Symplectic structures on moduli spaces of framed sheaves on surfaces

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Abstract. We provide generalizations of the notions of Atiyah class and the Kodaira-Spencer map to the case of framed sheaves. Moreover, we construct closed two-forms on the moduli spaces of framed sheaves on surfaces. As an application, we define a symplectic structure on the moduli spaces of framed sheaves on the second Hirzebruch surface. This generalizes a result of Bottacin for the locally free case.

Contents

1. Introduction

Let $\mathcal{M}(r,n)$ be the moduli space of *framed sheaves* on $\mathbb{CP}^2$, that is, the moduli space of pairs (E,α) modulo isomorphism, where E is a torsion free sheaf on $\mathbb{CP}^2$ of rank r with $c_2(E)=n$, locally trivial in a neighborhood of a fixed line l_∞, and $\alpha\colon E|_{l_\infty}\xrightarrow{\sim}\mathcal{O}_{l_\infty}^{\oplus r}$ is the *framing at infinity.* $\mathcal{M}(r,n)$ is a nonsingular quasi-projective variety of dimension $2rn$. This moduli space also admits a description in terms of linear data, the so-called *ADHM* data (see, for example, Chapter 2 of Nakajima's book [15]).

As described in Chapter 3 in [15], by using the ADHM data description, the moduli space $\mathcal{M}(r,n)$ can be realized as a hyper-Kähler quotient. By fixing a complex structure within

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the hyper-Kähler family of complex structures on $\mathcal{M}(r,n)$, one can define a holomorphic symplectic form on $\mathcal{M}(r,n)$.

Leaving aside the framed sheaves on $\mathbb{CP}^2$, the only other relevant result in the literature about symplectic structures on moduli spaces of framed sheaves is due to Bottacin [4].

Let X be a complex nonsingular projective surface, D an effective divisor such that $D = \sum_{i=1}^{n} C_i$, where C_i is an integral curve for $i = 1, \ldots, n$, and F_D a locally free $\mathcal{O}_D$-module. We call (D, F_D)-*framed vector bundle* on X a pair (E, α), where E is a locally free sheaf on X and $\alpha\colon E|_D \xrightarrow{\sim} F_D$ is an isomorphism. Let us fix a Hilbert polynomial P. Bottacin constructs Poisson structures on the moduli space $\mathcal{M}^*_{lf}(X; F_D, P)$ of framed vector bundles on X with Hilbert polynomial P, that are induced by global sections of the line bundle $\omega_X^{-1}(-2\,D)$. In particular, when X is the complex projective plane, $D = l_\infty$ is a line and F_D the trivial vector bundle of rank r on l_∞, this yields a symplectic structure on the moduli space $\mathcal{M}_{lf}(r,n)$ of framed vector bundles on $\mathbb{CP}^2$, induced by the standard holomorphic symplectic structure of $\mathbb{C}^2 = \mathbb{CP}^2 \setminus l_\infty$. It is not known if this symplectic structure is equivalent to that given by the ADHM construction.

Bottacin's result can be seen as a generalization to the framed case of the construction of Poisson brackets and holomorphic symplectic two-forms on the moduli spaces of Gieseker-stable torsion free sheaves on X. We recall briefly the main results for torsion free sheaves. In [13], Mukai proved that any moduli space of simple sheaves on a $K3$ surface or abelian surface has a non-degenerate holomorphic two-form. Its closedness was later proved by Mukai in [14]. Mukai's result was generalizated to moduli spaces of simple vector bundles on symplectic Kähler manifolds by Ran [19] and to moduli spaces of Gieseker-stable vector bundles over surfaces of general type and over Poisson surfaces by Tyurin [20]; a more thorough study of the Poisson case was made by Bottacin in [3]. In [18], by using these results O' Grady defined closed two-forms on algebraic varieties parametrizing flat families of coherent sheaves. In all these cases, the symplectic two-form is defined in terms of the Atiyah class.

In the present paper we define a modified Atiyah class of a family of framed sheaves, which allows us to describe a framed version of the Kodaira-Spencer map and to construct closed two-forms on the moduli spaces of framed sheaves, that under some conditions are symplectic.

More precisely, let X be a nonsingular projective surface over an algebraically closed field k of characteristic zero, $D \subset X$ a divisor, F_D a locally free $\mathcal{O}_D$-module and S a Noetherian k-scheme of finite type. A flat family of (D, F_D)-framed sheaves parametrized by S is a pair $\mathcal{E} := (E, \alpha)$ on $S \times X$, such that E is flat over S and all the restrictions to the fibres $\{s\} \times X$ are (D, F_D)-framed sheaves on X.

Let $\mathcal{E} = (E, \alpha)$ be a S-flat family of (D, F_D)-framed sheaves. We introduce the *framed sheaf of first jets* $\mathrm{J}^1_{fr}(\mathcal{E})$ as the subsheaf of the sheaf of first jets $\mathrm{J}^1(E)$ (introduced by Atiyah in [1]) consisting of those sections whose $p_S^*(\Omega^1_S)$-part vanishes along $S \times D$. We define the *framed Atiyah class* $at(\mathcal{E})$ of $\mathcal{E}$ as an extension class of $\mathrm{J}^1_{fr}(\mathcal{E})$ in

$$\mathrm{Ext}^1(E, \left(p_S^*(\Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \oplus p_X^*(\Omega^1_X)\right) \otimes E).$$

Starting from the framed Atiyah class $at(\mathcal{E})$, one can define a section $\mathcal{A}t_S(\mathcal{E})$ in

$$\mathrm{H}^0(S, \mathcal{E}xt^1_{p_S}(E, (p_S^*(\Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \otimes E)),$$

where $p_S\colon S \times X \to S$ is the projection.

In the same way as in the nonframed case (cf. Section 10.1.8 in [8]), by using $\mathcal{A}t_S(\mathcal{E})$ one can define the *framed* Kodaira-Spencer map associated to $\mathcal{E}$:

$$KS_{fr}\colon (\Omega^1_S)^\vee \longrightarrow \mathcal{E}xt^1_{p_S}(E, p_X^*(\mathcal{O}_X(-D))\otimes E).$$

(D, F_D)-framed sheaves are a particular case of *framed modules*, whose theory was developed by Huybrechts and Lehn [6, 7]. In order to construct moduli spaces parametrizing these objects, they define a notion of semistability depending on a polarization and a rational polynomial δ of degree less than the dimension of the ambient variety with positive leading coefficient. They prove that there exists a coarse moduli space parametrizing semistable framed modules and a fine moduli space parametrizing stables ones. Moduli spaces of stable (D, F_D)-framed sheaves turn out to be open subschemes of the fine moduli spaces of stable framed modules.

Let X be a nonsingular projective surface over k equipped with an ample line bundle $\mathcal{O}_X(1)$. We consider as above pairs (D, F_D). Let $\delta \in \mathbb{Q}[n]$ be a stability polynomial and P a numerical polynomial of degree two.

Let us denote by $\mathcal{M}^*_\delta(X; F_D, P)$ the moduli space of (D, F_D)-framed sheaves on X with Hilbert polynomial P that are stable with respect to $\mathcal{O}_X(1)$ and δ. Let $\mathcal{M}^*_\delta(X; F_D, P)^{sm}$ be the smooth locus of $\mathcal{M}^*_\delta(X; F_D, P)$ and $\tilde{\mathcal{E}} = (\tilde{E}, \tilde{\alpha})$ the *universal object* over $\mathcal{M}^*_\delta(X; F_D, P)^{sm}$.

The first result we obtained by using the framed Atiyah class is the following:

Theorem 1. *The framed Kodaira-Spencer map defined by $\tilde{\mathcal{E}}$ induces a canonical isomorphism*

$$KS_{fr}\colon T\mathcal{M}^*_\delta(X; F_D, P)^{sm} \xrightarrow{\sim} \mathcal{E}xt^1_p(\tilde{E}, \tilde{E}\otimes p_X^*(\mathcal{O}_X(-D))),$$

where p is the projection from $\mathcal{M}^*_\delta(X; F_D, P)^{sm} \times X$ to $\mathcal{M}^*_\delta(X; F_D, P)^{sm}$.

