On the Tutte-Krushkal-Renardy polynomial for cell complexes

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Avril 2012

IHES/M/12/07

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Abstract. Recently V. Krushkal and D. Renardy generalized the Tutte polynomial from graphs to cell complexes. We show that evaluating this polynomial at the origin gives the number of cellular spanning trees in the sense of A. Duval, C. Klivans, and J. Martin. Moreover, after a slight modification, the Tutte-Krushkal-Renardy polynomial evaluated at the origin gives a weighted count of cellular spanning trees, and therefore its free term can be calculated by the cellular matrix-tree theorem of Duval et al. In the case of cell decomposition of a sphere, this modified polynomial satisfies the same duality identity as before. We find that evaluating the Tutte-Krushkal-Renardy along a certain line is the Bott polynomial. Finally we prove skein relations for the Tutte-Krushkal-Renardy polynomial.

Introduction

We relate three invariants of cell complexes. The earliest one is the Bott polynomial introduced by Raoul Bott in 1952 [Bo1] and revived in [Bo2] . The second invariant we deal with is the number of cellular spanning trees from [DKM1, DKM2]. In these papers the classical matrix-tree theorem for graphs was generalized to higher arbitrary cell complexes. G. Kalai first noted [Ka] that in higher dimensions it makes sense to count spanning trees with weights equal to the square of the order of their codimension one homology groups. Exactly this weighted number of spanning trees is calculated as the determinant of an appropriate submatrix of the Laplacian in [DKM1, DKM2]. Our third invariant is a recent generalization of the Tutte polynomial from [KR], which we call *Tutte-Krushkal-Renardy polynomial*.

We introduce the Tutte-Krushkal-Renardy polynomial in Section 1 and show that its free term is the number of cellular spanning trees in Section 2. In Section 3, we modify the Tutte-Krushkal-Renardy polynomial so that its free term becomes the weighted number of cellular spanning trees. We prove the same duality identity for the modified polynomial as Krushkal and Renardy did for spheres in section 3.1. In Section 4, we show that the Bott polynomials can be obtained from the Tutte-Krushkal-Renardy polynomial by a substitution. We conclude the paper with the contraction-deletion relations for the Tutte-Krushkal-Renardy polynomial in Section 5.

This work has been done as a part of the Summer 2011 undergraduate research working group

`http://www.math.ohio-state.edu/~chmutov/wor-gr-su11/wor-gr.htm`

"Knots and Graphs" at the Ohio State University. We are grateful to all participants of the group for valuable discussions, to the OSU Honors Program Research Fund for the student financial support, and to the Summer Research Opportunities Program at OSU for supporting the first named author (C.B). The paper was finished during the third author (S.Ch) visit to the Institut des Hautes Études Scientifiques (IHES) in Bures-sur-Yvette, France. He thanks IHES for excellent working conditions and warm hospitality and the OSU MRI Stimulus Program for supporting his travel to France.

2010 *Mathematics Subject Classification.* 005C10, 05C31, 05C65, 57M15, 57Q15.

Key words and phrases. Cell complexes, Tutte polynomial, Krushkal-Renardy polynomial, cellular spanning trees, duality, Bott polynomial.

1. The Tutte-Krushkal-Renardy polynomial

Definition 1.1 ([KR])**.** Let K be a finite CW complex of dimension k. Define the j^{th} ($j \leqslant k$) Tutte-Krushkal-Renardy polynomial of K to be

$$T_K^j(X,Y) = \sum_{K_{(j-1)}\subseteq S\subseteq K_{(j)}} X^{\beta_{j-1}(S)-\beta_{j-1}(K)} Y^{\beta_j(S)} \ ,$$

where $K_{(j)}$ denotes the j-skeleton of K. Note that the sum is over all j-dimensional subcomplexes of K containing $K_{(j-1)}$. Such a complex is called a *spanning subcomplex*. Since every spanning complex contains $K_{(j-1)}$, it is useful to identify them with sets of j-cells. Thus there are 2^{f_j} summands, where f_j will denote the number of j-cells. We denote $\beta_j(S)$ the j^{th} Betti number of S, the rank of the homology group $H_j(S)$.

The first Tutte-Krushkal-Renardy polynomial essentially coincides with the Tutte polynomial of 1-skeleton $K_{(1)}$ considered as a graph: $T_K^1(X,Y) = T_{K_{(1)}}(X+1, Y+1)$.

Definition 1.2. Two cell structures K and K^* on an k-manifold M are dual to each other if there is a one to one correspondence between their open cells of complimentary dimensions such that the corresponding j-cell σ of K and $(k-j)$-cell σ^* of K^* intersect transversaly at a single point.

