A Juzvinskiĭ Addition Theorem for Finitely Generated Free Groups Actions

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Abstract. The classical *Juzvinskiĭ Addition Theorem* states that the entropy of an automorphism of a compact group decomposes along invariant subgroups. Thomas generalized the theorem to a skew-product setting. Using L. Bowen's *f-invariant* we prove the addition theorem for actions of finitely generated free groups on skew-products with compact totally disconnected groups or finitely dimensional Lie Groups (correcting an error from [Bo10c]) and discuss examples.

Contents

1. Introduction

The following result was proven independently by H. Li [Li11] and Lind-Schmidt [LS09].

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Theorem 1.1. *[Addition theorem for amenable groups] Let Γ be a countable discrete amenable group, G be a compact metrizable group and $\alpha : \Gamma \to Aut(G)$ an action of Γ on G by group-automorphisms. Suppose $N \lhd G$ is a closed normal $\alpha(\Gamma)$-invariant subgroup. Denote by $\alpha_N : \Gamma \to Aut(N)$ and $\alpha_{G/N} : \Gamma \to Aut(G/N)$ the induced actions and by $\mu_G, \mu_N, \mu_{G/N}$ the Haar probability measures on G, N and G/N respectively. Then the entropies of these actions satisfy:*

$$h_{\mu_G}(\alpha) = h_{\mu_N}(\alpha_N) + h_{\mu_{G/N}}(\alpha_{G/N}).$$

In the case $\Gamma = \mathbb{Z}$, this result is due to Juzvinskiĭ [Ju65] from which it receives its name. The case $\Gamma = \mathbb{Z}^d$ was proven in [LSW90]. Special cases were obtained by Miles [Mi08] and Björklund-Miles [BM09].

The paper [Bo10a] introduced a measure-conjugacy invariant, called the *f-invariant*, for probability-measure-preserving actions of finitely generated free groups. (Later a more general theory of sofic entropy was introduced in [Bo10b], of which we have little to say in the present article). In [Bo10c], a proof is claimed that the above addition formula extends to the case when Γ is a finitely generated free group, the entropy is replaced with the f-invariant, and G is either totally disconnected, a Lie group, or a connected finite-dimensional abelian group (whenever the f-invariant is well-defined). However, there is an error in the proof which this paper corrects (at least under a mild extra hypothesis). The main result is Theorem 2.3 below. We also prove a skew-product addition formula in Theorem 3.3 which may be of independent interest.

Organization: §2 reviews the f-invariant and states the main theorem; §3 reviews skew-products and proves Theorem 2.3 from Theorem 3.3. In §5 and §6 Theorem 3.3 is proven; §7 discuss examples, including the Ornstein-Weiss example. The appendix offers an erratum to [Bo10c].

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2. The f-invariant

Let $\Gamma = \langle s_1, \dots, s_r \rangle$ be a rank r free group. Let α be a measure-preserving action of Γ on a standard probability space $(X, \mathcal{B}_X, \nu)$. We consider α as a homomorphism from Γ to the group of automorphisms of $(X, \mathcal{B}_X, \nu)$ and write α_g for $\alpha(g)$ ($\forall g \in \Gamma$). Let $\mathcal{P} = \{P_1, P_2, \dots\}$

be a countable partition of X into measurable subsets. The Shannon-entropy of $\mathcal{P}$ is

$$H_\nu(\mathcal{P}) := -\sum_{P\in\mathcal{P}} \nu(P)\log(\nu(P)).$$

By convention $0\log(0) := 0$. If $\mathcal{P}, \mathcal{Q}$ are two partitions of X then their *join* is defined by $\mathcal{P}\vee\mathcal{Q} := \{P\cap Q : \ P\in\mathcal{P}, Q\in\mathcal{Q}\}$. If $W\subset\Gamma$ is finite, we let $\mathcal{P}^W := \bigvee_{w\in W}\alpha_w\mathcal{P}$. Note that α is only implicit in this notation. If $H(\mathcal{P})<\infty$ then define

$$\begin{aligned} F_\nu(\alpha,\mathcal{P}) &= (1-2r)H_\nu(\mathcal{P}) + \sum_{i=1}^r H_\nu(\mathcal{P}\vee\alpha_{s_i}\mathcal{P}) \\ f_\nu(\alpha,\mathcal{P}) &= \inf_{W\subset\Gamma} F_\nu\left(\alpha,\mathcal{P}^W\right) \end{aligned}$$

where the infimum is over all finite $W\subset\Gamma$.

For $g\in\Gamma$, let $h_\nu(\alpha_g,\mathcal{P})$ denote the entropy rate of $\mathcal{P}$ with respect to the $\mathbb{Z}$-action generated by α_g. To be precise,

$$h_\nu(\alpha_g,\mathcal{P}) = \lim_{n\to\infty}(2n+1)^{-1}H\left(\bigvee_{i=-n}^{n}\alpha_g^i\mathcal{P}\right).$$

The *entropy* of the action α_g is $h_\nu(\alpha_g) = \sup_{\mathcal{P}} h_\nu(\alpha_g,\mathcal{P})$ where the supremum is over all finite measurable partitions $\mathcal{P}$ of X. Define

$$\begin{aligned} F^*_\nu(\alpha,\mathcal{P}) &= (1-r)H_\nu(\mathcal{P}) + \sum_{i=1}^r h_\nu(\alpha_{s_i},\mathcal{P}) \\ f^*_\nu(\alpha,\mathcal{P}) &= \inf_{W\subset\Gamma} F^*_\nu\left(\alpha,\mathcal{P}^W\right) \end{aligned}$$

where the infimum is over all finite $W\subset\Gamma$.

The partition $\mathcal{P}$ is said to be *generating* (for the action α) if the smallest $\alpha(\Gamma)$-invariant sigma-algebra containing $\mathcal{P}$ is $\mathcal{B}_X$ (up to sets of measure zero).

Theorem 2.1. *Let α be a measure-preserving action of Γ on a standard probability space $(X,\mathcal{B}_X,\nu)$. If $\mathcal{P},\mathcal{Q}$ are any two finite-entropy generating partitions for α then $f_\nu(\alpha,\mathcal{P}) = f_\nu(\alpha,\mathcal{Q})$.*

Proof. Define

$$f'_\nu(\alpha,\mathcal{P}) = \inf_{n>0} F_\nu(\alpha,\mathcal{P}^{B(n)})$$

where $B(n)\subset\Gamma$ denotes the ball of radius n with respect to the word metric induced by $\{s_1^{\pm1},\ldots,s_r^{\pm1}\}$. This is the definition of the f-invariant given in [Bo10a] and [Bo10c]. Clearly, $f_\nu(\alpha,\mathcal{P})\le f'_\nu(\alpha,\mathcal{P})$. However, if $W\subset\Gamma$ is any finite set with $e\in W$ then it follows from [Bo10c, Propositions 4.3 and 5.1] that $F_\nu(\alpha, \mathcal{P}^{B(n)}) \le F_\nu(\alpha, \mathcal{P}^W)$ for all sufficiently large n. Thus $f'_\nu(\alpha, \mathcal{P}) \le f_\nu(\alpha, \mathcal{P})$ which implies $f'_\nu(\alpha, \mathcal{P}) = f_\nu(\alpha, \mathcal{P})$. The result now follows from the main theorem of [Bo10a]. □

Because of this theorem, we define $f_\nu(\alpha) := f_\nu(\alpha, \mathcal{P})$ where $\mathcal{P}$ is any finite-entropy generating partition for α. If there does not exist a finite-entropy generating partition for α then $f_\nu(\alpha)$ is undefined. One of the main results of [Bo10c] is:

Theorem 2.2. *Let α be a measure-preserving action of Γ on a standard probability space $(X, \mathcal{B}_X, \nu)$. Then for any finite-entropy generating partition $\mathcal{P}$ for α, $f_\nu(\alpha) = f^*_\nu(\alpha, \mathcal{P})$.*

The main result of this paper is:

Theorem 2.3. *Let $\Gamma = \langle s_1, \ldots, s_r \rangle$ be a rank r free group, G be a compact metrizable group and $\alpha : \Gamma \to Aut(G)$ an action of Γ on G by group-automorphisms. Suppose $N \lhd G$ is a closed normal $\alpha(\Gamma)$-invariant subgroup. Denote by $\alpha_N : \Gamma \to Aut(N)$ and $\alpha_{G/N} : \Gamma \to Aut(G/N)$ the induced actions and by $\mu_G, \mu_N, \mu_{G/N}$ the Haar probability measures on G, N and G/N respectively. Suppose there exists finite-entropy generating partitions for $\alpha, \alpha_N, \alpha_{G/N}$ and one of the following hold.*

1. *N is totally disconnected and there exists a clopen finite-index normal subgroup $N_0 \lhd N$ such that $\{gN_0 : g \in N\}$ is a generating partition for α_N.*
2. *G is a compact finite-dimensional Lie group and the action α is by smooth automorphisms.*

Then

$$f_{\mu_G}(\alpha) = f_{\mu_N}(\alpha_N) + f_{\mu_{G/N}}(\alpha_{G/N}).$$

Remark 2.1. The proof shows slightly more: if case (1) occurs and $\alpha_{G/N}$ has a finite-entropy generating partition, then α automatically has a finite-entropy generating partition. This follows from Lemmas 3.2 and 6.4. To be more precise, Lemma 3.2 shows that α is measurably conjugate to a skew-product action of the form $\alpha_{G/N} \times_\sigma \alpha_N$. If $\mathcal{P}$ is a finite-entropy generating partition for $\alpha_{G/N}$ and $\mathcal{Q} = \{gN_0 : g \in G\}$ is a generating partition for α_N of the kind described in case (1) above, then Lemma 6.4 shows that $\mathcal{P} \times \mathcal{Q}$ is generating for $\alpha_{G/N} \times_\sigma \alpha_N$. Because $\mathcal{P}$ has finite-entropy and $\mathcal{Q}$ is finite, $\mathcal{P} \times \mathcal{Q}$ has finite entropy as required.