Thus we get a generalization to the framed case of the corresponding statement for the moduli space of Gieseker-stable torsion free sheaves on X (cf. Theorem 10.2.1 in [8]).

From this theorem it follows that for any point $[(E, \alpha)]$ of $\mathcal{M}^*_\delta(X; F_D, P)^{sm}$, the vector space $\mathrm{Ext}^1(E, E(-D))$ is naturally identified with the tangent space $T_{[(E,\alpha)]}\mathcal{M}^*_\delta(X; F_D, P)$. For any $\omega \in \mathrm{H}^0(X, \omega_X(2\,D))$, we can define a skew-symmetric bilinear form

$$\begin{aligned}&\mathrm{Ext}^1(E, E(-D)) \times \mathrm{Ext}^1(E, E(-D)) \xrightarrow{\circ} \mathrm{Ext}^2(E, E(-2\,D))\\&\xrightarrow{tr} \mathrm{H}^2(X, \mathcal{O}_X(-2\,D)) \xrightarrow{\cdot\omega} \mathrm{H}^2(X, \omega_X) \cong k.\end{aligned}$$

By varying $[(E, \alpha)]$, these forms define an exterior two-form $\tau(\omega)$ on $\mathcal{M}^*_\delta(X; F_D, P)^{sm}$.

We prove that $\tau(\omega)$ is a closed form (cf. Theorem 21) and providd a criterion of its non-degeneracy (cf. Proposition 22). In particular, if the line bundle $\omega_X(2\,D)$ is trivial, the two-form $\tau(1)$ induced by $1 \in \mathrm{H}^0(X, \omega_X(2\,D)) \cong \mathbb{C}$ defines a holomorphic symplectic structure on $\mathcal{M}^*_\delta(X; F_D, P)^{sm}$.

As an application, we show that the moduli space of (D, F_D)-framed sheaves on the second Hirzebruch surface $\mathbb{F}_2$ has a symplectic structure, where D is a *conic* on $\mathbb{F}_2$ and F_D a Gieseker-semistable locally free $\mathcal{O}_D$-module. Thus for $X = \mathbb{F}_2$ we get a generalization to the non-locally free case of Bottacin's construction of symplectic structures on the moduli spaces of framed vector bundles on X with Hilbert polynomial P induced by non-degenerate Poisson structures (cf. [4]).

This paper is structured as follows. In Section 2, we briefly introduce the theory of framed modules and framed sheaves and state the main theorems about the existence of moduli spaces parametrizing these objects. In Section 3 we recall the definition of the Atiyah class for a flat family of coherent sheaves. In Section 4 we give the definitions of the framed version of the Atiyah class and of the Kodaira-Spencer map. In Section 5 we prove that the framed Kodaira-Spencer map is an isomorphism for the moduli space of stable (D, F_D)-framed sheaves and, in Section 6, we construct closed two-forms on it. Finally, in Section 7 we apply our results when the ambient space is the second Hirzebruch surface and define a symplectic structure on the moduli spaces of (stable) (D, F_D)-framed sheaves on it.

Conventions. All schemes are Noetherian of finite type over an algebraically closed field k of characteristic zero. A *polarized variety of dimension* d is a pair $(X, \mathcal{O}_X(1))$, where X is a nonsingular, projective, irreducible variety of dimension d, defined over k, and $\mathcal{O}_X(1)$ a very ample line bundle. As usual, a polarized variety of dimension two is called a *polarized surface*.

Let S be a scheme. We denote by $\mathcal{X}$ the cartesian product $S \times X$, and by p_S, p_X the projections from $\mathcal{X}$ to S and X respectively.

Finally, we denote by $\mathbb{E}\mathrm{xt}^1(E^\bullet, G^\bullet)$ the first hyper-Ext group of two finite complexes of locally free sheaves $E^\bullet$ and $G^\bullet$ on a scheme Y, that is, the first hyper-cohomology group of the total complex associated to the double complex $C^\bullet(\mathcal{H}om^\bullet(E^\bullet, G^\bullet))$ (see, e.g., Section 10.1.1 in [8]).

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2. Preliminaries on framed sheaves

In this section we introduce the notions of (D, F_D)-framed sheaves and framed modules. Moreover, we recall the main results about the construction of moduli spaces of these objects.

Let $(X, \mathcal{O}_X(1))$ be a polarized surface.

Definition 2. Let D be a divisor of X and F_D a coherent sheaf on X, which is a locally free $\mathcal{O}_D$-module. We say that a coherent sheaf E on X is (D, F_D)-*framable* if E is torsion free, locally free in a neighborhood of D, and there is an isomorphism $E|_D \xrightarrow{\sim} F_D$. An isomorphism $\alpha \colon E|_D \xrightarrow{\sim} F_D$ will be called a (D, F_D)-*framing* of E. A (D, F_D)-*framed sheaf* is a pair $\mathcal{E} := (E, \alpha)$ consisting of a (D, F_D)-framable sheaf E and a (D, F_D)-framing α. Two (D, F_D)-framed sheaves (E, α) and (E', α') are isomorphic if there is an isomorphism $f \colon E \to E'$ such that $\alpha' \circ f|_D = \alpha$.

Remark 3. Our notion of framing is the same as the one used in Lehn's Ph.D. thesis [11] and in Nevins' papers [16, 17]. △

Note that (D, F_D)-framed sheaves are a particular type of *framed modules*.

Definition 4 ([6, 7])**.** Let F be a coherent sheaf on X. A F-*framed module* on X is a pair (E, α), where E is a coherent sheaf on X and $\alpha \colon E \to F$ a morphism of coherent sheaves.

In [6, 7], Huybrechts and Lehn developed the theory of framed modules. In the case of polarized surfaces, they introduced a notion of (semi)stability depending on the polarization $\mathcal{O}_X(1)$ and rational polynomial $\delta(n) = \delta_1 n + \delta_0$, with $\delta_1 > 0$. We shall call δ a *stability polynomial.* When the framing α is zero, this reduces to Gieseker's (semi)stability condition for torsion free sheaves.

Fix a coherent sheaf F and a numerical polynomial P of degree two. Let us denote by $\underline{\mathcal{M}}_\delta^{ss}(X; F, P)$ (resp. $\underline{\mathcal{M}}_\delta^{s}(X; F, P)$) the contravariant functor from the category of schemes to the category of sets, that associates to every scheme S the set of isomorphism classes of flat families $(G, \beta\colon G \to p_X^*(F))$ of semistable (resp. stable) F-framed modules with Hilbert polynomial P parametrized by S. The main result in their papers is the following:

Theorem 5 ([6, 7])**.** *There exists a projective scheme $\mathcal{M}_\delta^{ss}(X; F, P)$ which corepresents the moduli functor $\underline{\mathcal{M}}_\delta^{ss}(X; F, P)$ of semistable F-framed modules. Moreover, there is an open subscheme $\mathcal{M}_\delta^{s}(X; F, P)$ of $\mathcal{M}_\delta^{ss}(X; F, P)$ that represents the moduli functor $\underline{\mathcal{M}}_\delta^{s}(X; F, P)$ of stable F-framed modules, i.e., $\mathcal{M}_\delta^{s}(X; F, P)$ is a fine moduli space for stable F-framed modules.*

"Fine" means the existence of a *universal* F-framed module over $\mathcal{M}_\delta^{s}(X; F, P)$, that is, a pair $(\tilde{E}, \tilde{\alpha}\colon \tilde{E} \to p_X^*(F))$, where $\tilde{E}$ is a coherent sheaf on $\mathcal{M}_\delta^{s}(X; F, P) \times X$, flat over $\mathcal{M}_\delta^{s}(X; F, P)$, such that for any scheme S and any family of stable F-framed modules $(G, \beta) \in \underline{\mathcal{M}}_\delta^{s}(X; F, P)(S)$ parametrized by S, there exists a unique morphim $g\colon S \to \mathcal{M}_\delta^{s}(X; F, P)$ such that the pull back of $(\tilde{E}, \tilde{\alpha})$ is isomorphic to (G, β).