The cell structure K^* dual to a triangulation K of M can be constructed by setting σ^* to be the union of all simplices of the barycentric subdivision of K intersecting σ only on its barycenter. Another way to construct dual cell structure is to use a handle decomposition of M. This construction is treated in detail in [RS].

Theorem 1.3 (Duality Theorem for Spheres [KR])**.** *Let K and K^* be dual cell structures on S^k, then*

$$T_K^j(X,Y) = T_{K^*}^{k-j}(Y,X) \ .$$

When K is a (planar) graph embedded in a sphere S^2, the theorem becomes the celebrated duality theorem for the Tutte polynomial of a planar graph and its planar dual.

Remark 1.4. If K is a compact orientable connected manifold without boundary, then the top Tutte-Krushkal-Renardy polynomial is not interesting. It can be computed explicitly

$$T_K^k(X,Y) = Y + \frac{(1+X)^{f_k}-1}{X} \ , \qquad (1)$$

where C_k is the number of k-cells of K. Here Y appears as the term with $S=K$, while the subcomplexes $S\neq K$ can be collapsed to $(k-1)$-dimensional complexes and therefore Y does not occur for them. Their contribution in X can be calculated by induction of the the number of k-cells. If K is a cycle graph, then K is a cell decomposition of S^1. Thus the above gives the Tutte polynomial for cycle graphs.

2. Cellular Spanning Trees

Definition 2.1. [DKM1, DKM2] A j-dimensional *Cellular Spanning Tree* (hence j-CST) S of a k-dimensional CW complex K is any j-dimensional subcomplex with $S_{(j-1)} = K_{(j-1)}$ that satisfies the following conditions:

(1) $\widetilde{H}_j(S)=0$, (2) $\widetilde{\beta}_{j-1}(S)=0$, (3) $f_j(S) = f_j(K) - \widetilde{\beta}_j(K_{(j)}) + \widetilde{\beta}_{j-1}(K_{(j)})$[1] ,

where $\widetilde{\beta}_j$ and $\widetilde{\beta}_{j-1}$ are reduced Betti numbers.

The condition (2) implies that the homology group $\widetilde{H}_{j-1}(S)$ is finite; we will use the notation $|\widetilde{H}_{j-1}(S)|$ for its order.

[1] There is a typo in [DKM1, DKM2] where in the condition (3) the whole complex K is used instead of $K_{(j)}$.

For $j = 0$, these conditions mean that S consists of a single point, a vertex (0-cell) of K. For graphs, $j = 1$, there is a classic graph theoretical theorem stating that any two of these conditions imply the third one.

Theorem 2.2 (The Two Out of Three Theorem [DKM1])**.** *Let S be a j-dimensional subcomplex with $S_{(j-1)} = K_{(j-1)}$, then any two of the conditions (1), (2), and (3) together imply the third one.*

For graphs, a spanning tree exists if and only if the graph is connected. We likewise need a condition to consider the existence of an CST.

Definition 2.3 ([DKM1])**.** A CW complex K of dimension k is called *acyclic in a positive codimension* (APC) if $\widetilde{\beta}_j = 0$ for all $j < k$.

For example a complex homotopy equivalent to a wedge of several homological spheres of the same dimension is APC. In particular, a connected graph is homotopy equivalent to a wedge of several circles and so is APC.

Theorem 2.4 ([DKM1])**.** *K is APC if and only if K has a j-CST for all $j \leqslant k$.*

Henceforth let $\tau_j(K)$ denote the set of j-CST's of K. For K being APC we are guaranteed that $|\tau_j(K)| \neq 0$, and we can introduce an invariant inspired by G. Kalai [Ka]: the number of j-CST's, S, counted with the weights $|\widetilde{H}_{j-1}(S)|^2$.

Definition 2.5 ([DKM2])**.**

$$\widetilde{\tau}_j(K) := \sum_{S \in \tau_j(K)} |\widetilde{H}_{j-1}(S)|^2.$$

For a connected graph K ($k = 1$), the invariant $\widetilde{\tau}_1(K)$ is equal to the number of its spanning trees because the group $\widetilde{H}_0(S)$ is always trivial for connected complexes. The classical matrix tree theorem states that for a graph the number of its spanning trees is equal to a cofactor of the Laplacian associated with a graph. This theorem was generalized to higher dimension in [DKM1, DKM2]. Thus $\widetilde{\tau}_j(K)$ can be calculated as a determinant of an appropriate matrix.

