Tempered automorphic representations of the unitary group

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Juin 2011

IHES/M/11/17

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Abstract. Following Arthur's study of the representations of the orthogonal and symplectic groups, we prove many cases of both the local and global Arthur conjectures for tempered representations of the unitary group. This completes the proof of Arthur's description of the discrete series representations of the quasi-split p-adic unitary group, and Arthur's description of the tempered discrete automorphic representations of the unitary group, satisfying certain technical conditions.

1. Introduction

Arthur [Art05, §30] has announced a proof of both the local and global Arthur conjectures for irreducible admissible representations of the quasi-split groups SO_{2n+1}, SP_{2n}, and SO_{2n}. Following Arthur, the aim of this article is to prove the analogous results for tempered representations of inner forms of the quasi-split unitary group. We warn the reader that, for technical reasons, we shall in fact work under more restrictive hypotheses. The tempered setting admits a number of simplifications over the general setting. An ulterior aim of this article to provide a first step towards the general result. Concerning anterior results in this direction, we remark that Rogawski [Rog90] studied irreducible admissible representations of unitary groups in two and three variables, and Clozel-Harris-Labesse [CHL09] were the first to study endoscopic automorphic representations of higher rank unitary groups.

Let us begin by describing our local results. Let k'/k be a quadratic extension of p-adic fields, and let $U_n^*(k'/k)$ denote the associated quasi-split unitary group in n-variables. We remind the reader that the classification of the discrete series representations of $U_n^*(k'/k)$ has been completed by Mœglin [Mœg07]. The classification proceeds in two stages.

1. Arrange the discrete series representations of $U_n^*(k'/k)$ into L-packets, and classify the L-packets.
2. Classify the discrete series representations inside a given L-packet.

Mœglin arranges the discrete series representations into L-packets by requiring that the representations appearing in a given L-packet have the same Langlands base change to $GL_n(k')$. Mœglin shows that the L-packets are finite and disjoint, and calculates their cardinality. Mœglin then classifies the representations of $GL_n(k')$ that appear as the Langlands base change of a discrete series representation of $U_n^*(k'/k)$. These representations are the tempered θ-discrete stable representations (cf. Definition 3.7). Using the local Langlands correspondence for GL_n, due to Harris-Taylor and Henniart, Mœglin assigns to each L-packet Π of discrete series representations of $U_n^*(k'/k)$, the L-parameter of GL_n/k'

$$\psi : L_{k'} \to GL_n(\mathbf{C})$$

Date: June 4, 2011.
2000 *Mathematics Subject Classification.* Primary 11F72; Secondary 22E50.
Key words and phrases. automorphic forms, endoscopy, unitary group, tempered representations, trace formula.

which is associated to the Langlands base change of Π.

Mœglin [Mœg07] completed the second stage of the classification by using certain properties of Jacquet modules. Mœglin associated to each discrete series representation inside a given L-packet a character of a certain abelian group. We are interested in obtaining an alternative description predicted by the local Arthur conjectures. Let S_ψ be the centraliser of the image of ψ in $GL_n(\mathbf{C})$, and let S_ψ^θ be the subgroup of θ-invariant elements where θ is the degree 2 automorphism defined in Section 2. We shall study the quotient group $\mathbf{S}_\psi = S_\psi^\theta / \{\pm 1\}$. The group $\mathbf{S}_\psi$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^r$ for some non-negative integer r. Using the endoscopic properties of the representations in the L-packet Π, Arthur defines a pairing (see Section 10.2)

$$\langle \cdot, \cdot \rangle : \mathbf{S}_\psi \times \Pi \to \mathbf{C}$$

This pairing is canonical, up to the arbitrary choice of a representation $\sigma^{\text{base}} \in \Pi$. The local Arthur conjectures predict the following.

Theorem (A)**.** *The pairing $\langle \cdot, \cdot \rangle$ takes values in ± 1, and induces a bijection between the representations in the L-packet Π and the characters of $\mathbf{S}_\psi$.*

Proof. Theorem 10.6. □

If instead $k'/k \simeq \mathbf{C}/\mathbf{R}$, then the L-packets of discrete series representations of a real unitary group $U(p,q)$ were parameterised, in terms of L-parameters, by Langlands who applied previous work of Harish-Chandra. Let Π be an L-packet of discrete series representations of $U(p,q)$, and let $\psi : L_{k'} \to {}^L U(p,q)$ be the L-parameter associated to Π. One can perform the analogous constructions to those in the non-archimedean case (cf. Section 10.1). The result in this case is due to Shelstad [She08b] (cf. Theorem 10.2).

Theorem (B)**.** *The pairing $\langle \cdot, \cdot \rangle$ takes values in ± 1, and induces an injection from the representations in the L-packet Π to the characters of $\mathbf{S}_\psi$.*

Let us now describe our global results, which relate certain discrete automorphic representations of unitary groups to automorphic representations of GL_n. Let E/F be a totally imaginary quadratic extension of a totally real field, let $U_n^*(E/F)$ be the associated quasi-split unitary group in n-variables, and let U be an inner form of $U_n^*(E/F)$ that is quasi-split at all finite places. We shall be interested in the discrete automorphic representations σ of $U(\mathbf{A}_F)$ that satisfy the following properties.

- For all archimedean places ν, σ_ν is a discrete series representation with the same infinitesimal character as an irreducible algebraic representation of GL_n whose highest weight is regular (cf. Section 3.3.1).
- For all non-archimedean places ν that remain inert in E, σ_ν is either unramified or a discrete series representation.

The first global result is a mild generalisation of a result of Labesse [Lab09, Theorem 5.1, Theorem 5.9].

Theorem (C)**.** *There exists an automorphic representation $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r$ of $GL_n(\mathbf{A}_E)$ such that*

- *for all places ν, Π_ν is the Langlands base change of σ_ν,*
- *for all $i = 1, \ldots, r$, Π_i is cuspidal and $\Pi_i \simeq \Pi_i \circ \theta$, and*
- *for all $i \neq j$, $\Pi_i \not\simeq \Pi_j$.*

Proof. This is a special case of Theorem 6.1. □

We shall now consider the converse problem. Let $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r$ be an automorphic representation of $GL_n(\mathbf{A}_E)$ that satisfies the following properties.

- For all archimedean places ν of F, Π_ν is the Langlands base change of a discrete series representation of $U(F_\nu)$ with the same infinitesimal character as an irreducible algebraic representation of GL_n whose highest weight is regular.
- For all non-archimedean places ν of F that are inert and unramified in E, Π_ν is either unramified, or the Langlands base change of a discrete series representation of $U_n^*(E_\nu/F_\nu)$.
- For all non-archimedean places ν of F that are ramified in E, Π_ν is the Langlands base change of a discrete series representation of $U_n^*(E_\nu/F_\nu)$.
- For all $i = 1, \ldots, r$, Π_i is cuspidal and $\Pi_i \simeq \Pi_i \circ \theta$.
- For all $i \neq j$, $\Pi_i \not\simeq \Pi_j$.

Let σ be an irreducible admissible representation of $U(\mathbf{A}_F)$ whose Langlands base change is Π at all places.

The global Arthur conjectures predict the multiplicity with which σ appears in the discrete automorphic spectrum of $U(\mathbf{A}_F)$, which we shall now describe. Let S be the set of places ν of F such that either

- ν is archimedean, or
- ν is non-archimedean, inert in E, and Π_ν is the Langlands base change of a discrete series representation of $U_n^*(E_\nu/F_\nu)$.

For all places ω of E, let $\varphi_\omega : L_{E_\omega} \to GL_n(\mathbf{C})$ be the L-parameter corresponding to Π_ω. Let S_Π be the group of elements of $GL_n(\mathbf{C})$ that commute with the image $\varphi_\omega(z)$ in $GL_n(\mathbf{C})$ for all $z \in L_{E_\nu}$ and for all ω. Let S_Π^θ be the subgroup of θ-invariant points of S_Π. We shall study the quotient group $\mathbf{S}_\Pi = S_\Pi^\theta / \{\pm 1\}$ There exists a natural embedding, for all $\nu \in S$,

$$\mathbf{S}_\Pi \hookrightarrow \mathbf{S}_{\psi_\nu}$$

where ψ_ν denotes the L-parameter associated to σ_ν. The characters $\langle \cdot, \sigma_\nu \rangle : \mathbf{S}_{\psi_\nu} \to \{\pm 1\}$, defined for all $\nu \in S$, induce by restriction a character

$$\langle \cdot, \sigma \rangle = \prod_{\nu \in S} \langle \cdot, \sigma_\nu \rangle : \mathbf{S}_\Pi \to \{\pm 1\}$$

The global Arthur conjectures predicts the following.

Theorem (D). *There exists a unique character*

$$\epsilon_\Pi : \mathbf{S}_\Pi \to \{\pm 1\}$$

such that σ appears in the discrete automorphic spectrum of $U(\mathbf{A}_F)$ with multiplicity equal to

$$\mathrm{m_{disc}}(\sigma) = \begin{cases} 1 & : \text{if } \langle \cdot, \sigma \rangle = \epsilon_\Pi \\ 0 & : \text{otherwise} \end{cases}$$

Proof. Theorem 11.1 □

By combining the proved local and global Arthur conjectures, we obtain the following result.

Theorem (E). *Assume, in addition to the previous assumptions, that either*

- Π *is cuspidal, or*
- *there exists a non-archimedean place $\nu \in S$.*

Then there exists a σ as above, such that, σ appears in the discrete automorphic spectrum of $U(\mathbf{A}_F)$ with multiplicity 1.

Proof. Corollary 11.2 □

Remark 1.1. In the case where Π is cuspidal and $[F:\mathbf{Q}]>1$, this result is due to Labesse [Lab09, Theorem 5.4, Theorem 5.9].

The method of proof of these conjectures follows the work of Arthur [Art05, §30] on the proof of these conjectures for general representations of the symplectic and orthogonal groups. The proofs are mostly global in nature, and rely upon the stabilisation of both Arthur's invariant trace formula for the unitary group U and Arthur's invariant twisted trace formula for $GL_n \rtimes \theta$. The stabilisation of the invariant trace formula for a general connected reductive group was completed by Arthur [Art02] [Art01] [Art03] under the assumption of the validity of the weighted fundamental lemma. This is now a theorem due to the work of Chaudouard-Laumon [CL10a] [CL10b], Ngô [Ngô10], and Walspurger [Wal09]. As of the time of this writing, the complete stabilisation of the twisted trace formula for $GL_n \rtimes \theta$ is unknown, however the stabilisation of a simple version of the twisted invariant trace formula for $GL_n \rtimes \theta$ has been completed by Labesse [Lab09] and Morel [Mor10]. This simple stable trace formula imposes a number of additional constraints upon the choice of test functions, and it is for this reason that we have been forced to work under the hypotheses described in the statements of our results. One would expect that the complete stabilisation of Arthur's invariant trace formula for $GL \rtimes \theta$ would enable one to treat the general case, however that is beyond the aim of this article. We should also mention that, as of the time of writing, the results of this article are conditional upon certain expected results on the inner product of elliptic tempered representations of $GL \rtimes \theta$ (cf. Hypothesis 8.0.1). These results would follow from the generalisation of previous results of Arthur [Art93] to the twisted setting.

Let us describe the contents of this article. In Section 2, we recall the groups of interest to us. In Section 3, we recall some known cases of the local Langlands correspondence. In Section 4, we recall the base change, and endoscopic L-homomorphisms. In Section 5, we recall the necessary properties of the trace formula. In Section 6, we follow Labesse and apply the trace formula to prove our base change result. In Section 7, we recall a result of Shin on the existence of discrete automorphic representations of the unitary group satisfying certain local conditions, and then combine this with our base change result. In Section 8, we prove that the sum of the characters of the representations appearing in an L-packet Π of discrete series representations of the quasi-split p-adic unitary group is a stable distribution. Mœglin had previously shown that a linear combination of the representations in Π is stable, as such, we are reduced to showing that Mœglin's coefficients are equal to 1. The result follows from two numerical constraints upon the possible values of Mœglin's coefficients. The first constraint is that the coefficients are non-negative integers. This is shown by judiciously choosing automorphic representations satisfying certain local properties, and considering their contribution to the trace formula. The second constraint relates to the norm of the coefficients, and follows from certain local character identities. Section 9 is the heart of this article in which we prove certain properties of the spectral transfer factors via arguments similar to those of Section 8. In Section 10, we recall and prove the local Arthur conjectures. Section 11 contains a statement and proof of the global Arthur conjectures.

1.1. **Notation.** The strictly positive (resp. non-negative integers) shall be denoted by $\mathbf{N}$ (resp. $\mathbf{N}^0$). The archimedean Weil groups shall be written as $W_{\mathbf{C}} = \mathbf{C}^\times$, and $W_{\mathbf{R}} = \mathbf{C}^\times \sqcup j\mathbf{C}^\times$ where $j^2=-1$ and $jzj^{-1}=\overline{z}$ for all $z \in \mathbf{C}^\times$. Unless stated otherwise, a representations shall be assumed to be irreducible and admissible with complex coefficients. The term induced representation shall refer to the unitarily normalised induced representation.

1.2. **Acknowledgements.** I would like to thank Don Blasius, Gaëtan Chenevier, Laurent Clozel, Michael Harris, Florian Herzig, Jean-Pierre Labesse, Alberto Minguez, Colette Mœglin, and Jacques Tilouine for helpful discussions and comments on earlier versions of this article.

2. Some groups

We shall recall here the groups that shall be of interest to us. Let k'/k be a quadratic extension of either local or global fields of characteristic 0. Consider the group

$$\operatorname*{Res}_{k'/k} GL_n$$

where Res denotes the Weyl restriction of scalars. This groups admits a degree 2 automorphism

$$\theta = \theta_n : x \mapsto \Phi_n{}^t \left(x^c\right)^{-1} \Phi_n^{-1}$$

where $c \in \mathrm{Gal}\left(k'/k\right)$ denotes the non-trivial element and

$$\Phi_n = \begin{pmatrix} & & -1 \\ & 1 & \\ \cdots & \cdots & \cdots \\ (-1)^n & & \end{pmatrix}$$

The *quasi-split unitary group* in n-variables, denoted as either $U_n^*\left(k'/k\right)$, U_n^* or U^*, is the k-algebraic group of θ_n-invariant points of $\mathrm{Res}_{k'/k} GL_n$.

Let $B = M_n\left(k'\right)$ be the algebra of k'-valued $n \times n$ matrices. Let $\ddagger : B \to B$ be an involution of the second kind, that is, $\ddagger|_{k'} = c$. We define $G_\ddagger$ to be the algebraic group whose R-valued points are given by

$$G_\ddagger\left(R\right) = \left\{ g \in \left(B \otimes_k R\right)^\times : g^\ddagger g = 1 \right\}$$

for all k-algebras R. The algebraic group $G_\ddagger$ is an inner form of $U_n^*\left(k'/k\right)$, and there exists a canonical, up to conjugation, isomorphism

$$G_\ddagger \times_k k' \xrightarrow{\sim} GL_n$$

Let us enumerate some of the possible groups that are obtained via this construction.

- Assume that $k'/k \simeq \mathbf{C}/\mathbf{R}$. Then $G_\ddagger$ is isomorphic to one of the real unitary groups $U\left(p,q\right)$ where $p+q=n$.
- Assume that k'/k is an extension of p-adic fields. If n is odd then $G_\ddagger$ is isomorphic to the quasi-split unitary group $U_n^*\left(k'/k\right)$. If n is even then $G_\ddagger$ is isomorphic to either $U_n^*\left(k'/k\right)$ or the unique inner form of $U_n^*\left(k'/k\right)$ that is not quasi-split.

Consider the global setting where $k'/k = E/F$ is a totally imaginary quadratic extension of a totally real field. The groups $G_\ddagger$ satisfy the Hasse principle (cf. [HL04, §1.2]), that is, they are determined, up to isomorphism, by their local forms $G_{\ddagger,\nu} = G_\ddagger \times_F F_\nu$ where ν runs through the places of F. If ν is either real or, finite and inert in E, then the possible local forms are enumerated above. Let us consider the case where ν is finite and splits in E.

- Assume that $\nu = \omega\omega^c$ is finite and splits in E. Then $G_\ddagger \times_F F_\nu$ is isomorphic to GL_n/F_ν. The isomorphism is non-canonical; it essentially depends upon a choice of either ω or ω^c. It will be important to distinguish between these isomorphisms. Observe that

$$B \otimes_{F_\nu} E_\nu = \left(B \otimes_{F_\nu} E_\omega\right) \oplus \left(B \otimes_{F_\nu} E_{\omega^c}\right)$$

and that $\ddagger$ induces, by restriction, an isomorphism

$$\ddagger : B \otimes_{F_\nu} E_\omega \xrightarrow{\sim} B \otimes_{F_\nu} E_{\omega^c}$$

By projection onto the ω (resp. ω^c) component, we obtain the isomorphism

$$\iota_\omega : G_\ddagger \times_F F_\nu \xrightarrow{\sim} GL_n \text{ (resp. } \iota_{\omega^c} : G_\ddagger \times_F F_\nu \xrightarrow{\sim} GL_n)$$

which is canonical up to conjugation.

In order to fix the choice of isomorphism $G_\ddagger \times_F F_\nu \xrightarrow{\sim} GL_n$ at finite split places, we choose a set of places Q of E containing either ω or ω^c for each finite split place $\nu = \omega\omega^c$ of F, and then define

$$\iota_\nu = \iota_{\omega'} : G_\ddagger \times_F F_\nu \xrightarrow{\sim} GL_n$$

where $\omega' \in Q$.

The groups $G_\ddagger$ have been classified in the cases of interest to us.

Proposition 2.1. *Let E/F be a totally imaginary quadratic extension of a totally real field. Let $n \in \mathbf{N}$, and for all real places ν of F, let $p_\nu, q_\nu \in \mathbf{N}^0$ such that $p_\nu + q_\nu = n$. Then there exists an involution of the second kind $\ddagger$ such that*

- *$G_\ddagger \times_F F_\nu \simeq U(p_\nu, q_\nu)$ for all archimedean ν, and*
- *$G_\ddagger \times_F F_\nu$ is quasi-split for all finite ν*

if and only if $\prod_{\nu|\infty} \epsilon(U(p_\nu, q_\nu)) = 1$ where

$$\epsilon(U(p_\nu, q_\nu)) = \begin{cases} 1 & : \text{if } n \text{ is odd} \\ (-1)^{n/2 - p_\nu} & : \text{if } n \text{ is even} \end{cases}$$

Proof. [HL04, Proposition 1.2.3]. □

3. The local Langlands correspondence

We shall recall here the local Langlands correspondence in the cases of interest to us.

3.1. *L*-groups.

We shall explicitly recall the L-groups of interest to us (cf. [Bor79]).

Let k be either a local or global field of characteristic 0. For G a connected reductive k-algebraic group, the *L-group* is defined to be

$${}^L G = \widehat{G} \rtimes W_k$$

where $\widehat{G}$ denotes the Langlands dual group, and W_k the absolute Weil group which acts on $\widehat{G}$ via its natural action on the root datum. The action of the Weil group is non-canonical; it depends upon a choice of splitting for the dual group $\widehat{G}$, however, different choices of splittings give rise to canonically isomorphic L-groups. We recall that inner forms give rise to isomorphic L-groups.

- Assume that $G = GL_n/k$. Then $\widehat{G} = GL_n(\mathbf{C})$, and
$${}^L GL_n = GL_n(\mathbf{C}) \times W_k$$
- Assume that k'/k is a quadratic extension and that $G = U_n^*(k'/k)$. Then $\widehat{G} = GL_n(\mathbf{C})$. The Weil group W_k acts on $\widehat{G}$ through its projection onto $\mathrm{Gal}(k'/k) = \{1, c\}$ where c acts as follows.
$$\begin{aligned} GL_n(\mathbf{C}) &\to GL_n(\mathbf{C}) \\ g &\mapsto \Phi_n {}^t g^{-1} \Phi_n^{-1} \end{aligned}$$

- Assume that k'/k is a quadratic extension and that $G = \mathrm{Res}_{k'/k}\, U_n^*\,(k'/k) \times_k k'$. Then $\widehat{G} = GL_n\,(\mathbf{C}) \times GL_n\,(\mathbf{C})$. The Weil group W_k acts on $\widehat{G}$ through its projection onto $\mathrm{Gal}\,(k'/k) = \{1, c\}$ where c acts as follows.

$$\begin{aligned} GL_n(\mathbf{C}) \times GL_n(\mathbf{C}) &\to GL_n\,(\mathbf{C}) \times GL_n\,(\mathbf{C}) \\ g_1 \times g_2 &\mapsto \Phi_n{}^t g_2^{-1} \Phi_n^{-1} \times \Phi_n{}^t g_1^{-1} \Phi_n^{-1} \end{aligned}$$

3.2. ***L*-parameters.** Let k be a local field of characteristic 0.

The *Langlands group* is defined to be

$$L_k = \begin{cases} W_k & : \text{if } \nu \text{ is archimedean} \\ W_k \times SU_2\,(\mathbf{R}) & : \text{if } \nu \text{ is non-archimedean} \end{cases}$$

Let G be a connected reductive k-algebraic group. An *L-parameter* for G is a continuous homomorphism

$$\psi : L_k \to {}^L G$$

that satisfies the following conditions.

- For all $w \in L_k$, the image of $\psi\,(w)$ in W_k is the same as the image of w in W_k
- For all $w \in L_k$, $\psi\,(w)$ is semi-simple.

Two L-parameters are said to be *equivalent* if they are conjugate by an element of $\widehat{G}$. We shall also define the following properties of L-parameters.

- ψ is said to be *unramified* if
  - G is unramified,
  - ψ is trivial on the $SU_2\,(\mathbf{R})$-component, and
  - the composite map $W_k \overset{\psi}{\to} {}^L G \overset{1\times\mathbf{v}}{\twoheadrightarrow} \widehat{G} \rtimes \mathbf{Z}$ factors through the valuation map $\mathbf{v} : W_k \twoheadrightarrow \mathbf{Z}$
- ψ is said to be *tempered* if the image of $\psi\,(L_k)$ in ${}^L G$ is bounded.
- ψ is said to be *discrete* if $C\,(\psi)^0 \subset Z\left(\widehat{G}\,(\mathbf{C})\right)$ where $C\,(\psi)^0$ denotes the identity component of

$$C\,(\psi) = \left\{ g \in \widehat{G}\,(\mathbf{C}) : g\psi\,(w) = \psi\,(w)\,g \;\forall w \in L_k \right\}$$

- ψ is said to be *relevant* if the image of $\psi\,(L_k)$ does not lie in any parabolic subgroup unless the corresponding parabolic subgroup of G is defined over k. If G is quasi-split, then all L-parameters are relevant.

