Box-Counting Fractal Strings, Zeta Functions, and Equivalent Forms of Minkowski Dimension

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Abstract. We discuss a number of techniques for determining the Minkowski dimension of bounded subsets of some Euclidean space of any dimension, including: the box-counting dimension and equivalent definitions based on various box-counting functions; the similarity dimension via the Moran equation (at least in the case of self-similar sets); the order of the (box-)counting function; the classic result on compact subsets of the real line due to Besicovitch and Taylor, as adapted to the theory of fractal strings; and the abscissae of convergence of new classes of zeta functions. Specifically, we define box-counting zeta functions of infinite bounded subsets of Euclidean space and discuss results from [**12**] pertaining to distance and tube zeta functions. Appealing to an analysis of these zeta functions allows for the development of theories of complex dimensions for bounded sets in Euclidean space, extending techniques and results regarding (ordinary) fractal strings obtained by the first author and van Frankenhuijsen.

1. Introduction

Motivated by the theory of complex dimensions of fractals strings (the main theme of [**14**]), we introduce box-counting fractal strings and box-counting zeta functions which, along with the distance and tube zeta functions of [**12**], provide possible foundations for the pursuit of theories of complex dimensions for arbitrary bounded sets in Euclidean space of any dimension. We also summarize a variety of well-known techniques for determining the box-counting dimension, or equivalently the Minkowksi dimension, of such sets. Thus, while new results are presented in this paper, it is partially expository and also partially tutorial.

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Our main result establishes line (iv) of the following theorem. (See also Theorem 6.1 below, along with the relevant definitions provided in this paper.) The other lines have been established elsewhere in the literature, as cited accordingly throughout the paper.

Theorem 1.1. *Let A denote a bounded infinite subset of $\mathbb{R}^m$ (equipped with the usual metric). Then the following quantities are equal:*

(i) *the* upper box-counting dimension *of A;*
(ii) *the* upper Minkowski dimension *of A;*
(iii) *the* asymptotic order of growth of the counting function *of the box-counting fractal string $\mathcal{L}_B$;*
(iv) *the* abscissa of convergence *of the box-counting zeta function ζ_B;*
(v) *the* abscissa of convergence *of the distance zeta function ζ_d.*

A summary of the remaining sections of this paper is as follows:

In Section 2, we discuss classical notions of dimension such as *similarity dimension*, *box-counting dimension*, and *Minkowski dimension* as well as their properties. (See [**1**, **2**, **4**–**12**, **14**, **16**, **17**, **20**–**23**].)

In Section 3, we summarize but a few of the interesting results on fractal strings and counting functions regarding, among other things, geometric zeta functions, complex dimensions, the order of a counting function, and connections with Minkowski measurability. (See [**1**, **11**, **14**, **15**, **21**].) The material in Sections 2 and 3 motivates the results presented in Sections 4 and 5.

In Section 4, we introduce *box-counting fractal strings* and *box-counting zeta functions* and, in particular, we show that the abscissa of convergence of the box-counting zeta function of a bounded infinite set is the upper box-counting dimension of the set. These topics are the focus of [**13**].

In Section 5, we share recent results from [**12**] on *distance, tube* and *relative zeta functions*, including connections between the corresponding complex dimensions and Minkowski content and measurability.

In Section 6, Theorem 1.1 is restated in Theorem 6.1 using notation and terminology discussed throughout the paper. We also propose several open problems for future work in this area.

2. Classic notions of dimension

We begin with a brief discussion of a classic method for constructing self-similar fractals and a famous fractal, the Cantor set C. (See [**2**, **5**].)

Definition 2.1. Let N be an integer such that $N \geq 2$. An *iterated function system* (IFS) $\mathbf{\Phi} = \{\Phi_j\}_{j=1}^N$ is a finite family of contractions on a complete metric space (X, d_X). Thus, for all $x, y \in X$ and each $j = 1, \ldots, N$ we have

$$d_X(\Phi_j(x), \Phi_j(y)) \leq r_j d_X(x, y), \tag{1}$$

where $0 < r_j < 1$ is the *scaling ratio* (or Lipschitz constant) of Φ_j for each $j = 1, \ldots, N$.