There is a different generalization of a notion of spanning tree to higher dimension suitable for so called "Pfaffian Matrix Tree Theorem" [MV1, MV2]. Their spanning trees are CST's in our sense, but the opposite, in general, is not true.

2.1. **Free term of the Tutte-Krushkal-Renardy polynomial.** For a graph G, a classical evaluation of the Tutte polynomial $T_G(1, 1)$ gives the number of spanning trees in the graph. Krushkal and Renardy define their polynomial as a generalization of the Tutte polynomial. Analogously for a higher dimensional cell complexes we have

Theorem 2.6. *For APC CW complex K and $j \geqslant 1$, $T^j_K(0, 0) = |\tau_j(K)|$.*

Proof. Note that the exponent of X in the Tutte-Krushkal-Renardy polynomial is equal to $\beta_{j-1}(S) - \beta_{j-1}(K) = \widetilde{\beta}_{j-1}(S) - \widetilde{\beta}_{j-1}(K)$. Since K is APC, $\widetilde{\beta}_{j-1}(K) = 0$. Then

$$T^j_K(X, Y) = \sum_{K_{(j-1)} \subseteq S \subseteq K_{(j)}} X^{\widetilde{\beta}_{j-1}(S)} Y^{\beta_j(S)}.$$

Now the evaluation $T^j_K(0, 0)$ is equal to the number of subcomplexes $K_{(j-1)} \subseteq S \subseteq K_{(j)}$ such that $\widetilde{\beta}_{j-1}(S) = 0$ and $\beta_j(S) = 0$. Note the S has dimension less than or equal to j, therefore its highest homology group $H_j(S)$ is a free abelian group of rank $\beta_j(S) = 0$. Thus it is trivial, and $\widetilde{H}_j(S) = 0$. Thus conditions (1) and (2) of Definition 2.1 are satisfied. By the Two Out of Three Theorem 2.2, S must be a cellular spanning tree. In the other direction, any j-CST S has $\widetilde{\beta}_{j-1}(S) = 0$ and $\widetilde{H}_j(S) = 0$. Since $j \geqslant 1$, the reduced homology group $\widetilde{H}_j(S)$ is isomorphic to the unreduced group $H_j(S)$, and therefore $\beta_j(S) = 0$. Thus S contributes 1 to the evaluation $T^j_K(0, 0)$. □

3. The Modified Tutte-Krushkal-Rendardy Polynomial

For an abelean group G, let $\mathrm{tor}(G)$ denote the torsion subgroup of G.

Definition 3.1. Let K be a CW complex of dimension $k \geqslant j$. Define the j^{th} *modified Tutte-Krushkal-Renardy polynomial* of K to be

$$\widetilde{T}^j_K(X,Y) = \sum_{K_{(j-1)} \subseteq S \subseteq K_{(j)}} |\mathrm{tor}(H_{j-1}(S))|^2 X^{\beta_{j-1}(S)-\beta_{j-1}(K)} Y^{\beta_j(S)} .$$

Theorem 3.2. *If K is APC and $j \geqslant 1$, then*

$$\widetilde{T}^j_K(0,0) = \widetilde{\tau}_j(K)$$

Proof. As in the proof of Theorem 2.6 only cellular spanning trees contribute to $\widetilde{T}^j_K(0,0)$. Only now the contribution of a CST S is equal to $|\mathrm{tor}(H_{j-1}(S))|^2 = |H_{j-1}(S)|^2$ since this group is finite. □

3.1. **Duality Theorem for the modified Tutte-Krushkal-Renardy polynomial.**

Theorem 3.3 (The Duality Theorem for $\widetilde{T}^j_K(X,Y)$)**.** *If K and K^* are dual cell decompositions of S^k and if $1 \leqslant j \leqslant k-1$, then*

$$\widetilde{T}^j_K(X,Y) = \widetilde{T}^{k-j}_{K^*}(Y,X).$$

In the proof of this theorem we use a technical lemma from Krushkal and Renardy.