By using Huybrechts and Lehn's result, Bruzzo and Markushevich constructed a moduli spaces for *all* (D, F_D)-framed sheaves on X, under some mild assumptions on the divisor D and on F_D. More precisely, they proved the following:

Theorem 6 ([5])**.** *Let X be a surface, $D \subset X$ a big and nef curve and F_D a Gieseker-semistable locally free $\mathcal{O}_D$-module. Then there exists a polarization $\mathcal{O}_X(1)$ and a stability polynomial $\delta(n) = \delta_1 n + \delta_0$ for which there is an open subset $\mathcal{M}^*(X; F_D, P) \subset \mathcal{M}_\delta^{s}(X; F_D, P)$ of the moduli space of stable F_D-framed modules, which is a fine moduli space for (D, F_D)-framed sheaves on X with Hilbert polynomial P.*

Moreover, if the surface X is rational and D is a smooth irreducible big and nef curve of genus zero, the moduli space $\mathcal{M}^*(X; F_D, P)$ of (D, F_D)-framed sheaves with Hilbert polynomial P is a nonsingular quasi-projective variety.

3. The Atiyah class

In this section we recall the notion of the *Atiyah class* for flat families of coherent sheaves. The Atiyah class was introduced by Atiyah in [1] for the case of coherent sheaves and by Illusie in [9, 10] for any complex of coherent sheaves (see Section 10.1.5 in [8] for a description of the Atiyah class in terms of Čech cocycles). Atiyah's approach involves the notion of the *sheaf of first jets* of a fixed coherent sheaf associated to the sheaf of one-forms (for a generalization of the sheaf of first jets to quotients of the sheaf of one-forms, see Maakestad's paper [12]).

Let Y be a scheme.

Definition 7. Let E be a coherent sheaf on Y. We call *sheaf of the first jets* $\mathrm{J}^1(E)$ of E the coherent sheaf of $\mathcal{O}_Y$-modules defined as follows:

- as a sheaf of $\mathbb{C}$-modules, we set $\mathrm{J}^1(E) := (\Omega_Y^1 \otimes E) \oplus E$,

- for any $y \in Y$, $a \in \mathcal{O}_{Y,y}$ and $(z \otimes e, f) \in \mathrm{J}^1(E)_y$, we define
$$a(z \otimes e, f) := (az \otimes e + d(a) \otimes f, af),$$
where d is the exterior differential of Y.

The sheaf $\mathrm{J}^1(E)$ fits into an exact sequence of coherent sheaves

$$0 \longrightarrow \Omega^1_Y \otimes E \longrightarrow \mathrm{J}^1(E) \longrightarrow E \longrightarrow 0. \tag{1}$$

Definition 8. Let E be a coherent sheaf on Y. We call *Atiyah class* of E the class $at(E)$ in $\mathrm{Ext}^1(E, \Omega^1_Y \otimes E)$ associated to the extension (1).

It is a very known fact that the Atiyah class $at(E)$ is the obstruction for the existence of an algebraic connection on E, that means the following:

Proposition 9 (Proposition 3.4 in [12])**.** *Let E be a coherent sheaf on Y. The Atiyah class $at(E)$ is zero if and only if there exists a connection on E.*

4. The Atiyah class for framed sheaves

In this section we turn to the framed case. First, we introduce the *framed sheaf of first jets* $\mathrm{J}^1_{fr}(\mathcal{E})$ of a S-flat family $\mathcal{E} = (E, \alpha)$ of (D, F_D)-framed sheaves as the subsheaf of the sheaf of first jets $\mathrm{J}^1(E)$ consisting of those sections whose $p_S^*(\Omega^1_S)$-part vanishes along $S \times D$. Then we define a *framed* version of the Atiyah class and of the Newton polynomials[1] for S-flat families. We use this *relative* approach, because later on we shall want to consider the framed Atiyah class of the *universal framed sheaf.*

From now on we fix the pair (D, F_D) and we just say *framed sheaf* for a (D, F_D)-framed sheaf.

Definition 10. Let S be a scheme. A *flat family of framed sheaves parametrized by* S is a pair $\mathcal{E} = (E, \alpha)$ where E is a coherent sheaf on $\mathcal{X}$, flat over S, and $\alpha\colon E \to p_X^*(F_D)$ is a morphism such that for any $s \in S$ the pair $(E|_{\{s\}\times X}, \alpha|_{\{s\}\times X})$ is a $\big(\{s\} \times D, p_X^*(F_D)|_{\{s\}\times D}\big)$-framed sheaf on $\{s\} \times X$.

Let $\mathcal{E} = (E, \alpha)$ be a flat family of framed sheaves parametrized by a scheme S. We define a subsheaf $\mathrm{J}^1_{fr}(\mathcal{E})$ of $\mathrm{J}^1(E)$, that we shall call *framed sheaf of first jets* of $\mathcal{E}$.

Denote by $\mathcal{D}$ the cartesian product $S \times D$. For a point $x \in \mathcal{X}$ such that $x \notin \mathcal{D}$, we set $\mathrm{J}^1_{fr}(\mathcal{E})_x := \mathrm{J}^1(E)_x$.

Fix $x \in \mathcal{D}$. By definition of a flat family of framed sheaves, $\big(E|_{\{s\}\times X}\big)_x$ is a free module for $s \in S$ such that $p_S(x) = s$, hence E_x is a free module (cf. Lemma 2.1.7 in [8]). Therefore there exists an open neighborhood $V \subset \mathcal{X}$ of x such that $E|_V$ is a locally free $\mathcal{O}_V$-module.

We denote by E_V the restriction of E to V. Let $\mathcal{D}_V := V \cap \mathcal{D}$ and $\mathcal{U} = \{U_i\}_{i\in I}$ a cover of $\mathcal{D}_V$ over which $p_X^*(F_D)|_{\mathcal{D}_V}$ trivializes, and choose on any U_i a set $\{e_i^0\}$ of basis sections of $\Gamma(p_X^*(F_D)|_{\mathcal{D}_V}, U_i)$. Let g_{ij}^0 be transition functions of $p_X^*(F_D)|_{\mathcal{D}_V}$ with respect to chosen local

[1] In the nonframed case, the Newton polynomials are introduced, for example, in Section 10.1.6 in [8].

basis sections (i.e., $e_i^0 = g_{ij}^0 e_j^0$), constant along S. Let us fix a cover $\mathcal{W} = \{W_i\}_{i\in I}$ of V over which E_V trivializes with sets $\{e_i\}$ of basis sections such that $W_i \cap \mathcal{D}_V = U_i$ for any $i \in I$ and

$$\begin{aligned} e_i|_{\mathcal{D}_V} &= e_i^0, \\ g_{ij}|_{\mathcal{D}_V} &= g_{ij}^0. \end{aligned}$$

Define $\mathrm{J}^1_{fr}(\mathcal{E})_x \subset \mathrm{J}^1(E)_x = (\Omega^1_{\mathcal{X},x} \otimes E_x) \oplus E_x$ as the $\mathcal{O}_{\mathcal{X},x}$-module spanned by the basis obtained by tensoring all the elements of the set $\{f_i dz_i^1, \dots, f_i dz_i^s, dz_i^{s+1}, \dots, dz_i^t\}$, where $\{dz_i^1, \dots, dz_i^t\}$ is a basis of $\Omega^1_{\mathcal{X},x}$, by the elements of the basis $\{e_i\} := \{e_i^1, \dots, e_i^r\}$ of E_x and then adding the elements of $\{e_i\}$, where we denote by $z_i^1, \dots, z_i^s$ and $z_i^{s+1}, \dots, z_i^t$ the local coordinates of S and X on W_i, respectively, and $f_i = 0$ is the local equation of $\mathcal{D}$ on W_i. Thus an arbitrary element of $\mathrm{J}^1_{fr}(\mathcal{E})_x$ is of the form

$$h_i + f_i \sum_{n=1}^{r} \sum_{k=1}^{s} \chi_{n,k}\, e_i^n \otimes dz_i^k + \sum_{m=1}^{r} \sum_{l=s+1}^{t} \psi_{m,l}\, e_i^m \otimes dz_i^l,$$

where $h_i \in E_x$ and $\chi_{n,k}, \psi_{m,l} \in \mathcal{O}_{X,x}$, for $m, n = 1, \dots r$, $k = 1, \dots, s$ and $l = s+1, \dots, t$.