Remark 2.2. Suppose N is totally disconnected and $N_0 \lhd N$ is a closed finite-index normal subgroup (the fact N_0 is closed and finite-index implies N_0 is clopen). Let $M = \cap_{g\in\Gamma}\alpha_g N_0$. Let $\alpha_{G/M}, \alpha_{N/M}$ be the induced actions on G/M and N/M respectively. Let $\mu_{G/M}, \mu_{N/M}$ be the respective Haar probability measures. Suppose that $\alpha_{G/M}$ and $\alpha_{G/N}$ admit finite-entropy generating partitions. Note that $\{gN_0/M : g \in G\}$ is a generating partition for $\alpha_{N/M}$. So the above theorem implies

$$f_{\mu_{G/M}}(\alpha_{G/M}) = f_{\mu_{N/M}}(\alpha_{N/M}) + f_{\mu_{G/N}}(\alpha_{G/N}).$$

By the previous remark, this formula holds as long as $\alpha_{G/N}$ admits a finite-entropy generating partition.

3. Skew-products

The proof of Theorem 2.3 is based on a more general skew-product theorem of independent interest, the construction of which we recall next.

Definition 3.1. Let Γ be a group. Let $(X, \mathcal{B}_X, \nu)$ be a Lebesgue space equipped with a Γ-action α. We consider α as a homomorphism from Γ to the group of automorphisms of $(X, \mathcal{B}_X, \nu)$ and write α_g for $\alpha(g)$ ($\forall g \in \Gamma$).

Let G be a compact group with Borel sigma-algebra $\mathscr{B}$ and Haar measure μ. Let β be a Γ-action by group-automorphisms on G. Let $\sigma : \Gamma \times X \to \mathrm{G}$ be a cocycle for β and α, i.e., σ is a measurable mapping so that for all $g, h \in \Gamma$, $x \in X$

$$\sigma(gh, x) = (\beta_g \sigma(h, x)) \cdot \sigma(g, \alpha_h x). \tag{3.1}$$

Define the *skew-product action* $\alpha \times_\sigma \beta$ *of* Γ *on* $X \times \mathrm{G}$ *by:*

$$(\alpha \times_\sigma \beta)_g(x, y) = (\alpha_g x, (\beta_g y) \cdot \sigma(g, x)) \ (g \in \Gamma,\ x \in X, y \in \mathrm{G}).$$

The connection between skew-product actions and the addition theorem is the following standard result (which we obtained from [Li11, Proof of Corollary 6.3]).

Lemma 3.2. *Let* Γ *be a countable group,* G *be a compact metrizable group,* $\alpha : \Gamma \to Aut(G)$ *an action of* Γ *on* G *by group-automorphisms and* $N \lhd G$ *a closed normal* $\alpha(\Gamma)$*-invariant subgroup. Denote by* $\alpha_N : \Gamma \to Aut(N)$ *and* $\alpha_{G/N} : \Gamma \to Aut(G/N)$ *the induced actions. Then there is a cocycle* $\sigma : \Gamma \times G/N \to N$ *such that* $\alpha_{G/N} \times_\sigma \alpha_N$ *is measurably conjugate with* α.

Proof. Let $\pi : G \to G/N$ denote the quotient map. Every continuous open surjective map between compact metrizable spaces has a Borel cross section by [Ar98, Theorem 3.4.1]. Thus we can find a Borel map

$\psi : G/N \to G$ such that $\pi\psi$ is the identity map on G/N. It is easily verified that the map $\phi : G/N \times N \to G$ sending (gN, h) to $\psi(gN)h$ is an isomorphism from the measurable space $(G/N \times N, \mathcal{B}_{G/N} \times \mathcal{B}_N)$ onto the measurable space $(G, \mathcal{B}_G)$ (where $\mathcal{B}_G, \mathcal{B}_N, \mathcal{B}_{G/N}$ denote the Borel sigma-algebras on G, N and G/N respectively). Furthermore, denoting the Haar probability measures on $G, N, G/N$ by $\mu_G, \mu_N, \mu_{G/N}$ respectively, one sees that $\phi_*(\mu_{G/N} \times \mu_N)$ is left-translation invariant and hence $\phi_*(\mu_{G/N} \times \mu_N) = \mu_G$. It is also readily checked that the map $\sigma : \Gamma \times G/N \to N$ defined by

$$\sigma(\gamma, gN) = \psi(\alpha_{G/N}(\gamma)(gN))^{-1}\alpha(\gamma)(\psi(gN))$$

is a cocycle for the actions $\alpha_{G/N}$ and α_N so that ϕ intertwines the actions $\alpha_{G/N} \times_\sigma \alpha_N$ with α. □

The main technical result of this paper is:

Theorem 3.3. *Let $\Gamma = \langle s_1, \ldots, s_r \rangle$ be a rank r free group, α a measure-preserving action of Γ on a standard probability space $(X, \mathcal{B}_X, \nu)$, G a compact metrizable group, β an action of Γ on G by group-automorphisms, and $\sigma : \Gamma \times X \to G$ a cocycle for these actions. Suppose that G is totally disconnected and there exists a finite-index clopen normal subgroup $N \lhd G$ such that $\{gN : g \in G\}$ is a generating partition for β. Let μ denote the Haar probability measure on G. Suppose also that there is a finite-entropy generating partition for α. Then*

$$f_{\nu\times\mu}(\alpha \times_\sigma \beta) = f_\nu(\alpha) + f_\mu(\beta).$$

The analog of this theorem for discrete countable amenable groups Γ when G is an arbitrary compact metrizable group was established in [Li11]. The case $\Gamma = \mathbb{Z}$ was proven earlier by Thomas [Th71] and the case $\Gamma = \mathbb{Z}^d$ is shown in [LSW90].

Theorem 3.3 is proven in the next section. Next we combine this result with the next two lemmas to complete the proof of Theorem 2.3.

Lemma 3.4. *Let M be a smooth compact Riemannian manifold. Let $T : M \to M$ be a diffeomorphism. Then $h_\mu(T) < \infty$ for any T-invariant probability measure μ.*

Proof. This is due to Kushnirenko [Ku65]. Alternatively, it follows from Ruelle's inequality (see e.g. [KH95, Corollary S.2.17]). □

Lemma 3.5. *Let $\Gamma = \langle s_1, \ldots, s_r \rangle$ be a rank r free group with $r > 1$, M be a smooth compact Riemannian manifold, α a measure-preserving action of Γ on M by diffeomorphisms and μ a non-atomic $\alpha(\Gamma)$-invariant probability measure on M. Then $f_\mu(\alpha) = -\infty$ if there is a finite-entropy generating partition for the action.*

Proof. Let $m = \max_{i=1}^r h_\mu(\alpha_{s_i})$. By the previous lemma, $m < \infty$. Let $\mathcal{P}$ be a finite-entropy generating partition for α. Let $N > 0$. Because μ is non-atomic, there is a finite partition $\mathcal{Q}$ of M with $H_\mu(\mathcal{Q}) > N$. So after replacing $\mathcal{P}$ with $\mathcal{P} \vee \mathcal{Q}$ if necessary, we may assume that $H_\mu(\mathcal{P}) > N$. By Theorem 2.2

$$\begin{aligned} f_\mu(\alpha) &= f^*_\mu(\alpha, \mathcal{P}) = \inf_{W \subset \Gamma} F^*_\mu(\alpha, \mathcal{P}^W) \\ &\le (1-r)H_\nu(\mathcal{P}) + \sum_{i=1}^r h_\nu(\alpha_{s_i}, \mathcal{P}) \\ &\le (1-r)N + rm. \end{aligned}$$

Because $N > 0$ is arbitrary and $r > 1$, this implies the lemma.

[Proof of Theorem 2.3 from Theorem 3.3] By Theorem 1.1, we may assume, without loss of generality, that $r > 1$. Because the case when G is trivial is clear, we assume G is non-trivial. Similarly, the case when $G = N$ is obvious, so we assume $G \neq N$. We also assume that the actions α, α_N and $\alpha_{G/N}$ all have finite-entropy generating partitions.

Suppose item (1) holds. By Lemma 3.2, α is measurable conjugate with $\alpha_{G/N} \times_\sigma \alpha_N$ for some cocycle σ. So Theorem 3.3 implies

$$f_{\mu_G}(\alpha) = f_{\mu_G}(\alpha_{G/N} \times_\sigma \alpha_N) = f_{\mu_{G/N}}(\alpha_{G/N}) + f_{\mu_N}(\alpha_N)$$

as required.

Suppose that item (2) holds; i.e., G is a finite-dimensional compact Lie group and α is an action by smooth group-automorphisms. If G is finite then the theorem is clear because

$$f_{\mu_G}(\alpha) = -(r-1)\log|G| = -(r-1)\log|G/N| + -(r-1)\log|N| = f_{\mu_{G/N}}(\alpha_{G/N}) + f_{\mu_N}(\alpha_N).$$

If G is infinite then, because it is compact, it has positive dimension. So μ_G is non-atomic. So the previous lemma implies $f_{\mu_G}(\alpha) = -\infty$.

Also, because G is infinite, either N or G/N is infinite. Therefore, either μ_N or $\mu_{G/N}$ is non-atomic. Of course, the actions α_N and $\alpha_{G/N}$ are smooth. It should be noted that the f-invariant does not take on the value $+\infty$. So the previous lemma implies $f_{\mu_{G/N}}(\alpha_{G/N}) + f_{\mu_N}(\alpha_N) = -\infty$. □

4. Relative entropy

The proof of Theorem 3.3 uses the relative f-invariant theory developed in [Bo10c], which we review here. So let $(X, \mathcal{B}_X, \nu)$ be a standard probability space. Let $\mathcal{P}$ be a countable measurable partition of X and let $\mathcal{F} \subset \mathcal{B}_X$ be a sub-sigma algebra. Recall that for a.e. $x \in X$, the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}](x)$ is a probability measure on $(X, \mathcal{B}_X)$ satisfying

(1) $x \mapsto \mathbb{E}[A|\mathcal{F}](x)$ is $\mathcal{F}$-measurable for any $A \in \mathcal{B}_X$;
(2) $\int \mathbb{E}[A|\mathcal{F}](x)\ d\nu(x) = \nu(A)$ for any $A \in \mathcal{B}_X$.