Lemma ([KR]). *If K is a cell decomposition of S^k and $S \subseteq K$ is a subcomplex of K, then S is homotopy equivalent to $S^k \setminus S^*$, where S^* is a subcomplex of the dual cell decomposition K^* forming by cells which do not intersect S.*

Proof. We invoke the universal coefficient theorem, the Alexander duality, and the above Lemma to get the following isomorphisms.

$$\widetilde{H}_j(S)/\mathrm{tor}(\widetilde{H}_j(S)) \oplus \mathrm{tor}(\widetilde{H}_{j-1}(S)) \cong \widetilde{H}^j(S) \cong \widetilde{H}_{k-j-1}(S^k \setminus S) \cong \widetilde{H}_{k-j-1}(S^*).$$

Which gives us the following equalities

$$\beta_j(S) = \widetilde{\beta}_j(S) = \widetilde{\beta}_{k-j-1}(S^*), \quad \widetilde{\beta}_{j-1}(S) = \widetilde{\beta}_{k-j}(S^*) = \beta_{k-j}(S^*), \quad \mathrm{tor}(\widetilde{H}_{j-1}(S)) \cong \mathrm{tor}(\widetilde{H}_{k-j-1}(S^*)).$$

Then

$$\begin{aligned}
\widetilde{T}^j_K(X,Y) &= \sum_{K_{(j-1)} \subseteq S \subseteq K_{(j)}} |\mathrm{tor}(H_{j-1}(S))|^2 X^{\widetilde{\beta}_{j-1}(S)-\widetilde{\beta}_{j-1}(K)} Y^{\beta_j(S)} \\
&= \sum_{K^*_{(k-j-1)} \subseteq S^* \subseteq K^*_{(k-j)}} |\mathrm{tor}(H_{k-j-1}(S^*))|^2 X^{\beta_{k-j}(S^*)} Y^{\widetilde{\beta}_{k-j-1}(S^*)-\widetilde{\beta}_{k-j-1}(K)} \\
&= \widetilde{T}^{k-j}_{K^*}(Y,X)
\end{aligned}$$

We used the equations $\widetilde{\beta}_{j-1}(K) = 0$ and $\widetilde{\beta}_{k-j-1}(K) = 0$ for $K = S^k$ and $1 \leqslant j \leqslant k-1$. □

4. The Bott polynomial

In 1952 Raoul Bott introduced two combinatorially invariant polynomials for CW complexes [Bo1, Bo2]. This means that they are invariant under subdivisions. He thought that they are independent. But he made a mistake in computing the second polynomial for a sphere. In fact, it was shown by Z. Wang [Wa, Proof of Theorem 4.2] that they are proportional to each other after a suitable change of variables as well as an entire class of invariant polynomials in [Wa]. Thus we essentially have only one Bott polynomial which for a finite k dimensional cell complex K can be defined as

$$R_K(\lambda) := \sum_{K_{(j-1)}\subseteq S\subseteq K_{(j)}} (-1)^{f_k(K)-f_k(S)}\lambda^{\beta_k(S)} ,$$

If K is an orientable manifold without boundary, then $R_K(\lambda) = \lambda - 1$, see [Bo1]. Z. Wang [Wa] observed that for graphs, $k = 1$, the coefficients of the Bott polynomial essentially coincide with the *Whitney numbers* [Wh1] and, in the case of planar graphs, the Bott polynomial is equal to the chromatic polynomial of the dual graph. Thus the Bott polynomial of a graph is equal to its *flow polynomial* (see its definition in [Bo]). Therefore one may regard the Bott polynomial as a higher dimensional generalization of the flow polynomial of graphs. A different approach to a higher dimensional flow polynomial was suggested in [BK]. It would be interesting to find a relation of this approach to the Bott polynomial.

Theorem 4.1. *Let K be a k-dimensional CW complex. Then*

$$R_K(\lambda) = (-1)^{\beta_k(K)} T^k_K(-1, -\lambda) .$$

Proof. The Euler characteristics of the $(k-1)$-skeleton $K_{(k-1)}$ (which is contained in both K and S) in terms of the numbers of cells gives the equation.

$$\chi(K_{(k-1)}) = \chi(K) - (-1)^k f_k(K) = \chi(S) - (-1)^k f_k(S) .$$

The same computation in terms of the Betti numbers gives the following.

$$\chi(K_{(k-1)}) = \chi(K) - (-1)^{k-1}\beta_{k-1}(K) - (-1)^k\beta_k(K) = \chi(S) - (-1)^{k-1}\beta_{k-1}(S) - (-1)^k\beta_k(S) .$$

Subtracting these two equations we get

$$(-1)^{k-1}\beta_{k-1}(K) + (-1)^k\beta_k(K) - (-1)^k f_k(K) = (-1)^{k-1}\beta_{k-1}(S) + (-1)^k\beta_k(S) - (-1)^k f_k(S) .$$

Therefore

$$\beta_{k-1}(S) - \beta_{k-1}(K) = f_k(K) - f_k(S) - \beta_k(K) + \beta_k(S) . \tag{2}$$