If x is also a point of the open subset W_j of $\mathcal{W}$, let us denote by $l_{ij} \in \mathcal{O}_V^*(W_i \cap W_j)$ the transition function on $W_i \cap W_j$ of the line bundle associated to the divisor $\mathcal{D}_V$ and by J_{ij} the Jacobian matrix of change of coordinates. Let us define the following matrices:

$$L_{ij} := \begin{pmatrix} l_{ij} I_s & 0_{s,t-s} \\ 0_{t-s,s} & I_{t-s} \end{pmatrix}$$

and

$$F_i := \begin{pmatrix} f_i I_s & 0_{s,t-s} \\ 0_{t-s,s} & I_{t-s} \end{pmatrix}$$

where I_k is the identity matrix of order k and $0_{k,l}$ is the k-by-l zero matrix.

The change of basis matrix of the two corresponding bases in $\mathrm{J}^1_{fr}(\mathcal{E})_x$ under changes of bases in E_x is:

$$\begin{pmatrix} L_{ij} \otimes g_{ij} & (F_i^{-1} \otimes \mathrm{id}) \cdot dg_{ij} \\ 0 & g_{ij} \end{pmatrix}$$

where the block at the position (1,2) is a regular matrix function, because g_{ij} is constant along $\mathcal{D}_V$. The change of basis matrix under changes of local coordinates is:

$$\begin{pmatrix} L_{ij} \cdot J_{ij} \otimes \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}$$

The framed sheaf of first jets $\mathrm{J}^1_{fr}(\mathcal{E})$ of $\mathcal{E}$ fits into an exact sequence of coherent sheaves of $\mathcal{O}_{\mathcal{X}}$-modules:

$$0 \longrightarrow \big(p_S^*(\Omega_S^1)(-\mathcal{D}) \oplus p_X^*(\Omega_X^1)\big) \otimes E \longrightarrow \mathrm{J}^1_{fr}(\mathcal{E}) \longrightarrow E \longrightarrow 0, \tag{2}$$

where we denote by $p_S^*(\Omega_S^1)(-\mathcal{D})$ the tensor product $p_S^*(\Omega_S^1) \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$.

Definition 11. Let $\mathcal{E} = (E, \alpha)$ be a flat family of framed sheaves parametrized by a scheme S. We call *framed Atiyah class* of the family $\mathcal{E}$ the class $at(\mathcal{E})$ in

$$\mathrm{Ext}^1(E, \big(p_S^*(\Omega_S^1)(-\mathcal{D}) \oplus p_X^*(\Omega_X^1)\big) \otimes E)$$

associated to the extension (2).

Let us consider the short exact sequence

$$0 \longrightarrow p_S^*(\Omega_S^1)(-\mathcal{D}) \oplus p_X^*(\Omega_X^1) \xrightarrow{i} \Omega_{\mathcal{X}}^1 \xrightarrow{q} p_S^*(\Omega_S^1)|_{\mathcal{D}} \longrightarrow 0.$$

After tensoring by E and applying the functor $\mathrm{Hom}(E, \cdot)$, we get the long exact sequence

$$\cdots \to \mathrm{Ext}^1(E, \big(p_S^*(\Omega_S^1)(-\mathcal{D}) \oplus p_X^*(\Omega_X^1)\big) \otimes E) \xrightarrow{i_*}$$
$$\mathrm{Ext}^1(E, \Omega_{\mathcal{X}}^1 \otimes E) \xrightarrow{q_*} \mathrm{Ext}^1(E, p_S^*(\Omega_S^1)|_{\mathcal{D}} \otimes E) \to \cdots .$$

By construction, the image of $at(E)$ under i_* is $at(\mathcal{E})$, which is equivalent to saying that we have the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \big(p_S^*(\Omega_S^1)(-\mathcal{D}) \oplus p_X^*(\Omega_X^1)\big) \otimes E & \longrightarrow & \mathrm{J}^1_{fr}(\mathcal{E}) & \longrightarrow & E & \longrightarrow & 0 \\ & \downarrow & & \downarrow & & \| & & \\ 0 \longrightarrow & \Omega_{\mathcal{X}}^1 \otimes E & \longrightarrow & \mathrm{J}^1(E) & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

Moreover, $q_*(at(E)) = q_*(i_*(at(\mathcal{E}))) = 0$, hence we get the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \Omega_{\mathcal{X}}^1 \otimes E & \longrightarrow & \mathrm{J}^1(E) & \longrightarrow & E & \longrightarrow & 0 \\ & \downarrow & & \downarrow & & \| & & \\ 0 \longrightarrow & p_S^*(\Omega_S^1)|_{\mathcal{D}} \otimes E & \longrightarrow & \big(p_S^*(\Omega_S^1)|_{\mathcal{D}} \otimes E\big) \oplus E & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

Example 12. Let F be a line bundle on D. Let $\mathcal{L} = (L, \alpha)$ be a flat family of framed sheaves parametrized by S with L line bundle on $\mathcal{X}$. In this case V is the whole $\mathcal{X}$ and we choose transition functions g_{ij}^0 and g_{ij} for $p_X^*(F_D)$ and L, respectively, such that

$$g_{ij}|_{\mathcal{D}} = g_{ij}^0.$$

Recall that $dg_{ij}g_{ij}^{-1}$ is a cocycle representing $at(L)$ (cf. Proposition 12 in [1]). By the choice of g_{ij}^0, we get that $d_S(g_{ij})$ vanishes along $\mathcal{D}$, where d_S is the exterior differential of S. Hence $dg_{ij}g_{ij}^{-1}$ can be also interpreted as a cocycle representing $at(\mathcal{L})$. Moreover, it vanishes under the restriction of the de Rham differential $\tilde{d} := d|_{p_S^*(\Omega_S^1)(-\mathcal{D}) \oplus p_X^*(\Omega_X^1)}$. $\triangle$

Now we shall provide another way to describe the framed Atiyah class of a flat family of framed sheaves $\mathcal{E} = (E, \alpha)$ by using finite locally free resolutions of E, but in this case the costruction is *local over the base*, as we will explain in the following. First, we recall a result due to Banica, Putinar and Schumacher that will be very useful later on.

Theorem 13 (Satz 3 in [2])**.** *Let* $p\colon R \to T$ *be a flat proper morphism of schemes of finite type over* k*,* T *smooth,* E *and* G *coherent* $\mathcal{O}_R$*-modules, flat over* T*. If the function* $y \mapsto \dim \mathrm{Ext}^l(E_y, G_y)$ *is constant for* l *fixed, then the sheaf* $\mathcal{E}xt_p^l(E, G)$ *is locally free on* T *and for any* $y \in T$ *we have*

$$\mathcal{E}xt_p^i(E, G)_y \otimes_{\mathcal{O}_{T,y}} (\mathcal{O}_{T,y}/m_y) \cong \mathrm{Ext}^i(E_y, G_y) \ \textit{for}\ i = l-1, l.$$

Moreover, the same statement is true for complexes.