In the case of GL_n, there is the obvious bijection between L-parameters and continuous homomorphisms

$$\psi' : L_k \to GL_n(\mathbf{C})$$

such that $\psi'\,(w)$ is semi-simple for all $w \in L_k$. We shall use this bijection without comment throughout this article.

3.3. **The local Langlands correspondence.**

3.3.1. *The archimedean case.* The local Langlands classification here is due to Langlands [Lan89] (see also [Kna94]).

Proposition 3.1. *Let $k \in \{\mathbf{R}, \mathbf{C}\}$. Let G be a connected reductive k-group. To each equivalence class of relevant L-parameters $\psi : L_k \to {}^L G$, one can naturally associate $\Pi\,(\psi)$, a finite non-empty set of infinitesimal equivalence classes of irreducible admissible representations of $G\,(k)$. The L-packets $\Pi\,(\psi)$ are disjoint and their union is equal to the set of infinitesimal equivalence classes of irreducible admissible representations of $G\,(k)$.*

Let us recall the correspondence for discrete series representations of the real unitary groups $G = U(p,q)$ (cf. [Kot90, §7]). The discrete series representations of G are parameterised by the tempered discrete relevant L-parameters of G. The tempered discrete relevant L-parameters of G are of the form

$$\begin{aligned}\psi : W_{\mathbf{R}} &\to GL_n(\mathbf{C}) \rtimes W_{\mathbf{R}} \\ z &\mapsto \operatorname{diag}\left((z/\overline{z})^{p_1+\frac{n+1}{2}-1}, \ldots, (z/\overline{z})^{p_n+\frac{n+1}{2}-n}\right) \times z \\ j &\mapsto \Phi_n \times j\end{aligned}$$

where $p_1 \geq \cdots \geq p_n$ are integers. Write V_ψ for the algebraic representation of GL_n of highest weight $(p_1, \ldots, p_n)$ (relative to the standard torus and Borel subgroups). The representation V_ψ is said to have *regular* highest weight if $p_1 > \cdots > p_n$. The L-packet $\Pi(\psi)$ contains the discrete series representations of G whose infinitesimal character is equal to the infinitesimal character of V_ψ. It will be convenient to also denote the L-packet $\Pi(\psi)$ by $\Pi(V_\psi)$.

The elements of the L-packet $\Pi(\psi)$ can be parameterised, using Harish-Chandra's character formula, by elements of

$$\Omega_{\mathbf{R}}/\Omega \xrightarrow{\sim} \mathfrak{S}_n/\mathfrak{S}_p \times \mathfrak{S}_q$$

where $\Omega_{\mathbf{R}}$ (resp. Ω) denotes the real (resp. complex) Weyl group of $U(p,q)$. In particular the cardinality of the L-packet is equal to $|\Pi(\psi)| = \frac{n!}{p!q!}$

3.3.2. *The unramified case.* The unramified local Langlands correspondence is due to Langlands (see [Bor79]).

Proposition 3.2. *Let k be a p-adic field, and let G be an unramified k-algebraic group. Then to each equivalence class of unramified L-parameters $\psi : L_k \to {}^L G$ and to each conjugacy class K of hyperspecial subgroups of $G(k)$, one can naturally associate an equivalence class of K-unramified representation $\pi(\psi, K)$ of $G(k)$. This correspondence induces a bijection between pairs (ψ, K) and equivalence classes of unramified representations of $G(k)$.*

Remark 3.3. Recall that an irreducible admissible representation π of $G(k)$ is said to be *K-unramified* if $\pi^K \neq 0$.

Remark 3.4. The unramified representations associated to an unramified L-parameter ψ are expected to form a subset of the conjectured L-packet associated to ψ. In general, the L-packet will contain additional representations that are not unramified.

Remark 3.5. If $G = GL_n$, then there exists a single conjugacy class of hyperspecial subgroups of $G(k)$. For general groups, see [Tit79].

3.3.3. GL_n. The local Langlands correspondence for GL_n is due to Harris-Taylor [HT01] and Henniart [Hen00].

Proposition 3.6. *Let k be a p-adic field. Then to each equivalence class of L-parameters $\psi : L_k \to {}^L G$, one can naturally associate $\Pi(\psi)$, a set consisting of the equivalence class of a single irreducible admissible representations of $G(k)$. Furthermore, the L-packets $\Pi(\psi)$ are disjoint and their union is equal to the set of equivalence classes of irreducible admissible representations of $G(k)$.*

3.4. $\mathrm{Res}_{E/F}\, U_n^*(E/F) \times_E F$. Let E/F be a totally imaginary quadratic extension of a totally real field, and let ν be a place of F. The canonical isomorphism

$$\operatorname*{Res}_{E/F}\left(U_n^*(E/F) \times_F E\right)(F_\nu) \xrightarrow{\sim} GL_n(E_\nu)$$

induces a bijection between the equivalence classes of representations of the two groups. There is a corresponding bijection between the equivalence classes of L-parameters of the groups $\mathrm{Res}_{E_\nu/F_\nu} U_n^*(E/F) \times_F E_\nu$ and GL_n/E_ν (cf. [Rog90, §4.7]).

3.4.1. $U_n^*(E/F)$. Let E/F be a totally imaginary quadratic extension of a totally real field, and let ν be a finite place of F.

- Assume that $\nu = \omega\omega^c$ splits in E. As recalled in Section 2, there is a non-canonical isomorphism
$$\iota_\nu : U_n^*(E/F) \times_F F_\nu \xrightarrow{\sim} GL_n/F_\nu$$
which depends upon the choice of the place ω or ω^c. This isomorphism induces a bijection between the equivalence classes of representations of the two groups; it also induces a bijection between the equivalence classes of L-parameters of the two groups.
- Assume that ν remains inert in E. Then $U_n^*(E/F) \times_F F_\nu \xrightarrow{\sim} U_n^*(E_\nu/F_\nu)$.

Let k'/k be a quadratic extension of p-adic fields. We shall consider the representations of the unitary group $U_n^*(k'/k)$.

Definition 3.7. Let $\psi : L_{k'} \to GL_n$ be an L-parameter for GL_n/k' that decomposes into a direct sum of irreducible representations of the form
$$\psi = \oplus_{(\rho,a)\in\mathcal{E}} \rho \otimes \sigma_a : L_{k'} \to GL_n$$
where ρ is an irreducible representation of $W_{k'}$ of dimension d_ρ, a is an integer and σ_a denotes the unique a-dimensional irreducible representation of $SU_2(\mathbf{R})$. We say that ψ is *tempered θ-discrete stable* if

- ψ is a tempered L-homomorphism, and
- for all $(\rho, a) \in \mathcal{E}$, $(\rho \otimes \sigma_a)^\vee \simeq (\rho \otimes \sigma_a)^c$ where $c \in \mathrm{Gal}(k'/k)$ denotes the non-trivial element, and c acts on $L_{k'}$ via conjugation of the $W_{k'}$-component, and
- the representations $\rho \otimes \sigma_a$ are pairwise non-isomorphic, and
- for all $(\rho, a) \in \mathcal{E}$,
  - if $n = ad_\rho \mod 2$, then the Asai-Shahidi L-function (cf. [Gol94]) associated to $\rho \otimes \sigma_a$ has a pole at $s = 0$
  - if $n \neq ad_\rho \mod 2$, then the Asai-Shahidi L-function associated to $\rho \otimes \sigma_a$ does not have a pole at $s = 0$

  We inform the reader that the property of the Asai-Shahidi L-function at $s = 0$ can be changed by twisting by a certain character (cf. Remark 3.9).

It will be useful to extend this definition to L-parameters of the groups $GL_a \times GL_b/k'$. An L-parameter $\psi = \psi_a \times \psi_b : L_{k'} \to GL_a \times GL_b$ of $GL_a \times GL_b/k'$ shall be said to be *tempered θ-discrete stable* if both ψ_a and ψ_b are tempered θ-discrete stable L-parameters of GL_a/k' and GL_b/k' respectively.

We shall say that an irreducible admissible representation $\pi_a \times \pi_b$ of $GL_a \times GL_b(k')$ is *tempered θ-discrete stable* if its L-parameter $\psi(\pi_a \times \pi_b) : L_{k'} \to GL_a \times GL_b$ is tempered θ-discrete stable.

Remark 3.8. Let π be an irreducible admissible representation of $GL_n(k')$, and write the cuspidal support of π as
$$\times_{(\rho,a)\in\mathcal{E}} \mathrm{St}(\rho, a)$$
where ρ is a supercuspidal representation of $GL_{d_\rho}(k')$, a is an integer, and $\mathrm{St}(\rho, a)$ denotes the generalised Steinberg representation. The condition that π be tempered θ-discrete stable is equivalent to requiring the following.

- π is tempered.

- For all $(\rho, a) \in \mathcal{E}$, $\mathrm{St}(\rho, a) \circ \theta \simeq \mathrm{St}(\rho, a)$.
- The representations $\mathrm{St}(\rho, a)$ are pairwise non-isomorphic.
- For all $(\rho, a) \in \mathcal{E}$,
  - if $n = ad_\rho \mod 2$, then the Asai-Shahidi L-function associated to $\mathrm{St}(\rho, a)$ has a pole at $s = 0$, and
  - if $n \neq ad_\rho \mod 2$, then the Asai-Shahidi L-function associated to $\mathrm{St}(\rho, a)$ does not have a pole at $s = 0$

Remark 3.9. If the Asai-Shahidi L-function associated to $\mathrm{St}(\rho, a)$ has a pole (resp. does not have a pole) at $s = 0$, then the Asai-Shahidi L-function associated to $\mathrm{St}(\rho, a) \cdot \mu_1$ does not have a pole (resp. has a pole) at $s = 0$ where μ_1 is the character defined in Section 4.2.1.

The local Langlands correspondence for discrete series representations of the quasi-split unitary group is due to Mœglin [Mœg07]. Mœglin classifies the discrete series representations in terms of their Langlands base change to the general linear group (cf. Remark 5.11). It is important to note that Mœglin implicitly works with a non-standard twist of the stable base change map, more precisely a twist by the character μ_1^n, whilst we have chosen to work with the stable base change map. It is for this reason that the normalisation of the correspondence recalled here differs from [Mœg07]. Consequently our definition of a tempered θ-discrete stable L-homomorphism also differs from the definition appearing in [Mœg07, p 161-162].

Proposition 3.10. *To each equivalence class of tempered θ-discrete stable L-parameters $\psi : L_{k'} \to GL_n/k'$, one can naturally associate $\Pi(\psi)$, a finite non-empty set of equivalence classes of discrete series representations of $U_n^*(k'/k)(k)$. The L-packets $\Pi(\psi)$ are disjoint and their union is equal to the set of equivalence classes of discrete series representations of $U_n^*(k'/k)(k)$. The cardinality of the L-packet $\Pi(\psi)$ is equal to $2^{l(\psi)-1}$ where $l(\psi)$ denotes the length of the representation ψ.*

4. Some L-homomorphisms

We shall recall here the L-homomorphisms that shall be of interest to us.

Let k be a local or global field of characteristic 0. Let H and G be connected reductive groups defined over k. An *L-homomorphism* is a group homomorphism

$$\xi : {}^L H \to {}^L G$$

such that

- ξ is a homomorphism over W_k,
- ξ is continuous, and
- the restriction of ξ to $\widehat{H}$ induces a complex analytic homomorphism

$$\xi|_{\widehat{H}} : \widehat{H} \to \widehat{G}$$

If k is a local field then ξ is said to be *unramified* if the groups H and G are unramified, and ξ induces a map from unramified L-parameters of H to unramified L-parameters of G. If k is a global field of characteristic 0, then the L-homomorphism $\xi : {}^L H \to {}^L G$, induces a family of L-homomorphisms

$$\xi : {}^L H_\nu \to {}^L G_\nu$$

where ν runs through the places of k.

If k is a local field of characteristic 0, then the L-homomorphism $\xi : {}^L H \to {}^L G$ induces a map from the L-parameters of H to the L-parameters of G, which, in the cases where the local Langlands correspondence is known, induces a correspondence of L-packets. It will be useful to introduce the following notation, in the cases where the local Langlands correspondence is known. Let π (resp. Π) be an irreducible admissible representation (resp. L-packet) of H. We shall write $\psi(\pi) : L_k \to {}^L H$ (resp. $\psi(\Pi) : L_k \to {}^L H$) for the L-parameter associated to π (resp. Π). An irreducible admissible representation π' (resp. L-packet Π') of G is said to be a *ξ-transfer* of π (resp. Π) if the L-parameters $\psi(\pi') \simeq \xi \circ \psi(\pi)$ (resp. $\psi(\Pi') \simeq \xi \circ \psi(\Pi)$) are equivalent. We shall also write either $\Pi(\pi)$ or $\Pi(\psi(\pi))$ for the L-packet containing π.

4.1. **Base change.** Let k'/k be a quadratic extension of local or global fields of characteristic 0. The base change L-homomorphism for unitary groups is defined as follows.

$$\mathrm{BC} : {}^L U_n^*(k'/k) \to {}^L \operatorname*{Res}_{k'/k} U_n^*(k'/k) \times_k k'$$
$$g \times w \mapsto g \times g \times w$$

Assume now that $k'/k = E/F$ is a totally imaginary quadratic extension of a totally real field. Consider the induced map of L-parameters from a group U appearing in Proposition 2.1 to GL_n/E. Let ν be a place of F, and let $\rho_\nu : L_{F_\nu} \to {}^L U_\nu$ be a relevant L-parameter.

- Assume that $\nu = \omega\omega^c$ splits in E where $\omega \in Q$ (cf. Section 2). Then, identifying $F_\nu = E_\omega = E_{\omega^c}$, we have that
$$\mathrm{BC}(\rho_\nu) \simeq \rho_\nu \times \rho_\nu^\vee$$
seen as an L-parameter of $GL_n/E_\omega \times E_{\omega^c}$. In terms of L-packets, writing $\Pi(\rho_\nu) = \{\pi_\nu\}$, we have that $\Pi(\mathrm{BC}(\rho_\nu)) = \{\pi_\nu \times \pi_\nu^\vee\}$.
- Assume that ν remains inert in E. Then
$$\mathrm{BC}(\rho_\nu) \simeq \rho_\nu|_{L_{E_\nu}} : L_{E_\nu} \to {}^L GL_n/E_\nu$$
If ρ_ν is unramified, then the correspondence of unramified representations can be explicitly described in terms of Satake parameters (cf. [Min09, Theorem 4.1]). If we consider Mœglin's reformulation of the local Langlands correspondence (cf. Section 3.4.1) where $\rho_\nu : L_{E_\nu} \to GL_n$ is a tempered θ-discrete stable L-parameter. Then, by definition,
$$\mathrm{BC}(\rho_\nu) = \rho_\nu$$
seen as an L-parameter of GL_n/E_ν.
- Assume that ν is real. Then
$$\mathrm{BC}(\rho_\nu) \simeq \rho_\nu|_{W_{\mathbf{C}}} : W_{\mathbf{C}} \to {}^L GL_n/\mathbf{C}$$

4.2. **Endoscopic transfer.**

4.2.1. *Some Hecke characters.* Let k'/k be a quadratic extension of either local or global fields of characteristic 0. If k is local (resp. global) let $\eta : k^\times \to \mathbf{C}^\times$ (resp. $\eta : k^\times/\mathbf{A}_k^\times \to \mathbf{C}^\times$) be the character associated to the extension k'/k via class field theory. For all $a \in \mathbf{N}^0$, fix a character $\mu_a : k'^\times \to \mathbf{C}^\times$ (resp. $\mu_a : k'^\times/\mathbf{A}_{k'}^\times \to \mathbf{C}^\times$) that extends η^a. We remark that μ_a can be seen, via class field theory, as a character of the Weil group $W_{k'}$. The Hecke characters are easily seen to satisfy the following properties (cf. [BC09, §6.9.2]).

- μ_a is unitary.
- $\mu_a \circ \theta \simeq \mu_a$

- For all complex places ω of k', if a is even (resp. odd) the L-parameter of $\mu_{a,\omega}$ is of the form
$$\begin{aligned} W_{\mathbf{C}} &\to \mathbf{C}^{\times} \times W_{\mathbf{C}} \\ z &\mapsto (z/\overline{z})^{\alpha_{a,\omega}} \times z \end{aligned}$$
for some integer (resp. half integer) $\alpha_{a,\omega}$.

4.2.2. *Endoscopic transfer.* Let $a, b \in \mathbf{N}^0$ and let $n = a + b$. We shall consider the endoscopic L-homomorphism
$$\begin{aligned} \xi_{a,b} : {}^{L}U_a^*(k'/k) \times U_b^*(k'/k) &\to {}^{L}U_n^*(k'/k) \\ g_1 \times g_2 \times 1 &\mapsto \operatorname{diag}(g_1, g_2) \times 1 \\ I_{n_1} \times I_{n_2} \times w &\mapsto \operatorname{diag}(\mu_b(w) I_a, \mu_a(w) I_b) \times w \ \forall w \in W_E \\ I_{n_1} \times I_{n_2} \times w_c &\mapsto \operatorname{diag}(\Phi_a, \Phi_b) \Phi_n^{-1} \times w_c \end{aligned}$$
where w_c denotes a chosen lift of c, the non-trivial element of $\operatorname{Gal}(k'/k) = \{1, c\}$.

It is often simplest to study the endoscopic L-homomorphism in tandem with the base change L-homomorphism. The next lemma follows immediately from the respective definitions.

Lemma 4.1. *Let $H = U_a^*(k'/k) \times U_b^*(k'/k)$. Let ν be a place of k, and let $\rho_\nu = \rho_{a,\nu} \times \rho_{b,\nu} : L_{k_\nu} \to {}^{L}H_\nu$ be an L-parameter. Then*
$$\mathrm{BC}(\xi_{a,b}(\rho_\nu)) \simeq \mu_{b,\nu} \cdot \mathrm{BC}(\rho_{a,\nu}) \times \mu_{a,\nu} \cdot \mathrm{BC}(\rho_{b,\nu})$$

5. The Arthur-Selberg trace formula

We shall recall, in this section, the stable base change identity and the stable trace formula for the unitary group.

We begin by introduction some notation. Let G^+ be a reductive algebraic group defined over a local or global field k of characteristic 0. Let G^0 be the connected component containing the identity element of G^+, and let G be any connected component of G^+. An element $\gamma \in G$ is said to be *semisimple* (resp. *strongly regular*) if, viewed as an element of G^+, γ is semisimple (resp. strongly regular). Recall that an element $\gamma \in G^+$ is said to be *strongly regular* if the centraliser of γ in G^0 is a torus. We shall denote the connected component of the centraliser of γ in G^0 by G^0_γ. We shall define $\Gamma_{\mathrm{ss}}(G)$ (resp. $\Gamma_{\mathrm{reg,ss}}(G)$) to be the set of semisimple (resp. strongly regular semisimple) elements of G. Two elements $\gamma, \gamma' \in G(k)$ are said to be *conjugates* if they are conjugate by an element of $G^0(k)$. Two elements $\gamma, \gamma' \in \Gamma_{\mathrm{reg,ss}}(G)$ are said to be *stable conjugates* if they are conjugate by an element of $G^0(\overline{k})$.

Assume momentarily that k is local. Let $\gamma \in \Gamma_{\mathrm{reg,ss}}(G)$, and let $f \in \mathcal{C}_c^\infty(G(k))$ (resp. $f \in \mathcal{S}(G(k))$) if k is non-archimedean (resp. archimedean). We remind the reader that $C_c^\infty(G(k))$ denotes the space of smooth functions with compact support on $G(k)$, and $\mathcal{S}(G(k))$ denotes the space of Schwartz functions on $G(k)$. The *orbital integral* of f at γ is defined to be
$$\Phi(\gamma, f) = \int_{G^0_\gamma(k) \backslash G^0(k)} f(g^{-1}\gamma g)\, dg$$
The *stable orbital integral* of f at γ is defined to be
$$\Phi^{\mathrm{st}}(\gamma, f) = \sum_{\gamma'} \Phi(\gamma', f)$$
where the summation is taken over a set of representatives γ' of the conjugacy classes inside the stable conjugacy class of γ.

If k is local, then a *distribution*

$$A : \mathcal{C}_c^{\infty}(G(k)) \to \mathbf{C}$$

is said to be *stable* if, for all $f \in \mathcal{C}_c^{\infty}(G(k))$, $A(f)$ depends only upon the values of the stable orbital integrals $\Phi^{\mathrm{st}}(\gamma, f)$ for $\gamma \in \Gamma_{\mathrm{reg,ss}}(G)$. If k is global, then a *distribution*

$$A = \bigotimes_{\nu} A_{\nu} : \mathcal{C}_c^{\infty}(G(\mathbf{A}_k)) \to \mathbf{C}$$

is said to be *stable* if the individual A_{ν} are stable.

Throughout this article, we shall normalise our Haar measures such that

- the Haar measures satisfy the usual compatibility conditions (cf. [LS87]),
- if G is a connected reductive unramified p-adic group, then the measure of any hyperspecial subgroup of G is equal to 1, and
- if G is a connected reductive group defined over a number field k, then the product measure $dg = \prod_{\nu} dg_{\nu}$ on $G(\mathbf{A}_k)$ is equal to the Tamagawa measure.

5.1. **Stable base change.** Throughout this section, we shall denote by k'/k either a quadratic extension of local fields of characteristic 0, or a totally imaginary quadratic extension of a totally real number field.

Let $H = U_a^*(k'/k) \times U_b^*(k'/k)$ where $a, b \in \mathbf{N}^0$. Define the connected reductive group $G^0 = GL_a \times GL_b/k'$. The group G^0 admits a degree 2 automorphism

$$\theta = \theta_a \times \theta_b : GL_a \times GL_b \to GL_a \times GL_b$$

where θ_a and θ_b are defined in Section 2. We define the non-connected group $G^+ = G^0 \rtimes \langle \theta \rangle$, and the connected component $G = G^0 \times \theta$. There exists a natural bijection

$$\begin{gathered} \mathcal{C}_c^{\infty}\left(G^0(k')\right) \to \mathcal{C}_c^{\infty}(G(k')) \\ f \mapsto f \times \theta \end{gathered}$$

which identifies the two spaces of functions.