The information function $I(\mathcal{P}|\mathcal{F})$ is a function on X defined by

$$I(\mathcal{P}|\mathcal{F})(x) = -\mathbb{E}[P_x|\mathcal{F}](x)\log(\mathbb{E}[P_x|\mathcal{F}](x))$$

where $P_x \in \mathcal{P}$ is the unique partition element with $x \in P_x$. The Shannon entropy of $\mathcal{P}$ *relative* to $\mathcal{F}$ is

$$H_\nu(\mathcal{P}|\mathcal{F}) = \int I(\mathcal{P}|\mathcal{F})(x)\ d\nu(x).$$

If T is a measure-preserving transformation of $(X, \mathcal{B}_X, \nu)$ then the entropy rate of $(T, \mathcal{P})$ *relative* to $\mathcal{F}$ is

$$h_\nu(T, \mathcal{P}|\mathcal{F}) = \lim_{n\to\infty}(2n+1)^{-1} H_\nu\left(\bigvee_{i=-n}^{n} T^i\mathcal{P}|\ \mathcal{F}\right).$$

This is well-defined whenever $\mathcal{F}$ is T-invariant. We also define the entropy rate of T *relative* to $\mathcal{F}$ by

$$h_\nu(T|\mathcal{F}) = \sup_{\mathcal{P}} h_\nu(T, \mathcal{P}|\mathcal{F})$$

where the supremum is over all finite-entropy partitions $\mathcal{P}$ of X.

Now suppose $\Gamma = \langle s_1, \ldots, s_r\rangle$ and α is a measure-preserving action of Γ on $(X, \mathcal{B}_X, \nu)$. Define

$$\begin{aligned} F_\nu(\alpha, \mathcal{P}|\mathcal{F}) &= (1-2r)H_\nu(\mathcal{P}|\mathcal{F}) + \sum_{i=1}^{r} H_\nu(\mathcal{P} \vee \alpha_{s_i}\mathcal{P}|\mathcal{F}) \\ f_\nu(\alpha, \mathcal{P}|\mathcal{F}) &= \inf_{W\subset\Gamma} F_\nu\left(\alpha, \mathcal{P}^W|\mathcal{F}\right) \end{aligned}$$

where the infimum is over all finite $W \subset \Gamma$. Also define

$$\begin{aligned} F^*_\nu(\alpha, \mathcal{P}|\mathcal{F}) &= (1-r)H_\nu(\mathcal{P}|\mathcal{F}) + \sum_{i=1}^{r} h_\nu(\alpha_{s_i}, \mathcal{P}|\mathcal{F}) \\ f^*_\nu(\alpha, \mathcal{P}|\mathcal{F}) &= \inf_{W\subset\Gamma} F^*_\nu\left(\alpha, \mathcal{P}^W|\mathcal{F}\right) \end{aligned}$$

where the infimum is over all finite $W \subset \Gamma$.

Theorem 4.1. *Let α be a measure-preserving action of Γ on a standard probability space $(X, \mathcal{B}_X, \nu)$. If $\mathcal{P}, \mathcal{Q}$ are any two finite-entropy generating partitions for α and $\mathcal{F} \subset \mathcal{B}_X$ is an $\alpha(\Gamma)$-invariant sub-sigma-algebra then $f_\nu(\alpha, \mathcal{P}|\mathcal{F}) = f_\nu(\alpha, \mathcal{Q}|\mathcal{F})$.*

Proof. Define

$$f'_\nu(\alpha, \mathcal{P}|\mathcal{F}) = \inf_{n>0} F_\nu(\alpha, \mathcal{P}^{B(n)}|\mathcal{F})$$

where $B(n) \subset \Gamma$ denotes the ball of radius n with respect to the word metric induced by $\{s_1^{\pm 1}, \ldots, s_r^{\pm 1}\}$. This is the definition of the relative f-invariant given in [Bo10c]. Clearly, $f_\nu(\alpha, \mathcal{P}|\mathcal{F}) \le f'_\nu(\alpha, \mathcal{P}|\mathcal{F})$. However, if $W \subset \Gamma$ is any finite set with $e \in W$ then it follows from [Bo10c, Propositions 4.3 and 5.1] that $F_\nu(\alpha, \mathcal{P}^{B(n)}|\mathcal{F}) \le F_\nu(\alpha, \mathcal{P}^W|\mathcal{F})$ for all sufficiently large n. Thus $f'_\nu(\alpha, \mathcal{P}|\mathcal{F}) \le f_\nu(\alpha, \mathcal{P}|\mathcal{F})$ which implies $f'_\nu(\alpha, \mathcal{P}|\mathcal{F}) = f_\nu(\alpha, \mathcal{P}|\mathcal{F})$. The result now follows from [Bo10c, Theorem 5.3]. □

Because of this theorem, we define $f_\nu(\alpha|\mathcal{F}) := f_\nu(\alpha, \mathcal{P}|\mathcal{F})$ where $\mathcal{P}$ is any finite-entropy generating partition for α. If there does not exist a finite-entropy generating partition for α then $f_\nu(\alpha|\mathcal{F})$ is undefined. One of the main results of [Bo10c] is:

Theorem 4.2. *Let α be a measure-preserving action of Γ on a standard probability space $(X, \mathcal{B}_X, \nu)$. Let $\mathcal{P}, \mathcal{Q}$ be finite-entropy partitions of X. Let $\Sigma(\mathcal{Q})$, $\Sigma(\mathcal{P})$ be the smallest Γ-invariant sub-sigma-algebras containing $\mathcal{Q}$ and $\mathcal{P}$ respectively. Assume $\mathcal{Q} \subset \mathcal{P}$. Then*

$$f_\nu(\alpha, \mathcal{P}|\Sigma(\mathcal{Q})) = f^*_\nu(\alpha, \mathcal{P}|\Sigma(\mathcal{Q})).$$

Proof. This is [Bo10c, Theorem 9.1]. The proof requires a small correction; see §A. □

Theorem 4.3. *[The f-invariant Abramov-Rokhlin Addition Formula] Let α be a measure-preserving action of Γ on a standard probability space $(X, \mathcal{B}_X, \nu)$. Let $\mathcal{P}, \mathcal{Q}$ be finite-entropy partitions of X. Let $\Sigma(\mathcal{Q})$ be the smallest Γ-invariant sub-sigma-algebra containing $\mathcal{Q}$. Then*

$$f_\nu(\alpha, \mathcal{P} \vee \mathcal{Q}) = f_\nu(\alpha, \mathcal{Q}) + f_\nu(\alpha, \mathcal{P}|\Sigma(\mathcal{Q})).$$

Proof. This is [Bo10c, Theorem 1.3]. The proof requires a small correction; see §A. □

5. A key Lemma

The purpose of this section is to prove the key lemma below for skew-products of $\mathbb{Z}$-actions. So let $(X, \mathcal{B}_X, \nu)$ be a Lebesgue space, $T \in \mathrm{Aut}(X, \mathcal{B}_X, \nu)$, G be a compact metrizable group, equipped with Haar measure μ and S be a group-automorphism of G. A cocycle for T and S is a cocycle for the actions of $\mathbb{Z}$ induced by T and S. That is, it is a measurable map $\sigma : \mathbb{Z} \times X \to \mathrm{G}$ such that

$$\sigma(n+m, x) = (S^n \sigma(m, x)) \cdot \sigma(n, T^m x). \tag{5.1}$$

Let $T \times_\sigma S$ be the automorphism of $(X \times G, \nu \times \mu)$ defined by

$$T \times_\sigma S(x, g) = (Tx, S(g)\sigma(x)).$$

This is the *skew product* of T and S with respect to σ.

Lemma 5.1. *Let $(X, \mathcal{B}_X, \nu), \mathrm{G}, T, S, \sigma$ be as above. Let $\mathcal{Q}$ be a finite-entropy partition of* G. *Let*

$$K(\mathcal{Q}) = \sup_{g \in \mathrm{G}} H(\mathcal{Q}g|\mathcal{Q}) + H(\mathcal{Q}|\mathcal{Q}g).$$

Then

$$\Big| h_{\nu\times\mu}\big(T \times_\sigma S, X \times \mathcal{Q}|\mathcal{B}_X\big) - h_\mu(S, \mathcal{Q}) \Big| \leq K(\mathcal{Q}).$$

Proof. By the definition of conditional entropy :

$$h_{\nu\times\mu}(T \times_\sigma S, X \times \mathcal{Q}|\mathcal{B}_X) = \lim_{m\to\infty} \frac{1}{m} h_{\nu\times\mu}\left(\bigvee_{k=0}^{m-1} (T \times_\sigma S)^{-k} X \times \mathcal{Q}|\mathcal{B}_X \right)$$

where

$$h_{\nu\times\mu}\left(\bigvee_{k=0}^{m-1} (T \times_\sigma S)^{-k} X \times \mathcal{Q}|\mathcal{B}_X \right) = \int I\left(\bigvee_{k=0}^{m-1} (T \times_\sigma S)^{-k} X \times \mathcal{Q}|\mathcal{B}_X \right)(x,y)d\nu(x)d\mu(y)$$

and the *conditional information* is given by:

$$I\left(\bigvee_{k=0}^{m-1} (T \times_\sigma S)^{-k} X \times \mathcal{Q}|\mathcal{B}_X \right)(x,y) = -\mathbb{E}[P_{x,y}|\mathcal{B}_X](x,y)\log(\mathbb{E}[P_{x,y}|\mathcal{B}_X](x,y))$$

where $P_{x,y} \in \bigvee_{k=0}^{m-1}(T \times_\sigma S)^{-k} X \times \mathcal{Q}$ is the partition element containing (x,y). Observe that the conditional expectation $\mathbb{E}[\cdot|\mathcal{B}_X](x,y)$ is the probability measure $\mu_x := \delta_x \times \mu$ (where δ_x is the Dirac measure concentrated on $\{x\}$). Thus

$$\int_{\mathrm{G}} I\left(\bigvee_{k=0}^{m-1} (T \times_\sigma S)^{-k} X \times \mathcal{Q}|\mathcal{B}_X \right)(x,y)\, d\mu(y) = H_{\mu_x}\left(\bigvee_{k=0}^{m-1} (T \times_\sigma S)^{-k} X \times \mathcal{Q} \right).$$

We claim that for any set $P \subset \mathrm{G}$,

$$\{x\} \times \mathrm{G} \cap (T \times_\sigma S)^{-k}(X \times P) = \{x\} \times S^{-k}(P\sigma(k,x)^{-1}).$$

Indeed, (x,y) is contained in $(T \times_\sigma S)^{-k}(X \times P)$ if and only if

$$(T \times_\sigma S)^k(x,y) = (T^k x, (S^k y)\sigma(k,x)) \in X \times P$$

which occurs if and only if

$$y \in S^{-k}(P\sigma(k,x)^{-1}).$$

So if

$$\mathcal{Q}_x^m = \bigvee_{k=0}^{m-1} S^{-k}(\mathcal{Q}\sigma(k,x)^{-1}).$$

then

$$H_{\mu_x}\left(\bigvee_{k=0}^{m-1}(T\times_\sigma S)^{-k}X\times\mathcal{Q}\right) \;=\; H_\mu(\mathcal{Q}_x^m).$$