Let $\mathcal{E} = (E, \alpha)$ be a flat family of framed sheaves parametrized by a smooth scheme S. Since the projection morphism $p_S\colon \mathcal{X} \longrightarrow S$ is smooth and projective, there exists a finite locally free resolution $E^\bullet \to E$ of E (see, e.g., Proposition 2.1.10 in [8]).

Let us fix a point $s_0 \in S$. By the flatness property, the complex $(E^\bullet)|_{\{s_0\}\times D}$ is a finite resolution of locally free sheaves of $E|_{\{s_0\}\times D} \cong F_D$. Let us denote by $F^\bullet$ the complex $(E^\bullet)|_{\{s_0\}\times D}$. Define $\mathcal{F}^\bullet := F^\bullet \boxtimes \mathcal{O}_S$. The complex $\mathcal{F}^\bullet$ is S-flat since $(E^\bullet)|_{\{s_0\}\times D}$ is a complex of locally free $\mathcal{O}_D$-modules and the sheaf $\mathcal{O}_\mathcal{D}$ is a S-flat $\mathcal{O}_\mathcal{X}$-module. Moreover, for any $s \in S$, the complex $(\mathcal{F}^\bullet)|_{\{s\}\times X}$ is quasi-isomorphic to F, hence we get

$$\mathrm{Hom}((E^\bullet)|_{\{s\}\times X}, (\mathcal{F}^\bullet)|_{\{s\}\times X}) = \mathrm{Hom}(E|_{\{s\}\times X}, F_D) \cong \mathrm{End}(F_D).$$

By applying Theorem 13, we get that the the natural morphism of complexes between $E^\bullet$ and $\mathcal{F}^\bullet$ on $\{s_0\} \times X$ extends to a morphism of complexes

$$\alpha_\bullet \colon E^\bullet \longrightarrow \mathcal{F}^\bullet.$$

Let $U \subset S$ be a neighborhood of s_0 such that the following condition holds

$$(\alpha_\bullet)|_{\{s\}\times D} \textit{ is an isomorphism for any } s \in U.$$

Let $\mathcal{X}_U = U \times X$ and $\mathcal{D}_U = U \times D$. For any i, the pair $\mathcal{E}^i_U := (E^i|_{\mathcal{X}_U}, \alpha_i|_{\mathcal{X}_U} \colon E^i|_{\mathcal{X}_U} \to \mathcal{F}^i|_{\mathcal{X}_U})$ is a flat family $\mathcal{E}^i_U$ of $(D, E^i|_{\{s_0\}\times D})$-framed sheaves parametrized by U.

Thus we proved the following:

Proposition 14. *Let $\mathcal{E} = (E, \alpha)$ be a flat family of framed sheaves parametrized by a smooth scheme S and $E^\bullet \to E$ a finite locally free resolution of E. Let s_0 be a point in S. Then there exists a complex $\mathcal{F}^\bullet$, a morphism of complexes $\alpha_\bullet \colon E^\bullet \to \mathcal{F}^\bullet$ and an open neighborhood $U \subset S$ of s_0 with the following property: for any i the sheaf $\mathcal{F}^i|_{\{s_0\}\times D}$ is a locally free $\mathcal{O}_D$-module and the pair $\mathcal{E}^i_U := (E^i|_{\mathcal{X}_U}, \alpha_i|_{\mathcal{X}_U})$ is a flat family of $(D, \mathcal{F}^i|_{\{s_0\}\times D})$-framed sheaves parametrized by U.*

If for any i, we consider the short exact sequence associated to $\mathrm{J}^1_{fr}(\mathcal{E}^i_U)$

$$0 \longrightarrow \left(p_U^*(\Omega^1_U)(-\mathcal{D}_U) \oplus p_X^*(\Omega^1_X)\right) \otimes E^i|_{Y_U} \longrightarrow \mathrm{J}^1_{fr}(\mathcal{E}^i_U) \longrightarrow E^i|_{Y_U} \longrightarrow 0,$$

we get a class $at_U(\mathcal{E})$ in

$$\begin{aligned}&\mathbb{E}\mathrm{xt}^1\left(E^\bullet|_{Y_U}, \left(p_U^*(\Omega^1_U)(-\mathcal{D}_U) \oplus p_X^*(\Omega^1_X)\right) \otimes E^\bullet|_{Y_U}\right) \cong \\ &\cong \mathrm{Ext}^1\left(E|_{Y_U}, \left(p_U^*(\Omega^1_U)(-\mathcal{D}_U) \oplus p_X^*(\Omega^1_X)\right) \otimes E|_{Y_U}\right).\end{aligned}$$

By construction, $at_U(\mathcal{E})$ is independent of the resolution and it is the image of $at(\mathcal{E})$ with respect to the map on Ext-groups induced by the natural morphism $i^* \colon \Omega^1_S \to \Omega^1_U$, where $i \colon U \hookrightarrow S$ is the inclusion morphism.

4.1. **Framed Newton polynomials.** Let $\mathcal{E} = (E, \alpha)$ be a flat family of framed sheaves parametrized by a smooth scheme S. Let $at(\mathcal{E})^i$ denote the image in $\mathrm{Ext}^1\left(E, \tilde{\Omega}^i_\mathcal{X} \otimes E\right)$ of the i-th product

$$at(\mathcal{E}) \circ \cdots \circ at(\mathcal{E}) \in \mathrm{Ext}^1\left(E, (\tilde{\Omega}^1_\mathcal{X})^{\otimes i} \otimes E\right)$$

under the morphism induced by $(\tilde{\Omega}^1_\mathcal{X})^{\otimes i} \to \tilde{\Omega}^i_\mathcal{X}$, where $\tilde{\Omega}^1_\mathcal{X} := p_S^*(\Omega^1_S)(-\mathcal{D}) \oplus p_X^*(\Omega^1_X)$ and $\tilde{\Omega}^i_\mathcal{X} := \Lambda^i(\tilde{\Omega}^1_\mathcal{X})$ is the i-th exterior power of $\tilde{\Omega}^1_\mathcal{X}$.

Definition 15. The i-th *framed Newton polynomial* of $\mathcal{E}$ is

$$\gamma^i(\mathcal{E}) := tr(at(\mathcal{E})^i) \in \mathrm{H}^i(\mathcal{X}, \tilde{\Omega}^i_\mathcal{X}).$$

Fix a finite locally free resolution $E^\bullet \to E$ of E and a point $s_0 \in S$. Let $U \subset S$ be a neighborhood of s_0 as in Proposition 14. We define the i-th *framed Newton polynomial* of $\mathcal{E}$ on U as

$$\gamma^i_U(\mathcal{E}) := tr(at_U(\mathcal{E})^i) \in \mathrm{H}^i(\mathcal{X}_U, \tilde{\Omega}^i_{\mathcal{X}_U}).$$

Note that $\gamma^i(\mathcal{E})|_{\mathcal{X}_U} = \gamma^i_U(\mathcal{E})$ by construction.