5.1.1. *The norm map.* Assume that k is local throughout this section.

Labesse [Lab99, §2.4] defines the *norm map*

$$\mathcal{N} : \Gamma_{\mathrm{reg,ss}}(G) \to \Gamma_{\mathrm{ss}}(H)$$

which canonically maps conjugacy classes of $G(k')$ to stable conjugacy classes of $H(k)$. An element $\gamma_G \in \Gamma_{\mathrm{reg,ss}}(G)$ is said to be *H-strongly regular semisimple* if $\mathcal{N}(\gamma_G)$ is strongly regular semisimple. We define $\Gamma_{\text{H-reg,ss}}(G)$ to be the set of H-strongly regular semisimple elements of $G(k')$. An element $\gamma_H \in \Gamma_{\mathrm{reg,ss}}(H)$ is said to be a *norm* of an element $\gamma_G \in \Gamma_{\text{H-reg,ss}}(G)$ if γ_H and $\mathcal{N}(\gamma_G)$ are stable conjugates.

5.1.2. *Intertwining operators.* Assume that k is local throughout this section.

Let V be a complex vector space, and let $\pi : G^0(k') \to GL(V)$ be an irreducible admissible representation such that $\pi \simeq \pi \circ \theta$. There exists an intertwining operator

$$A_{\pi} : \pi \to \pi \circ \theta$$

By Schur's lemma, A_{π} is uniquely determined, up to a non-zero constant, and A_{π}^2 is a non-zero constant. The operator A_{π} is said to be *normalised* if $A_{\pi}^2 = 1$. A normalised intertwining operator is uniquely determined up to a sign.

If π is generic, then a canonical choice for the intertwining operator A_{π} can be made via Whittaker models. Let

$$\lambda : V \to \mathbf{C}$$

be a Whittaker functional on π (which depends upon the choice of a non-trivial additive character of k'). The intertwining operator A_π is said to be *Whittaker normalised* if

$$A_\pi \lambda = \lambda$$

where A_π acts, via its dual action, on the space of Whittaker functionals of π. We shall denote the Whittaker normalised intertwining operator by A^W.

5.1.3. *The transfer.* Assume that k is local throughout this section.

Following Labesse [Lab99, §3.2], two functions $\phi \in \mathcal{C}_c^\infty(G(k'))$ and $f^H \in \mathcal{C}_c^\infty(H(k))$ are said to be *associated* if, for all $\gamma_H \in \Gamma_{\text{reg,ss}}(H)$,

$$\Phi^{\text{st}}\left(\gamma_H, f^H\right) = \begin{cases} \Phi(\gamma_G, \phi) & : \text{if } \gamma_H \text{ is a norm of some } \gamma_G \in \Gamma_{\text{H-reg,ss}}(G) \\ 0 & : \text{otherwise} \end{cases}$$

We shall now recall some results on the existence and properties of associated functions. In what follows, we shall assume that $k'/k = E/F$ is a totally imaginary quadratic extension of a totally real field.

5.1.4. *The transfer: archimedean places.* Assume that ν is a real place of F.

We begin by recalling the important class of twisted Euler-Poincaré functions. Let V be an irreducible algebraic representation of $GL_a \times GL_b$. Consider the algebraic representation $V \otimes V^\theta$ of

$$\operatorname*{Res}_{E_\nu/F_\nu} G^0 \times_E E_\nu \xrightarrow{\sim} (GL_a \times GL_b) \times (GL_a \times GL_b)$$

and the intertwining operator

$$\begin{aligned} A_{V\otimes V^\theta} : V \otimes V^\theta &\to V^\theta \otimes V \\ v_1 \otimes v_2 &\mapsto v_2 \otimes v_1 \end{aligned}$$

Let K be a maximal compact θ-invariant subgroup of $\operatorname{Res}_{E_\nu/F_\nu}\left(G^0 \times_E E_\nu\right)(F_\nu)$, and let $\mathfrak{g} = \operatorname{Lie}\left(\operatorname{Res}_{E_\nu/F_\nu} G^0 \times_E E_\nu\right)$. For all irreducible admissible representations π of $\operatorname{Res}_{E_\nu/F_\nu}\left(G^0 \times_E E_\nu\right)(F_\nu) \simeq G^0(E_\nu)$ such that $\pi \simeq \pi \circ \theta$, equipped with normalised intertwining operator A_π, the *twisted Euler-Poincaré characteristic* is defined to be

$$\operatorname{ep}\left(\mathfrak{g}, K; \pi \otimes V \otimes V^\theta, A_\pi \otimes A_\theta\right) = \sum_i (-1)^i \operatorname{Tr}\left(A_\pi \otimes A_\theta | H^i\left(\mathfrak{g}, K, \pi \otimes V \otimes V^\theta\right)\right)$$

Lemma 5.1. *Let π be an irreducible admissible representation of $G^0(E_\nu)$ such that $\pi \simeq \pi \circ \theta$. Assume that*

$$\operatorname{ep}\left(\mathfrak{g}, K; \pi \otimes V \otimes V^\theta, A_\pi \otimes A_\theta\right) \neq 0$$

Then the infinitesimal characters of π and $\left(V \otimes V^\theta\right)^\vee$ are equal.

Proof. This follows from well-known properties of the relative Lie-algebra cohomology (cf. [Lab91, §7]). □

Lemma 5.2. *There exists a unique irreducible generic unitary representation π of $G^0(E_\nu)$ such that $\pi \simeq \pi \circ \theta$ and*

$$\operatorname{ep}\left(\mathfrak{g}, K; \pi \otimes V \otimes V^\theta, A_\pi \otimes A_\theta\right) \neq 0$$

For this π,

$$\operatorname{ep}\left(\mathfrak{g}, K; \pi \otimes V \otimes V^\theta, A^W \otimes A_\theta\right) = (-1)^{q(H_\nu)} 2^n$$

where $q(H_\nu) = \frac{1}{2}\dim(H_\nu(F_\nu)/K)$; furthermore, π is the Langlands base change of the L-packet of discrete series representations $\Pi(V^\vee)$ of $H(F_\nu)$ (cf. Section 3.3.1).

Proof. Labesse [Lab09, Lemma 4.7] shows the existence and uniqueness of such a π and calculates the twisted Euler-Poicaré characteristic up to a sign. The determination of the sign for the Whittaker normalised intertwining operator is due to Clozel [Clo09, Corollary 2.2]. Finally, the fact that π is the Langlands base change of the L-packet $\Pi(V^\vee)$ follows from the properties of the local Langlands correspondence (cf. [Kna94]). $\square$

Lemma 5.3. *There exists a* twisted Euler-Poincaré *function* $\phi_{V\otimes V^\theta} \in \mathcal{C}_c^\infty(G(E_\nu))$ *such that*

- $\phi_{V\otimes V^\theta}$ *is K-finite and cuspidal (cf.* [Art88, §7]*), and*
- *for all irreducible admissible representations π of $G^0(E_\nu)$ such that $\pi \simeq \pi\circ\theta$,*

$$\operatorname{Tr}\pi\circ A_\pi(\phi_{V\otimes V^\theta}) = \operatorname{ep}\left(\mathfrak{g}, K; \pi\otimes V\otimes V^\theta, A_\pi\otimes A_\theta\right)$$

Proof. [Lab91, Proposition 12] $\square$

Lemma 5.4. *The twisted Euler-Poincaré function $\phi_{V\otimes V^\theta} \in \mathcal{C}_c^\infty(G(E_\nu))$ is associated to $f_V \in \mathcal{C}_c^\infty(H(F_\nu))$ where f_V denotes the Euler-Poincaré function associated to V (cf. Section 5.2.5).*

Proof. [Lab09, Lemma 4.4] $\square$

5.1.5. *The transfer: unramified case.* Assume that ν is a finite place of F that is unramified in E.

Let K_{G^0} (resp. K_H) be a hyperspecial subgroup of $G^0(E_\nu)$ (resp. $H(F_\nu)$). The base change L-homomorphism $BC : {}^LH \to {}^L\operatorname{Res}_{E_\nu/F_\nu} G^0 \times_E E_\nu$ is unramified and induces a map from the K_H-unramified representations of $H(F_\nu)$ to the K_{G^0}-unramified representations of $G^0(E_\nu)$ (cf. Section 4). Dual to this transfer, there exists a morphism of spherical Hecke algebras (see Minguez [Min09, §4] for an explicit description)

$$\operatorname{BC} : \mathcal{C}_c^\infty\left(G^0(E_\nu), K_{G^0}\right) \to \mathcal{C}_c^\infty(H(F_\nu), K_H)$$

Lemma 5.5. *For all $\phi \in \mathcal{C}_c^\infty\left(G^0(E_\nu), K_{G^0}\right)$, the function $\operatorname{BC}(\phi) \in \mathcal{C}_c^\infty(H(F_\nu), K_H)$ is associated to ϕ.*

Proof. If ν splits in E then the result is straight forward (cf. [Lab99, §3.4]). Assume now that ν remains inert in E. If $\phi = \mathbf{1}_{K_{G^0}} \times \theta$, then the result is due to Kottwitz [Kot86]. For general ϕ, the result is due to Clozel [Clo90] and Labesse [Lab90]. $\square$

Lemma 5.6. *Let $\phi \in \mathcal{C}_c^\infty\left(G^0(E_\nu), K_{G^0}\right)$, and let let $f^H \in \mathcal{C}_c^\infty(H(F_\nu), K_H)$. Assume that ϕ and f^H are associated. Let π_H be a K_H-unramified representation of $H(F_\nu)$, and let π be an unramified representation of $G^0(E_\nu)$. Assume that π is the Langlands base change of π_H, that is, $\psi(\pi) \simeq \operatorname{BC}(\psi(\pi_H))$. Then*

$$\operatorname{Tr}\pi_H\left(f^H\right) = \pm\operatorname{Tr}\pi\circ A_\pi(\phi)$$

where the sign depends upon the choice of the normalised intertwining operator A_π. If π is generic and A_π is chosen to be the Whittaker normalised intertwining operator, then

$$\operatorname{Tr}\pi_H\left(f^H\right) = \operatorname{Tr}\pi\circ A^W(\phi)$$

Proof. It follows from Lemma 5.5 that

$$\operatorname{Tr}\pi_H\left(f^H\right) = \pm\operatorname{Tr}\pi(\phi)$$

The result then follows from the following observations.

- A normalised intertwining operator acts on the 1-dimensional vector space $\pi^{K_{G^0}}$ by multiplication by ± 1.

- The Whittaker normalised intertwining operator A^W acts as the identity on $\pi^{K_{G^0}}$.

□

5.1.6. *The transfer: split places.* Assume that $\nu = \omega\omega^c$ is a finite place of F that splits in E. The results are well known in this case (cf. [Lab99, §3.4]).

Let $\pi_\nu = \pi_\omega \times \pi_{\omega^c}$ be an irreducible admissible representation of $G^0(E_\nu) = G^0(E_\omega) \times G^0(E_{\omega^c})$ such that $\pi_\nu \simeq \pi_\nu \circ \theta$. There is a natural choice for the normalised intertwining operator

$$A : \pi_\omega \times \pi_{\omega^c} \to \pi_\omega \times \pi_{\omega^c}$$
$$v_1 \times v_2 \mapsto v_2 \times v_1$$

If π is generic, then A coincides with the Whittaker normalised intertwining operator A^W.

Lemma 5.7. *For all $\phi \in \mathcal{C}_c^\infty(G(E_\nu))$, there exists $f^H \in \mathcal{C}_c^\infty(H(F_\nu))$ such that ϕ and f^H are associated.*

Lemma 5.8. *Let $\phi \in \mathcal{C}_c^\infty(G(E_\nu))$, and let $f^H \in \mathcal{C}_c^\infty(H(F_\nu))$. Assume that ϕ and f^H are associated. Then for all irreducible admissible representations π_H of $H(F_\nu)$,*

$$\operatorname{Tr}\pi_H\left(f^H\right) = \operatorname{Tr}\pi \circ A(\phi)$$

where π is the irreducible admissible representation of $G^0(E_\nu)$ which is the Langlands base change of π_H, that is, $\psi(\pi) \simeq \mathrm{BC}(\psi(\pi_H))$.

5.1.7. *The transfer: inert places.* Assume that ν is a finite place of F that remains inert in E.

Lemma 5.9. *For all $\phi \in \mathcal{C}_c^\infty(G(E_\nu))$, there exists $f^H \in \mathcal{C}_c^\infty(H(F_\nu))$ such that ϕ and f^H are associated.*

Proof. [Lab99, Theorem 3.3.1] □

Lemma 5.10. *Let $\psi : L_{E_\nu} \to {}^LG_\nu^0$ be a tempered θ-discrete stable L-parameter. Then there exist unique complex numbers $n(\psi,\sigma)$ for all irreducible admissible representations σ of $H(F_\nu)$, such that, for all associated $\phi \in \mathcal{C}_c^\infty(G(E_\nu))$ and $f^H \in \mathcal{C}_c^\infty(H(F_\nu))$,*

$$\operatorname{Tr}\pi \circ A^W(\phi) = \sum_\sigma n(\psi,\sigma)\operatorname{Tr}\sigma\left(f^H\right)$$

where π is the irreducible admissible representation of $G^0(E_\nu)$ such that $\psi(\pi) \simeq \psi$, seen as L-parameters of G_ν^0.

Proof. [Mœg07, §5.7] □

Remark 5.11. One defines the L-packet $\Pi(\psi)$ to be the set of σ such that $n(\psi,\sigma) \neq 0$, which are shown to be discrete series representations (cf. [Mœg07, §5.5]).

Lemma 5.12. *Keeping the notation and the assumptions of Lemma 5.10. The distribution*

$$\sum_{\sigma\in\Pi(\psi)} n(\psi,\sigma)\operatorname{Tr}\sigma$$

is stable, and is the unique, up to a scalar, linear combination of representations in the L-packet $\Pi(\psi)$ which is stable.

Proof. [Mœg07, §5.5] □

5.1.8. *Stable base change.* Let E/F be a totally imaginary quadratic extension of a totally real number field. Let $H = U_a^* (E/F) \times U_b^* (E/F)$, let $G^0 = GL_a \times GL_b/E$, let $G^+ = G^0 \rtimes \langle\theta\rangle$, and let $G = G^0 \times \theta$. Let $\mathrm{S}_{\mathrm{ram}}$ denote the finite set of places ν of F such that either ν is archimedean, or ν is non-archimedean and ramified in E.

Proposition 5.13. *Let $S \supset \mathrm{S}_{\mathrm{ram}}$ be a finite set of places of F. Let $\phi_S = \otimes_{\nu\in S}\phi_\nu \in \mathcal{C}_c^\infty\left(G^0(\mathbf{A}_S)\right)$, and let $\phi = \phi_S \otimes \mathbf{1}_{K^S}$ where $K^S = \prod_{\nu\notin S} K_\nu$ is a product of hyperspecial subgroups K_ν of $G^0(F_\nu)$. Assume that $f^H = \otimes_\nu f_\nu^H \in \mathcal{C}_c^\infty(H(\mathbf{A}))$ is associated to ϕ at all places ν. Assume that for all archimedean places ν, f_ν^H and ϕ_ν are, up to a multiple, those functions appearing in Lemma 5.4. Then*

$$S^H\left(f^H\right) = \begin{cases} 2\cdot I(\phi): & \textit{if } ab = 0 \\ 4\cdot I(\phi): & \textit{otherwise}\end{cases}$$

where I denotes Arthur's twisted invariant trace formula for G (cf. [Art88]*), and S^H denotes Arthur's stable trace formula for $H_{a,b}$ (cf.* [Art02]*).*

Proof. If $[F:\mathbf{Q}] \geq 2$, then this result is due to Labesse [Lab09]. It is important to note that the constant (either 2 or 4) does not explicitly appear in [Lab09] as it is subsumed in Labesse's chosen normalisation of Arthur's twisted invariant trace formula, which differs by, in the notation of [Lab09], the constant $J\left(\widetilde{G}\right)$. This constant $J\left(\widetilde{G}\right)$ is equal to 2 if $ab = 0$ and otherwise 4.

If $F = \mathbf{Q}$, then the result is due to Morel [Mor10, Proposition 8.3.1]. Morel demonstrates the result, up to a constant, which in the case of the unitary group is equal to the desired constant (cf. proof of [Lab09, Theorem 4.12]) □

We remark that Arthur's stable trace formula for connected reductive groups (cf. [Art02] [Art01] [Art03]) is now unconditional due to the proof of the generalised fundamental lemma by Chaudouard-Laumon [CL10a] [CL10b], Ngô [Ngô10], and Walspurger [Wal09]. Arthur's [Art88] twisted invariant trace formula is also unconditional due to the work of Kottwitz-Rogawski [KR00] and Delorme-Mezo [DM08]. Furthermore, the twisted invariant trace formula admits a simple expression here due to the fact that our chosen ϕ is cuspidal at infinity (cf. [Art88, Theorem 7.1]).

$$I(f) = I_{\mathrm{disc}}(f) = \sum_{L_0\in\mathcal{L}^0} \frac{|W_0^{L_0}|}{|W_0^{G^0}|} \sum_{s\in W^G\left(\mathfrak{a}_{L_0}\right)_{\mathrm{reg}}} |\det(s-1)_{\mathfrak{a}_{L_0}^G}|^{-1}$$
$$\sum_{\widetilde{\pi}\in\Pi_{\mathrm{disc}}(L_0\rtimes\langle s\rangle)} \mathrm{m}_{\mathrm{disc}}(\widetilde{\pi})\,\mathrm{Tr}\left(M_{Q_0|sQ_0}(0)\,\rho_{Q_0,t}(s,0,f)\,|_{\mathrm{Ind}_{Q_0}^{G^0}\pi}\right)$$

where the notation is that of [Art88], in particular,

- M_0 is a minimal θ-invariant Levi subgroup of G^0,
- $\mathcal{L}^0$ is the set of Levi subgroups of G^0 containing M_0,
- A_{L_0} is the maximal split torus contained in the restriction of scalars to $\mathbf{Q}$ of the centre of L_0,
- $\mathfrak{a}_{L_0}$ is the Lie algebra of A_{L_0},
- $\mathfrak{a}_{L_0}^G$ is the quotient of $\mathfrak{a}_{L_0}$ by the subgroup of θ-invariant points of $\mathfrak{a}_{G^0}$,
- the Weyl group $W_0^{L_0}$ is the group of isomorphisms of $\mathfrak{a}_{L_0}$ induced by G,
- $W^G(\mathfrak{a}_{L_0})$ is the quotient of $W_0^{G^0}$ by $W_0^{L_0}$,
- $W^G(\mathfrak{a}_{L_0})_{\mathrm{reg}} = \left\{s\in W^G(\mathfrak{a}_{L_0}) : \det(s-1)\,|_{\mathfrak{a}_{L_0}^G} \neq 0\right\}$
- $\Pi_{\mathrm{disc}}(L_0\rtimes\langle s\rangle)$ is the set of irreducible unitary representations $\widetilde{\pi}$ of $(L_0\times\langle s\rangle)(\mathbf{A})$ whose restriction π to L_0 remains irreducible and appears in the discrete automorphic spectrum of L_0 with non-zero multiplicity,

- $\mathrm{m}_{\mathrm{disc}}(\widetilde{\pi})$ is the multiplicity of $\widetilde{\pi}$ in the discrete automorphic spectrum of $L_0 \rtimes \langle s\rangle$, which in our situation, due to the multiplicity 1 theorem for GL_n, is equal to $\mathrm{m}_{\mathrm{disc}}(\pi)$ the multiplicity of π in the discrete automorphic spectrum of L_0, and
- $M_{Q_0|sQ_0}(0)$ and $\rho_{Q_0,t}(s,0,f)$ are the operators defined by Arthur [Art88] where Q_0 denotes the standard parabolic subgroup of G^0 containing L_0.

For our choice of f, the stable trace formula admits a spectral expansion,

$$S^H\left(f^H\right) = S^H_{\mathrm{disc}}\left(f^H\right) = \sum_{\sigma} n(\sigma)\,\mathrm{Tr}\,\sigma\left(f^H\right)$$

where $n(\sigma)$ is a rational number, and σ ranges over a set of irreducible admissible representations of $H(\mathbf{A}_F)$. The σ for which $n(\sigma)\neq 0$ are called the *stable discrete automorphic representations* of H.

Remark 5.14. We warn the reader that a stable discrete automorphic representation of H need not be automorphic.

Remark 5.15. The work of Muller [Mül98] on the traceability of the discrete spectrum has allowed us to omit the summation over 't' utilised by Arthur.

Lemma 5.16. *Let $s=\theta_{n_1}\times\cdots\times\theta_{n_r}$. Then*

$$\left|\det(s-1)_{\mathfrak{a}^G_{L_0}}\right| = 2^r$$

Proof. We see that $\mathfrak{a}^G_{L_0}=\mathfrak{a}_{L_0}\simeq\mathbf{R}^r$, and s acts by multiplication by -1. The result follows. □

Lemma 5.17. *Let $s\in W^G(\mathfrak{a}_{L_0})_{\mathrm{reg}}$, and let $\widetilde{\pi}\in\Pi_{\mathrm{disc}}(L_0\rtimes\langle s\rangle)$. Write $\pi=\pi_1\times\pi_2\times\cdots\times\pi_r$ where each $\pi_i\in\Pi_{\mathrm{disc}}(GL_{n_i})$ for some $n_i\in\mathbf{N}$. Then $\pi_i\circ\theta\simeq\pi_i$ for all $i=1,\dots,r$. Furthermore, if $\pi_i\not\simeq\pi_j$ for all $i\neq j$, then $s\simeq\theta_{n_1}\times\cdots\times\theta_{n_r}$.*

Proof. [Lab09, Lemma 3.8] □

Lemma 5.18. *Let $s\in W^G(\mathfrak{a}_{L_0})_{\mathrm{reg}}$, and let $\widetilde{\pi}\in\Pi_{\mathrm{disc}}(L_0\rtimes\langle s\rangle)$. Write $\pi=\pi_1\times\pi_2\times\cdots\times\pi_r$ where each π_i is a cuspidal automorphic representation of GL_{n_i} for some $n_i\in\mathbf{N}$. Assume that $\pi_i\not\simeq\pi_j$ for all $i\neq j$. Then Arthur's implicit normalisation of the intertwining operators is compatible with the Whittaker normalisation in the sense that*

$$\mathrm{Tr}\left(M_{Q_0|sQ_0}(0)\,\rho_{Q_0,t}(s,0,f)\,|_{\mathrm{Ind}^{G^0}_{Q_0}\pi}\right) = \mathrm{Tr}\,\mathrm{Ind}^{G^0}_{Q_0}\pi\circ A^W(f)$$

where $A^W=\otimes_\nu A^W$ is the product of the local Whittaker normalised intertwining operator at each place ν.