Now the monomial of the right hand side corresponding to a subcomplex S is

$$(-1)^{\beta_k(K)}(-1)^{\beta_{k-1}(S)-\beta_{k-1}(K)}(-\lambda)^{\beta_k(S)} = (-1)^{f_k(K)-f_k(S)}\lambda^{\beta_k(S)} ,$$

which coincides with the corresponding monomial of the left hand side. □

5. Skein relations for the Tutte-Krushkal-Renardy polynomial

For graphs, the Tutte polynomial can be define via contraction-deletion relations, which we call here *skein relations* following the knot theoretic terminology. Z. Wang [Wa] found skein (contraction/deletion) relations for the Bott polynomial. Here we generalize them to the Tutte-Krushkal-Renardy polynomial.

Let K be a finite CW complex of dimension k and σ be an open k-cell of K. We will denote σ's closure in K and its boundary in K by $\overline{\sigma}$ and $\partial\sigma$ respectively. The following definitions generalize the standard definitions for graphs.

Definition 5.1.
- σ is a *loop* in K if $H_k(\overline{\sigma}) \cong \mathbb{Z}$;
- σ is a *bridge* in K if $\beta_{k-1}(K \setminus \sigma) = \beta_{k-1}(K) + 1$;
- σ is *contractible* if $H_{k-1}(\partial\sigma) \cong \mathbb{Z}$.

Lemma 5.2. *For a top dimensional k-cell $\sigma \subset K$ we have*

$$H_k(\overline{\sigma}) = \begin{cases} \mathbb{Z} & \text{if } \sigma \text{ is a loop,} \\ 0 & \text{otherwise .} \end{cases}$$

$$\beta_{k-1}(K \setminus \sigma) = \begin{cases} \beta_{k-1}(K) + 1 & \text{if } \sigma \text{ is a bridge,} \\ \beta_{k-1}(K) & \text{otherwise .} \end{cases}$$

Proof. The k-th cellular chain group for $\overline{\sigma}$ is isomorphic to $\mathbb{Z}$ and generated by σ. If $\partial_k(\sigma) = 0$, then $H_k(\overline{\sigma}) = \mathbb{Z}$. If $\partial_k(\sigma) \neq 0$, then $H_k(\overline{\sigma}) = 0$.

The cellular chain complexes for K and for $K \setminus \sigma$ differ by the k-th chain groups $C_k(K)$ and $C_k(K \setminus \sigma)$ which has rank one less. Thus $(k-1)$-st boundary maps ∂_{k-1} are the same in both complexes. Then $\beta_{k-1}(K \setminus \sigma) = \mathrm{rank}(\mathrm{Ker}(\partial_{k-1})) - \mathrm{rank}(\partial_k(C_k(K \setminus \sigma)))$. The rank of the image $\partial_k(C_k(K \setminus \sigma))$ differs from the rank of $\partial_k(C_k(K))$ at most by 1. □

Corollary 5.3. (a) *A loop is not a bridge.*

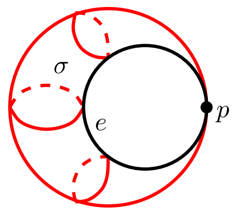
(b) *If σ is a bridge for K, then it is a bridge for any spanning ($S \supseteq K_{(k-1)}$) subcomplex $S \ni \sigma$.*

Proof. Let σ be a loop. Then by the proof of the previous lemma $\partial_k(\sigma) = 0$. So $\partial_k(C_k(K \setminus \sigma)) = \partial_k(C_k(K))$. Thus, from the same proof, $\beta_{k-1}(K \setminus \sigma) = \beta_{k-1}(K)$ and σ is not a bridge.

If σ is a bridge for K, then $\mathrm{rank}(\partial_k(C_k(K \setminus \sigma))) = \mathrm{rank}(\partial_k(C_k(K))) - 1$. This means that $\partial_k(\sigma)$ is independent from the images of all other k-cells of K. In particular, it is independent from the images of the k-cells of S different from σ. Consequently, $\mathrm{rank}(\partial_k(C_k(S \setminus \sigma))) = \mathrm{rank}(\partial_k(C_k(S))) - 1$, which means that σ is a bridge for S. □

Example 5.4. To the contrary of the graph situation, a loop in cell complexes could be contractible.

Let K be a 2-sphere with two points identified to a single point p. It has a CW structure consisting of one 2-cell σ, one 1-cell (edge) e, and one 0-cell p. The closure $\overline{\sigma}$ coincides with the whole complex K which has a homotopy type of the wedge $S^2 \vee S^1$. All its homology groups are isomorphic, $H_2(K) = H_1(K) = H_0(K) = \mathbb{Z}$. Thus σ is a loop.