The restricted de Rham differential $\tilde{d}$ introduced in Example 12 induces k-linear maps

$$\tilde{d}\colon \mathrm{H}^i(\mathcal{X}, \tilde{\Omega}^i_{\mathcal{X}}) \longrightarrow \mathrm{H}^{i+1}(\mathcal{X}, \tilde{\Omega}^i_{\mathcal{X}}(\mathcal{D})).$$

Proposition 16. *The i-th* framed Newton polynomial *of $\mathcal{E}$ is $\tilde{d}$-closed.*

Proof. Let $E^\bullet \to E$ be a finite locally free resolution of E and $s_0 \in S$. Let $U \subset S$ be a neighborhood of s_0 as in Proposition 14. The cohomology class $\gamma^i_U(\mathcal{E})$ is $\tilde{d}|_U$-closed by the same arguments as in the nonframed case (cf. Section 10.1.6 in [8]), in particular the fact we can reduce to the case of line bundles by using the splitting principle, and by Example 12. Since the restriction of $\gamma^i(\mathcal{E})$ to $\mathcal{X}_U$ is $\gamma^i_U(\mathcal{E})$, we get that $\gamma^i(\mathcal{E})$ is closed with respect to $\tilde{d}$. □

4.2. **The Kodaira-Spencer map for framed sheaves.** Let $\mathcal{E} = (E, \alpha)$ be a flat family of framed sheaves parametrized by scheme S. Consider the framed Atiyah class $at(\mathcal{E})$ in $\mathrm{Ext}^1\left(E, \left(p_S^*(\Omega^1_S)(-\mathcal{D}) \oplus p_X^*(\Omega^1_X)\right) \otimes E\right)$ and the induced section $\mathcal{A}t(\mathcal{E})$ under the global-relative map

$$\mathrm{Ext}^1\left(E, \left(p_S^*(\Omega^1_S)(-\mathcal{D}) \oplus p_X^*(\Omega^1_X)\right) \otimes E\right) \longrightarrow \mathrm{H}^0(S, \mathcal{E}xt^1_{p_S}(E, \left(p_S^*(\Omega^1_S)(-\mathcal{D}) \oplus p_X^*(\Omega^1_X)\right) \otimes E)),$$

coming from the relative-to-global spectral sequence

$$\mathrm{H}^i(S, \mathcal{E}xt^j_{p_S}(E, \left(p_S^*(\Omega^1_S)(-\mathcal{D}) \oplus p_X^*(\Omega^1_X)\right) \otimes E)) \Rightarrow \mathrm{Ext}^{i+j}(E, \left(p_S^*(\Omega^1_S)(-\mathcal{D}) \oplus p_X^*(\Omega^1_X)\right) \otimes E).$$

By considering the S-part $\mathcal{A}t_S(\mathcal{E})$ of $\mathcal{A}t(\mathcal{E})$ in

$$\mathrm{H}^0(S, \mathcal{E}xt^1_{p_S}(E, p_S^*(\Omega^1_S)(-\mathcal{D}) \otimes E)) = \mathrm{H}^0(S, \mathcal{E}xt^1_{p_S}(E, p_S^*(\Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \otimes E)),$$

we define the framed version of the Kodaira-Spencer map.

Definition 17. The *framed Kodaira-Spencer map* associated to the family $\mathcal{E}$ is the composition

$$\begin{aligned} KS_{fr}\colon (\Omega^1_S)^\vee &\xrightarrow{\mathrm{id}\otimes\mathcal{A}t_S(\mathcal{E})} (\Omega^1_S)^\vee \otimes \mathcal{E}xt^1_{p_S}(E, p_S^*(\Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \otimes E) \to \\ &\longrightarrow \mathcal{E}xt^1_{p_S}(E, p_S^*((\Omega^1_S)^\vee \otimes \Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \otimes E) \to \\ &\longrightarrow \mathcal{E}xt^1_{p_S}(E, p_X^*(\mathcal{O}_X(-D)) \otimes E). \end{aligned}$$

4.3. **Closed two-forms via the framed Atiyah class.** Let S be a smooth affine scheme and $\mathcal{E} = (E, \alpha)$ a flat family of framed sheaves parametrized by S. Let $\gamma^{0,2}$ denote the component of $\gamma^2(\mathcal{E})$ in $\mathrm{H}^0(S, \Omega^2_S) \otimes \mathrm{H}^2(X, \mathcal{O}_X(-2\,D))$.

Definition 18. Let τ_S be the homomorphism given by

$$\tau_S\colon \mathrm{H}^0(X, \omega_X(2\,D)) \cong \mathrm{H}^2(X, \mathcal{O}_X(-2\,D))^\vee \xrightarrow{\cdot\gamma^{0,2}} \mathrm{H}^0(S, \Omega^2_S),$$

where $\cong$ denotes Serre's duality.

Proposition 19. *For any $\omega \in \mathrm{H}^0(X, \omega_X(2\,D))$, the associated two-form $\tau_S(\omega)$ on S is closed.*

Proof. We can write

$$\gamma^{0,2} = \sum_l \mu_l \otimes \nu_l,$$

for elements $\mu_l \in \mathrm{H}^0(S, \Omega_S^2)$ and $\nu_l \in \mathrm{H}^2(X, \mathcal{O}_X(-2\,D))$. Since $\tilde{d}(\gamma^2(\mathcal{E})) = 0$ (cf. Proposition 16), the component of $\tilde{d}(\gamma^{0,2})$ in $\mathrm{H}^0(S, \Omega_S^3) \otimes \mathrm{H}^2(X, \mathcal{O}_X(-2\,D))$ is zero, which means

$$\sum_l d_S(\mu_l) \otimes \nu_l = 0.$$

Therefore

$$d_S(\tau_S(\omega)) = d_S\left(\sum_l \mu_l \cdot \omega(\nu_l)\right) = \sum_l d_S(\mu_l) \cdot \omega(\nu_l) = 0.$$

□

Fix $\omega \in \mathrm{H}^0(X, \omega_X(2\,D))$. For any point $s_0 \in S$, we obtained a skew-symmetric bilinear form $\tau_S(\omega)_{s_0}$ on $T_{s_0}S$:

$$T_{s_0}S \times T_{s_0}S \xrightarrow{KS\times KS} \mathrm{Ext}^1(E|_{\{s_0\}\times X}, E|_{\{s_0\}\times X}(-D)) \times \mathrm{Ext}^1(E|_{\{s_0\}\times X}, E|_{\{s_0\}\times X}(-D))$$
$$\xrightarrow{\circ} \mathrm{Ext}^2(E|_{\{s_0\}\times X}, E|_{\{s_0\}\times X}(-2\,D)) \xrightarrow{tr} \mathrm{H}^2(X, \mathcal{O}_X(-2\,D)) \xrightarrow{\cdot\omega} \mathrm{H}^2(X, \omega_X) \cong k.$$

5. The tangent bundle of moduli spaces of framed sheaves

Let $\mathcal{M}^s(X;P)$ be the moduli space of Gieseker-stable torsion free sheaves on X with Hilbert polynomial P. The open subset $\mathcal{M}_0(X;P) \subset \mathcal{M}^s(X;P)$ of points $[E]$ such that $\mathrm{Ext}_0^2(E,E)$ vanishes is smooth according to Theorem 4.5.4 in [8]. Suppose there exists a universal family $\tilde{E}$ on $\mathcal{M}_0(X;P) \times X$. By using the Atiyah class of $\tilde{E}$, one can define the Kodaira-Spencer map for $\mathcal{M}_0(X;P)$:

$$KS\colon T_{\mathcal{M}_0(X;P)} \longrightarrow \mathcal{E}xt_p^1(\tilde{E}, \tilde{E}),$$

where $p\colon \mathcal{M}_0(X;P) \times X \to \mathcal{M}_0(X;P)$ is the projection. Moreover, it is possible to prove that KS is an isomorphism (this result holds also when a universal family for $\mathcal{M}_0(X;P)$ does not exist, cf. Theorem 10.2.1 in [8]). In this section we shall prove the framed analogue of this result for the moduli spaces of stable framed sheaves on X.

Let $\delta \in \mathbb{Q}[n]$ be a stability polynomial and P a numerical polynomial of degree two. Let $\mathcal{M}_\delta^*(X;F_D,P)$ be the moduli space of framed sheaves on X with Hilbert polynomial P that are stable with respect to δ. This is an open subset of the fine moduli space $\mathcal{M}_\delta^s(X;F_D,P)$ of stable F_D-framed modules with Hilbert polynomial P.

Let us denote by $\mathcal{M}_\delta^*(X;F_D,P)^{sm}$ the smooth locus of $\mathcal{M}_\delta^*(X;F_D,P)$ and by $\tilde{\mathcal{E}} = (\tilde{E}, \tilde{\alpha})$ the *universal framed sheaf* over $\mathcal{M}_\delta^*(X;F_D,P)^{sm}$.