Proof. Firstly by Lemma 5.17, we see that $s\simeq\theta_{n_1}\times\cdots\times\theta_{n_r}$. Arthur's operators implicitly define an intertwining operator $A_\pi=\otimes_\nu A_{\pi_\nu}$ via the identity

$$\mathrm{Tr}\left(M_{Q_0|sQ_0}(0)\,\rho_{Q_0,t}(s,0,f)\,|_{\mathrm{Ind}^{G^0}_{Q_0}\pi}\right) = \mathrm{Tr}\,\mathrm{Ind}^{G^0}_{Q_0}\pi\circ A_\pi(f)$$

We recall that Arthur's operators act on the following representations

$$\mathrm{Ind}^{G^0}_{Q_0}\pi \xrightarrow{\rho_{Q_0,t}(s,0,f)} \left(\mathrm{Ind}^{G^0}_{Q_0^s}\pi\right)\circ s \xrightarrow{M_{Q_0|sQ_0}(0)} \left(\mathrm{Ind}^{G^0}_{Q_0}\pi\right)\circ s$$

The operators act in a componentwise on $G^0=GL_a\times GL_b$, as such, it will suffice to demonstrate this result when $G^0=GL_n$.

Let $\lambda=\otimes_\nu\lambda_\nu$ be a Whittaker functional on $\pi=\otimes_\nu\pi_\nu$ (this depends upon the choice of a non-trivial additive character of E). The Whittaker functional λ induces Whittaker functionals on both the induced representations $\mathrm{Ind}^{G^0}_{Q_0}\pi$ and

$\left(\mathrm{Ind}_{Q_0}^{G^0}\pi\right)\circ s$. We shall also denote these Whittaker functionals by λ. The result will follow upon confirmation that the operator $M_{Q_0|sQ_0}(0)\,\rho_{Q_0,t}(s,0,f)$ preserves the Whittaker functional, that is, maps λ to λ.

The operator $\rho_{Q_0,t}(s,0,f)$ preservers the Whittaker functional (cf. [CHL09, §4.4]), that is, it maps λ to λ. Consider now the operator $M_{Q_0|sQ_0}(0)$. Decompose $s = s_{N-1}\cdot\cdots\cdot s_1$ where the s_i are simple reflections and the decomposition is reduced. Shahidi [Sha81] [Sha83] shows that $M_{Q_0|sQ_0}(0)$ maps λ to $c(\pi,s)\cdot\lambda$ where where $c(\pi,s)$ is equal to the value at $\alpha=0$ of

$$c(\pi,s,\alpha)=\prod_{i=1}^{N-1}\epsilon\left(\pi_{i,1}\times\pi_{i,2}^{\vee},\alpha\right)\frac{L\left(\pi_{i,1}^{\vee}\times\pi_{i,2},1-\alpha\right)}{L\left(\pi_{i,1}\times\pi_{i,2}^{\vee},\alpha\right)}$$

where $\pi_{i,1}$ and $\pi_{i,2}$ are the representations of the adjacent Levi-blocks of L_0 that are interchanged by s_i. Since $\pi_{i,1}\not\simeq\pi_{i,2}$, the L-functions extend holomorphically to the entire complex plane. Furthermore they satisfy the functional equation

$$L\left(\pi_{i,1}\times\pi_{i,2}^{\vee},\alpha\right)=\epsilon\left(\pi_{i,1}\times\pi_{i,2}^{\vee},\alpha\right)L\left(\pi_{i,1}^{\vee}\times\pi_{i,2},1-\alpha\right)$$

It follows that $c(\pi,s)=1$, that is, $M_{Q_0|sQ_0}(0)$ preserves the Whittaker functional. □

5.2. **The Stable Trace Formula for the Unitary Group.** Throughout this section k'/k shall denote either a quadratic extension of local fields of characteristic 0 or a totally imaginary quadratic extension of a totally real field. Let U denote an inner form of $U_n^*(k'/k)$ that is quasi-split at all finite places. Let G denote either GL_n/k or U.

5.2.1. *Endoscopic data.* We recall that an *endoscopic data* for G is a quadruple $H=(H,\mathcal{H},s,\xi)$ where

- H is a quasi-split k-group,
- $\mathcal{H}$ is a split extension of W_k by $\widehat{H}$,
- s is a semisimple element of $\widehat{G}$, and
- $\xi:\mathcal{H}\to {}^LG$ is an L-homomorphism

such that ξ induces an isomorphism of $\widehat{H}$ with the connected component of the centraliser of s in $\widehat{G}$, and the conditions of [LS87, §1.2] are satisfied. There is a notion of *equivalence* for endoscopic data, and also a notion of an *elliptic* endoscopic data (cf. [LS87, §1.2]).

For unitary groups, the classification of the elliptic endoscopic data is due to Rogawski [Rog90, Proposition 4.6.1].

Definition 5.19. For all $a,b\in\mathbf{N}^0$ such that $a+b=n$, we define the quadruple

$$H_{a,b}=\left(H_{a,b},{}^LH_{a,b},s_{a,b},\xi_{a,b}\right)$$

where

- $H_{a,b}=U_a^*(k'/k)\times U_b^*(k'/k)$,
- $s_{a,b}=\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$ where 1 (resp. -1) appears with multiplicity a (resp. b), and
- $\xi_{a,b}$ is the endoscopic L-homomorphism of Section 4.

Lemma 5.20. *The $H_{a,b}$ are endoscopic data for U, and $H_{a,b}$ is equivalent to $H_{b,a}$. If k is global, then the $H_{a,b}$ are elliptic, and*

$$\{H_{a,b}:a\leq b\}$$

is a set of representations of the equivalence classes of elliptic endoscopic data for U.

5.2.2. *The norm map.* Assume that k is local throughout this section.

Let $H = \left(H, {}^{L}H, s, \xi\right)$ be an endoscopic data for G. Then there exists a canonical map of semisimple conjugacy classes (cf. [LS87, §1.3])

$$\mathcal{A}_{H/G} : \Gamma_{\mathrm{ss}}\left(H\left(\overline{k}\right)\right) \to \Gamma_{\mathrm{ss}}\left(G\left(\overline{k}\right)\right)$$

An element $\gamma_H \in H(k)$ is said to be *semisimple G-strongly regular* if $\mathcal{A}_{H/G}(\gamma_H)$ is semisimple strongly regular. We define $\Gamma_{\text{G-reg,ss}}(H)$ to be the set of semisimple G-strongly regular elements of $H(k)$. The map $\mathcal{A}_{H/G}$ induces a canonical map (cf [LS87, §1.3])

$$\mathcal{A}_{H/G} : \Gamma_{\text{G-reg,ss}}(H) \to \Gamma_{\text{reg,ss}}(G)$$

An element $\gamma_H \in \Gamma_{\text{G-reg,ss}}(H)$ is said to be a *norm* of an element $\gamma_G \in G(k)$ if γ_G lies in the conjugacy class of $\mathcal{A}_{H/G}(\gamma_H)$.

5.2.3. *The Langlands-Shelstad geometric transfer factors.* Assume that k is local throughout this section. The *geometric transfer factors* of Langlands-Shelstad [LS87] are functions

$$\Delta : \Gamma_{\text{G-reg,ss}}(H) \times \Gamma_{\text{reg,ss}}(G) \to \mathbf{C}$$

defined for all endoscopic data $H = \left(H, {}^{L}H, s, \xi\right)$ of G. They are canonically defined up to a constant, and are given a specific normalisation as follows. Choose $\overline{\gamma}_H \in \Gamma_{\text{G-reg,ss}}(H)$ and $\overline{\gamma}_G \in \Gamma_{\text{reg,ss}}(G)$ such that $\overline{\gamma}_H$ is a norm of $\overline{\gamma}_G$. The *relative geometric transfer factor*

$$\Delta\left(\gamma_H, \gamma_G : \overline{\gamma}_H, \overline{\gamma}_G\right) = \frac{\Delta\left(\gamma_H, \gamma_G\right)}{\Delta\left(\overline{\gamma}_H, \overline{\gamma}_G\right)}$$

is canonically defined for all $\gamma_H \in \Gamma_{\text{G-reg,ss}}(H)$ and for all $\gamma_G \in \Gamma_{\text{reg,ss}}(G)$. To specify a normalisation of the geometric transfer factors, one arbitrarily fixes the value of $\Delta\left(\overline{\gamma}_H, \overline{\gamma}_G\right)$ as a complex number of norm 1, and then defines

$$\Delta\left(\gamma_H, \gamma_G\right) = \Delta\left(\overline{\gamma}_H, \overline{\gamma}_G\right)\Delta\left(\gamma_H, \gamma_G : \overline{\gamma}_H, \overline{\gamma}_G\right)$$

for all $\gamma_H \in \Gamma_{\text{G-reg,ss}}(H)$ and for all $\gamma_G \in \Gamma_{\text{reg,ss}}(G)$.

5.2.4. *The transfer.* Assume that k is local throughout this section.

Let $H = (H, {}^{L}H, s, \xi)$ be an endoscopic data for G. If k is archimedean (resp. non-archimedean) then let $f \in \mathcal{S}(G(k))$ (resp. $f \in \mathcal{C}_c^{\infty}(G(k))$ and let $f^H \in \mathcal{S}(H(k))$ (resp. $f \in \mathcal{C}_c^{\infty}(H(k))$). The function f^H is said to be a Δ-*transfer* of f if

$$\Phi^{\mathrm{st}}\left(\gamma_H, f^H\right) = \sum_{\gamma \in \Gamma_{\text{reg,ss}}(G)} \Delta\left(\gamma_H, \gamma\right) \Phi\left(\gamma, f\right)$$

for all $\gamma_H \in \Gamma_{\text{G-reg,ss}}(H(k))$.

We shall now recall some results on the existence and properties of the transfer. In what follows E/F shall denote a totally imaginary quadratic extension of a totally real field, U shall denote a unitary group appearing in Proposition 2.1, and H shall denote an endoscopic data appearing in Definition 5.19.

5.2.5. *The transfer: archimedean places.* Assume that ν is a real place of F.

Lemma 5.21. *Let $f \in \mathcal{S}(U(F_\nu))$. Then there exists $f^H \in \mathcal{S}(H(F_\nu))$ such that f^H is a Δ-transfer of f.*

Proof. [She08a, Theorem 14.3]. □

The geometric transfer induces a dual spectral transfer of tempered representations. In studying the spectral transfer, Shelstad [She10] explicitly defines complex valued *spectral transfer factors* $\Delta_{\mathrm{spec}}(\psi_H, \pi)$ for all tempered L-parameters ψ_H of H_ν and tempered representations π of $U(F_\nu)$. They satisfy the following properties.

- $\Delta_{\mathrm{spec}}(\psi_H, \pi) = 0$ if $\pi \notin \Pi(\xi \circ \psi_H)$
- $|\Delta_{\mathrm{spec}}(\psi_H, \pi)| = 1$ if $\pi \in \Pi(\xi \circ \psi_H)$

The spectral transfer factors are canonically defined up to a constant. They are in a sense dual to the geometric transfer factors, and choosing a normalisation of one induces a normalisation of the other.

Lemma 5.22. *Let $\psi_H : L_{F_\nu} \to {}^L H_\nu$ be a tempered L-parameter. Let $f \in \mathcal{S}(U(F_\nu))$ and let $f^H \in \mathcal{S}(H(F_\nu))$. Assume that f^H is a Δ-transfer of f. Then*

$$\sum_{\pi_H \in \Pi(\psi_H)} \operatorname{Tr} \pi_H \left(f^H\right) = \sum_{\pi \in \Pi_{\mathrm{temp}}(U_\nu)} \Delta_{\mathrm{spec}}(\psi_H, \pi) \operatorname{Tr} \pi(f)$$

Proof. [She10, Theorem 5.1] □

An important class of test functions are the pseudo-coefficients and Euler-Poincaré functions. Let L denote either $U \times_F F_\nu$ or $H \times_F F_\nu$, which is a real connected reductive group. Let π be a discrete series representation of $L(\mathbf{R})$. A function $f \in \mathcal{S}(L(\mathbf{R}))$ is said to be a *pseudo-coefficient* of π if for all tempered representations σ of $L(\mathbf{R})$,

$$\operatorname{Tr} \sigma(f) = \begin{cases} 1 & : \text{if } \sigma \simeq \pi \\ 0 & : \text{otherwise} \end{cases}$$

A function $f \in \mathcal{S}(L(\mathbf{R}))$ is said to be an *Euler-Poincaré* function if for all irreducible admissible representations σ of $L(\mathbf{R})$,

$$\operatorname{Tr} \sigma(f) = \operatorname{ep}(\mathfrak{g}, K; \sigma \otimes V) = \sum_i (-1)^i \dim H^i(\mathfrak{g}, K; \sigma \otimes V)$$

where

- $\mathfrak{g} = \operatorname{Lie} L$,
- K is a maximal compact subgroup of $L(\mathbf{R})$, and
- V is an irreducible algebraic representation of L.

Lemma 5.23. *Let π be a discrete series representation of $L(\mathbf{R})$. Then there exists a pseudo-coefficient $f_\pi \in \mathcal{C}_c^\infty(L(\mathbf{R}))$ of π. The pseudo-coefficient f_π is K-finite and cuspidal. Furthermore, if $\operatorname{Tr} \sigma(f_\pi) \neq 0$ for some irreducible admissible representation σ of $L(\mathbf{R})$, then the infinitesimal characters of σ and π are equal.*

Proof. The existence of pseudo-coefficients is due to Clozel-Delorme [CD90]. Labesse [Lab91] has shown that these functions can be chosen to be cuspidal. □

Lemma 5.24. *Let π be a discrete series representation of $L(\mathbf{R})$ whose infinitesimal character is equal to that of an irreducible algebraic representation V whose highest weight is regular (cf. Section 3.3.1). Then for all irreducible admissible representations σ of $L(\mathbf{R})$,*

$$\operatorname{Tr} \sigma(f_\pi) = \begin{cases} 1 & : \textit{if } \sigma \simeq \pi \\ 0 & : \textit{otherwise} \end{cases}$$

Proof. By Lemma 5.23, we only have to consider the case where σ is non-tempered. By the local Langlands classification, such a σ can be realised as a constituent of an induced representation $\operatorname{Ind}_{P(\mathbf{R})}^{L(\mathbf{R})} \rho$ where ρ is a discrete series representation of some Levi-subgroup. It can be seen that ρ can not have the same infinitesimal character as V. The result then follows by Lemma 5.23. □

Lemma 5.25. *Let V be an irreducible algebraic representation of L. We define the test function*

$$f_V = \sum_{\pi \in \Pi(V^\vee)} (-1)^{q(L)} f_\pi$$

where $q(L) = \frac{1}{2}\dim(L(\mathbf{R})/K)$. *Then* f_V *is an Euler-Poincaré function, and, for all irreducible admissible representations* σ *of* $L(\mathbf{R})$,

$$\operatorname{Tr}\sigma(f) = \operatorname{ep}(\mathfrak{g}, K; \sigma \otimes V)$$

Proof. [Lab91, §6] □

Lemma 5.26. *Let* V *be an irreducible algebraic representation of* L *whose highest weight is regular. Then for all irreducible admissible representations* σ *of* $L(\mathbf{R})$,

$$\operatorname{Tr}\sigma(f_V) = \begin{cases} (-1)^{q(L)} & : \textit{if } \sigma \in \Pi(V^\vee) \\ 0 & : \textit{otherwise} \end{cases}$$

Proof. The result follows Lemma 5.24, and Lemma 5.25 □

When the highest weight of V is no longer assumed regular, we have the following result of Kottwitz.

Lemma 5.27. *Let* $f_\nu \in \mathcal{C}_c^\infty(F_\nu)$ *be an Euler-Poincaré function for all archimedean places* ν *of* F. *Let* S *be a finite set of places of* F *including all archimedean places. Let* τ_0^S *be an irreducible admissible representation of* $U(\mathbf{A}^S)$. *Then there exists a sign* $\epsilon \in \{1, -1\}$ *such that if*

$$\prod_{\nu|\infty} \operatorname{Tr}\tau(f_\nu) \neq 0$$

then its sign is ϵ, *for all discrete automorphic representations* τ *of* U *such that* $\tau^S \simeq \tau_0^S$.

Proof. Kottwitz [Kot92, Theorem 1] proved the analogous result for the group GU using Shimura varieties. Clozel-Labesse [CL99, §A.4] have shown that Kottwitz's argument extends to the setting of unitary groups. □

Lemma 5.28. *Let* σ *be a discrete series representation of* $U(F_\nu)$. *Let*

$$f^H = \sum_{\xi\circ\psi_H \simeq \psi(\sigma)} \frac{\Delta_{\text{spec}}(\psi_H, \sigma)}{|\Pi(\psi_H)|} \sum_{\pi \in \Pi(\psi_H)} f_\pi$$

where ψ_H *runs through the tempered* L*-parameter for* H_ν. *Then* f^H *is a* Δ*-transfer of* f_σ.

Proof. We remark that

$$\sum_{\pi'_H \in \Pi(\psi'_H)} \operatorname{Tr}\pi'_H\left(f^H\right) = \sum_{\sigma' \in \Pi_{\text{temp}}(U_\nu)} \Delta_{\text{spec}}(\psi'_H, \sigma') \operatorname{Tr}\sigma'(f_\sigma)$$

for all tempered L-parameters $\psi'_H : L_{F_\nu} \to {}^L H_\nu$. It follows by [She10, Theorem 5.1] that f^H is a Δ-transfer of f_σ. □

5.2.6. *The transfer: unramified case.* Assume that ν is a finite place of F that is unramified in E.

Let K_U (resp. K_H) be a hyperspecial subgroup of $U(F_\nu)$ (resp. $H(F_\nu)$). The L-homomorphism ξ is unramified at ν, and induces a transfer of K_H-unramified representations to K_U-unramified representations (cf. Section 4). Dual to this transfer, there is a morphism of spherical Hecke algebras (see [Min09, §4] for an explicit description)

$$\xi : \mathcal{C}_c^\infty(U(F_\nu), K_U) \to \mathcal{C}_c^\infty(H(F_\nu), K_H)$$

Lemma 5.29. *There exists a complex number* $c(\Delta, K_U, K_H)$ *of norm* 1, *depending only upon the chosen normalisation of the geometric transfer factor,* K_U, *and* K_H, *such that* $c(\Delta, K_U, K_H) \cdot \xi(f)$ *is a* Δ*-transfer of* f *for all* $f \in \mathcal{C}_c^\infty(U(F_\nu), K_U)$.

Proof. If ν splits in E, then the result is well known (cf. [Shi10b, §3.3]). Assume that ν remains inert in E. When $f = \mathbf{1}_{K_U}$, the characteristic function on K_U, this is the fundamental lemma for unitary groups, which was proved by Laumon-Ngô [LN08] and Waldspurger [Wal06]. Hales [Hal95] deduced the result for general f from the fundamental lemma. □

This allows us to define the *spectral transfer factors*

$$\Delta_{\text{spec}}(\pi_H, \pi) = c(\Delta, K_U, K_H)$$

for all K_H-unramified representations π_H and for all K_U-unramified representations π such that $\psi(\pi) \simeq \xi \circ \psi(\pi_H)$.

Lemma 5.30. *Let $f \in \mathcal{C}_c^\infty(U(F_\nu), K_U)$ and let $f^H \in \mathcal{C}_c^\infty(H(F_\nu), K_H)$ be a Δ-transfer of f. Let π_H be K_H-unramified and let π be K_U-unramified. If $\psi(\pi) \simeq \xi \circ \psi(\pi_H)$, then*

$$\operatorname{Tr} \pi_H \left(f^H\right) = \Delta_{\text{spec}}(\pi_H, \pi) \operatorname{Tr} \pi(f)$$

Proof. This is a direct consequence of Lemma 5.29. □

5.2.7. *The transfer: split places.* Assume that ν is a finite place of F that splits in E. The existence and properties of the transfer are well known in this case (cf. [Shi10b, §3.3]).

Lemma 5.31. *Let $f \in \mathcal{C}_c^\infty(U(F_\nu))$. There exists $f^H \in \mathcal{C}_c^\infty(H(F_\nu))$ such that f^H is a Δ-transfer of f.*

Lemma 5.32. *There exists a complex number $c(\Delta)$ of norm 1, depending only upon the chosen normalisation of the geometric transfer factor, such that, if we define the spectral transfer factors*

$$\Delta_{\text{spec}}(\psi_H, \pi) = \begin{cases} 0 & : \text{if } \pi \notin \Pi(\xi \circ \psi_H) \\ c(\Delta) & : \text{if } \pi \in \Pi(\xi \circ \psi_H) \end{cases}$$

for all L-parameters $\psi_H : L_{F_\nu} \to {}^L H_\nu$ and all irreducible admissible representations π of $U(F_\nu)$, then the spectral transfer factors satisfy the following identity. For all $f \in \mathcal{C}_c^\infty(U(F_\nu))$ and $f^H \in \mathcal{C}_c^\infty(H(F_\nu))$ such that f^H is a Δ-transfer of f, for all L-parameters $\psi_H : L_{F_\nu} \to {}^L H_\nu$,

$$\operatorname{Tr} \pi_H \left(f^H\right) = \sum_\pi \Delta_{\text{spec}}(\psi_H, \pi) \operatorname{Tr} \pi(f)$$

where $\Pi(\psi_H) = \{\pi_H\}$.

Remark 5.33. It is a consequence of Lemma 5.32 and Lemma 5.30 that the definition of spectral transfer factors at split places is consistent with the definition given for unramified representations, that is, $\Delta_{\text{spec}}(\pi_H, \pi) = \Delta_{\text{spec}}(\psi(\pi_H), \pi)$ for all K_U-unramified π and for all K_H-unramified π_H such that $\psi(\pi) \simeq \xi \circ \psi(\pi_H)$.

5.2.8. *The transfer: inert places.* Assume that ν is a finite place of F of that remains inert in E.