So $I\left(\bigvee_{k=0}^{m-1}(T\times_\sigma S)^{-k}X\times\mathcal{Q}|\mathcal{B}_X\right)(x,y) = H_\mu(\mathcal{Q}_x^m)$ which implies:

$$h_{\nu\times\mu}((T\times_\sigma S), X\times\mathcal{Q}|\mathcal{B}_X) = \lim_{m\to\infty}\frac{1}{m}\int_X H_\mu(\mathcal{Q}_x^m)d\nu(x) \tag{5.2}$$

Define:

$$\mathcal{Q}^m = \bigvee_{k=0}^{m-1} S^{-k}\mathcal{Q}$$

By the definition of entropy:

$$h_\mu(S,\mathcal{Q}) = \lim_{m\to\infty}\frac{1}{m}\int_X H_\mu(\mathcal{Q}^m)d\nu(x) \tag{5.3}$$

Note $|H_\mu(\mathcal{Q}^m) - H_\mu(\mathcal{Q}_x^m)| \le H_\mu(\mathcal{Q}^m|\mathcal{Q}_x^m) + H_\mu(\mathcal{Q}_x^m|\mathcal{Q}^m)$. Thus:

$$\begin{aligned}|H_\mu(\mathcal{Q}^m) - H_\mu(\mathcal{Q}_x^m)| &\le \sum_{k=0}^{m-1} H_\mu(S^{-k}\mathcal{Q}|S^{-k}(\mathcal{Q}\sigma(k,x)^{-1})) + H_\mu(S^{-k}(\mathcal{Q}\sigma(k,x)^{-1})|S^{-k}\mathcal{Q})\\ &= \sum_{k=0}^{m-1} H_\mu(\mathcal{Q}|\mathcal{Q}\sigma(k,x)^{-1}) + H_\mu(\mathcal{Q}\sigma(k,x)^{-1})|\mathcal{Q}) \le mK(\mathcal{Q}).\end{aligned}$$

Finally (5.2) and (5.3) imply $|h_{\nu\times\mu}((T\times_\sigma S), X\times\mathcal{Q}|\mathcal{B}_X) - h_\mu(S,\mathcal{Q})| \le K(\mathcal{Q})$. □

6. Proof of Theorem 3.3

For the rest of this section, let $\Gamma, (X,\mathcal{B}_X,\nu), (\mathrm{G},\mathcal{B}_G,\mu), \alpha, \beta, \sigma$ be as Theorem 3.3. A *special partition* of G is a partition $\mathcal{Q}$ such that there exists a finite-index normal clopen subgroup $N < \mathrm{G}$ such that $\mathcal{Q} = \{gN :\ g \in \mathrm{G}\}$.

Lemma 6.1. *If $\mathcal{Q}$ is special and $T_1,\ldots,T_n$ are automorphisms of* G *then $\bigvee_{i=1}^n T_i\mathcal{Q}$ is also special.*

Proof. Let $\mathcal{Q} = \{gN :\ g\in\mathrm{G}\}$ where N is a finite-index normal clopen subgroup. Because each T_i is an automorphism, $M := \cap_{i=1}^n T_iN$ is a finite-index normal clopen subgroup. So $\mathcal{Q}_M := \{gM :\ g \in G\}$ is special. Because each $T_i\mathcal{Q}$ coarsens $\mathcal{Q}_M$, it follows that $\bigvee_{i=1}^m T_i\mathcal{Q} \geq \mathcal{Q}_M$.

On the other hand, $M \in \bigvee_{i=1}^m T_i\mathcal{Q}$. Because each $T_i\mathcal{Q}$ is G-invariant (i.e., $gT_i\mathcal{Q} = T_i\mathcal{Q}$ for every $g \in G$), $\bigvee_{i=1}^m T_i\mathcal{Q}$ is G-invariant. Hence $gM \in \bigvee_{i=1}^m T_i\mathcal{Q}$ for every $g \in G$. So $\bigvee_{i=1}^m T_i\mathcal{Q} \leq \mathcal{Q}_M$. Thus $\bigvee_{i=1}^m T_i\mathcal{Q} = \mathcal{Q}_M$ is special. □

Lemma 6.2. *If $\mathcal{P}$ is any finite-entropy partition of X and $\mathcal{Q}$ is a special partition of G then*

$$F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, \mathcal{P} \times \mathcal{Q}|\mathcal{B}_X) = F^*_\mu(\beta, \mathcal{Q}).$$

Proof. Because $\mathcal{Q}g = \mathcal{Q}$ for any $g \in G$, it follows that $K(\mathcal{Q}) = 0$ where $K(\cdot)$ is as defined in Lemma 5.1. So that Lemma implies

$$\begin{aligned} F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, \mathcal{P} \times \mathcal{Q}|\mathcal{B}_X) &= (1-r)H_{\nu\times\mu}(\mathcal{P} \times \mathcal{Q}|\mathcal{B}_X) + \sum_{i=1}^r h_{\nu\times\mu}((\alpha \times_\sigma \beta)_{s_i}, \mathcal{P} \times \mathcal{Q}|\mathcal{B}_X) \\ &= (1-r)H_\mu(\mathcal{Q}) + \sum_{i=1}^r h_\mu(\beta_{s_i}, \mathcal{Q}) = F^*_\mu(\beta, \mathcal{Q}). \end{aligned}$$

□

Lemma 6.3. *Let $\mathcal{Q}$ be a special partition of G, $g \in \Gamma$ and let $\mathcal{P}_g$ denote the partition of X obtained by pulling $\beta_g(\mathcal{Q})$ back under the cocycle $\sigma(g, \cdot)$. Also, let $\mathcal{P}'$ be an arbitrary measurable partition of X. Then*

$$(\alpha \times_\sigma \beta)_g((\mathcal{P}_g \vee \mathcal{P}') \times \mathcal{Q}) = \alpha_g(\mathcal{P}_g \vee \mathcal{P}') \times \beta_g(\mathcal{Q})$$

(up to sets of measure zero).

Proof. Let N be the finite-index clopen normal subgroup of G such that $\mathcal{Q} = \{qN : q \in G\}$. Let $P \in \mathcal{P}_g$, $P' \in \mathcal{P}'$ and $qN \in \mathcal{Q}$. It suffices to show that there exists some $q'' \in G$ such that

$$(\alpha \times_\sigma \beta)_g((P \cap P') \times qN) = \alpha_g(P \cap P') \times q''\beta_g(N)$$

up to sets of measure zero. By definition of $\mathcal{P}_g$, there exists a coset $q'\beta_g(N) \in G/\beta_g(N)$ such that for every $y \in P$, $\sigma(g, y) \in q'\beta_g(N)$.

Let $x \in P \cap P'$ and $n \in N$. Then there exists some $m \in N$ such that

$$(\alpha \times_\sigma \beta)_g(x, qn) = (\alpha_g x, \beta_g(qn)\sigma(x, g)) = (\alpha_g x, \beta_g(qn)q'\beta_g(m)).$$

Because N is normal, $\beta_g(qn)q'\beta_g(m) \in \beta_g(q)q'\beta_g(N)$. Thus $(\alpha \times_\sigma \beta)_g(x, qn) \in \alpha_g(P \cap P') \times \beta_g(q)q'\beta_g(N)$. Since x, n are arbitrary, this implies $(\alpha \times_\sigma \beta)_g((P \cap P') \times qN) \subset \alpha_g(P \cap P') \times q''\beta_g(N)$ where $q'' = \beta_g(q)q'$. Because

$$\nu\times\mu((P\cap P')\times qN) = \nu\times\mu((\alpha\times_\sigma\beta)_g((P\cap P')\times qN)) = \nu\times\mu(\alpha_g(P\cap P')\times q''\beta_g(N))$$

it follows that $(\alpha \times_\sigma \beta)_g((P \cap P') \times qN) = \alpha_g(P \cap P') \times q''\beta_g(N)$ up to sets of measure zero. Because P, P', qN are arbitrary, $(\alpha \times_\sigma \beta)_g((\mathcal{P}_g \vee \mathcal{P}') \times \mathcal{Q}) = (\alpha_g(\mathcal{P}_g \vee \mathcal{P}')) \times (\beta_g \mathcal{Q})$ as claimed. □

Lemma 6.4. *Let $\mathcal{P}, \mathcal{Q}$ be measurable partitions for α, β respectively. Suppose $\mathcal{Q}$ is special and $\mathcal{P}$ is generating. Let $\Sigma(\mathcal{P}, \mathcal{Q})$ be the smallest $\alpha \times_\sigma \beta(\Gamma)$-invariant sigma-algebra containing $\mathcal{P} \times \mathcal{Q}$. Similarly, let $\Sigma(\mathcal{Q})$ be the smallest β-invariant sigma-subalgebra of $\mathcal{B}_G$ which contains $\mathcal{Q}$.*

Then $\Sigma(\mathcal{P}, \mathcal{Q})$ is the smallest sigma-algebra containing $\mathcal{B}_X \times \Sigma(\mathcal{Q})$ (up to sets of measure zero).

Proof. Clearly, $\mathcal{P} \times G$ is contained in $\Sigma(\mathcal{P}, \mathcal{Q})$. Because

$$(\alpha \times_\sigma \beta)_g(\mathcal{P} \times \mathrm{G}) = (\alpha_g \mathcal{P}) \times G, \quad \forall g \in \Gamma,$$

it follows that $(\alpha_g \mathcal{P}) \times G \subset \Sigma(\mathcal{P}, \mathcal{Q})$ for every $g \in \Gamma$. Because $\mathcal{P}$ is generating, this implies $\mathcal{B}_X \times G \subset \Sigma(\mathcal{P}, \mathcal{Q})$ (up to sets of measure zero).