From the other hand, $\partial\sigma = e \cup p = S^1$. So $H_1(\partial\sigma) = \mathbb{Z}$, and σ is contractible.

Theorem 5.5. *The Tutte-Krushkal-Renardy polynomial satisfies the following relations:*

(i) *If σ is neither a bridge nor a loop and is contractible, then*

$$T^k_K(X,Y) = T^k_{K/\overline{\sigma}}(X,Y) + T^k_{K\setminus\sigma}(X,Y) \ .$$

(ii) *If σ is a loop, then*

$$T^k_K(X,Y) = (Y+1)T^k_{K\setminus\sigma}(X,Y) \ .$$

(iii) *If σ is a bridge and contractible, then we can specialize the case (i) to*

$$T^k_K(X,Y) = (X+1)T^k_{K/\overline{\sigma}}(X,Y) \ .$$

We use standard basic tools from algebraic topology such as the long exact sequence of a pair and the fact that a CW complex X and its subcomplex A form a "good pair", so $H_i(X,A) \cong \widetilde{H}_i(X/A)$. We refer to [Ha] for all of these facts. We additionally will use the following lemma about contraction.

Lemma 5.6. *For $S \ni \sigma$, if σ is not a loop and contractible, then $\beta_k(S) = \beta_k(S/\sigma)$ and $\beta_{k-1}(S) - \beta_{k-1}(K) = \beta_{k-1}(S/\sigma) - \beta_{k-1}(K/\sigma)$.*

Proof. Since σ is not a loop, $H_k(\overline{\sigma}) = 0$. Since σ is contractable, $H_{k-1}(\partial\sigma) \cong \mathbb{Z}$. These two conditions are equivalent to the condition (C) of [Wa, Theorem 4.1].

Consider the long exact sequence of a pair $(\overline{\sigma}, \partial\sigma)$

$$H_k(\partial\sigma) \longrightarrow H_k(\overline{\sigma}) \longrightarrow \widetilde{H}_k(\overline{\sigma}/\partial\sigma) \longrightarrow H_{k-1}(\partial\sigma) \longrightarrow H_{k-1}(\overline{\sigma}) \longrightarrow \widetilde{H}_{k-1}(\overline{\sigma}/\partial\sigma) \ .$$

We have $H_k(\partial\sigma) = 0$, $H_k(\overline{\sigma}) = 0$, $H_{k-1}(\partial\sigma) \cong \mathbb{Z}$, $\widetilde{H}_k(\overline{\sigma}/\partial\sigma) \cong \mathbb{Z}$, and $\widetilde{H}_{k-1}(\overline{\sigma}/\partial\sigma) = 0$ because $\overline{\sigma}/\partial\sigma$ is a k-sphere. So the sequence becomes

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow H_{k-1}(\overline{\sigma}) \longrightarrow 0 \ .$$

The exactness of this sequence implies that $H_{k-1}(\overline{\sigma})$ is a finite group. Thus $\beta_{k-1}(\overline{\sigma}) = 0$.

Now for a subcomplex S, consider the long exact sequence of a pair $(S, \overline{\sigma})$

$$H_k(\overline{\sigma}) \longrightarrow H_k(S) \longrightarrow H_k(S/\overline{\sigma}) \longrightarrow H_{k-1}(\overline{\sigma})$$

Tensoring it by a field of real number $\mathbb{R}$ we get

$$0 \longrightarrow H_k(S;\mathbb{R}) \longrightarrow H_k(S/\overline{\sigma};\mathbb{R}) \longrightarrow 0 \ .$$

Which means that $\beta_k(S) = \beta_k(S/\sigma)$ for all $S \ni \sigma$, and in particular for $S = K$. Then, using the equation (2), we have

$$\begin{aligned}\beta_{k-1}(S) - \beta_{k-1}(K) &= f_k(K) - f_k(S) - \beta_k(K) + \beta_k(S) \\ &= f_k(K/\sigma) - f_k(S/\sigma) - \beta_k(K/\sigma) + \beta_k(S/\sigma) = \beta_{k-1}(S/\sigma) - \beta_{k-1}(K/\sigma) \ .\end{aligned}$$

□

Proof of Theorem 5.5. To prove the theorem we split the set of all top dimensional subcomplexes S according to the property $S \ni \sigma$ or $S \not\ni \sigma$.