Lemma 5.34. *Let $f \in \mathcal{C}_c^\infty(U(F_\nu))$. Then there exists $f^H \in \mathcal{C}_c^\infty(H(F_\nu))$ such that f^H is a Δ-transfer of f.*

Proof. The existence of the transfer is due to Laumon-Ngô [LN08] and Waldspurger [Wal06] [Wal97]. □

Lemma 5.35. *There exist* spectral transfer factors $\Delta_{\text{spec}}(\psi_H, \pi)$ *defined for all L-parameters $\psi_H : L_{F_\nu} \to {}^L H_\nu$ such that $\xi \circ \psi_H$ is a tempered θ-discrete stable L-parameter, and discrete series representations π of $U(F_\nu)$, such that,*

- $\Delta_{\text{spec}}(\psi_H, \pi) = 0$ *if* $\pi \notin \Pi(\xi \circ \psi_H)$*, and*
- $|\Delta_{\text{spec}}(\psi_H, \pi)| \in \mathbf{C}^\times$ *if* $\pi \in \Pi(\xi \circ \psi_H)$*.*

The spectral transfer factors satisfy the following identity. For all $f \in \mathcal{C}_c^\infty(U(F_\nu))$ *and* $f^H \in \mathcal{C}_c^\infty(H(F_\nu))$ *such that* f^H *is a* Δ*-transfer of* f*,*

$$\sum_{\pi_H \in \Pi(\psi_H)} n(\psi_H, \pi_H) \operatorname{Tr} \pi_H \left(f^H\right) = \sum_{\pi \in \Pi_{\text{disc}}(U_\nu)} \Delta_{\text{spec}}(\psi_H, \pi) \operatorname{Tr} \pi(f)$$

where the $n(\psi_H, \pi_H)$ *are defined in Lemma 5.10.*

Proof. [Mœg07, §7] □

Remark 5.36. If π_H is a K_H-unramified discrete series representation of $H(F_\nu)$, then it follows by Lemma 5.6 and Lemma 5.10 that $n(\psi(\pi_H), \pi_H) = 1$. It is then a consequence of Lemma 5.35 and Lemma 5.30 that the definition of spectral transfer factors at inert places is consistent with the definition given for unramified representations, that is, $\Delta_{\text{spec}}(\pi_H, \pi) = \Delta_{\text{spec}}(\psi(\pi_H), \pi)$ for all K_U-unramified discrete series π and for all K_H-unramified discrete series π_H such that $\psi(\pi) \simeq \xi \circ \psi(\pi_H)$.

Lemma 5.37. *Assume that there exists an identity consisting of finite linear combinations of irreducible admissible representations*

$$\sum_{\pi_H} a(\pi_H) \operatorname{Tr} \pi_H \left(f^H\right) = \sum_{\pi} b(\pi) \operatorname{Tr} \pi(f)$$

for all $f \in \mathcal{C}_c^\infty(U(F_\nu))$ *and* $f^H \in \mathcal{C}_c^\infty(H(F_\nu))$ *such that* f^H *is a* Δ*-transfer of* f*. Furthermore, assume that the LHS of the identity is a stable distribution. Then for all tempered* θ*-discrete stable L-parameters* $\psi : L_{F_\nu} \to {}^L U_\nu$*,*

$$\sum_{\xi \circ \psi(\pi_H) \simeq \psi} a(\pi_H) \operatorname{Tr} \pi_H \left(f^H\right) = \sum_{\psi(\pi) \simeq \psi} b(\pi) \operatorname{Tr} \pi(f)$$

for all $f \in \mathcal{C}_c^\infty(U(F_\nu))$ *and* $f^H \in \mathcal{C}_c^\infty(H(F_\nu))$ *such that* f^H *is a* Δ*-transfer of* f*.*

Proof. By [Art96, Theorem 6.2], we can deduce the identity

$$\sum a(\pi_H) \operatorname{Tr} \pi_H \left(f^H\right) = \sum b(\pi) \operatorname{Tr} \pi(f)$$

where the summations are taken over the subset of representations that are elliptic tempered. The result then follows from [Mœg07, §7]. □

5.2.9. *Normalisation of the transfer factors.* Let us now describe our specific normalisation of the geometric transfer factors. This will consequently fix the normalisation of the spectral transfer factors. Fix $\overline{\gamma}_H \in H(F)$ and $\overline{\gamma}_U \in U(F)$ such that for all places ν of F, $\overline{\gamma}_H \in \Gamma_{\text{G-reg,ss}}(H(F_\nu))$ is a norm of $\overline{\gamma}_U \in \Gamma_{\text{reg,ss}}(U(F_\nu))$. Writing Δ_ν for the local geometric transfer factor at a place ν of F, we fix the value of $\Delta_\nu(\overline{\gamma}_H, \overline{\gamma}_U)$ as a complex number of norm 1 such that the following conditions are satisfied.

- If the groups H_ν and U_ν are unramified, then we hyperspecial subgroups K_{H_ν} and K_{U_ν} of $H(F_\nu)$ and $U(F_\nu)$ respectively. We require that $c(\Delta_\nu, K_{U_\nu}, K_{H_\nu}) = 1$ (cf. Section 5.2.6).
- If ν is finite and splits in E, then we require that $c(\Delta_\nu) = 1$ (cf. Section 5.2.7).
- $\prod_\nu \Delta_\nu(\overline{\gamma}_H, \overline{\gamma}_U) = 1$

If $U \simeq U^*$, then the geometric transfer factors for the principal endoscopic group $H = U^*$ are, up to constant, trivial (cf. [Lab09, §4.2]). In this case, we shall normalise the geometric transfer factors to be trivial, that is, identifying the groups

$U \xrightarrow{\sim} U^* \xrightarrow{\sim} H$, we have that for all places ν, for all $\gamma_{H,\nu} \in \Gamma_{\text{G-reg,ss}}(H(F_\nu))$, and for all $\gamma_{U,\nu} \in \Gamma_{\text{reg,ss}}(U(F_\nu))$,

$$\Delta_\nu(\gamma_{H,\nu}, \gamma_{U,\nu}) = \begin{cases} 1 & : \text{if } \gamma_{H,\nu} \text{ and } \gamma_{U,\nu} \text{ are stable conjugates} \\ 0 & : \text{otherwise} \end{cases}$$

Lemma 5.38. *Assume that $U \simeq U^* \simeq H$, and that the transfer factors are normalised to be trivial. Then the dual spectral transfer factors satisfy the following properties*

- *If H_ν and U_ν are unramified, then $c(\Delta_\nu, K_{U_\nu}, K_{H_\nu}) = 1$ for all hyperspecial subgroups K_{H_ν} and K_{U_ν} of $H(F_\nu)$ and $U(F_\nu)$ respectively.*
- *If ν is finite and splits in E, then $c(\Delta_\nu) = 1$.*
- *If ν is archimedean, then*

$$\Delta_{\text{spec}}(\psi_H, \pi) = \begin{cases} 1 & : \text{if } \pi \in \Pi(\psi_H) \\ 0 & : \text{otherwise} \end{cases}$$

  for all tempered L-parameters ψ_H of H_ν and tempered representations π of $U^*(F_\nu)$.
- *If ν is finite and inert in E, then*

$$\Delta_{\text{spec}}(\psi_H, \pi) = \begin{cases} n(\psi_H, \pi) & : \text{if } \pi \in \Pi(\psi_H) \\ 0 & : \text{otherwise} \end{cases}$$

  for all tempered θ-discrete stable L-parameters $\psi_H : L_{F_\nu} \to {}^L H_\nu$ and for all discrete series representations π of $U(F_\nu)$.

Proof. We shall prove here the last property. The other properties can be proved by using the same argument, and the details are left to the reader. Firstly, we recall that $\Delta_{\text{spec}}(\psi_H, \pi) = 0$ if $\pi \notin \Pi(\psi_H)$. We are left to consider the case where $\pi \in \Pi(\psi_H)$. The spectral transfer factors satisfy the identity (cf. Lemma 5.35),

$$\sum_{\pi \in \Pi(\psi_H)} n(\psi_H, \pi) \operatorname{Tr} \pi\left(f^H\right) = \sum_{\pi \in \Pi(\psi_H)} \Delta_{\text{spec}}(\psi_H, \pi) \operatorname{Tr} \pi(f) \tag{5.1}$$

for all $f \in \mathcal{C}_c^\infty(U(F_\nu))$ and $f^H \in \mathcal{C}_c^\infty(H(F_\nu))$ such that f^H is a Δ-transfer of f. By Lemma 5.12, the distribution on the LHS is stable. Since the geometric transfer factors are trivial, this implies that the distribution on the RHS is also stable. By Lemma 5.12, there exists a constant C such that

$$\Delta_{\text{spec}}(\psi_H, \pi) = C \cdot n(\psi_H, \pi)$$

for all $\pi \in \Pi(\psi_H)$. By considering Equation 5.1, we see that $C = 1$. □

5.2.10. *The Stable Trace Formula.* Let E/F be a totally imaginary quadratic extension of a totally real number field, and let U denote a group appearing in Proposition 2.1. Let S_{ram} denote the finite set of places ν of F such that either ν is archimedean, or ν is non-archimedean and ramified in E.

Proposition 5.39. *Let $S \supset \mathrm{S}_{\text{ram}}$ be a finite set of places of F. Let $f_S = \otimes_{\nu \in S} f_\nu \in \mathcal{C}_c^\infty(U(\mathbf{A}_S))$, and assume that f_ν is as in Lemma 5.23 for all archimedean ν. Let $f = f_S \otimes \mathbf{1}_{K^S}$ where $K^S = \prod_{\nu \notin S} K_\nu$ is a product of hyperspecial subgroups K_ν of $U(F_\nu)$. Assume that for all $a \le b$, $f^{H_{a,b}} = \otimes_\nu f_\nu^{H_{a,b}} \in \mathcal{C}_c^\infty(H_{a,b}(\mathbf{A}))$ is a Δ-transfer of f at all places ν. Then*

$$I(f) = \sum_{a \le b} \iota(U, H_{a,b}) S^{H_{a,b}}\left(f^{H_{a,b}}\right)$$

where I denotes the invariant trace formula for U (cf. [Art88]*), $S^{H_{a,b}}$ denotes the stable trace formula for $H_{a,b}$ (cf.* [Art02]*), and*

$$\iota(U, H_{a,b}) = \begin{cases} 1 & : \text{if } a = 0 \\ \frac{1}{4} & : \text{if } a = b \\ \frac{1}{2} & : \text{otherwise} \end{cases}$$

Proof. The stabilisation of the invariant trace formula for a connected reductive group has been completed by Arthur [Art02] [Art01] [Art03] under the assumption of the validity of the weighted fundamental lemma. This is now proven due to the work of Chaudouard-Laumon [CL10a] [CL10b], Ngô [Ngô10], and Walspurger [Wal09]. For the evaluation of the constants $\iota(U, H_{a,b})$ see [Lab09, Proposition 4.11]. □

The invariant trace formula admits a simple expression due to the fact that f is cuspidal at infinity (cf. [Art88, Theorem 7.1]).

$$I(f) = I_{\text{disc}}(f) = \sum_{\pi \in \Pi_{\text{disc}}(U)} \mathrm{m}_{\text{disc}}(\pi) \operatorname{Tr} \pi(f)$$

where $\mathrm{m}_{\text{disc}}(\pi)$ denotes the multiplicity of π in the discrete automorphic spectrum of U.

6. Base change

In this section, we obtain a result on Langlands base change which mildly improves upon a previous result of Labesse [Lab09, Theorem 5.1, Theorem 5.9].

In order to succinctly state the results, we begin by recalling the definition of a cohomological representation. Let G be a connected reductive algebraic group defined over k where k is the field of either real or complex numbers. Let $G' = \operatorname{Res}_{k/\mathbf{R}} G$, let $\mathfrak{g}' = \operatorname{Lie} G'$, and let K' be a maximal compact subgroup of $G'(\mathbf{R})$. A *system of coefficients* for G is an irreducible algebraic representation V of G'. An irreducible admissible representation σ of $G'(\mathbf{R})$ is said to have *cohomology* (for the system of coefficients V) if

$$H^i(\mathfrak{g}', K'; \sigma \otimes V) \neq 0$$

for some integer i. This is equivalent to demanding that the Euler-Poincaré characteristic

$$\operatorname{ep}(\mathfrak{g}', K'; \sigma \otimes V) \neq 0$$

be non-zero (cf. [CL99, Lemma A.4.1]).

Theorem 6.1. *Let E/F be a totally imaginary quadratic extension of a totally real field. Let U be a unitary group appearing in Proposition 2.1. Let σ be a discrete automorphic representation of $U(\mathbf{A}_F)$. Assume that σ_ν has cohomology for a system of coefficients V_ν for all archimedean places ν of F. Then there exists an automorphic representation Π of $GL_n(\mathbf{A}_E)$ such that*

- *for all archimedean places ν, Π_ν has cohomology for the system of coefficients $\left(V_\nu \otimes V_\nu^\theta\right)$, and*
- *Π_ν is the Langlands base change of σ_ν at finite places ν for which either*
  - *ν splits in E, or*
  - *σ_ν is unramified, or*
  - *σ_ν is a discrete series representation.*

The automorphic representation Π can be written as an isobaric sum

$$\Pi = \Pi_1 \boxplus \Pi_2 \cdots \boxplus \Pi_r$$

where each Π_i *is a discrete automorphic representation of some* $GL_{n_i}(\mathbf{A}_E)$ *such that*

- $\Pi_i \simeq \Pi_i \circ \theta$ *for all* i*, and*
- $\Pi_i \not\simeq \Pi_j$ *for all* $i \neq j$*.*

Furthermore, if each Π_i *is cuspidal, then* Π_ν *is the Langlands base change of* σ_ν *at archimedean places* ν *where* σ_ν *is a discrete series representation.*

Remark 6.2. If there exists an archimedean place ν such that the highest weight of V_ν is regular, then by considering the infinitesimal characters of the Π_i in conjunction with the Mœglin-Waldspurger [MW89] description of the discrete spectrum of GL, we see that the Π_i are cuspidal.

Proof. Let S denote the finite set of places ν of F for which σ_ν is not unramified. Let S' denote the subset of places $\nu \in S$ such that

- ν is archimedean, or
- ν is finite inert in E, and σ_ν is not a discrete series representation.

For all $\nu \notin S$ and for all H, fix hyperspecial subgroups K_{U_ν} and K_{H_ν} of $U(F_\nu)$ and $H(F_\nu)$ respectively such that $\sigma_\nu^{K_{U_\nu}} \neq 0$.

The stable trace formula of Proposition 5.39 provides us with the identity

$$I^U(f) = \sum_H \iota(U,H)\, S^H\left(f^H\right)$$

where

- $f^H = \otimes_\nu f_\nu^H$ is a Δ-transfer of $f = \otimes_\nu f_\nu$ at each place ν, and
- for all $\nu|\infty$, f_ν is the Euler-Poincaré function associated to $V_\nu^\vee$ as in Lemma 5.25, and f_ν^H is a linear combination of the Euler-Poincaré functions appearing in Lemma 5.28.

We shall further require that f_ν (resp. f_ν^H) is bi-invariant under K_{U_ν} (resp. K_{H_ν}) for all $\nu \notin S$.

By Lemma 5.30, Lemma 5.32, Lemma 5.37 and using the linear independence of characters, we can separate the chain of representations

$$\sum_{\sigma'} \mathrm{m}_{\mathrm{disc}}(\sigma')\, \mathrm{Tr}\,\sigma(f) = \sum_H \iota(U,H) \sum_{\tau_H} n(\tau_H)\, \mathrm{Tr}\,\tau_H\left(f^H\right) \tag{6.1}$$

where

- σ' runs through the discrete automorphic representations of U such that
  - for all archimedean ν, σ'_ν has cohomology for the system of coefficients V_ν,
  - for all $\nu \notin S$, $\sigma'_\nu \simeq \sigma_\nu$, and
  - for all $\nu \notin S'$, σ'_ν and σ_ν are elements of the same L-packet, that is, $\psi(\sigma'_\nu) \simeq \psi(\sigma_\nu)$
- τ_H runs through the stable discrete automorphic representations of H such that
  - for all archimedean ν, $\tau_{H,\nu}$ has cohomology for a system of coefficients $V_{H,\nu}$ such that $\xi \circ \psi(\Pi(V_{H,\nu})) \simeq \psi(\Pi(V_\nu))$,
  - for all $\nu \notin S$, $\tau_{H,\nu}$ is K_{H_ν}-unramified and $\xi \circ \psi(\tau_{H_\nu}) \simeq \psi(\sigma_\nu)$, and
  - for all $\nu \notin S'$, $\xi \circ \psi(\tau_{H_\nu}) \simeq \psi(\sigma_\nu)$

By Lemma 5.27 and using the linear independence of characters, we see that the distribution is non-trivial. This implies that there exists a $H = U_a^* \times U_b^*$ for which the distribution

$$\sum_{\tau_H} n(\tau_H)\, \mathrm{Tr}\,\tau_H\left(f^H\right) \tag{6.2}$$

is non-trivial.

The stable base change identity of Proposition 5.13 provides us with the identity

$$C \cdot I^{G_H}\left(\phi^H\right) = ST^H\left(f'^H\right)$$

where

- $G_H = GL_a \times GL_b \times \theta$,
- $$C = \begin{cases} 2 & : \text{ if } H = U_n^* \\ 4 & : \text{otherwise} \end{cases}$$
- $f'^H = \otimes_\nu f'^H_\nu$ and $\phi^H = \otimes_\nu \phi^H_\nu$ are associated at each place ν, and
- for all archimedean ν, $f'^H_\nu = f^H_\nu$ as in Equation 6.1, and ψ^H_ν is a linear combination of the twisted Euler-Poincaré functions appearing in Lemma 5.4.

We shall further require that f'^H_ν is bi-invariant under K_{H_ν} for all $\nu \notin S$.

By Lemma 5.6, Lemma 5.8, Lemma 5.10, Lemma 5.12, and using the linear independence of characters, we can separate the chain of representations

(6.3)

$$C \cdot \sum_{L_0 \in \mathcal{L}^0} \frac{|W_0^{L_0}|}{|W_0^{G^0}|} \sum_{s \in W^G\left(\mathfrak{a}_{L_0}\right)_{\text{reg}}} |\det(s-1)_{\mathfrak{a}^G_{L_0}}|^{-1}$$

$$\sum_{\widetilde{\pi}} \mathrm{m_{disc}}\left(\widetilde{\pi}\right) \mathrm{Tr}\left(M_{Q_0|sQ_0}(0)\, \rho_{Q_0,t}\left(s,0,\phi^H\right)|_{\mathrm{Ind}^{G^0}_{Q_0}\pi}\right) = \sum_{\tau_H} n\left(\tau_H\right) \mathrm{Tr}\, \tau_H\left(f^H\right)$$

where

- the τ_H are those appearing in Equation 6.2, and
- $\widetilde{\pi}$ runs through the $\widetilde{\pi} \in \Pi_{\text{disc}}\left(L_0 \rtimes \langle s\rangle\right)$ such that
  - for all archimedean ν, $\mathrm{Ind}^{GL_a \times GL_b}_{Q_0} \pi_\nu$ is cohomological for a system of coefficients $V_{H_\nu} \otimes V^\theta_{H_\nu}$ where V_{H_ν} is one of the previously described algebraic representations (cf. Equation 6.1),
  - for all $\nu \notin S$, $\mathrm{Ind}^{GL_a \times GL_b}_{Q_0} \pi_\nu$ is unramified, and is the Langlands base change of $\tau_{H,\nu}$ where τ_H is a representation appearing in Equation 6.2,
  - for all $\nu \notin S'$, $\mathrm{Ind}^{GL_a \times GL_b}_{Q_0} \pi_\nu$ is the Langlands base change of $\tau_{H,\nu}$ where τ_H is a representation appearing in Equation 6.2.

Invoking the Mœglin-Waldspurger [MW89] and Jacquet-Shalika [JS81] description of the automorphic spectrum of $GL_n(\mathbf{A}_E)$, we see that there exists a partition $n_1 + \cdots + n_r = n$, and a corresponding set of discrete automorphic representations Π_i of $GL_{n_i}(\mathbf{A}_E)$, such that, the representations $\widetilde{\pi}$ appearing in Equation 6.3 are exactly those for which

$$\pi \simeq \left(\Pi_{j_1}\mu_b^{-1} \times \cdots \times \Pi_{j_{r'}}\mu_b^{-1}\right) \times \left(\Pi_{j_{r'+1}}\mu_a^{-1} \times \cdots \times \Pi_{j_r}\mu_a^{-1}\right)$$

where $\{j_1, \ldots, j_r\} = \{1, \ldots, r\}$ and $n_{j_1} + \cdots + n_{j_{r'}} = a$. Furthermore, by Lemma 5.17, $\Pi_i \simeq \Pi_i \circ \theta$ for all $i = 1, \ldots r$.

It follows from Lemma 4.1 that the automorphic representation of GL_n

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r$$

is the Langlands base change of σ outside of S', and that Π_ν has cohomology in the system of coefficients $V_\nu \otimes V^\theta_\nu$ for all archimedean ν. By considering infinitesimal characters, we see that $\Pi_i \not\simeq \Pi_j$ for all $i \neq j$. Finally if the Π_i are cuspidal, then by Lemma 5.2, Π_ν is the Langlands base change of σ_ν at the archimedean places ν for which σ_ν is a discrete series representation. □

7. Certain representations

In this section, we shall combine some results of Shin [Shi10a] on the limit multiplicity with our results on base change. This allows us to deduce the existence of automorphic representations π of GL_n such that $\pi \simeq \pi \circ \theta$, and which satisfy certain imposed local conditions. These representations shall be extensively used throughout this article.

Lemma 7.1. *Let E/F be a totally imaginary quadratic extension of a totally real field, and let U be a unitary group appearing in Proposition 2.1. Let S be a finite set of places of F including all archimedean places, and all non-archimedean places which are ramified in E. For all $\nu \in S$,*

- *if ν is archimedean, let τ_ν be a discrete series representation of $U(F_\nu)$ that has cohomology for a system of coefficients V_ν whose highest weight is regular, and*
- *if ν is non-archimedean, let τ_ν be a discrete series representation of $U(F_\nu)$.*

Then there exists a discrete automorphic representation σ of $U(\mathbf{A}_F)$ such that

- *if $\nu \in S$ is archimedean, then σ_ν is a discrete series representation which appears in the same L-packet as τ_ν, that is, $\psi(\sigma_\nu) \simeq \psi(\tau_\nu)$,*
- *if $\nu \in S$ is non-archimedean, then σ_ν is a discrete series representation which is isomorphic to a twist of τ_ν by some unitary character χ_ν, and*
- *if $\nu \notin S$ is non-archimedean and inert in E, then σ_ν is unramified.*

Proof. [Shi10a, Theorem 5.7] □

Remark 7.2. By applying the result of Shin [Shi10a, Theorem 5.7] to a finite product of unitary groups, we can demand the following slightly stronger result. For $i = 1, \ldots, t$, let U_i be a unitary group associated to E/F and let $\{t_{i,\nu} : \nu \in S\}$ be a finite collection of representations as in Lemma 7.1. Then there exist discrete automorphic representations $\sigma_1, \ldots, \sigma_t$ as in Lemma 7.1 such that for all $\nu \in S$, the characters $\chi_{1,\nu} \simeq \cdots \simeq \chi_{t,\nu}$ are isomorphic.