Theorem 20. *The framed Kodaira-Spencer map defined by $\tilde{\mathcal{E}}$ induces a canonical isomorphism*

$$KS_{fr}\colon T\mathcal{M}_\delta^*(X;F_D,P)^{sm} \xrightarrow{\sim} \mathcal{E}xt_p^1(\tilde{E}, \tilde{E} \otimes p_X^*(\mathcal{O}_X(-D))),$$

where p is the projection from $\mathcal{M}_\delta^*(X;F_D,P)^{sm} \times X$ to $\mathcal{M}_\delta^*(X;F_D,P)^{sm}$.

Proof. First note that $\mathcal{M}^*_\delta(X; F_D, P)^{sm}$ is a reduced separated scheme of finite type over k. Hence it suffices to prove that the framed Kodaira-Spencer map is an isomorphism on the fibres over closed points. Let $[(E, \alpha)]$ be a closed point. We want to show that the following diagram commutes

$$\begin{array}{ccc}
T_{[(E,\alpha)]}\mathcal{M}^*_\delta(X; F_D, P)^{sm} & \xrightarrow{\sim} & \mathrm{Ext}^1(E, E(-D)) \\
\Big\downarrow {\scriptstyle KS_{fr}([(E,\alpha)])} & \diagup\!\!\diagup & \\
\mathrm{Ext}^1(E, E(-D)) & &
\end{array}$$

where the horizontal isomorphism comes from deformation theory (see proof of Theorem 4.1 in [7]).

Let w be an element in $\mathrm{Ext}^1(E, E(-D))$. Consider the long exact sequence

$$\cdots \to \mathrm{Ext}^1(E, E(-D)) \xrightarrow{j_*} \mathrm{Ext}^1(E, E) \xrightarrow{\alpha_*} \mathrm{Ext}^1(E, F) \to \cdots$$

obtained by applying the functor $\mathrm{Hom}(E, \cdot)$ to the exact sequence

$$0 \longrightarrow E(-D) \xrightarrow{j} E \xrightarrow{\alpha} F \longrightarrow 0.$$

Let $v = j_*(w) \in \mathrm{Ext}^1(E, E)$. We get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E(-D) & \xrightarrow{\tilde{i}} & \tilde{G} & \xrightarrow{\tilde{\pi}} & E & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle j} & & \Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & E & \xrightarrow{i} & G & \xrightarrow{\pi} & E & \longrightarrow & 0
\end{array}$$

where the first arrow is a representative for w and the second one a representative for v.

Let $S = \mathrm{Spec}(k[\varepsilon])$ be the spectrum of the ring of dual numbers, where $\epsilon^2 = 0$. We can think G as a S-flat family by letting ε act on G as the morphism $i \circ \pi$. Since $\varepsilon G' = E(-D)$ and $\varepsilon G = E$, by applying the snake lemma to the previous diagram we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E(-D) & \xrightarrow{\tilde{i}} & \tilde{G} & \xrightarrow{\tilde{\pi}} & E & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{id}_E} & & \\
0 & \longrightarrow & E & \xrightarrow{i} & G & \xrightarrow{\pi} & E & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow {\scriptstyle \beta} & & \Big\downarrow & & \\
0 & \longrightarrow & \varepsilon F & \longrightarrow & \varepsilon F & \longrightarrow & 0 & & \\
 & & \Big\downarrow & & \Big\downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Moreover $\alpha_*(v) = 0$, hence we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \xrightarrow{i} & G & \xrightarrow{\pi} & E & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & \varepsilon F & \longrightarrow & E \oplus \varepsilon F & \longrightarrow & E & \longrightarrow & 0
\end{array}$$

Thus we obtain a framing $\gamma\colon G \to F \oplus \varepsilon F$ induced by α and β. Moreover $\gamma|_{\mathcal{D}}$ is an isomorphism. We denote by $\mathcal{G}$ the corresponding S-flat family of (D, F_D)-framed sheaves on X.

In the nonframed case, one can define a relative Atiyah class for families of coherent sheaves parametrized by a scheme S and also its S-part (see Section 10.1.8 in [8]). As it is explained in Example 10.1.9 in [8], since S is affine, the relative S-part $\mathcal{A}t_S(G)$ of G is an element of

$$\mathrm{Ext}^1_{\mathcal{X}}(G, p_S^*\Omega^1_S \otimes G) \cong \mathrm{Ext}^1_{\mathcal{X}}(G, E).$$

Consider the short exact sequence of coherent sheaves over $\mathrm{Spec}\left({}^{k[\varepsilon_1,\varepsilon_2]}/{}_{(\varepsilon_1,\varepsilon_2)^2}\right) \times X$

$$0 \longrightarrow E \xrightarrow{i'} G' \xrightarrow{\pi'} G \longrightarrow 0, \tag{3}$$

where ε_1 and ε_2 act trivially on E and by $i \circ \pi$ on G, and $G' \cong {}^{k[\varepsilon_1] \otimes_k G}/{}_{\varepsilon_1\varepsilon_2 G} \cong G \oplus E$, with actions

$$\varepsilon_1 = \begin{pmatrix} 0 & \pi \\ 0 & 0 \end{pmatrix} \text{ and } \varepsilon_2 = \begin{pmatrix} i\pi & 0 \\ 0 & 0 \end{pmatrix}.$$

By definition of Atiyah class, $\mathcal{A}t_S(G)$ is precisely the extension class of the short exact sequence (3), considered as a sequence of $k[\varepsilon_1] \otimes \mathcal{O}_X$-modules.

The morphims π induces a pull-back morphism $\pi^*\colon \mathrm{Ext}^1_X(E,E) \to \mathrm{Ext}^1_{\mathcal{X}}(G,E)$, which is an isomorphism. As it is proved in Example 10.1.9 in [8], $\pi^*(v) = \mathcal{A}t_S(G)$, indeed we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \xrightarrow{i'} & G' & \xrightarrow{\pi'} & G & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle t'} & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & E & \xrightarrow{i} & G & \xrightarrow{\pi} & E & \longrightarrow & 0 \end{array}$$

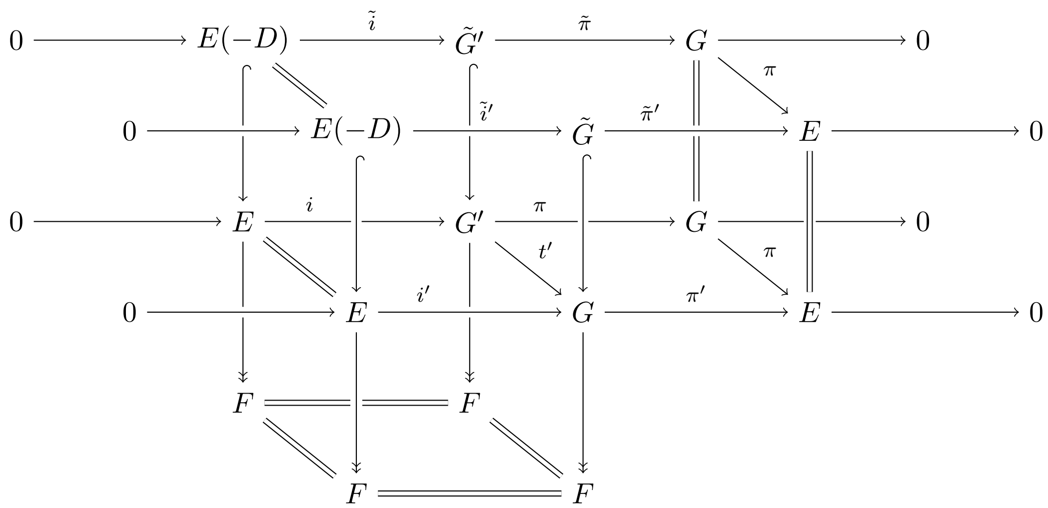
Thus G' is the sheaf of first jets of G relative to the quotient $\Omega^1_{\mathcal{X}} \to p_S^*(\Omega^1_S) \to 0$. By following Maakestad's construction of Atiyah classes of coherent sheaves relative to quotients of $\Omega^1_{\mathcal{X}}$ (cf. Section 3 in [12]) and by readapting to this particular case our construction of the framed sheaf of first jets given in Section 4, we can define a *framed sheaf of first jets* $\tilde{G}'$ of the framed sheaf $\mathcal{G}$ relative to $p_S^*(\Omega^1_S)$. Thus we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(-D) & \xrightarrow{\tilde{i}'} & \tilde{G}' & \xrightarrow{\tilde{\pi}'} & G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & E & \xrightarrow{i'} & G' & \xrightarrow{\pi'} & G & \longrightarrow & 0 \end{array}$$