In case (i), the sum over all $S \not\ni \sigma$ gives $T^k_{K\setminus\sigma}(X,Y)$ because $\beta_{k-1}(K\setminus\sigma) = \beta_{k-1}(K)$ since σ is not a bridge. It remain to prove that the sum over all $S \ni \sigma$ is equal to $T^k_{K/\overline{\sigma}}(X,Y)$. By the above lemma, it is true. This proves part (i) of the theorem.

In case (ii), the sum over all $S \not\ni \sigma$ again gives $T^k_{K\setminus\sigma}(X,Y)$ because of the same reason, $\beta_{k-1}(K\setminus\sigma) = \beta_{n-1}(K)$ since a loop σ is not a bridge according to Corollary 5.3 and Lemma 5.2.

For $S \ni \sigma$ we have $\partial_k(\sigma) = 0$ by the proof of Lemma 5.2. So $\partial_k(C_k(S\setminus\sigma)) = \partial_k(C_k(S))$. Therefore the chain complex for S is isomorphic to the direct sum of the chain complex for $S\setminus\sigma$ and the chain complex $0 \longrightarrow \mathbb{Z} \longrightarrow 0$ with $\mathbb{Z}$ at the grading k. Thus we get $\beta_{k-1}(S) = \beta_{k-1}(S\setminus\sigma)$ and $\beta_k(S) = \beta_k(S\setminus\sigma)+1$. Consequently the sum over $S \ni \sigma$ is equal to $YT^k_{K\setminus\sigma}(X,Y)$ which proves part (ii).

For case (iii), the lemma gives us that the sum over all $S \ni \sigma$ is equal to $T^k_{K/\overline{\sigma}}(X,Y)$. The subcomplexes $S \not\ni \sigma$ are in 1-to-1 correspondence with the subcomplexes $(S\cup\sigma) \ni \sigma$. We prove that under this correspondence the sum over all $S \not\ni \sigma$ is equal to X times the sum all $(S\cup\sigma) \ni \sigma$ which is $T^k_{K/\overline{\sigma}}(X,Y)$.

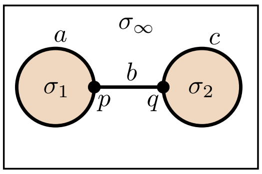
According to Corollary 5.3, if σ is a bridge for K it is a bridge for any subcomplex containing σ, in particular for $S\cup\sigma$. Then Lemma 5.2 gives $\beta_{k-1}(S) = \beta_{k-1}(S\cup\sigma)+1$ for any subcomplex $S \not\ni \sigma$. This gives extra X in the sum over all such S comparably to the sum over $S\cup\sigma$. To compare the exponents of Y consider the cell chain complexes of S and $S\cup\sigma$:

$$\begin{tikzcd} 0 \arrow[r] & C_k(S) \arrow[d, hook] \arrow[dr, "\partial_k"] & & \\ & & C_{k-1}(K) \arrow[r] & \dots \\ 0 \arrow[r] & C_k(S\cup\sigma) \arrow[ur, "\partial_k"] & & \end{tikzcd}$$

Since σ is a bridge of S, its image $\partial_k(\sigma)$ is independent from the images of all other k-cells of $S\cup\sigma$. Therefore any element of the kernel $\mathrm{Ker}(\partial_k|_{C_k(S\cup\sigma)})$ actually belongs to $C_k(S)$. This means that $H_k(S\cup\sigma) = \mathrm{Ker}(\partial_k|_{C_k(S\cup\sigma)}) = \mathrm{Ker}(\partial_k|_{C_k(S)}) = H_k(S)$. Consequently, $\beta_k(S\cup\sigma) = \beta_k(S)$ which proves case (iii) of the Theorem. □

Example 5.7. Consider the following cell structure K on a 2-sphere which we will be represented as a plane together with a point at infinity. It has three 2-cells

$\sigma_1, \sigma_2, \sigma_\infty$, three 1-cells a, b, c, and two vertices (0-cells) p, q. Note that σ_∞ is not contractible since its boundary $\partial\sigma$ coincides with 1-skeleton of K and has a homotopy type of a wedge of two circles. Thus $H_1(\partial\sigma) = \mathbb{Z}^2$. It is also neither a bridge nor a loop. The next table shows the contribution of the various subcomplexes S into the Tutte-Krushkal-Renardy polynomial $T^2_K(X,Y)$.