Lemma 7.3. *Let E/F be a totally imaginary quadratic extension of a totally real field, and let U be a unitary group appearing in Proposition 2.1. Let S be a finite set of places of F including all archimedean places, and all non-archimedean places which are ramified in E. For all $\nu \in S$,*

- *if ν is archimedean, let τ_ν be a discrete series representation of $U(F_\nu)$ that has cohomology for a system of coefficients V_ν whose highest weight is regular, and*
- *if ν is non-archimedean, let τ_ν be a discrete series representation of $U(F_\nu)$.*

Then there exists a cuspidal automorphic representation π of $GL_n(\mathbf{A}_E)$ such that

- *$\pi \simeq \pi \circ \theta$,*
- *if $\nu \in S$ is archimedean, then π_ν is the Langlands base change of τ_ν, that is, $\psi(\pi_\nu) \simeq \mathrm{BC}(\psi(\tau_\nu))$,*
- *if $\nu \in S$ is non-archimedean, then π_ν is isomorphic to the Langlands base change of a discrete series representation τ'_ν of $U(F_\nu)$ which is isomorphic to a twist of τ_ν by some unitary character χ_ν, and*
- *if $\nu \notin S$ is non-archimedean and inert in E, then π_ν is unramified.*

Proof. Let $\omega \notin S$ be a non-archimedean place of F that splits in E, and let τ_ω be a supercuspidal representation of $U(F_\omega) \xrightarrow{\sim} GL_n(F_\omega)$. Let σ be a discrete automorphic representation of $U(\mathbf{A}_F)$ obtained by applying Lemma 7.1 at the set of places $S \cup \{\omega\}$. Let π be the Langlands base change of σ given by Theorem 6.1. The result will follow upon confirmation that π is cuspidal. However, π_ω is a supercuspidal representation, and it follows that π is cuspidal. □

8. L-packets of discrete series representations of the p-adic quasi-split unitary group

The aim of this section is to show that the non-trivial coefficients $n(\psi,\pi)$ of Mœglin appearing in Lemma 5.10 are equal to 1. In order to do so, we shall assume the existence of twisted analogues of some results of Arthur [Art93] on the inner product of elliptic tempered representations of p-adic groups (cf. Hypothesis 8.0.1).

Let k'/k be a quadratic extension of p-adic fields. Let $\psi : L_{k'} \to GL_n$ be a tempered θ-discrete stable L-parameter of GL_n/k'. Let $\pi_{k'}$ denote the irreducible admissible representation of $GL_n(k')$ corresponding to ψ viewed as an L-parameter of GL_n/k'. There exists a natural inner product on the space of tempered θ-discrete stable representations of $GL_n(k')$ (cf. [Mœg07, §1]). We shall admit the following hypothesis.

Hypothesis 8.0.1.
$$\langle \pi_{k'}, \pi_{k'} \rangle = 2^{l(\psi)-1}$$

Remark 8.1. This would follow from the existence of the twisted analogues of some results of Arthur [Art93].

Lemma 8.2. *The coefficients $n(\psi,\sigma_k)$ of Mœglin appearing in Lemma 5.10 are equal to*
$$n(\psi,\sigma_k) = \begin{cases} 1 & : \text{if } \sigma_k \in \Pi(\psi) \\ 0 & : \text{otherwise} \end{cases}$$
for all irreducible admissible representations σ_k of $U^*(k'/k)$.

Proof. We remark that if $\sigma_k \notin \Pi(\psi)$, then the result is trivial (cf. Remark 5.11). There exists for any connected reductive p-adic group, a natural inner product on the space of elliptic tempered representations (cf. [Art93]). We also know that (cf [Mœg07])
$$\langle \pi_{k'}, \pi_{k'} \rangle = \left\langle \sum_{\sigma_k \in \Pi(\psi)} n(\psi,\sigma_k)\,\sigma_k, \sum_{\sigma_k \in \Pi(\psi)} n(\psi,\sigma_k)\,\sigma_k \right\rangle$$
Arthur [Art93, Corollary 6.2] has shown that the discrete series representations of $U^*(k'/k)$ are orthonormal for this inner product. Thus
$$\langle \pi_{k'}, \pi_{k'} \rangle = \sum_{\sigma_k \in \Pi(\psi)} |n(\psi,\sigma_k)|^2$$
By admitting Hypothesis 8.0.1, we deduce that
$$2^{l(\psi)-1} = |\Pi(\psi)| = \sum_{\sigma_k \in \Pi(\psi)} |n(\psi,\sigma_k)|^2$$
Since the $n(\psi,\sigma_k)$ are non-zero, the result will then follow upon confirmation that the $n(\psi,\sigma_k)$ are non-negative integers. This is accomplished by the following lemma. □

Lemma 8.3. $n(\psi,\sigma_k) \in \mathbf{N}^0$ *for all* $\sigma_k \in \Pi(\psi)$.

Proof. Choose a totally imaginary quadratic extension of a totally real field E/F, and a place v' of F such that

- $E_{\nu'}/F_{\nu'} \simeq k'/k$, and
- E/F is unramified outside of v'.

By Lemma 7.3, we can find a cuspidal automorphic representation Π of $GL_n(\mathbf{A}_E)$ such that

- $\Pi \circ \theta \simeq \Pi$,

- for all archimedean ν, Π_ν has cohomology in a system of coefficients $V_\nu \otimes V_\nu^\theta$ where V_ν is an irreducible algebraic representation of GL_n with regular highest weight,
- $\Pi_{\nu'}$ is the Langlands base change of a discrete series representation which is isomorphic to a twist of some discrete series representation in the L-packet $\Pi(\psi)$ by some unitary character $\chi_{\nu'}$, and
- for all non-archimedean places $\nu \neq \nu'$ that are inert in E, Π_ν is unramified.

Let $\sigma = \otimes_\nu \sigma_\nu$ be an irreducible admissible representation of $U^*(\mathbf{A}_F)$ such that

- Π is the Langlands base change of σ at all places ν, and
- for all non-archimedean $\nu \neq \nu'$ inert in E, σ_ν is unramified.

Let S be the finite set of places ν of F such that either

- ν is archimedean, or
- $\nu = \nu'$, or
- ν is non-archimedean and σ_ν is not unramified.

For all $\nu \notin S$ and for all H, let $K_{U_\nu^*}$ and K_{H_ν} be hyperspecial subgroups of $U^*(F_\nu)$ and $H(F_\nu)$ respectively such that $\sigma_\nu^{K_{U_\nu^*}} \neq 0$.

The stable trace formula of Proposition 5.39 provides us with the identity

$$I(f) = \sum_H \iota(U,H)\, S^H\left(f^H\right)$$

where

- $f^H = \otimes_\nu f_\nu^H$ is a Δ-transfer of $f = \otimes_\nu f_\nu$ at each place ν, and
- for all archimedean ν, f_ν is a pseudo-coefficient of σ_ν and f_ν^H is as in Lemma 5.28.

We shall further require that f_ν (resp. f_ν^H) is bi-invariant under $K_{U_\nu^*}$ (resp. K_{H_ν}) for all $\nu \notin S$.

By Lemma 5.30, Lemma 5.32, Lemma 5.37, and using the linear independence of characters, we can separate the chain of representations

$$\sum_{\sigma'} \mathrm{m_{disc}}(\sigma')\,\mathrm{Tr}\,\sigma'(f) = \sum_H \iota(U,H) \sum_{\tau_H} n(\tau_H)\,\mathrm{Tr}\,\tau_H\left(f^H\right) \tag{8.1}$$

where

- σ' runs through the discrete automorphic representations of U^* such that
  - for all $\nu|\infty$, $\sigma'_\nu \simeq \sigma_\nu$,
  - for all $\nu \notin S$, σ'_ν is $K_{U_\nu^*}$-unramified, and
  - for all ν, σ_ν and σ'_ν are in the same L-packet, that is, $\psi(\sigma_\nu) \simeq \psi(\sigma'_\nu)$.
- τ_H runs through the stable discrete automorphic representations of H such that
  - for all $\nu \notin S$, $\tau_{H,\nu}$ is K_{H_ν}-unramified, and
  - for all ν, $\psi(\sigma_\nu) \simeq \xi \circ \psi(\tau_{H,\nu})$.

For each H, we have the stable base change identity of Proposition 5.13

$$C \cdot I^{G_H}\left(\phi^H\right) = ST^H\left(f'^H\right)$$

where

- $G_H = GL_a \times GL_b \times \theta$,
- $$C = \begin{cases} 2 & : \text{if } H = U_n^* \\ 4 & : \text{otherwise} \end{cases}$$
- $f'^H = \otimes_\nu f_\nu'^H$ and $\phi^H = \otimes_\nu \phi_\nu^H$ are associated at each place ν, and
- for all archimedean ν, $f_\nu'^H = f_\nu^H$ as in Equation 8.1, and ϕ_ν^H is the associated twisted Euler-Poincaré function given by Lemma 5.4.

We shall further require that $f_\nu'^H$ is bi-invariant under K_{H_ν} for all $\nu \notin S$.

By Lemma 5.6, Lemma 5.8, Lemma 5.10, Lemma 5.12, and using the linear independence of characters, we can separate the chain of representations

(8.2)

$$C \cdot \sum_{L_0 \in \mathcal{L}^0} \frac{|W_0^{L_0}|}{|W_0^{G^0}|} \sum_{s \in W^G(\mathfrak{a}_{L_0})_{\mathrm{reg}}} |\det(s-1)_{\mathfrak{a}_{L_0}^G}|^{-1}$$

$$\sum_{\widetilde{\pi}} \mathrm{m_{disc}}(\widetilde{\pi}) \operatorname{Tr}\left(M_{Q_0|sQ_0}(0)\, \rho_{Q_0,t}\left(s,0,\phi^H\right)|_{\mathrm{Ind}_{Q_0}^{G^0}\pi}\right) = \sum_{\tau_H} n(\tau_H) \operatorname{Tr} \tau_H\left(f^H\right)$$

where

- the τ_H are those appearing in Equation 8.1, and
- $\widetilde{\pi}$ runs through the $\widetilde{\pi} \in \Pi_{\mathrm{disc}}(L_0 \rtimes s)$ such that
  - for all $\nu \notin S$, $\mathrm{Ind}_{Q_0}^{GL_a \times GL_b} \pi_\nu$ is unramified, and
  - for all ν, $\mathrm{Ind}_{Q_0}^{GL_a \times GL_b} \pi_\nu$ is the Langlands base change $\tau_{H,\nu}$ where τ_H is a representation appearing in Equation 8.1

It follows from Lemma 4.1 that, writing $\pi = \pi_a \times \pi_b$, seen as a representation of a Levi-subgroup of $GL_a \times GL_b$, we have that

$$\mathrm{Ind}_{Q_0}^{GL_n} \pi_{a,\nu} \cdot \mu_{b,\nu} \times \pi_{b,\nu} \cdot \mu_{a,\nu} \simeq \Pi_\nu$$

for all $\nu \notin S$. Now Π is cuspidal, and it follows from the Mœglin-Waldspurger [MW89] and Jacquet-Shalika [JS81] description of the automorphic spectrum of $GL_n(\mathbf{A}_E)$ that $H = U_n^*$ and $\pi \simeq \Pi$. It then follows from Lemma 5.16, Lemma 5.17, and Lemma 5.18 that Equation 8.2 is equal to

$$\begin{cases} 0 & : \text{if } H \neq U_n^* \\ \operatorname{Tr} \Pi \circ A^W\left(\phi^H\right) & : \text{if } H = U_n^* \end{cases}$$

By Lemma 5.2, Lemma 5.6, Lemma 5.8, Lemma 5.10, and Lemma 5.24, we see that the τ_{U^*} appearing in Equation 8.2 (and hence also Equation 8.1) are exactly the representations

$$\otimes_{\nu|\infty} \tau'_{U^*,\nu} \otimes \tau'_{U^*,\nu'} \otimes \otimes_{\nu \notin \{\infty \cup \nu'\}} \tau'_{U^*,\nu}$$

where

- for all archimedean ν, $\tau'_{U^*,\nu}$ is in the same L-packet as σ_ν,
- $\tau'_{U^*,\nu'}$ is in the same L-packet as $\sigma_{\nu'}$, and
- for all non-archimedean $\nu \neq \nu'$, $\tau'_{U^*,\nu} \simeq \sigma_\nu$.

We also see that

$$n\left(\otimes_{\nu|\infty} \tau'_{U^*,\nu} \otimes \tau'_{U^*,\nu'} \otimes \otimes_{\nu \notin \{\infty \cup \nu'\}} \tau'_{U^*,\nu}\right) = n\left(\psi(\Pi_{\nu'}), \tau'_{U^*,\nu'}\right)$$

It follows that Equation 8.1 can be written as

$$\sum_{\sigma'_{\nu'} \in \Pi(\psi(\Pi_{\nu'}))} \mathrm{m_{disc}}\left(\sigma^{\nu'} \otimes \sigma'_{\nu'}\right) \operatorname{Tr}\left(\sigma^{\nu'} \otimes \sigma'_{\nu'}\right)(f)$$

$$= \sum n\left(\psi(\Pi_{\nu'}), \tau'_{U^*,\nu'}\right) \operatorname{Tr}\left(\otimes_{\nu|\infty} \tau'_{U^*,\nu} \otimes \tau'_{U^*,\nu'} \otimes \otimes_{\nu \notin \{\infty \cup \nu'\}} \tau'_{U^*,\nu}\right)\left(f^{U^*}\right)$$

It then follows from the spectral transfer results of Lemma 5.22, Lemma 5.30, Lemma 5.32, and Lemma 5.35 that

$$\prod_{\nu|\infty} \Delta_{\mathrm{spec}}\left(\psi\left(\tau'_{U^*,\nu}\right), \sigma_\nu\right) \cdot \Delta_{\mathrm{spec}}\left(\psi(\Pi_{\nu'}), \sigma'_{\nu'}\right) = \mathrm{m_{disc}}\left(\sigma^{\nu'} \otimes \sigma'_{\nu'}\right)$$

for all $\sigma'_{\nu'} \in \Pi(\psi(\Pi_{\nu'}))$. Due to our normalisation of the transfer factors for the principal endoscopic group (cf. Lemma 5.38), this reduces to give

$$n(\psi(\Pi_{\nu'}), \sigma'_{\nu'}) = \mathrm{m}_{\mathrm{disc}}\left(\sigma^{\nu'} \otimes \sigma'_{\nu'}\right)$$

It follows that $n(\psi(\Pi_{\nu'}), \sigma'_{\nu'})$ is a non-negative integer for all $\sigma'_{\nu'} \in \Pi(\psi(\Pi_{\nu'}))$ as the multiplicity of a representation in the discrete automorphic spectrum of U is a non-negative integer. The result then follows since for all $\sigma_k \in \Pi(\psi)$,

$$n(\psi, \sigma_k) = n(\psi(\Pi_{\nu'}), \sigma'_{\nu'})$$

where $\sigma'_{\nu'} \in \Pi(\psi(\Pi_{\nu'}))$ is the twist of σ_k by the unitary character $\chi_{\nu'}$. □

9. Some properties of the spectral transfer factors

The aim of this section is to prove some properties of the spectral transfer factors. These properties shall form the basis of our proof of the Arthur conjectures.

Let E/F be a totally imaginary quadratic extension of a totally real field, and let U be a unitary group in n-variables appearing in Proposition 2.1. Let

$$\Pi^0 = \Pi_1^0 \boxplus \cdots \boxplus \Pi_r^0$$

be an automorphic representation of $GL_n(\mathbf{A}_E)$ where

- for all $i = 1, \ldots, r$, Π_i^0 is a cuspidal automorphic representation of $GL_{n_i}(\mathbf{A}_E)$ such that $\Pi_i^0 \simeq \Pi_i^0 \circ \theta$,
- for all archimedean places ν of F, Π_ν^0 has cohomology for a system of coefficients $V_\nu^0 \otimes V_\nu^{0\theta}$ where the highest weight of V_ν^0 is regular,
- for all non-archimedean places ν of F that are inert and unramified in E, Π_ν^0 is either unramified or tempered θ-discrete stable, and
- for all non-archimedean places ν of F that are ramified in E, Π_ν^0 is tempered θ-discrete stable.

Let σ^0 be an irreducible admissible representation of $U(\mathbf{A}_F)$ whose Langlands base change is Π at all places ν. Let $\mathcal{S}$ be a finite set of places of F such that

- $\mathcal{S}$ contains all archimedean places,
- $\mathcal{S}$ contains all non-archimedean places ν that are ramified in E,
- if $\nu \in \mathcal{S}$ is non-archimedean, then ν is inert E and Π_ν is tempered θ-discrete stable, and
- if $\nu \notin \mathcal{S}$ is non-archimedean and inert in E, then σ_ν^0 is unramified.

9.1. **Properties involving a single endoscopic data.**

Lemma 9.1. $\prod_{\nu \in \mathcal{S}} \Delta_{\mathrm{spec}}\left(\psi_\nu^0, \sigma_\nu^0\right) = 1$ *where ψ_ν^0 denotes the L-parameter associated to σ_ν^0, viewed as an L-parameter of $U^*(F_\nu)$.*

Proof. By Lemma 7.3, we can find a cuspidal automorphic representation Π of $GL_n(\mathbf{A}_E)$ such that

- $\Pi \simeq \Pi \circ \theta$,
- for all archimedean ν, $\Pi_\nu \simeq \Pi_\nu^0$, and
- for all non-archimedean $\nu \in \mathcal{S}$, Π_ν is the Langlands base change of a discrete series representation σ_ν of $U(F_\nu)$ which is isomorphic to a twist of σ_ν^0 by a unitary character χ_ν, and
- for all non-archimedean places $\nu \notin \mathcal{S}$ that are inert in E, Π_ν is unramified.

Let σ be an irreducible admissible representation of $U(\mathbf{A}_F)$ such that

- Π is the Langlands base change of σ at all places ν,
- for all archimedean ν, $\sigma_\nu \simeq \sigma_\nu^0$,
- for all non-archimedean $\nu \in \mathcal{S}$, σ_ν is the twist of σ_ν^0 by the character χ_ν, and

- for all non-archimedean $\nu \notin \mathcal{S}$ that are inert in E, σ_ν is unramified.

We see that

$$\prod_{\nu\in\mathcal{S}} \Delta_{\text{spec}}\left(\psi_\nu, \sigma_\nu\right) = \prod_{\nu\in\mathcal{S}} \Delta_{\text{spec}}\left(\psi^0_\nu, \sigma^0_\nu\right)$$

where ψ_ν denotes the L-parameter associated to σ_ν, viewed as an L-parameter of $U^*\left(F_\nu\right)$. By arguing as in the proof of Lemma 8.3, we deduce that

$$\prod_{\nu\in\mathcal{S}} \Delta_{\text{spec}}\left(\psi_\nu, \sigma_\nu\right) \in \mathbf{N}^0$$

By Lemma 5.38 and Lemma 8.2, $\left|\Delta_{\text{spec}}\left(\psi_\nu, \sigma_\nu\right)\right| = 1$ for all $\nu \in \mathcal{S}$. The result follows. □

9.2. **Properties involving two endoscopic data.** Let $j_1, \dots, j_r \in \mathbf{N}$ such that

$$\{j_1, \dots, j_{r'}\} \cup \{j_{r'+1}, \dots, j_r\} = \{1, \dots, r\}$$

Let $a = n_{j_1} + \cdots + n_{j_{r'}}$ and let $b = n_{j_{r'+1}} + \cdots n_{j_r}$. We define the following automorphic representations Π^0_a and Π^0_b of $GL_a\left(\mathbf{A}\right)$ and $GL_b\left(\mathbf{A}\right)$ respectively.

$$\Pi^0_a = \Pi^0_{j_1}\mu_b^{-1} \boxplus \cdots \boxplus \Pi^0_{j_{r'}}\mu_b^{-1}$$
$$\Pi^0_b = \Pi^0_{j_{r'+1}}\mu_a^{-1} \boxplus \cdots \boxplus \Pi^0_{j_r}\mu_a^{-1}$$

Lemma 9.2. $\prod_{\nu\in\mathcal{S}} \Delta_{\text{spec}}\left(\psi^0_{a,b,\nu}, \sigma^0_\nu\right) = \pm 1$ *where* $\psi^0_{a,b,\nu}$ *denotes the L-parameter of* $H_{a,b}$ *for which* $\mathrm{BC}\left(\psi^0_{a,b,\nu}\right) \simeq \psi\left(\Pi^0_{a,\nu}\right) \times \psi\left(\Pi^0_{b,\nu}\right)$.

Proof. This will follow from the next two results: Lemma 9.3 and Lemma 9.4. □

Lemma 9.3. $\prod_{\nu\in\mathcal{S}} \Delta_{\text{spec}}\left(\psi^0_{a,b,\nu}, \sigma^0_\nu\right) \in \mathbf{Z}$

Proof. By Lemma 7.3, we can find cuspidal automorphic representations Π_a and Π_b of $GL_a\left(\mathbf{A}_E\right)$ and $GL_b\left(\mathbf{A}_E\right)$ respectively such that

- $\Pi_a \simeq \Pi_a \circ \theta$ and $\Pi_b \simeq \Pi_b \circ \theta$,
- for all archimedean ν, $\Pi_{a,\nu} \simeq \Pi^0_{a,\nu}$ (resp. $\Pi_{b,\nu} \simeq \Pi^0_{b,\nu}$),
- for all non-archimedean $\nu \in \mathcal{S}$, $\Pi_{a,\nu}$ (resp. $\Pi_{b,\nu}$) is the Langlands base change of a discrete series representation $\sigma_{a,\nu}$ (resp. $\sigma_{b,\nu}$) and there exists a unitary character χ_ν such that $\Pi^0_{a,\nu}$ (resp. $\Pi^0_{b,\nu}$) is the Langlands base change of $\sigma_{a,\nu} \cdot \chi_\nu^{-1}$ (resp. $\sigma_{b,\nu} \cdot \chi_\nu^{-1}$), and
- for all non-archimedean places $\nu \notin \mathcal{S}$ that are inert in E, $\Pi_{a,\nu}$ (resp. $\Pi_{b,\nu}$) is unramified.