For each $g \in \Gamma$, recall that $\mathcal{P}_g$ is the partition of X obtained by pulling $\beta_g(\mathcal{Q})$ back under the cocycle $\sigma(g, \cdot)$. Because $\sigma(g, \cdot)$ is $\mathcal{B}_X$-measurable, $\mathcal{P}_g \times \mathcal{Q}$ is contained in $\Sigma(\mathcal{P}, \mathcal{Q})$. By Lemma 6.3,

$$(\alpha \times_\sigma \beta)_g(\mathcal{P}_g \times \mathcal{Q}) = (\alpha_g \mathcal{P}_g) \times (\beta_g \mathcal{Q}) \subset \Sigma(\mathcal{P}, \mathcal{Q})$$

(up to sets of measure zero). Because $X \times \beta_g \mathcal{Q}$ coarsens $(\alpha_g \mathcal{P}_g) \times (\beta_g \mathcal{Q})$, it follows that $X \times \beta_g \mathcal{Q} \subset \Sigma(\mathcal{P}, \mathcal{Q})$ for every $g \in \Gamma$. By definition of $\Sigma(\mathcal{Q})$, this implies $X \times \Sigma(\mathcal{Q}) \subset \Sigma(\mathcal{P}, \mathcal{Q})$. Because $X \times \Sigma(\mathcal{Q})$ and $\mathcal{B}_X \times G$ generate $\mathcal{B}_X \times \Sigma(\mathcal{Q})$ (up to sets of measure zero), this implies $\Sigma(\mathcal{P}, \mathcal{Q}) \supset \mathcal{B}_X \times \Sigma(\mathcal{Q})$.

To show the opposite inclusion, it suffices to show that $(\alpha \times_\sigma \beta_g)(\mathcal{P} \times \mathcal{Q}) \in \mathcal{B}_X \times \Sigma(\mathcal{Q})$ for any $g \in \Gamma$. By the previous lemma,

$$(\alpha \times_\sigma \beta_g)(\mathcal{P} \times \mathcal{Q}) \le (\alpha \times_\sigma \beta)_g((\mathcal{P}_g \vee \mathcal{P}) \times \mathcal{Q}) = (\alpha_g(\mathcal{P}_g \vee \mathcal{P})) \times (\beta_g \mathcal{Q}) \in \mathcal{B}_X \times \Sigma(\mathcal{Q}).$$

This shows the opposite inclusion.

[Proof of Theorem 3.3] By Theorem 4.3

$$f_{\nu \times \mu}(\alpha \times_\sigma \beta) = f_\nu(\alpha) + f_{\nu \times \mu}(\alpha \times_\sigma \beta | \mathcal{B}_X).$$

Let $\mathcal{P}$ be a finite-entropy generating partition for α and $\mathcal{Q}$ be a special generating partition for β. By the previous lemma, $\mathcal{P} \times \mathcal{Q}$ is generating for $\alpha \times_\sigma \beta$. So Theorem 4.2 implies

$$f_{\nu \times \mu}(\alpha \times_\sigma \beta | \mathcal{B}_X) = \inf_{W \subset \Gamma} F^*_{\nu \times \mu}(\alpha \times_\sigma \beta, (\mathcal{P} \times \mathcal{Q})^W | \mathcal{B}_X)$$

where

$$(\mathcal{P} \times \mathcal{Q})^W = \bigvee_{w \in W} (\alpha \times_\sigma \beta)_w \mathcal{P} \times \mathcal{Q}$$

and we take the infimum over all finite sets $W \subset \Gamma$. More generally, if $\mathcal{L}$ is any partition of $X \times G$, we let $\mathcal{L}^W = \bigvee_{w\in W}(\alpha \times_\sigma \beta)_w \mathcal{L}$. If $\mathcal{L}$ is a partition of X, we let $\mathcal{L}^W = \bigvee_{w\in W} \alpha_w \mathcal{L}$ and if $\mathcal{L}$ is a partition of G then we let $\mathcal{L}^W = \bigvee_{w\in W} \beta_w \mathcal{L}$.

For each $g \in \Gamma$, let $\mathcal{P}_g$ be the partition of X obtained by pulling $(\beta_g \mathcal{Q})$ back under $\sigma(g, \cdot)$. By Lemma 6.3, for any partition $\mathcal{P}'$ of X,

$$(\alpha \times_\sigma \beta)_g((\mathcal{P}_g \vee \mathcal{P}') \times \mathcal{Q}) = (\alpha_g(\mathcal{P}_g \vee \mathcal{P}')) \times (\beta_g \mathcal{Q}).$$

Let $\mathcal{R}_W = \bigvee_{g\in W} \mathcal{P}_g$ and $\mathcal{R}_W^W = \bigvee_{w\in W} \alpha_w \mathcal{R}_W$. By Lemma 6.3,

$$\begin{aligned} (\mathcal{P} \vee \mathcal{R}_W) \times \mathcal{Q})^W &= \bigvee_{w\in W} (\alpha \times_\sigma \beta)_w((\mathcal{P} \vee \mathcal{R}_W) \times \mathcal{Q}) \\ &= \bigvee_{w\in W} \alpha_w(\mathcal{P} \vee \mathcal{R}_W) \times \beta_w(\mathcal{Q}) \\ &= (\mathcal{P} \vee \mathcal{R}_W)^W \times \mathcal{Q}^W. \end{aligned}$$

Because we are conditioning on $\mathcal{B}_X$ and $(\mathcal{R}_W \times G)^W = (\mathcal{R}_W^W \times G)$,

$$\begin{aligned} F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, (\mathcal{P} \times \mathcal{Q})^W | \mathcal{B}_X) &= F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, (\mathcal{P} \times \mathcal{Q})^W \vee \mathcal{R}_W^W \times G | \mathcal{B}_X) \\ &= F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, ((\mathcal{P} \vee \mathcal{R}_W) \times \mathcal{Q})^W | \mathcal{B}_X) \\ &= F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, ((\mathcal{P} \vee \mathcal{R}_W)^W \times \mathcal{Q}^W | \mathcal{B}_X). \end{aligned}$$

By Lemma 6.2,

$$F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, ((\mathcal{P} \vee \mathcal{R}_W)^W \times \mathcal{Q}^W | \mathcal{B}_X) = F^*_\mu(\beta, \mathcal{Q}^W).$$

So we now have

$$\begin{aligned} f_{\nu\times\mu}(\alpha \times_\sigma \beta) &= f_\nu(\alpha) + f_{\nu\times\mu}(\alpha \times_\sigma \beta | \mathcal{B}_X) \\ &= f_\nu(\alpha) + \inf_{W\subset\Gamma} F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, (\mathcal{P} \times \mathcal{Q})^W | \mathcal{B}_X) \\ &= f_\nu(\alpha) + \inf_{W\subset\Gamma} F^*_\mu(\beta, \mathcal{Q}^W) \\ &= f_\nu(\alpha) + f_\mu(\beta). \end{aligned}$$

The last equality holds by Theorem 2.2. □

7. Examples

It is convenient to introduce the following notation. Let $\Gamma = \langle s_1, \ldots, s_r \rangle$ be the rank r free group. If K is a set then K^Γ is the set of all functions $x : \Gamma \to K$. The *shift-action* of Γ on K^Γ is defined as follows. For $g, f \in \Gamma$ and $x \in K^\Gamma$, $gx \in K^\Gamma$ is the map $(gx)(f) = x(g^{-1}f)$.

If Γ acts on a compact group G and the action is understood, we write $f(\Gamma \curvearrowright G)$ to mean the f-invariant of the action of G with respect to Haar measure.

7.1. **The Ornstein-Weiss Example.** This example comes from the appendix to [OW87]. To explain its relevance, let us recall some basic facts from classical entropy theory. Let Δ be an amenable group, K a finite set and u the uniform probability measure on K. It is straightforward to compute the entropy of the shift action of Δ on (K^Δ, u^Δ): it is $\log |K|$. Because entropy never increases under a factor map, it follows that if $|K| > 1$ then the action $\Delta \curvearrowright (K^\Delta, u^\Delta)$ cannot factor onto the action $\Delta \curvearrowright ((K \times K)^\Delta, (u \times u)^\Delta)$.

By contrast, Ornstein and Weiss showed that if Γ is the rank 2 free group then $\Gamma \curvearrowright (\mathbb{Z}/2\mathbb{Z})^\Gamma$ factors onto $\Gamma \curvearrowright (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^\Gamma$. This convinced many researchers that there could not be an entropy theory for free groups.

The factor map is defined by

$$\phi : (\mathbb{Z}/2\mathbb{Z})^\Gamma \to (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^\Gamma,$$

$$\phi(x)(g) = (x(g) + x(gs_1), x(g) + x(gs_2)), \forall x \in (\mathbb{Z}/2\mathbb{Z})^\Gamma, g \in \Gamma.$$

We consider $(\mathbb{Z}/2\mathbb{Z})^\Gamma$ and $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^\Gamma$ as compact groups under pointwise addition. It is a straightforward exercise to show that ϕ is a surjective homomorphism which is equivariant with respect to the shift-actions of Γ and therefore, defines a factor map. Moreover, the kernel of ϕ consists of two elements, x_0, x_1, where $x_i : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ is defined by $x_i(g) = i$. Let $N = \{x_0, x_1\}$. Because N is finite, it clearly satisfies the conditions of Theorem 2.3. So that result implies

$$f(\Gamma \curvearrowright (\mathbb{Z}/2\mathbb{Z})^\Gamma) = f(\Gamma \curvearrowright N) + f(\Gamma \curvearrowright (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^\Gamma).$$

In [Bo10a], it is shown that $f(\Gamma \curvearrowright (\mathbb{Z}/2\mathbb{Z})^\Gamma) = \log(2)$ and $f(\Gamma \curvearrowright (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^\Gamma) = \log(4)$ as expected. Therefore, $f(\Gamma \curvearrowright N) = -\log(2)$. This is easy to verify by direct computation.

7.2. **A generalization.** The example above can be generalized with the help of [MRV11, proof of Theorem B] which states the following: if $\Gamma = \langle s_1, \ldots, s_r \rangle$ is any finite rank free group, K is any compact second countable group, K^Γ is the group of all functions $x : \Gamma \to K$ under pointwise multiplication and K is identified with the constant functions in K^Γ then the action $\Gamma \curvearrowright K^\Gamma/K$ is measurably conjugate to $\Gamma \curvearrowright (K^r)^\Gamma$ (where the measures involved are the Haar measures and the actions are the shift actions).

When K is finite, we can apply Theorem 2.3 to obtain

$$f(\Gamma \curvearrowright K^\Gamma) = f(\Gamma \curvearrowright K) + f(\Gamma \curvearrowright (K^r)^\Gamma).$$

This is easy to check: $f(\Gamma \curvearrowright K^\Gamma) = \log(|K|)$ and $f(\Gamma \curvearrowright (K^r)^\Gamma) = r\log(|K|)$ by [Bo10a]. By a straightforward computation, $f(\Gamma \curvearrowright K) = -(r-1)\log|K|$.