S	$\varnothing$	$\{\sigma_1\}$	$\{\sigma_2\}$	$\{\sigma_1,\sigma_2\}$	$\{\sigma_\infty\}$	$\{\sigma_1,\sigma_\infty\}$	$\{\sigma_2,\sigma_\infty\}$	$\{\sigma_1,\sigma_2,\sigma_\infty\}$
	X^2	X	X	1	X	1	1	Y

Therefore, $T^2_K(X,Y) = X^2 + 3X + 3 + Y$ which agrees with the equation (1) since $K = S^2$ is a manifold. The deletion $K\setminus\sigma_\infty$ consists of two discs connected by a segment b. Its Tutte-Krushkal-Renardy polynomial is equal to $T^2_{K\setminus\sigma_\infty}(X,Y) = X^2+2X+1$. The contraction $K/\overline{\sigma}_\infty$ is a wedge of two spheres, so $T^2_{K/\overline{\sigma}_\infty}(X,Y) = 1+2Y+Y^2$. So the contractability condition is essential for the skein relation. From the other hand, σ_1 and σ_2 satisfy the contritions (i) of Theorem, and one may check that $T^2_K(X,Y) = T^2_{K/\overline{\sigma}_1}(X,Y) + T^2_{K\setminus\sigma_1}(X,Y)$.

Example 5.8. Let $K = \mathbb{R}P^2$ with the standard CW structure: one 2-cell σ, one 1-cell e, and one 0-cell p. In this case σ is a bridge since its deletion gives the circle $\mathbb{R}P^1$. It is also contractible because $\partial\sigma$ is the same circle. We have $T^2_K(X,Y) = X+1$, while $T^2_{K/\overline{\sigma}}(X,Y) = 1$ since $K/\overline{\sigma}$ is a point. And so $T^2_K(X,Y) = (X+1)T^2_{K/\overline{\sigma}}(X,Y)$.

Remark 5.9. Theorem 5.5 gives the skein relation for the top dimensional, k-th, Tutte-Krushkal-Renardy polynomial. However, a lower dimensional polynomial $T^j_K(X,Y)$ is proportional to $T^j_{K_{(j)}}(X,Y)$, so essentially it depends only on the j-th skeleton of K. Thus we have a skein relation for them as well. The only thing is that one has to be careful with the deletion of a j-cell σ for $j < k$: the resulting topological space $K \setminus \sigma$ might not be a cell complex anymore.

Remark 5.10. As it was indicated in [KR], the Tutte-Krushkal-Renardy polynomial is the Tutte polynomial of a matroid $\mathcal{M}$ obtained in the following way. Consider the j-th chain group of K with real coefficients, $C_j(K;\mathbb{R})$. As a vector space over $\mathbb{R}$ it has a distinguished basis formed by the j-cells σ_i. Let us consider the $(j-1)$-st chain group $C_{j-1}(K;\mathbb{R})$ and the images of σ_i's under the boundary map $\partial_j(\sigma_i)$. The matroid $\mathcal{M}$ is a *vectorial matroid* of linear dependences of vectors $\partial_j(\sigma_i) \in C_{j-1}(K;\mathbb{R})$. On matroid theory we refer to two excellent books [Ox, Wel] and a pioneering paper [Wh2]. Krushkal and Renardy [KR] showed that $T^j_K(X,Y) = T_{\mathcal{M}}(X,Y)$.

Our definitions 5.1 of a loop and of a bridge are designed in such a way that the corresponding element of the matroid $\mathcal{M}$ will be a loop or a bridge respectively. Moreover, if a cell σ is contractible, then the matroid of the chain complex of $K/\overline{\sigma}$ is the matroid obtained from $\mathcal{M}$ by the matroid theoretic contraction of the corresponding element. The contractability condition in Definition 5.1 guarantees that the topological contraction of a cell would agree with the matroid theoretical contraction.

In Example 5.7 the ground set of the matroid $\mathcal{M}$ consist of three vectors $\partial_2(\sigma_1) = a$, $\partial_2(\sigma_2) = c$, and $\partial_2(\sigma_\infty) = -a-c$ in 3-space $\mathbb{R}^3 = \langle a,b,c\rangle$, and one relations between them: $\partial_2(\sigma_1)+\partial_2(\sigma_2)+\partial_2(\sigma_\infty) = 0$. The matroid theoretical contraction of an element $\partial_2(\sigma_\infty) \in \mathcal{M}$ would give a matroid on two elements which are dependent. So the rank $\mathcal{M}/\partial_2(\sigma_\infty)$ is equal to 1. Meanwhile, the topological contraction of the cell σ_∞ would give a wedge of two spheres. The corresponding matroid would consist of two zero vectors, since the boundary of each of the two cells consist of the single point. Its rank would be 0.

This correspondence between cell complexes and matroids provides a different way to prove Theorem 5.5.

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