We define the automorphic representation Π of $GL_n\left(\mathbf{A}_E\right)$:

$$\Pi = \Pi_a\mu_b \boxplus \Pi_b\mu_a$$

Let σ be an irreducible admissible representation of $U\left(\mathbf{A}_F\right)$ such that

- Π is the Langlands base change of σ at all places ν,
- for all archimedean ν, $\sigma_\nu \simeq \sigma^0_\nu$,
- for all non-archimedean $\nu \in \mathcal{S}$, $\sigma_\nu \simeq \sigma^0_\nu \cdot \chi_\nu$, and
- for all non-archimedean $\nu \notin \mathcal{S}$ that are inert in E, σ_ν is unramified.

We see that

$$\prod_{\nu\in\mathcal{S}} \Delta_{\text{spec}}\left(\psi_{a,b,\nu}, \sigma_\nu\right) = \prod_{\nu\in\mathcal{S}} \Delta_{\text{spec}}\left(\psi^0_{a,b,\nu}, \sigma^0_\nu\right)$$

where $\psi_{a,b,\nu}$ denotes the L-parameter of $H_{a,b}$ for which $\mathrm{BC}\left(\psi_{a,b,\nu}\right) \simeq \psi\left(\Pi_{a,\nu}\right) \times \psi\left(\Pi_{a,\nu}\right)$. The result will follow upon confirmation that

$$\prod_{\nu\in\mathcal{S}} \Delta_{\text{spec}}\left(\psi_{a,b,\nu}, \sigma_\nu\right) \in \mathbf{Z}$$

Let S be a finite set of places ν such that either

- $\nu \in \mathcal{S}$, or
- σ_ν is not unramified.

For all H and for all $\nu \notin S$, let K_{U_ν} and K_{H_ν} be hyperspecial subgroups of $U(F_\nu)$ and $H(F_\nu)$ respectively such that $\sigma_\nu^{K_{U_\nu}} \neq 0$. By arguing as in the proof of Lemma 8.3, we can deduce the following identity.

$$\sum_{\sigma'_\nu \in \Pi(\sigma_\nu)} \mathrm{m}_{\mathrm{disc}}\left(\sigma_\infty \otimes \otimes_{\nu \in \{\mathcal{S}-\infty\}} \sigma'_\nu \otimes \sigma^{\mathcal{S}}\right) \mathrm{Tr}\left(\sigma_\infty \otimes \otimes_{\nu \in \{\mathcal{S}-\infty\}} \sigma'_\nu \otimes \sigma^{\mathcal{S}}\right)(f)$$
$$= \frac{1}{2} \sum_{\tau'_{U^*_\nu} \in \Pi(\sigma_\nu)} \mathrm{Tr}\left(\otimes_{\nu \in \mathcal{S}} \tau'_{U^*_\nu} \otimes \otimes_{\nu \notin \mathcal{S}} \tau'_{U^*_\nu}\right)\left(f^{U^*}\right)$$
$$+ \frac{1}{2} \sum_{\tau'_{H_{a,b,\nu}} \in \Pi(\psi_{a,b,\nu})} \mathrm{Tr}\left(\otimes_{\nu \in \mathcal{S}} \tau'_{H_{a,b,\nu}} \otimes \otimes_{\nu \notin \mathcal{S}} \tau'_{H_{a,b,\nu}}\right)\left(f^{H_{a,b}}\right)$$

where

- $f^H = \otimes_\nu f^H_\nu$ is a Δ-transfer of $f = \otimes_\nu f_\nu$ for all ν,
- for all archimedean ν, f_ν is a pseudo-coefficient of σ_ν and f^H_ν is the associated Euler-Poincaré function appearing in Lemma 5.4,
- for all $\nu \notin S$, f_ν (resp. f^H_ν) is bi-invariant under K_{U_ν} (resp. K_{H_ν}),
- for all $\nu \notin S$, $\tau'_{U^*_\nu}$ is the unique $K_{U^*_\nu}$-unramified representation whose Langlands base change is Π_ν,
- for all $\nu \in S - \mathcal{S}$, $\tau'_{U^*_\nu}$ is the unique representation whose Langlands base change is Π_ν,
- for all $\nu \notin S$, $\tau'_{H_{a,b,\nu}}$ is the unique $K_{H_{a,b,\nu}}$-unramified representation whose Langlands base change is $\Pi_{a,\nu} \times \Pi_{b,\nu}$, and
- for all $\nu \in S - \mathcal{S}$, $\tau'_{H_{a,b,\nu}}$ is the unique representation whose Langlands base change is $\Pi_{a,\nu} \times \Pi_{b,\nu}$

Applying the results of Section 5, we deduce that

$$\mathrm{m}_{\mathrm{disc}}(\sigma) = \frac{1}{2} \prod_{\nu \in \mathcal{S}} \Delta_{\mathrm{spec}}(\psi_\nu, \sigma_\nu) + \frac{1}{2} \prod_{\nu \in \mathcal{S}} \Delta_{\mathrm{spec}}(\psi_{a,b,\nu}, \sigma_\nu)$$

where ψ_ν denotes the L-parameter associated to σ_ν, viewed as an L-parameter of $U^*(F_\nu)$. By Lemma 9.1, we see that $\prod_{\nu \in \mathcal{S}} \Delta_{\mathrm{spec}}(\psi_\nu, \sigma_\nu) = 1$. It follows that $\prod_{\nu \in \mathcal{S}} \Delta_{\mathrm{spec}}(\psi_{a,b,\nu}, \sigma_\nu)$ is integral. □

Lemma 9.4. *Let k'/k be a quadratic extension of p-adic fields. Let $\psi_H : L_{k'} \to GL_a \times GL_b$ be an L-parameter such that $\xi \circ \psi_H$ is a tempered θ-discrete stable L-parameter. Then*

$$|\Delta_{\mathrm{spec}}(\psi_H, \sigma_k)| = \begin{cases} 1 & : \textit{if } \xi \circ \psi_H \simeq \psi(\sigma_k) \\ 0 & : \textit{otherwise} \end{cases}$$

for all discrete series representations σ_k of $U^*_n(k'/k)$.

Proof. We remark that if $\xi \circ \psi_H \not\simeq \psi(\sigma_k)$, then the result is part of Lemma 5.35. Let $\psi_H : L_{k'} \to GL_a \times GL_b$ be an L-parameter such that $\xi \circ \psi_H$ is a tempered

θ-discrete stable L-parameter. We know that (cf. [Mœg07, §7])

$$\left\langle \sum_{\sigma_H\in\Pi(\psi_H)} n(\psi_H,\sigma_H)\,\sigma_H, \sum_{\sigma_H\in\Pi(\psi_H)} n(\psi_H,\sigma_H)\,\sigma_H \right\rangle$$
$$= \left\langle \sum_{\sigma_k\in\Pi(\xi\circ\psi_H)} \Delta_{\text{spec}}(\psi_H,\sigma_k)\,\sigma_k, \sum_{\sigma_k\in\Pi(\xi\circ\psi_H)} \Delta_{\text{spec}}(\psi_H,\sigma_k)\,\sigma_k \right\rangle$$

By Lemma 8.2, $n(\psi_H,\sigma_H)=1$ for all $\sigma_H\in\Pi(\psi_H)$. By a result of Arthur [Art93, Corollary 6.2], the discrete series representations of $H(k)$ and $U^*(k)$ are orthonormal. It follows that

$$|\Pi(\psi_H)| = \sum_{\sigma_k\in\Pi(\xi\circ\psi_H)} |\Delta_{\text{spec}}(\psi_H,\sigma_k)|^2$$

Now

$$|\Pi(\psi_H)| = 2^{l(\psi_H)-1} = 2^{l(\xi\circ\psi_H)-1} = |\Pi(\xi\circ\psi_H)|$$

and by Lemma 5.35, $\Delta_{\text{spec}}(\psi_H,\sigma_k)\neq 0$ for all $\sigma_k\in\Pi(\xi\circ\psi_H)$. The result will thus follow upon confirmation that $|\Delta_{\text{spec}}(\psi_H,\sigma_k)|\in\mathbf{N}^0$ for all $\sigma_k\in\Pi(\xi\circ\psi_H)$.

Let $\sigma_k\in\Pi(\xi\circ\psi_H)$. Choose a totally imaginary quadratic extension of a totally real field E/F, and a place ν' such that

- $E_{\nu'}/F_{\nu'}\simeq k'/k$, and
- E/F is unramified outside of ν'.

Write $\psi_H=\psi_a\times\psi_b$, and let $\pi_{a,k}$ (resp. $\pi_{b,k}$) be the tempered θ-discrete stable representation of $GL_a(k')$ (resp. $GL_b(k')$) corresponding to ψ_a (resp. ψ_b) viewed as an L-parameter of GL_a (resp. GL_b). By Lemma 7.3, we can find cuspidal automorphic representations Π_a^0 and Π_b^0 of $GL_a(\mathbf{A}_E)$ and $GL_b(\mathbf{A}_E)$ respectively such that, writing $\Pi^0=\Pi_a^0\mu_b\boxplus\Pi_b^0\mu_a$, we have that

- $\Pi_a^0\simeq\Pi_a^0\circ\theta$ and $\Pi_b^0\simeq\Pi_b^0\circ\theta$,
- for all archimedean ν, Π_ν^0 has cohomology for a system of coefficients $V_\nu\otimes V_\nu^\theta$ where V is an algebraic representation with regular highest weight,
- $\Pi_{a,\nu'}^0$ (resp. $\Pi_{b,\nu'}^0$) is the Langlands base change of a discrete series representation $\sigma_{a,\nu'}^0$ (resp. $\sigma_{b,\nu'}^0$) and there exists a unitary character $\chi_{\nu'}$ such that $\pi_{a,k}$ (resp. $\pi_{b,k}$) is the Langlands base change of the discrete series representation $\sigma_{a,\nu'}\cdot\chi_{\nu'}^{-1}$ (resp. $\sigma_{b,\nu'}\cdot\chi_{\nu'}^{-1}$), and
- for all non-archimedean places $\nu\neq\nu'$ that are inert in E, Π_ν^0 is unramified.

Let σ^0 be an irreducible admissible representation of $U(\mathbf{A}_F)$ such that

- Π^0 is the Langlands base change of σ^0 at all places ν,
- $\sigma_{\nu'}^0\simeq\sigma_k\cdot\chi_{\nu'}$, and
- for all non-archimedean $\nu\neq\nu'$ that remain inert in E, σ_ν^0 is unramified.

We see that

$$\Delta_{\text{spec}}\left(\psi_{a,b,\nu'}^0,\sigma_{\nu'}^0\right)=\Delta_{\text{spec}}(\psi_H,\sigma_k)$$

where $\psi_{a,b,\nu'}^0$ denotes the L-parameter of $H_{a,b}$ for which

$$\text{BC}\left(\psi_{a,b,\nu'}^0\right)\simeq\psi\left(\Pi_{a,\nu'}^0\right)\times\psi\left(\Pi_{b,\nu'}^0\right)$$

Thus the result will follow upon confirmation that $|\Delta_{\text{spec}}\left(\psi_{a,b,\nu'}^0,\sigma_{\nu'}^0\right)|\in\mathbf{N}^0$.

By Lemma 9.3,

$$\Delta_{\text{spec}}\left(\psi_{a,b,\nu'}^0,\sigma_{\nu'}^0\right)\cdot\prod_{\nu|\infty}\Delta_{\text{spec}}\left(\psi_{a,b,\nu}^0,\sigma_\nu^0\right)\in\mathbf{Z}$$

where for archimedean ν, $\psi_{a,b,\nu}^0$ denotes the L-parameter of $H_{a,b}$ for which

$$\text{BC}\left(\psi_{a,b,\nu}^0\right)\simeq\psi\left(\Pi_{a,\nu}^0\right)\times\psi\left(\Pi_{b,\nu}^0\right)$$

The result then follows as $|\Delta_{\mathrm{spec}}\left(\psi^0_{a,b,\nu},\sigma^0_\nu\right)|=1$ for all archimedean ν. □

9.3. **Properties involving multiple endoscopic data.** Let $j_1,\dots,j_r\in\mathbf{N}$ such that

$$\{j_1,\dots,j_{r'}\}\cup\{j_{r'+1},\dots,j_{r''}\}\cup\{j_{r''+1},\dots,j_r\}=\{1,\dots,r\}$$

Let $a_1=n_{j_1}+\cdots+n_{j_{r'}}$ and let $b_1=n-a_1$. We define the automorphic representations of $GL_{a_1}(\mathbf{A}_E)$ and $GL_{b_1}(\mathbf{A}_E)$:

$$\begin{aligned}\Pi^0_{a_1}&=\Pi^0_{j_1}\mu^{-1}_{b_1}\boxplus\cdots\boxplus\Pi^0_{j_{r'}}\mu^{-1}_{b_1}\\ \Pi^0_{b_1}&=\Pi^0_{j_{r'+1}}\mu^{-1}_{a_1}\boxplus\cdots\boxplus\Pi^0_{j_r}\mu^{-1}_{a_1}\end{aligned}$$

Let $a_2=n_{j_{r'+1}}+\cdots n_{j_{r''}}$ and let $b_2=n-a_2$. Define the automorphic representations of $GL_{a_2}(\mathbf{A}_E)$ and $GL_{b_2}(\mathbf{A}_E)$:

$$\begin{aligned}\Pi^0_{a_2}&=\Pi^0_{j_{r'+1}}\mu^{-1}_{b_2}\boxplus\cdots\boxplus\Pi^0_{j_{r''}}\mu^{-1}_{b_2}\\ \Pi^0_{b_2}&=\Pi^0_{j_1}\mu^{-1}_{a_2}\boxplus\cdots\boxplus\Pi^0_{j_{r'}}\mu^{-1}_{a_2}\boxplus\Pi^0_{j_{r''+1}}\mu^{-1}_{a_2}\boxplus\cdots\boxplus\Pi^0_{j_r}\mu^{-1}_{a_2}\end{aligned}$$

Let $a_3=n_{j_{r''+1}}+\cdots n_{j_r}$ and let $b_3=n-a_3$. Define the representations $GL_{a_3}(\mathbf{A}_E)$ and $GL_{b_3}(\mathbf{A}_E)$:

$$\begin{aligned}\Pi^0_{a_3}&=\Pi^0_{j_{r''+1}}\mu^{-1}_{b_3}\boxplus\cdots\boxplus\Pi^0_{j_r}\mu^{-1}_{b_3}\\ \Pi^0_{b_3}&=\Pi^0_{j_1}\mu^{-1}_{a_3}\boxplus\cdots\boxplus\Pi^0_{j_{r''}}\mu^{-1}_{a_3}\end{aligned}$$

Lemma 9.5.

$$\prod_{\nu\in\mathcal{S}}\Delta_{\mathrm{spec}}\left(\psi^0_{a_1,b_1,\nu},\sigma^0_\nu\right)=\prod_{\nu\in\mathcal{S}}\Delta_{\mathrm{spec}}\left(\psi^0_{a_2,b_2,\nu},\sigma^0_\nu\right)\cdot\prod_{\nu\in\mathcal{S}}\Delta_{\mathrm{spec}}\left(\psi^0_{a_3,b_3,\nu},\sigma^0_\nu\right)$$

where for $i=1,2,3$, $\psi^0_{a_i,b_i,\nu}$ is the L-parameter of H_{a_i,b_i} for which

$$\mathrm{BC}\left(\psi^0_{a_i,b_i,\nu}\right)\simeq\psi\left(\Pi^0_{a_i,\nu}\right)\times\psi\left(\Pi^0_{b_i,\nu}\right)$$

Proof. By Lemma 7.3, we can find cuspidal automorphic representations Π_{a_1}, Π_{a_2} and Π_{a_3} of $GL_{a_1}(\mathbf{A}_E)$, $GL_{a_2}(\mathbf{A}_E)$, and $GL_{a_3}(\mathbf{A}_E)$ respectively such that for all $i=1,2,3$,

- $\Pi_{a_i}\simeq\Pi_{a_i}\circ\theta$,
- for all archimedean ν, $\Pi_{a_i,\nu}\simeq\Pi^0_{a_i,\nu}$,
- for all non-archimedean $\nu\in\mathcal{S}$, $\Pi_{a_i,\nu}$ is the Langlands base change of a discrete series representation $\sigma_{a_i,\nu}$ and there exists a unitary character, independent of i, χ_ν such that $\Pi^0_{a_i,\nu}$ is the Langlands base change of the twist $\sigma_{a_i,\nu}\cdot\chi_\nu^{-1}$, and
- for all non-archimedean places $\nu\notin\mathcal{S}$ that are inert in E, $\Pi_{a_i,\nu}$ is unramified.

We then define the following automorphic representations.

$$\begin{aligned}\Pi_{b_1}&=\Pi_{a_2}\mu_{b_2}\mu^{-1}_{a_1}\boxplus\Pi_{a_3}\mu_{b_3}\mu^{-1}_{a_1}\\ \Pi_{b_2}&=\Pi_{a_1}\mu_{b_1}\mu^{-1}_{a_2}\boxplus\Pi_{a_3}\mu_{b_3}\mu^{-1}_{a_2}\\ \Pi_{b_3}&=\Pi_{a_1}\mu_{b_1}\mu^{-1}_{a_3}\boxplus\Pi_{a_2}\mu_{b_2}\mu^{-1}_{a_3}\end{aligned}$$

We also define the automorphic representation of $GL_n(\mathbf{A}_E)$,

$$\Pi=\Pi_{a_1}\mu_{b_1}\boxplus\Pi_{b_1}\mu_{a_1}\simeq\Pi_{a_2}\mu_{b_2}\boxplus\Pi_{b_2}\mu_{a_2}\simeq\Pi_{a_3}\mu_{b_3}\boxplus\Pi_{b_3}\mu_{a_3}$$

Let σ be an irreducible admissible representation of $U(\mathbf{A}_F)$ such that

- Π is the Langlands base change of σ at all places,
- for all archimedean ν, $\sigma_\nu\simeq\sigma^0_\nu$,
- for all non-archimedean $\nu\in\mathcal{S}$, $\sigma_\nu\simeq\sigma^0_\nu\cdot\chi_\nu$, and
- for all non-archimedean $\nu\notin\mathcal{S}$ that are inert in E, σ_ν is unramified.

We see that, for $i = 1, 2, 3$,

$$\prod_{\nu \in \mathcal{S}} \Delta_{\text{spec}}\left(\psi_{a_i,b_i,\nu}, \sigma_\nu\right) = \prod_{\nu \in \mathcal{S}} \Delta_{\text{spec}}\left(\psi^0_{a_i,b_i,\nu}, \sigma^0_\nu\right)$$

where for $i = 1, 2, 3$, $\psi^0_{a_i,b_i,\nu}$ is the L-parameter of H_{a_i,b_i} for which $\mathrm{BC}\left(\psi_{a_i,b_i,\nu}\right) \simeq \psi\left(\Pi_{a_i,\nu}\right) \times \psi\left(\Pi_{b_i,\nu}\right)$. The result will follow upon confirmation that,

$$\prod_{\nu \in S} \Delta_{\text{spec}}\left(\psi_{a_1,b_1,\nu}, \sigma_\nu\right) = \prod_{\nu \in S} \Delta_{\text{spec}}\left(\psi_{a_2,b_2,\nu}, \sigma_\nu\right) \cdot \prod_{\nu \in S} \Delta_{\text{spec}}\left(\psi_{a_3,b_3,\nu}, \sigma_\nu\right)$$

Let S be the set of places ν such that either

- $\nu \in \mathcal{S}$, or
- σ_ν is not unramified.