The *attractor* of $\mathbf{\Phi}$ is the nonempty compact set $F \subset X$ defined as the unique fixed point of the contraction mapping

$$\mathbf{\Phi}(\cdot) := \bigcup_{j=1}^{N} \Phi_j(\cdot) \tag{2}$$

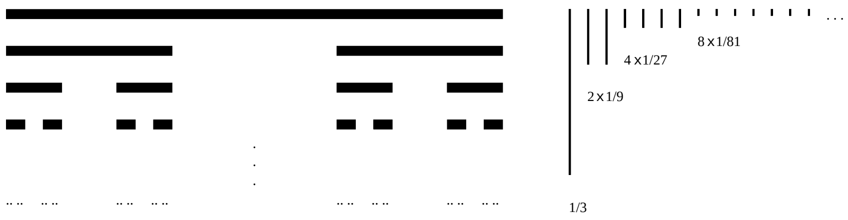

FIGURE 1. The classic "middle-third removal" construction of the Cantor set C is depicted on the left. The Cantor string $\mathcal{L}_{CS}$ is the nondecreasing sequence comprising the lengths of the removed intervals which are depicted on the right as a fractal harp.

on the space of compact subsets of X equipped with the Hausdorff metric. That is, $F = \mathbf{\Phi}(F)$. If

$$d_X(\Phi_j(x), \Phi_j(y)) = r_j d_X(x, y) \tag{3}$$

for each $j = 1, \ldots, N$ (i.e., if the contraction maps Φ_j are similarities with scaling ratios r_j), then the attractor F is the *self-similar set* associated with $\mathbf{\Phi}$.

Remark 2.2. We focus our attention on Euclidean spaces of the form $X = \mathbb{R}^m$, where m is a positive integer and $d_X = d_m$ is the classic m-dimensional Euclidean distance. Furthermore, we consider only iterated functions systems which satisfy the *open set condition* (see [**2**, **5**]). Roughly speaking, an IFS $\mathbf{\Phi}$ satisfies the open set condition if there is a nonempty open set $V \subset \mathbb{R}^m$ for which the images $\Phi_j(V)$ are pairwise disjoint.

Example 2.3 (The dimension of the Cantor set)**.** The Cantor set C can be constructed in various ways. For instance, we have the classic "middle-third removal" construction of C as depicted in Figure 1. A more elegant construction shows C to be the unique nonempty attractor of the iterated function system $\mathbf{\Phi}_C$ on $[0, 1]$ given by the two contracting similarities $\varphi_1(x) = x/3$ and $\varphi_2(x) = x/3 + 2/3$. The box-counting dimension of C is $\log_3 2$, a fact which can be established with any of the myriad of formulas presented in this paper. Notably, $\log_3 2$ is equivalently found to be: the *order of the geometric counting function* (Definition 3.23) of the *box-counting fractal string* $\mathcal{L}_B$ of C (which is related but not equal to the Cantor string $\mathcal{L}_{CS}$, see Definition 4.1, Equation (16), and [**14**, Ch.1]); the *abscissa of convergence* of either the *geometric zeta function* of $\mathcal{L}_{CS}$ (Definition 3.6), the *box-counting zeta function* of C (Definition 4.8), the *distance zeta function* of C (Definition 5.1), or the *tube zeta function* of C (Equation (36) in Section 5.2); or else the unique real-valued solution of the corresponding Moran equation (cf. Equation (4)): $2 \cdot 3^{-s} = 1$.

2.1. Similarity dimension.
The first notion of dimension we consider is the similarity dimension of a self-similar set.

Definition 2.4. Let $\mathbf{\Phi}$ be an iterated function system that satisfies the open set condition with scaling ratios $\{r_j\}_{j=1}^N$, with $N \geq 2$. Then the *similarity dimension* of the attractor of $\mathbf{\Phi}$ (that is, of the self-similar set associated with $\mathbf{\Phi}$) is the unique real solution $D_{\mathbf{\Phi}}$ of the equation

$$\sum_{j=1}^{N} r_j^{\sigma} = 1, \quad \sigma \in \mathbb{R}. \tag{4}$$

Remark 2.5. Equation (4) is known as Moran's equation. Moran's Theorem is a well-known result which states that the similarity dimension $D_{\mathbf{\Phi}}$ is equal to the box-counting (and Hausdorff) dimension of the attractor of $\mathbf{\Phi}$.[1] In fact, $D_{\mathbf{\Phi}}$ is positive, a fact that can be verified directly from Equation (4). For details regarding iterated functions systems, the open set condition, and Moran's Theorem, see [**2**, Ch.9] as well as [**5**] and [**17**].

