiss every resort to intuition, every reference to the object which would not be a mathematical ideality, but there is a reification, encoded reasoning is a reification, it is a new object itself submitted to physical constraints.

Whatever the definition of the most elementary object in the universe, the number of these objects is lesser than 10^{100}. Basically, all estimates are consistent. Now, with the stacked powers play, 10 power 10 power 10, and so forth, we are vastly beyond. It is not difficult to imagine a mathematical reasoning which would require a lot more symbols to be encoded properly.

[5]I listened recently to a statistician whose data base comprises 6×10^{14} units of information (the "character" or "byte").

We arrive in a new world of virtuality. The whole effort of formal logic has been to give this reality observable by the mind. But we go beyond all possibilities of real apprehension. The question which arises in logic is how to ensure construction against contradiction, that is, we do not want to be caught with our hands in the cookie jar, saying yes *and* no. We do not want to be called a liar. And what if our logical system contains a flaw which can be explained only through a text of a gigantic size, containing 10^{100} characters? It is possible that our logical system contains a contradiction, yet invisible, in the same meaning as the visible and invisible numbers: a contradiction beyond the horizon. This does not mean that we will never reach it. It is beyond the current horizon, which can even count for a vastly important time. On the one hand, we have no insurance that our logical systems contain no contradiction. On the other hand, we can try to cheat, that is, to deliberately work with logics which we know to be formally contradictory, provided the contradiction may not be manifested. What is the use of knowing that there is a trap beyond the horizon, since I will never go beyond the horizon anyway? The trap is so far that it does not disturb me. This has practical consequences: inconsistent, para-consistent logics. They are logics which conceal a contradiction sufficiently distant to be not reached. As the goal of logic is to speak reasonably, if we are not placed in front of its contradiction, then it does not exist.

Conclusion

To conclude, if we want to found non-standard analysis, if we want to handle infinitely large integers, there is a possibility of doing it by saying that we refine the way of seeing things. Within everyone's integers, we distinguish those which may be reached in rising from the finite and those which may be obtained only in descending from the infinite. We refine between the sky and the earth. Fermat's principle must be violated. We cannot accept that every set has a smallest element, because this would be the smallest infinite or the largest finite? We adapt Fermat's principle. It is true under certain conditions and not under others. Basically, Fermat's principle is applicable to every sufficiently explicit construction and it is not in a virtual or implicit construction. Among integers, we distinguish those which we can actually reach through a set in advance stock of operations. We can write them on a sheet of paper or on a computer screen. They are the accessible numbers.

There is the theoretical finite and the true finite. If we take this distinction into account, we allow ourselves a vast new freedom.

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