7.3. **An algebraic example.** As above, let $\Gamma = \langle s_1, \ldots, s_r \rangle$ be a finite rank free group. Let $p > 1$ be a prime number and $h \in (\mathbb{Z}/p\mathbb{Z})\Gamma$. We consider h as a function from Γ to $\mathbb{Z}$ such that $h(s) = 0$ for all but finitely many $s \in \Gamma$. Define the convolution operator $\phi_h : (\mathbb{Z}/p\mathbb{Z})^\Gamma \to (\mathbb{Z}/p\mathbb{Z})^\Gamma$ by

$$\phi_h(x)(g) = \sum_{s \in \Gamma} x(gs)h(s^{-1}), \quad \forall g \in \Gamma.$$

This is a Γ-equivariant homomorphism. Let $X_{h,p}$ denote the kernel of ϕ_h. Let $X^*_{h,p} < X_{h,p}$ be the subgroup consisting of all elements $x \in X_{h,p}$ with $x(e) = 0$. This is a finite-index normal clopen subgroup and $\{gX^*_{h,p} : \ g \in X_{h,p}\}$ is a generating partition for the shift-action of Γ. Therefore, we can apply Theorem 2.3 to obtain

$$f(\Gamma \curvearrowright (\mathbb{Z}/p\mathbb{Z})^\Gamma) = f(\Gamma \curvearrowright X_{h,p}) + f(\Gamma \curvearrowright \phi_h((\mathbb{Z}/p\mathbb{Z})^\Gamma)).$$

Theorem 7.1. *ϕ_h is onto if h is nonzero.*

Therefore,

$$f(\Gamma \curvearrowright \phi_h((\mathbb{Z}/p\mathbb{Z})^\Gamma)) = f(\Gamma \curvearrowright (\mathbb{Z}/p\mathbb{Z})^\Gamma).$$

Thus $f(\Gamma \curvearrowright X_{h,p}) = 0$.

To prove Theorem 7.1, we need a little preparation.

Definition 7.2. Let C_r be the Cayley graph of Γ. It has vertex set Γ and edges $\{g, gs_i\}$ for all $g \in \Gamma$ and $1 \le i \le r$. Given a set $F \subset \Gamma$, the *induced subgraph* of F is the subgraph $C_r(F) \subset C_r$ which has vertex set F and contains every edge of C_r which has both endpoints in F. A subset $F \subset \Gamma$ is said to be *connected* if its induced subgraph in C_r is connected. The *convex hull* of a set $F \subset \Gamma$ is the smallest connected set $F' \subset \Gamma$ with $F \subset F'$. An *extreme point* of F is an element $f \in F$ that has degree 1 in $C_r(F)$. We let $\mathrm{Ex}(F)$ denote the set of extreme points of F. Note that if F' is the convex hull of F then $\mathrm{Ex}(F') \subset F$.

Lemma 7.3. *Let $F = \{g \in \Gamma : \ h(g^{-1}) \neq p\mathbb{Z}\}$. Let $\overline{F}$ be the convex hull of F. Suppose there exists an ordering $\gamma_0, \gamma_1, \gamma_2, \ldots$ of Γ such that for every $n \ge 1$ $\{\gamma_0, \ldots, \gamma_n\}$ is connected and*

$$\gamma_n \overline{F} \nsubseteq \cup_{i=0}^{n-1} \gamma_i \overline{F}.$$

Then ϕ_h is onto.

Proof. By compactness of $(\mathbb{Z}/p\mathbb{Z})^\Gamma$ and continuity of ϕ_h, it suffices to show that for every $y \in (\mathbb{Z}/p\mathbb{Z})^\Gamma$ and every $n \ge 0$, there exists an $x \in (\mathbb{Z}/p\mathbb{Z})^\Gamma$ such that $\phi_h(x)(\gamma_i) = y(\gamma_i)$ for every $0 \le i \le n$. We will prove this statement by induction on n. It is clearly true for $n = 0$. So suppose there is an $n \ge 0$ for which the statement is true. Fix

$y \in (\mathbb{Z}/p\mathbb{Z})^\Gamma$ and let $x \in (\mathbb{Z}/p\mathbb{Z})^\Gamma$ be such that $\phi_h(x)(\gamma_i) = y(\gamma_i)$ for every $0 \le i \le n$.

By hypothesis, $\gamma_{n+1}\overline{F} \nsubseteq \cup_{i=0}^n \gamma_n \overline{F}$. Because $\cup_{i=0}^n \gamma_n \overline{F}$ and $\gamma_{n+1}\overline{F}$ are connected and the convex hull of the exreme points set of a connected set is the connected set itself , there must be an extremal point $f \in \mathrm{Ex}(\overline{F})$ such that $\gamma_{n+1} f \notin \cup_{i=0}^n \gamma_n \overline{F}$. However, $\mathrm{Ex}(\overline{F}) \subset F$. So $f \in F$. By definition, this means that $h(f^{-1}) \neq p\mathbb{Z}$. Because p is prime, we may therefore define an element $m \in \mathbb{Z}/p\mathbb{Z}$ by

$$m = h(f^{-1})^{-1} \left(y(\gamma_{n+1}) - \sum_{g \in \Gamma \setminus \{f\}} x(\gamma_{n+1} g) h(g^{-1}) \right).$$

Define $x' \in (\mathbb{Z}/p\mathbb{Z})^\Gamma$ by $x'(g) = x(g)$ if $g \neq \gamma_{n+1} f$ and $x'(\gamma_{n+1} f) = m$. Because $\gamma_{n+1} f \notin \cup_{i=0}^n \gamma_n \overline{F}$, it follows that $\phi_h(x')(\gamma_i) = \phi_h(x)(\gamma_i)$ for all $0 \le i \le n$. Also a straightforward computation shows $\phi_h(x')(\gamma_{n+1}) = y(\gamma_{n+1})$. So $\phi_h(x')(\gamma_i) = y(\gamma_i)$ for all $0 \le i \le n+1$. This completes the inductive step and the claim. □

Definition 7.4. Let $S = \{s_1, \ldots, s_r\}$. For $g \in \Gamma$, let $|g|$ be the smallest number $n \ge 0$ such that there exist elements $t_1, \ldots, t_n \in S \cup S^{-1}$ with $g = t_1 \cdots t_n$. We also let $d(g_1, g_2) = |g_1^{-1} g_2|$ for any $g_1, g_2 \in \Gamma$. For $g \in \Gamma$ and $n \ge 0$, let $B(g, n) = \{k \in \Gamma : \ d(k, g) \le n\}$ be the ball of radius n centered at g.

Let $K \subset \Gamma$ be a finite set. The *radius* of K is the smallest number $r \ge 0$ such that there exists a $v \in \Gamma$ such that $B(v, r) \supset K$. An element $v \in \Gamma$ is called a *center* of K if $B(v, r) \supset K$ where r is the radius of K. For any $v, w \in \Gamma$, we let $[v, w] \subset \Gamma$ be the set of all $g \in \Gamma$ such that the shortest path from v to w in the Cayley graph C_r contains g.

Lemma 7.5. *Let K be a connected finite set with radius $r \ge 1$. Suppose the identity element e is a center of K. Then there exist elements $v, w \in K$ such that $[e, v] \cap [e, w] = \{e\}$, $|v| = r$ and $|w| \in \{r-1, r\}$.*

Proof. Because K has radius r and center e, there is an element v with $|v| = r$. To obtain a contradiction, suppose that there is no $w \in K$ with $|w| \in \{r-1, r\}$ and $[e, v] \cap [e, w] = \{e\}$. Let $v_1 \in S \cup S^{-1}$ be the unique element with $|v_1^{-1} v| = r - 1$. We claim that $B(v_1, r-1) \supset K$. To see this, let $w \in K$. If $|w| \le r - 2$ then $w \in B(e, r-2) \subset B(v_1, r-1)$. If $|w| > r - 2$ then, because K has center e and radius r, $|w| \in \{r-1, r\}$. By assumption, this implies $[e, v] \cap [e, w] \neq \{e\}$. So let $y \in [e, v] \cap [e, w]$ with $y \neq e$. Then $[e, y] \subset [e, v]$. This implies that $v_1 \in [e, y]$. In particular, $v_1 \in [e, v] \cap [e, w]$, so $v_1 \in [e, w]$. Because $|w| \le r$, this implies $d(v_1, w) \le r - 1$ as claimed. So we have shown that in all cases, if $w \in K$ then $w \in B(v_1, r-1)$. This shows that the radius of K is at most $r-1$, a contradiction. This contradiction proves the lemma. $\square$

Lemma 7.6. *Let K be a connected finite set with radius $r \geq 1$. Suppose the identity element e is a center of K. Suppose $g_1, \ldots, g_n \in \Gamma \setminus \{e\}$ are elements with*

$$K \subset \cup_{i=1}^n g_n K.$$

Then e is contained in the convex hull of $\{g_1, \ldots, g_n\}$.

Proof. Let $v, w \in K$ be elements such that $[e, v] \cap [e, w] = \{e\}$, $|v| = r$ and $|w| \in \{r-1, r\}$. Let $g_i, g_j \in \{g_1, \ldots, g_n\}$ be such that $v \in g_i K$ and $w \in g_j K$. Let $x, y \in K$ be such that $v = g_i x$ and $w = g_j y$.

Let $v_1, v_2, x_1, x_2 \in \Gamma$ be such that $v = v_1 v_2, |v| = |v_1| + |v_2|$, $x_2 = v_2$, $x = x_1 x_2$, $|x| = |x_1| + |x_2|$ and $|v_2| = |x_2|$ is as large as possible. Thus $g_i = vx^{-1} = v_1 x_1^{-1}$ and $|vx^{-1}| = |v_1| + |x_1|$. Because r is the radius of K, e is a center and $x \in K$ we have $|x| \leq r$. Also, we cannot have $v = x$ (since this would imply $g_i = vx^{-1} = e$, a contradiction). So we must have $|v_1| \geq 1$. Thus $[e, v] \cap [e, g_i] \neq \{e\}$.

Let $w_1, w_2, y_1, y_2 \in \Gamma$ be such that $w = w_1 w_2, |w| = |w_1| + |w_2|$, $y_2 = w_2$, $y = y_1 y_2$, $|y| = |y_1| + |y_2|$ and $|w_2| = |y_2|$ is as large as possible. Thus $g_j = wy^{-1} = w_1 y_1^{-1}$ and $|wy^{-1}| = |w_1| + |y_1|$. Because r is the radius of K, e is a center and $y \in K$ we have $|y| \leq r$.