The first arrow is a representative for the S-part $\mathcal{A}t_S(\mathcal{G})$ of $\mathcal{G}$ in

$$\mathrm{Ext}^1_{\mathcal{X}}(G, p_S^*\Omega^1_S(-\mathcal{D}) \otimes G) \cong \mathrm{Ext}^1_{\mathcal{X}}(G, E(-D)).$$

Consider the diagram



By diagram chasing, one can define a morphism $\tilde{G}' \to \tilde{G}$ such that the corresponding diagram commutes. Thus the image of w through the map $\mathrm{Ext}^1_X(E, E(-D)) \to \mathrm{Ext}^1_{\mathcal{X}}(G, E(-D))$ is exactly $\mathcal{A}t_S(\mathcal{G})$. This completes the proof. □

6. Closed two-forms on moduli spaces of framed sheaves

In this section we show how to construct closed two-forms on the moduli space $\mathcal{M}^*_\delta(X; F_D, P)^{sm}$ by using global sections of the line bundle $\omega_X(2\,D)$. Moreover, we establish a criterion of non-degeneracy for these two-forms.

Let us fix a point $[(E, \alpha)]$ in $\mathcal{M}^*_\delta(X; F_D, P)^{sm}$. By Theorem 20, the vector space $\mathrm{Ext}^1(E, E(-D))$ is naturally identified with the tangent space $T_{[(E,\alpha)]}\mathcal{M}^*_\delta(X; F_D, P)$.

For any $\omega \in \mathrm{H}^0(X, \omega_X(2\,D))$, we can define a skew-symmetric bilinear form

$$\begin{aligned} &\mathrm{Ext}^1(E, E(-D)) \times \mathrm{Ext}^1(E, E(-D)) \xrightarrow{\circ} \mathrm{Ext}^2(E, E(-2\,D)) \\ &\xrightarrow{tr} \mathrm{H}^2(X, \mathcal{O}_X(-2\,D)) \xrightarrow{\cdot\,\omega} \mathrm{H}^2(X, \omega_X) \cong k. \end{aligned}$$

By varying the point $[(E, \alpha)]$, these forms fit into a exterior two-form $\tau(\omega)$ on $\mathcal{M}^*_\delta(X; F_D, P)^{sm}$.

Theorem 21. *For any $\omega \in \mathrm{H}^0(X, \omega_X(2\,D))$, the two-form $\tau(\omega)$ is closed on $\mathcal{M}^*_\delta(X; F_D, P)^{sm}$.*

Proof. It suffices to prove that given a smooth affine variety S, for any S-flat family $\mathcal{E} = (E, \alpha)$ of framed sheaves on X defining a classifying morphism

$$\begin{aligned} \psi\colon S &\longrightarrow \mathcal{M}^*_\delta(X; F_D, P), \\ s &\longmapsto [\mathcal{E}|_{\{s\}\times X}], \end{aligned}$$

the pullback $\psi^*(\tau(\omega)) \in \mathrm{H}^0(S, \Omega^2_S)$ is closed. Since $\psi^*(\tau(\omega)) = \tau_S(\omega)$ by construction, this follows from Proposition 19. □

Thus we have constructed closed two-forms $\tau(\omega)$ on the moduli space $\mathcal{M}^*_\delta(X; F_D, P)^{sm}$ depending on a choice of $\omega \in \mathrm{H}^0(X, \omega_X(2\,D))$. In general, these forms may be degenerate.

Now we want to give a criterion to check when the two-form is non-degenerate.

Proposition 22. *Let* $\omega \in \mathrm{H}^0(X, \omega_X(2\,D))$ *and* $[(E,\alpha)]$ *a point in* $\mathcal{M}^*_\delta(X; F_D, P)^{sm}$. *The closed two-form* $\tau(\omega)_{[(E,\alpha)]}$ *is non-degenerate at the point* $[(E,\alpha)]$ *if and only if the multiplication by* ω *induces an isomorphism*

$$\omega_* \colon \mathrm{Ext}^1(E, E(-D)) \longrightarrow \mathrm{Ext}^1(E, E \otimes \omega_X(D)).$$

Proof. The proof is similar to that of Proposition 10.4.1 in [8]. □

Obviously, if the line bundle $\omega_X(2\,D)$ is trivial, for any point $[(E,\alpha)]$ in $\mathcal{M}^*_\delta(X; F_D, P)^{sm}$ the pairing

$$\tau(1) \colon \mathrm{Ext}^1(E, E(-D)) \times \mathrm{Ext}^1(E, E(-D)) \longrightarrow k$$

is a non-degenerate alternating form, where $1 \in \mathrm{H}^0(X, \omega_X(2\,D)) \cong \mathbb{C}$.

7. An example of symplectic structure (the second Hirzebruch surface)

We denote by $\mathbb{F}_p$ the p-th Hirzebruch surface $\mathbb{F}_p := \mathbb{P}(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(-p))$, which is the projective closure of the total space of the line bundle $\mathcal{O}_{\mathbb{CP}^1}(-p)$ on $\mathbb{CP}^1$. One can describe explicitly $\mathbb{F}_p$ as the divisor in $\mathbb{CP}^2 \times \mathbb{CP}^1$

$$\mathbb{F}_p := \{([z_0 : z_1 : z_2], [z : w]) \in \mathbb{CP}^2 \times \mathbb{CP}^1 \mid z_1 w^p = z_2 z^p\}.$$

Let us denote by $p \colon \mathbb{F}_p \to \mathbb{CP}^2$ the projection onto $\mathbb{CP}^2$. Let D be the inverse image of a generic line of $\mathbb{CP}^2$ through p. D is a smooth connected big and nef curve of genus zero.

Let F denote the fibre of the projection $\mathbb{F}_p \to \mathbb{CP}^1$. Then the Picard group of $\mathbb{F}_p$ is generated by D and F. One has

$$D^2 = p, \ D \cdot F = 1, \ F^2 = 0.$$

In particular, the canonical divisor K_p can be expressed as

$$K_p = -2\,D + (p-2)F.$$

Let us consider the second Hirzebruch surface $\mathbb{F}_2$. It is the projective closure of the cotangent bundle $T^*\mathbb{CP}^1$ of the complex projective line $\mathbb{CP}^1$.

Let D be as before and F_D a Gieseker-semistable locally free $\mathcal{O}_D$-module. By Theorem 6 there exists a fine moduli space $\mathcal{M}^*(\mathbb{F}_2; F_D, P)$ of framed sheaves on $\mathbb{F}_2$ with Hilbert polynomial P. Moreover, it is a smooth projective variety.

The canonical divisor of $\mathbb{F}_2$ is $K_2 = -2\,D$, hence the line bundle $\omega_{\mathbb{F}_2}(2\,D)$ is trivial and, for $1 \in \mathrm{H}^0(\mathbb{F}_2, \omega_{\mathbb{F}_2}(2\,D)) \cong \mathbb{C}$, the two-form $\tau(1)$ defines a symplectic structure on $\mathcal{M}^*(\mathbb{F}_2; F_D, P)$.

Remark 23. If one compares our costruction of symplectic structures on the moduli spaces of framed sheaves on $\mathbb{F}_2$ with Hilbert polynomial P to that of Bottacin (cf. [4]), one can see that they are equivalent on the locally free part of the moduli space. △

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