For all H and for all $\nu \notin S$, let K_{U_ν} and K_{H_ν} be hyperspecial subgroups of $U\left(F_\nu\right)$ and $H\left(F_\nu\right)$ respectively such that $\sigma_\nu^{K_{U_\nu}} \neq 0$. By arguing as in the proof of Lemma 8.3, we can deduce the following identity.

$$\begin{aligned}\sum_{\sigma'_\nu \in \Pi(\sigma_\nu)} & \mathrm{m}_{\text{disc}}\left(\sigma_\infty \otimes \otimes_{\nu \in \{\mathcal{S}-\infty\}} \sigma'_\nu \otimes \sigma^{\mathcal{S}}\right) \mathrm{Tr}\left(\sigma_\infty \otimes \otimes_{\nu \in \{\mathcal{S}-\infty\}} \sigma'_\nu \otimes \sigma^{\mathcal{S}}\right)(f) \\ &= \frac{1}{4} \sum_{\tau'_{U^*_\nu} \in \Pi(\sigma_\nu)} \mathrm{Tr}\left(\otimes_{\nu \in \mathcal{S}} \tau'_{U^*_\nu} \otimes \otimes_{\nu \notin \mathcal{S}} \tau'_{U^*_\nu}\right)\left(f^{U^*}\right) \\ &+ \frac{1}{4} \sum_{i=1}^{3} \sum_{\tau'_{H_{a_i,b_i,\nu}} \in \Pi\left(\psi_{a_i,b_i,\nu}\right)} \mathrm{Tr}\left(\otimes_{\nu \in \mathcal{S}} \tau'_{H_{a_i,b_i,\nu}} \otimes \otimes_{\nu \notin \mathcal{S}} \tau'_{H_{a_i,b_i,\nu}}\right)\left(f^{H_{a_i,b_i}}\right)\end{aligned}$$

where

- $f^H = \otimes_\nu f^H_\nu$ is a Δ-transfer of $f = \otimes_\nu f_\nu$ for all ν,
- for all archimedean ν, f_ν is a pseudo-coefficient of σ_ν and f^H_ν is the associated Euler-Poincaré function as appearing in Lemma 5.4,
- for all $\nu \notin S$, f_ν (resp. f^H_ν) is bi-invariant under K_{U_ν} (resp. K_{H_ν}),
- for all $\nu \notin S$, $\tau'_{U^*_\nu}$ is the unique $K_{U^*_\nu}$-unramified representation whose Langlands base change is Π_ν,
- for all $\nu \in S - \mathcal{S}$, $\tau'_{U^*_\nu}$ is the unique representation whose Langlands base change is Π_ν,
- for $i = 1, 2, 3$, for all $\nu \notin S$, $\tau'_{H_{a_i,b_i},\nu}$ is the unique $K_{H_{a_i,b_i},\nu}$-unramified representation whose Langlands base change is $\Pi_{a_i,\nu} \times \Pi_{b_i,\nu}$, and
- for $i = 1, 2, 3$, for all $\nu \in S - \mathcal{S}$, $\tau'_{H_{a_i,b_i},\nu}$ is the unique representation whose Langlands base change is $\Pi_{a_i,\nu} \times \Pi_{b_i,\nu}$

Applying the results of Section 5, we deduce that

$$\mathrm{m}_{\text{disc}}(\sigma) = \frac{1}{4} \prod_{\nu \in \mathcal{S}} \Delta_{\text{spec}}\left(\psi_\nu, \sigma_\nu\right) + \frac{1}{4} \sum_{i=1}^{3} \prod_{\nu \in \mathcal{S}} \Delta_{\text{spec}}\left(\psi_{a_i,b_i,\nu}, \sigma_\nu\right)$$

By Lemma 9.1,

$$\prod_{\nu \in \mathcal{S}} \Delta_{\text{spec}}\left(\psi_\nu, \sigma_\nu\right) = 1$$

and by Lemma 9.2, for $i = 1, 2, 3$,

$$\prod_{\nu \in \mathcal{S}} \Delta_{\text{spec}}\left(\psi_{a_i,b_i,\nu}, \sigma_\nu\right) = \pm 1$$

The multiplicity $\mathrm{m}_{\mathrm{disc}}(\sigma)$ is a non-negative integer, and it follows that $\mathrm{m}_{\mathrm{disc}}(\sigma)$ is equal to either 0 or 1. By considering the possible values of the terms, we see that

$$\prod_{\nu\in\mathcal{S}} \Delta_{\mathrm{spec}}\left(\psi_{a_1,b_1,\nu},\sigma_\nu\right) = \prod_{\nu\in\mathcal{S}} \Delta_{\mathrm{spec}}\left(\psi_{a_2,b_2,\nu},\sigma_\nu\right) \cdot \prod_{\nu\in\mathcal{S}} \Delta_{\mathrm{spec}}\left(\psi_{a_3,b_3,\nu},\sigma_\nu\right)$$

□

10. The local Arthur conjectures

The main aim of this section is to prove a formulation of the local Arthur conjectures for discrete series representations of the quasi-split p-adic unitary group. Our proof of the local Arthur conjectures follows the work of Arthur [Art05, Theorem 30.1] who has proved these conjectures for general representations of the orthogonal and symplectic groups. For tempered representations of real groups, the local Arthur conjectures are due to Shelstad [She08b]. We shall also recall a formulation of Shelstad's result for discrete series representations of real unitary groups.

10.1. **Discrete series representations of real unitary groups.** Let $U = U(p,q)$ be a real unitary group. Let $\psi : L_{\mathbf{R}} \to {}^L U$ be a tempered discrete L-parameter; we shall also have need of its Langlands base change $\mathrm{BC}(\psi) : L_{\mathbf{C}} \to GL_n(\mathbf{C})$. We shall denote by S_ψ the centraliser of the image of $\mathrm{BC}(\psi)(z)$ in $GL_n(\mathbf{C})$ for all $z \in \mathbf{C}$, and we shall denote by S_ψ^θ the subgroup of θ-invariant points of S_ψ. We shall study the quotient group $\mathbf{S}_\psi = S_\psi^\theta / \{\pm 1\}$. By Schur's lemma, we see that

$$\mathbf{S}_\psi \simeq (\mathbf{Z}/2\mathbf{Z})^{n-1}$$

For all $s \in \mathbf{S}_\psi$, one associates to s an endoscopic data $H_s = H_{a,b}$ and a tempered discrete L-parameter

$$\psi_s : L_{\mathbf{R}} \to {}^L H_s$$

such that $\psi \simeq \xi \circ \psi_s$ via the following construction. The centraliser of a representative of s in $GL_n(\mathbf{C})$ is of the following form.

$$C\left(s, GL_n(\mathbf{C})\right) \xrightarrow{\sim} \left(GL_a \times GL_b\right)(\mathbf{C})$$

for a unique $a, b \in \mathbf{N}^0$ such that $a \le b$. The endoscopic data H_s is then defined to be $H_s = H_{a,b}$ as in Definition 5.19. The L-homomorphism ψ_s is defined to be the L-homomorphism whose Langlands base change

$$\mathrm{BC}(\psi_s) : L_{\mathbf{C}} \to \left(GL_a \times GL_b\right)(\mathbf{C})$$

is the pull back of $\mathrm{BC}(\psi)$ through the morphism

$$\left(GL_a \times GL_b\right)(\mathbf{C}) \xrightarrow{\sim} \left(GL_a \times GL_b\right)(\mathbf{C}) \xrightarrow{\sim} C\left(s, GL_n(\mathbf{C})\right) \hookrightarrow GL_n(\mathbf{C})$$

where the first isomorphism is defined as follows.

$$\left(GL_a \times GL_b\right)(\mathbf{C}) \to \left(GL_a \times GL_b\right)(\mathbf{C})$$
$$g_a \times g_b \mapsto g_a \cdot \mu_b(\det g_a) \times g_b \cdot \mu_a(g_b)$$

Remark 10.1. The ψ_s are easily seen to be in bijection with the equivalence classes of L-parameters $\psi_H : L_{\mathbf{R}} \to {}^L H$ such that $\xi \circ \psi_H \simeq \psi$.

We fix a representation $\sigma^{\mathrm{base}} \in \Pi(\psi)$ in the L-packet of discrete series representations of $U(\mathbf{R})$. Arthur defines the pairing

$$\langle \cdot, \cdot \rangle : \mathbf{S}_\psi \times \Pi(\psi) \to \mathbf{C}^\times$$
$$s \times \sigma \mapsto \frac{\Delta_{\mathrm{spec}}(\psi_s, \sigma)}{\Delta_{\mathrm{spec}}(\psi_s, \sigma^{\mathrm{base}})}$$

This pairing is canonical: it depends only upon the equivalence class of the L-homomorphism ψ and the choice of a base point representation $\sigma^{\mathrm{base}} \in \Pi(\psi)$.

In particular, the pairing does not depend upon the chosen normalisation of the transfer factors.

Theorem 10.2. *The pairing $\langle\cdot,\cdot\rangle$ takes values in $\{\pm 1\}$, and induces an injection from the elements of the L-packet $\Pi(\psi)$ to the characters of $\mathbf{S}_\psi$.*

Proof. This is a simple reformulation of a result of Shelstad [She08b]. □

Remark 10.3. The results of Shelstad [She08b] are in fact stronger, and include a study of the characters that appear in the image of this injective map.

Remark 10.4. If we restrict ourselves to L-packets of discrete series representations $\Pi(V)$ where V is an irreducible algebraic representation with regular highest weight, then our method of proof of the local Arthur conjectures for discrete series representations of the p-adic quasi-split unitary group can be adapted to give a global proof of Theorem 10.2.

10.2. **Discrete series representations of the p-adic quasi-split unitary group.** Let k'/k be a quadratic extension of p-adic fields. We shall study the discrete series representations of the p-adic quasi-split unitary group $U^*(k'/k)$. Let $\psi : L_{k'} \to {}^L GL_n/k'$ be a tempered θ-discrete stable L-parameter. We shall denote by S_ψ the centraliser of the image of ψ in $GL_n(\mathbf{C})$, and we shall denote by S_ψ^θ the subgroup of θ-invariant points of S_ψ. We shall study the quotient group $\mathbf{S}_\psi = S_\psi^\theta / \{\pm 1\}$. By Schur's lemma, we see that

$$\mathbf{S}_\psi \simeq (\mathbf{Z}/2\mathbf{Z})^{l(\psi)-1}$$

For all $s \in \mathbf{S}_\psi$, one associates to s an endoscopic data $H_s = H_{a,b}$, and a tempered θ-discrete L-parameter

$$\psi_s : L_{k'} \to {}^L GL_a \times GL_b/k'$$

such that $\psi \simeq \xi \circ \psi_s$ via the following construction. The centraliser of a representative of s in $GL_n(\mathbf{C})$ is of the following form.

$$C(s, GL_n(\mathbf{C})) \xrightarrow{\sim} (GL_a \times GL_b)(\mathbf{C})$$

for a unique $a, b \in \mathbf{N}^0$ such that $a \leq b$. The endoscopic data H_s is then defined to be $H_s = H_{a,b}$ as in Definition 5.19. The L-homomorphism ψ_s is defined to be the pull back of ψ through the morphism

$$(GL_a \times GL_b)(\mathbf{C}) \xrightarrow{\sim} (GL_a \times GL_b)(\mathbf{C}) \xrightarrow{\sim} C(s, GL_n(\mathbf{C})) \hookrightarrow GL_n(\mathbf{C})$$

where the first isomorphism is defined as follows.

$$\begin{aligned} (GL_a \times GL_b)(\mathbf{C}) &\to (GL_a \times GL_b)(\mathbf{C}) \\ g_a \times g_b &\mapsto g_a \cdot \mu_b(\det g_a) \times g_b \cdot \mu_a(g_b) \end{aligned}$$

Remark 10.5. The ψ_s are easily seen to be in bijection with the equivalence classes of L-parameters $\psi_H : L_{k'} \to {}^L GL_a \times GL_b/k'$ such that $\xi \circ \psi_H \simeq \psi$.

We fix a representation $\sigma^{\text{base}} \in \Pi(\psi)$ in the L-packet of discrete series representations of $U^*(k'/k)$. Arthur defines the pairing

$$\begin{aligned} \langle\cdot,\cdot\rangle : \mathbf{S}_\psi \times \Pi(\psi) &\to \mathbf{C}^\times \\ s \times \sigma &\mapsto \frac{\Delta_{\text{spec}}(\psi_s, \sigma)}{\Delta_{\text{spec}}(\psi_s, \sigma^{\text{base}})} \end{aligned}$$

This pairing is canonical: it depends only upon the equivalence class of the L-homomorphism ψ and the choice of base point representation $\sigma^{\text{base}} \in \Pi(\psi)$. In particular, the pairing does not depend upon the chosen normalisation of the transfer factors.

Theorem 10.6. *The pairing $\langle \cdot, \cdot \rangle$ takes values in $\{\pm 1\}$, and induces a bijection between the elements of the L-packet $\Pi(\psi)$ and the characters of $\mathbf{S}_\psi$.*

Proof. Choose a totally imaginary quadratic extension of a totally real field E/F, and a place v' of F such that

- $E_{\nu'}/F_{\nu'} \simeq k'/k$, and
- E/F is unramified outside of v'.

By Lemma 7.3, we can find $r = l(\psi)$ cuspidal automorphic representations Π_i^0 of $GL_{n_i}(\mathbf{A}_E)$ such that, writing $\Pi^0 = \Pi_1^0 \boxplus \cdots \boxplus \Pi_r^0$, the following conditions are satisfied.

- For all archimedean ν, Π_ν^0 has cohomology in a system of coefficients $V_\nu^0 \otimes {V_\nu^0}^\theta$ where V_ν^0 is an irreducible algebraic representation of GL_n with regular highest weight.
- For all $i = 1, \ldots, r$, $\Pi_i^0 \circ \theta \simeq \Pi_i^0$.
- $\Pi_{\nu'}^0$ is the Langlands base change of an L-packet of discrete series representations whose elements are isomorphic to a twist of the representations in the L-packet $\Pi(\psi)$ by some unitary character $\chi_{\nu'}$.
- For all non-archimedean places $\nu \neq \nu'$ that are inert in E, Π_ν^0 is unramified.

We can identify the L-packets via the natural bijection

$$\begin{aligned} \Pi(\psi) &\to \Pi\left(\psi^0\right) \\ \sigma_k &\mapsto \sigma_k \chi_{\nu'} \end{aligned}$$

where $\psi^0 = \psi\left(\Pi_{\nu'}^0\right)$. Similarly, we can identify the groups $\mathbf{S}_\psi \xrightarrow{\sim} \mathbf{S}_{\psi^0}$. We see that, for all $s \in \mathbf{S}_\psi$ and for all $\sigma_k \in \Pi(\psi)$,

$$\Delta_{\mathrm{spec}}(\psi_s, \sigma_k) = \Delta_{\mathrm{spec}}\left(\psi_s^0, \sigma_k \chi_{\nu'}\right)$$

Thus the theorem will follow from the analogous statement concerning the L-packet $\Pi\left(\psi^0\right)$.

Let $\sigma^0 = \otimes_\nu \sigma_\nu^0$ be an irreducible admissible representation of $U^*(\mathbf{A}_F)$ such that

- for all non-archimedean $\nu \neq \nu'$ that are inert in E, σ_ν^0 is unramified, and
- Π^0 is the Langlands base change of σ^0 at all places ν.

We remark that we have the natural injections

$$\mathbf{S}_{\psi(\sigma_\nu^0)} \hookrightarrow \mathbf{S}_{\psi^0}$$

for all $\nu|\infty$. By the results of Section 9, we see that the function

$$\begin{aligned} \mathbf{S}_{\psi^0} &\to \mathbf{C}^\times \\ s &\mapsto \Delta_{\mathrm{spec}}\left(\psi_s^0, \sigma_{\nu'}^0\right) \prod_{\nu|\infty} \Delta_{\mathrm{spec}}\left(\psi\left(\sigma_\nu^0\right)_s, \sigma_\nu^0\right) \end{aligned}$$

is a ± 1 valued character. Write $\sigma_{\nu'}^{\mathrm{base}} \in \Pi\left(\psi^0\right)$ for the chosen base point representation. Consider now the function associated to the representation ${\sigma^0}^{\nu'} \otimes \sigma_{\nu'}^{\mathrm{base}}$

$$\begin{aligned} \mathbf{S}_{\psi^0} &\to \mathbf{C}^\times \\ s &\mapsto \Delta_{\mathrm{spec}}\left(\psi_s^0, \sigma_{\nu'}^{\mathrm{base}}\right) \prod_{\nu|\infty} \Delta_{\mathrm{spec}}\left(\psi\left(\sigma_\nu^0\right)_s, \sigma_\nu^0\right) \end{aligned}$$

which by the results of Section 9, is also a ± 1 valued character. By multiplying the first character by the inverse of the second character, we see that the function induced by Arthur's pairing

$$\begin{aligned}\langle\cdot,\sigma^0_{\nu'}\rangle : \mathbf{S}_{\psi^0} &\to \mathbf{C}^\times\\ s &\mapsto \frac{\Delta_{\mathrm{spec}}\left(\psi^0_s,\sigma^0_{\nu'}\right)}{\Delta_{\mathrm{spec}}\left(\psi^0_s,\sigma^{\mathrm{base}}_{\nu'}\right)}\end{aligned}$$

is a ± 1 valued character. Since $\sigma^0_{\nu'} \in \Pi\left(\psi^0\right)$ can be chosen to be any element of the L-packet, we see that Arthur's pairing $\langle\cdot,\cdot\rangle$ induces a map from the elements of the L-packet $\Pi\left(\psi^0\right)$ to the characters of $\mathbf{S}_{\psi^0}$. The map is known to be injective; this can be seen by inverting the spectral transfer factors and expressing the distribution of a representation in the L-packet $\Pi\left(\psi^0\right)$ in terms of the stable distributions associated to the L-packets $\Pi\left(\psi^0_s\right)$ (cf. [Mœg07, §8.1]). The result follows as the sets have the same cardinality

$$|\Pi\left(\psi^0\right)| = 2^{l\left(\psi^0\right)-1} = |\widehat{\mathbf{S}}_{\psi^0}|$$

where $\widehat{\mathbf{S}}_{\psi^0}$ denotes the group of characters of $\mathbf{S}_{\psi^0}$. □

11. The global Arthur conjectures

The main aim of this section is to prove a formulation of the global Arthur conjectures for certain representations of the unitary group. Our proof follows the work of Arthur [Art05, Theorem 30.2] who has proved these conjectures for general representations of the orthogonal and symplectic groups.

Let E/F be a totally imaginary quadratic extension of a totally real field, and let U be a unitary group appearing in Proposition 2.1. Let $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r$ be an automorphic representation of $GL_n\left(\mathbf{A}_E\right)$ that satisfies the following properties.

- For all $i = 1,\ldots,r$, Π_i is cuspidal and $\Pi_i \simeq \Pi_i \circ \theta$.
- For all archimedean places ν of F, Π_ν is the Langlands base change of a discrete series representation of $U\left(F_\nu\right)$ with the same infinitesimal character as an irreducible algebraic representation of GL_n whose highest weight is regular.
- For all non-archimedean places ν of F that are inert and unramified in E, Π_ν is either unramified, or tempered θ-discrete stable.
- For all non-archimedean places ν of F that are ramified in E, Π_ν is tempered θ-discrete stable.

Let σ be an irreducible admissible representation of $U\left(\mathbf{A}_F\right)$ whose Langlands base change is Π at all places.

The global Arthur conjectures predict the multiplicity with which σ appears in the discrete automorphic spectrum of $U\left(\mathbf{A}_F\right)$, which we shall now describe. Let S be the set of places ν of F such that either

- ν is archimedean, or
- ν is non-archimedean, inert in E, and Π_ν is tempered θ-discrete stable.

For all places ω of E, let $\psi_\omega : L_{E_\nu} \to GL_n\left(\mathbf{C}\right)$ be the L-parameter corresponding to Π_ω. Let S_Π be the group of elements of $GL_n\left(\mathbf{C}\right)$ that commute with the image of $\psi_\omega\left(z\right)$ in $GL_n\left(\mathbf{C}\right)$ for all $z \in L_{E_\omega}$ and for all ω. Let S^θ_Π be the subgroup of θ-invariant points. We shall study the quotient group $\mathbf{S}_\Pi = S^\theta_\Pi / \{\pm 1\}$. There exists a natural embedding for all $\nu \in S$,

$$\mathbf{S}_\Pi \hookrightarrow \mathbf{S}_{\psi_\nu}$$

The local characters $\langle\cdot,\sigma_\nu\rangle : \mathbf{S}_{\psi_\nu} \to \{\pm 1\}$, defined for all $\nu \in S$, induce by restriction a character

$$\langle\cdot,\sigma\rangle = \prod_{\nu\in S}\langle\cdot,\sigma_\nu\rangle|_{\mathbf{S}_\Pi} : \mathbf{S}_\Pi \to \{\pm 1\}$$

The global Arthur conjectures predicts the following.

Theorem 11.1. *There exists a unique character*

$$\epsilon_\Pi : \mathbf{S}_\Pi \to \{\pm 1\}$$

such that σ *appears in the discrete automorphic spectrum of* $U(\mathbf{A}_F)$ *with multiplicity equal to*

$$\mathrm{m}_{\mathrm{disc}}(\sigma) = \begin{cases} 1 & : \textit{if } \langle \cdot, \sigma \rangle = \epsilon_\Pi \\ 0 & : \textit{otherwise} \end{cases}$$

Proof. By considering the trace formula, and arguing as in the proof of Lemma 8.3, we deduce that

$$\mathrm{m}_{\mathrm{disc}}(\sigma) = \frac{1}{|\mathbf{S}_\Pi|} \sum_{s \in \mathbf{S}_\Pi} \prod_{\nu \in S} \Delta_{\mathrm{spec}}\left(\psi(\sigma_\nu)_s, \sigma_\nu\right)$$

By Lemma 9.2, we see that each term in the summation is equal to ± 1. It follows that

$$\mathrm{m}_{\mathrm{disc}}(\sigma) = \begin{cases} 1 & : \text{if } \forall s \in \mathbf{S}_\Pi,\ \prod_{\nu \in S} \Delta_{\mathrm{spec}}\left(\psi(\sigma_\nu)_s, \sigma_\nu\right) = 1 \\ 0 & : \text{otherwise} \end{cases}$$

We define the function

$$\begin{aligned} \Xi_\Pi : \mathbf{S}_\Pi &\to \mathbf{C}^\times \\ s &\mapsto \prod_{\nu \in S} \Delta_{\mathrm{spec}}\left(\psi\left(\sigma_\nu^{\mathrm{base}}\right)_s, \sigma_\nu^{\mathrm{base}}\right) \end{aligned}$$

where $\sigma_\nu^{\mathrm{base}} \in \Pi(\psi(\sigma_\nu))$ denotes the chosen base point representation of the L-packet. It follows from the results of Section 9 that Ξ_Π is a ± 1-valued character. We define the character $\epsilon_\Pi = \Xi_\Pi^{-1}$. The result now follows from the definition of the local pairings $\langle \cdot, \cdot \rangle$. □

Corollary 11.2. *Assume in addition to the previous assumptions that either*

- Π *is cuspidal, or*
- *there exists a non-archimedean place* $\nu' \in S$.

Then there exists a σ *as above such that* σ *appears in the discrete automorphic spectrum of* $U(\mathbf{A}_F)$ *with multiplicity* 1.

Remark 11.3. In the case where Π is cuspidal and $[F : \mathbf{Q}] > 1$, this result is due to Labesse [Lab09, Theorem 5.4, Theorem 5.9].

Proof. Firstly if Π is cuspidal, then the group $\mathbf{S}_\Pi = \{1\}$ is trivial. It follows by Theorem 11.1 that $\mathrm{m}_{\mathrm{disc}}(\sigma) = 1$ for any representation σ as above.

Consider now the second case, that is, assume that there exists a non-archimedean place $\nu' \in S$. Let σ be any irreducible admissible representation as above. Let $\sigma'_{\nu'}$ be a discrete series representation of $U(F_\nu)$ lying in the same L-packet of $\sigma_{\nu'}$ such that the following characters are equal

$$\langle \cdot, \sigma'_{\nu'} \rangle|_{\mathbf{S}_\Pi} = \epsilon_\Pi \prod_{\nu \in \{S - \nu'\}} \langle \cdot, \sigma_\nu \rangle^{-1}|_{\mathbf{S}_\Pi} : \mathbf{S}_\Pi \to \mathbf{C}^\times$$

This is possible by Theorem 10.6. Define the irreducible admissible representation $\sigma' = \sigma^{\nu'} \otimes \sigma'_{\nu'}$ of $U(\mathbf{A}_F)$. We see that $\langle \cdot, \sigma' \rangle = \epsilon_\Pi$. It follows by Theorem 11.1 that $\mathrm{m}_{\mathrm{disc}}(\sigma') = 1$.

□

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