Case 1. If $|w| = r$, then, as in the previous paragraph, we must have $[e, w] \cap [e, g_j] \neq \{e\}$. Because $[e, v] \cap [e, w] = \{e\}$, this implies $e \in [g_i, g_j]$ which implies the lemma.

Case 2. Suppose $|w| = r-1$ and $|w_1| \geq 1$. Thus $[e, w] \cap [e, g_j] \neq \{e\}$. Because $[e, v] \cap [e, w] = \{e\}$, this implies $e \in [g_i, g_j]$ which implies the lemma.

Case 3. Suppose $|w| = r-1$ and $|w_1| = 0$. Then $w = w_2$, so $|w_2| = r - 1$. Because $g_j = wy^{-1} = w_1 y_1^{-1} = y_1^{-1} \neq e$, we must $y_1 \neq e$. Thus $|y| = |y_1| + |y_2| = |y_1| + |w_2| = |y_1| + r - 1$. Because $y \in K$ and K has radius r and center e, we must have $|y_1| = 1$ and $|y| = r$. If $[e, y] \cap [e, v] = \{e\}$ then, after replacing w with y we are in the situation of Case 1 (note $y = g_k y'$ for some $1 \leq k \leq n$ and $y' \in K$). So we may assume $[e, y] \cap [e, v] \neq \{e\}$ which implies $y_1 \in [e, v]$. Because $g_j = y_1^{-1}$, and $[e, v] \cap [e, g_i] \neq \{e\}$, we have $[e, g_i] \cap [e, g_j] = \{e\}$ which implies $e \in [g_i, g_j]$ which implies the lemma.

[Proof of Theorem 7.1] Let $F = \{g \in \Gamma : \ h(g^{-1}) \neq p\mathbb{Z}\}$. Let $\overline{F}$ be the convex hull of F. For any $g \in \Gamma$, ϕ_h is onto if and only if ϕ_{gh} is onto. So after replacing h with gh for some $g \in \Gamma$, we may assume that e is a center of $\overline{F}$.

Let $g_0, g_1, \dots$ be an ordering of Γ such that for every $n \ge 0$, $\{g_0, \dots, g_n\}$ is connected. We claim that for every $n \ge 1$,

$$\gamma_n \overline{F} \not\subseteq \cup_{i=0}^{n-1} \gamma_i \overline{F}.$$

To obtain a contradiction, suppose that the claim is false for some $n \ge 1$. Then $\overline{F} \subset \cup_{i=0}^{n-1} \gamma_n^{-1} \gamma_i \overline{F}$, $\gamma_n^{-1}\gamma_i \ne e$ for any $0 \le i \le n-1$ and because $\{\gamma_0, \dots, \gamma_{n-1}\}$ is connected, $\{\gamma_n^{-1}\gamma_0, \dots, \gamma_n^{-1}\gamma_{n-1}\}$ is connected which implies that e is not in the convex hull of $\{\gamma_n^{-1}\gamma_0, \dots, \gamma_n^{-1}\gamma_{n-1}\}$. This contradicts the previous lemma. So we must have that for every $n \ge 1$,

$$\gamma_n \overline{F} \not\subseteq \cup_{i=0}^{n-1} \gamma_i \overline{F}.$$

The theorem now follows from Lemma 7.3. □

Appendix A. Erratum to [Bo10c]

[Bo10c, Lemma 9.3] is incorrect because the support of ν is not contained in the image of ϕ in general. However, the proof of [Bo10c, Lemma 9.3] remains correct when $\beta = \alpha^n$ (see justification below). This special case is the only case used to prove [Bo10c, Theorem 9.1] and the Abramov-Rokhlin Addition Formula [Bo10c, Theorem 1.3]. So those theorems hold as stated.

Proof. [Justification of a key step in the proof of Lemma 9.3] We now justify the claim that the proof of [Bo10c, Lemma 9.3] remains correct when $\beta = \alpha^n$. Recall that K is a finite set, $G = \langle s_1, \dots, s_r \rangle$ is a finitely generated free group and $n \ge 0$. Let $B(e,n) \subset G$ denote the ball of radius n centered at the identity element (with respect to the word metric). Let $L = K^{B(e,n)}$. Let $\phi : K^G \to L^G$ be the map

$$\phi(x)(g)(f) = x(gf), \quad x \in K^G, g \in G, f \in B(e,n).$$

Let μ be a shift-invariant probability measure on K^G and let ν be the Markov measure on L^G induced from $\phi_*\mu$. Then ν is supported on the set $Z \subset L^G$ of all $z : G \to L$ with the property that, for any $g \in G$ and $s \in S \cup S^{-1}$ (where $S = \{s_1, \dots, s_r\}$), there exists a $y \in K^G$ with $\phi(y)(g) = z(g)$ and $\phi(y)(gs) = z(gs)$. We claim that $Z \subset \phi(K^G)$ (which implies that the proof of [Bo10c, Lemma 9.3] remains correct when $\beta = \alpha^n$).

To prove the claim, define $\psi : Z \to K^G$ by $\psi(z)(g) = z(g)(e)$. It suffices to show that $\phi\psi$ is the identity map on Z. Because ϕ and ψ are G-equivariant, it suffices to prove that $\phi(\psi(z))(e) = z(e)$ for any $z \in Z$. Equivalently, it suffices to show that for every $f \in B(e,n)$, $\phi(\psi(z))(e)(f) = z(e)(f)$ which, by definition of ϕ, is equivalent to

$\psi(z)(f) = z(e)(f)$. By definition of ψ, this is equivalent to $z(f)(e) = z(e)(f)$.

So let $f \in B(e,n)$. Let us write $f = t_1 \cdots t_m$ where $t_i \in S \cup S^{-1}$ and m is the word length of f. Let $f_i = t_1 \cdots t_i$ for $1 \le i \le m$. Also let $f_0 = e$ the identity element. Let z_i be the map from $B(f_i, n)$ (the ball of radius n centered at f_i) to K defined by $z_i(f_i g) = z(f_i)(g)$ for $g \in B(e,n)$. By the definition of Z, we must have that z_i and z_{i+1} agree on $B(f_i,n) \cap B(f_{i+1},n)$ for $0 \le i \le m-1$. Therefore, z_0 and z_m agree on the set $\bigcap_{i=0}^m B(f_i,n)$. It is easy to see that $f \in \bigcap_{i=0}^m B(f_i,n)$. Therefore, $z_0(f) = z_m(f)$ which implies $z(e)(f) = z(f)(e)$ as claimed. □

The proof of [Bo10c, Proposition 12.1] relies on the incorrect [Bo10c, Lemma 9.3]. Moreover, the statement is incorrect even when $G = \mathbb{Z}$ because of the next result.

Theorem A.1. *There exists an ergodic automorphism $T \in \mathrm{Aut}(\mathrm{X},\mu)$ (where (X,μ) is a standard probability space), a finite generating partition α of X and an increasing sequence $\{\mathcal{P}_n\}_{n=1}^\infty$ of finite partitions such that $\bigvee_{n=1}^\infty \mathcal{P}_n$ is the partition into points and $f_\mu(\alpha) = h_\mu(T) \neq \liminf_{n\to\infty} H_\mu(\mathcal{P}_n|T^{-1}\mathcal{P}_n) = \liminf_{n\to\infty} F_\mu(\mathcal{P}_n)$.*

To prove this, we need the next few lemmas.

Lemma A.2. *Let $x > 0$. Then for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $x_1, \ldots, x_n > 0$ are such that $\sum_{i=1}^n x_i = x$ and $x_i < \delta$ $\forall i$ then*

$$\sum_{i=1}^n x_i^2 \le \epsilon \text{ and } \quad \sum_{i=2}^n x_i x_{i-1} \le \epsilon.$$

Proof. Let $\delta < \frac{\epsilon}{x}$. Notice $\sum_{i=1}^n x_i^2 \le \max_i x_i \sum_{i=1}^n x_i < \delta x < \frac{\epsilon}{x} x = \epsilon$. Similarly $\sum_{i=2}^n x_i x_{i-1} \le \max_i x_i \sum_{i=2}^n x_i < \epsilon$. □

Let $[n] = \{1, \ldots, n\}$.

Lemma A.3. *Let (X,μ) be a standard probability space and $B \subset X$ a set of positive measure. Let $\epsilon > 0$. For any measurable partition $\{B_1, \ldots, B_n\}$ of B and function $\phi : [n] \to [2]$, for $i,j \in [2]$, let*

$$C_{ij}(\phi) = \mu(\cup\{B_r : \ \phi(r-1) = j, \phi(r) = i\}).$$

Then there exists a $\delta > 0$ such that if $\{B_1, \ldots, B_n\}$ is any measurable partition of B with $\mu(B_i) \le \delta$ for every i and

$$Y_{ij} := \{\phi : [n] \to [2] : \ |C_{ij}(\phi) - (1/4)\mu(B)| < \epsilon\}$$

then $2^{-n}|Y_{ij}| > 1 - \epsilon$ (for every $i,j \in [2]$).

Proof. Fix $i, j \in [2]$. Let $\phi : [n] \to [2]$ be chosen uniformly at random. To prove the lemma, by Chebyshev's inequality, it suffices to show that, as $\delta \searrow 0$, the expected value of $C_{ij}(\phi)$ tends to $1/4$ and the variance of $C_{ij}(\phi)$ tends to 0. Let $Z_k = 1$ if $\phi(k) = i$ and $Z_k = 0$ otherwise.

Case 1. Let us assume $i = j$. Then

$$C_{ij}(\phi) = \sum_{k=2}^{n} Z_{k-1} Z_k \mu(B_k).$$

The expected value of $Z_{k-1}Z_k$ is $1/4$. So, the expected value of $C_{ij}(\phi)$ is $(1/4)(\mu(B) - \mu(B_1))$. This implies that, as $\delta \searrow 0$, the expected value of $C_{ij}(\phi)$ tends to $1/4$.

The variance of $C_{ij}(\phi)$ is

$$\mathrm{Var}(\mathrm{C_{ij}}(\phi)) = \sum_{\mathrm{k}=2}^{\mathrm{n}} \mu(\mathrm{B_k})^2 \mathrm{Var}(\mathrm{Z_{k-1}Z_k}) + 2\sum_{\mathrm{j<k}} \mu(\mathrm{B_j})\mu(\mathrm{B_k})\mathrm{Cov}(\mathrm{Z_{j-1}Z_j}, \mathrm{Z_{k-1}Z_k}).$$

Note that

$$\mathrm{Var}(\mathrm{Z_{k-1}Z_k}) = \mathbb{E}[\mathrm{Z_{k-1}^2 Z_k^2}] - [\mathbb{E}\mathrm{Z_{k-1}Z_k}]^2 = (1/4) - (1/16) = (3/16).$$

If $j < k - 1$ then $Z_{j-1}Z_j$ and $Z_{k-1}Z_k$ are indepedent which implies $\mathrm{Cov}(\mathrm{Z_{j-1}Z_j}, \mathrm{Z_{k-1}Z_k}) = 0$. On the other hand, if $j = k - 1$ then

$$\mathrm{Cov}(\mathrm{Z_{j-1}Z_j}, \mathrm{Z_{k-1}Z_k}) = \mathbb{E}[\mathrm{Z_{k-2}Z_{k-1}^2 Z_k}] - \mathbb{E}[\mathrm{Z_{k-2}Z_{k-1}}]\mathbb{E}[\mathrm{Z_{k-1}Z_k}] = (1/8) - (1/16) = (1/16).$$

Therefore,

$$\mathrm{Var}(\mathrm{C_{ij}}(\phi)) \leq (3/16)\sum_{\mathrm{k}=1}^{\mathrm{n}} \mu(\mathrm{B_i})^2 + (2/16)\sum_{\mathrm{k}=2}^{\mathrm{n}} \mu(\mathrm{B_{k-1}})\mu(\mathrm{B_k}).$$

By the previous lemma, $\mathrm{Var}(\mathrm{C_{ij}}(\phi))$ tends to zero as $\delta \searrow 0$. This finishes Case 1.

Case 2. Let us assume $i \neq j$. Then

$$C_{ij}(\phi) = \sum_{k=2}^{n} (1 - Z_{k-1}) Z_k \mu(B_k).$$

The expected value of $(1 - Z_{k-1})Z_k$ is $1/4$. So, the expected value of $C_{ij}(\phi)$ is $(1/4)(\mu(B) - \mu(B_1))$. This implies that, as $\delta \searrow 0$, the expected value of $C_{ij}(\phi)$ tends to $1/4$.

The variance of $C_{ij}(\phi)$ is

$$\mathrm{Var}(\mathrm{C_{ij}}(\phi)) = \sum_{\mathrm{k}=2}^{\mathrm{n}} \mu(\mathrm{B_k})^2 \mathrm{Var}((1-\mathrm{Z_{k-1}})\mathrm{Z_k}) + 2\sum_{\mathrm{j<k}} \mu(\mathrm{B_j})\mu(\mathrm{B_k})\mathrm{Cov}((1-\mathrm{Z_{j-1}})\mathrm{Z_j}, (1-\mathrm{Z_{k-1}})\mathrm{Z_k}).$$

Note that

$$\mathrm{Var}((1-\mathrm{Z_{k-1}})\mathrm{Z_k}) = \mathbb{E}[(1-\mathrm{Z_{k-1}})^2 \mathrm{Z_k^2}] - [\mathbb{E}(1-\mathrm{Z_{k-1}})\mathrm{Z_k}]^2 = (1/4) - (1/16) = (3/16).$$

If $j < k-1$ then $(1-Z_{j-1})Z_j$ and $(1-Z_{k-1})Z_k$ are independent which implies $\mathrm{Cov((1-Z_{j-1})Z_j, (1-Z_{k-1})Z_k)} = 0$. On the other hand, if $j = k-1$ then

$$\begin{aligned} & \mathrm{Cov((1-Z_{j-1})Z_j, (1-Z_{k-1})Z_k)} \\ = \ & \mathbb{E}[(1-Z_{k-2})Z_{k-1}(1-Z_{k-1})Z_k] - \mathbb{E}[(1-Z_{k-2})Z_{k-1}]\mathbb{E}[(1-Z_{k-1})Z_k] \\ = \ & 0 - (1/16). \end{aligned}$$

Therefore,

$$\mathrm{Var(C_{ij}(\phi))} \leq (3/16)\sum_{\mathrm{k=1}}^{\mathrm{n}} \mu(\mathrm{B_i})^2.$$

By the previous lemma, $\mathrm{Var(C_{ij}(\phi))}$ tends to zero as $\delta \searrow 0$. This finishes Case 2. □

Lemma A.4. *Let $T \in \mathrm{Aut(X,\mu)}$ be a free ergodic automorphism of a standard probability space. Let $\mathcal{P}$ be a finite measurable partition of X. Let $\epsilon > 0$. Then there exists a finite measurable partition $\mathcal{Q} \geq \mathcal{P}$ such that $H_\mu(\mathcal{Q}|T^{-1}\mathcal{Q}) \geq H_\mu(\mathcal{P}|T^{-1}\mathcal{P}) + \log(2) - \epsilon$.*

Proof. Let $\delta > 0$ and $N > 0$ be an integer. By the Rokhlin Lemma, there exists a measurable set $B \subset X$ and an $n \geq N$ such that $B, TB, \ldots, T^{n-1}B$ are pairwise disjoint and

$$\mu\left(\bigcup_{i=0}^{n-1} T^i B\right) > 1 - \delta.$$

Let $\phi : [n] \to [2]$ be chosen at random and for $i \in [2]$, let $C_i = \cup_{j\in\phi^{-1}(i)} B_j$, and $\mathcal{Q} = \mathcal{P} \vee \{C_1, C_2, X \setminus (C_1 \cup C_2)\}$. Let $\epsilon' > 0$. By the previous lemma, it follows that, by choosing δ small enough and N large enough, with high probability, for every $P, P' \in \mathcal{P}$ and $i, j \in [2]$,

$$\left|\mu(P \cap T^{-1}P' \cap C_i \cap T^{-1}C_j) - (1/4)\mu(P \cap T^{-1}P')\right| < \epsilon'.$$

By choosing ϵ' to be sufficiently small, we see that there exists such a ϕ so that $H_\mu(\mathcal{Q}|T^{-1}\mathcal{Q}) \geq H_\mu(\mathcal{P}|T^{-1}\mathcal{P}) + \log(2) - \epsilon$ as required. □

Proof of Theorem A.1. Let (X,μ) be a standard probability space, $T \in \mathrm{Aut(X,\mu)}$ a free and ergodic automorphism such that there exists a finite generating partition for T. Let $\{\mathcal{P}_n\}_{n=1}^\infty$ be a sequence of increasing finite partitions such that $\bigvee_{n=1}^\infty \mathcal{P}_n$ is the partition into points. Using the previous lemma and an inductive argument, we see that there exists a sequence $\{\mathcal{Q}_n\}_{n=1}^\infty$ of increasing finite partitions such that $\mathcal{P}_n \leq \mathcal{Q}_n$

and $H_\mu(\mathcal{Q}_n|T^{-1}\mathcal{Q}_n) \geq H_\mu(\mathcal{P}_n|T^{-1}\mathcal{P}_n)+\log(2)-\frac{1}{n}$. Therefore, $\bigvee_{n=1}^\infty \mathcal{Q}_n$ is the partition into points and

$$\liminf_{n\to\infty} H_\mu(\mathcal{Q}_n|T^{-1}\mathcal{Q}_n) \geq \log(2) + \liminf_{n\to\infty} H_\mu(\mathcal{P}_n|T^{-1}\mathcal{P}_n).$$

So either $\{\mathcal{P}_n\}_{n=1}^\infty$ or $\{\mathcal{Q}_n\}_{n=1}^\infty$ satisfies the theorem.

□

The proof of the addition theorem, [Bo10c, Theorem 13.1], relies on the incorrect [Bo10c, Proposition 12.1] (however, nothing else in [Bo10c] relies on this proposition). We conjecture that the statement of [Bo10c, Theorem 13.1] is correct. The proof also relies on [Bo10c, Theorem 13.2], a result which is assumed to follow from minor modifications of [Th71, Theorem 2.3]. It now appears that [Bo10c, Theorem 13.2] does not so follow and we do not know whether it remains true.

References

[Ar98] W. Arveson. *An Invitation to C*-algebras*. Graduate Texts in Mathematics, No. 39. Springer- Verlag, New York, reprinted 1998.

[BM09] M. Björklund and R. Miles. *Entropy range problems and actions of locally normal groups*. Discrete Contin. Dyn. Syst. 25 (2009), no. 3, 981–989.

[Bo10a] L. Bowen. *A measure-conjugacy invariant for actions of free groups. Ann. of Math. (2)* **171** (2010), no. 2, 1387–1400.

[Bo10b] L. Bowen. *Measure conjugacy invariants for actions of countable sofic groups*. J. Amer. Math. Soc. 23 (2010), 217–245.

[Bo10c] L. Bowen. *Nonabelian free group actions: Markov processes, the Abramov-Rohlin formula and Yuzvinskii's formula*. Ergodic Theory Dynam. Systems 30 (2010), no. 6, 1629–1663.

[Ju65] S. A. Juzvinskii. *Metric properties of the endomorphisms of compact groups*. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 29 1965 1295–1328.

[KH95] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems, with a supplement by Anatole Katok and Leonardo Mendoza*. Encyclopedia of Mathematics and its Applications 54, Cambridge University Press, 1995, 822 pp.

[Ku65] A. G. Kushnirenko, *An upper bound of the entropy of classical dynamical systems*. (Russian), Sov. Math. Dokl. 6 (1965), 360–362.

[Li11] H. Li. *Compact group automorphisms, addition formulas and Fuglede-Kadison determinants*. arXiv:1001.0419.

[LS09] D. Lind and K. Schmidt, preprint.

[LSW90] D. Lind, K. Schmidt and T. Ward. *Mahler measure and entropy for commuting automorphisms of compact groups*. Invent. Math. 101 (1990), no. 3, 593–629.

[MRV11] N. Meesschaert, S. Raum and S. Vaes. *Stable orbit equivalence of Bernoulli actions of free groups and isomorphism of some of their factor actions*. arXiv:1107.1357

[Mi08] R. Miles. *The entropy of algebraic actions of countable torsion-free abelian groups*. Fund. Math. 201 (2008), 261Ð282.

[OW87] D. Ornstein and B. Weiss. *Entropy and isomorphism theorems for actions of amenable groups.* J. Analyse Math. 48 (1987), 1–141.

[Th71] R. K. Thomas. *The addition theorem for the entropy of transformations of G-spaces.* Trans. Amer. Math. Soc. 160 (1971), 119–130.

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