2.2. Box-counting dimension.

In this section we discuss the central notion of box-counting dimension and some of its properties.

Definition 2.6. Let A be a subset of $\mathbb{R}^m$. The *box-counting function* of A is the function $N_B(A,\cdot) : (0,\infty) \to \mathbb{N} \cup \{0\}$, where (for a given $x > 0$) $N_B(A,x)$ denotes the maximum number of disjoint closed balls $B(a,x^{-1})$ with centers $a \in A$ of radius x^{-1}.

Definition 2.7. For a set $A \subset \mathbb{R}^m$, the *lower* and *upper box-counting dimensions* of A, denoted $\underline{\dim}_B A$ and $\overline{\dim}_B A$, respectively, are given by

$$\begin{aligned} \underline{\dim}_B A &:= \liminf_{x\to\infty} \frac{\log N_B(A,x)}{\log x}, \\ \overline{\dim}_B A &:= \limsup_{x\to\infty} \frac{\log N_B(A,x)}{\log x}. \end{aligned} \tag{5}$$

When $\underline{\dim}_B A = \overline{\dim}_B A$, the following limit exists and is called the *box-counting dimension* of A, denoted $\dim_B A$:

$$\dim_B A := \lim_{x\to\infty} \frac{\log N_B(A,x)}{\log x}.$$

In most applications the set A is such that $N_B(A,x) \asymp x^d$ as $x \to \infty$, for some constant $d \in [0,m]$ (the relation $\asymp$ is explained at the end of Notation 2.11 below). It is easy to see that then, $\dim_B A = d$.

Remark 2.8. There are many equivalent definitions of the box-counting dimension (see [**2**, Ch.3]). For instance, the box-counting function $N_B(A,x)$ given in Definition 2.6 may be replaced by:

(i) the minimum number of sets of diameter at most x^{-1} required to cover A;
(ii) the minimum number of closed balls of radius x^{-1} required to cover A;
(iii) the minimum number of closed cubes with side length x^{-1} required to cover A; or
(iv) the number of x^{-1}-mesh cubes that intersect A.

One may also define the box-counting function in terms of $\varepsilon > 0$, where ε plays the role of x^{-1}.[2] Although this may be a more natural way to describe a

[1] Moran's original result in [**17**] was established in $\mathbb{R}$ (i.e., for $m = 1$) but is valid for $m \geq 1$; cf. [**2**, **5**].

[2] Indeed, note that given $\varepsilon > 0$, $N_B(A,\varepsilon^{-1})$ is the maximum number of disjoint balls $B(a,\varepsilon)$ with center $a \in A$ and radius ε (or any of its counterparts given in (i)–(iv) of Remark 2.8).

box-counting function, the results relating box-counting functions and *geometric counting functions* (see Definition 3.9) presented in Section 4 are stated and analyzed in terms of $x > 0$.

If the box-counting function $N_B(A, x)$ is given as in Definition 2.6 or one of the alternatives in Remark 2.8, then the upper and lower box-counting dimensions have the following properties (cf. [**2**, Ch. 3] and [**3**]):

(i) Let V be a bounded n-dimensional submanifold of $\mathbb{R}^m$ which is *rectifiable* in the sense that $V \subset f(\mathbb{R}^n)$, where $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function. Then $\dim_B V = n$.

(ii) Both $\overline{\dim}_B$ and $\underline{\dim}_B$ are monotonic. That is, if $A_1 \subset A_2 \subset \mathbb{R}^m$, then
$$\overline{\dim}_B A_1 \leq \overline{\dim}_B A_2, \quad \underline{\dim}_B A_1 \leq \underline{\dim}_B A_2.$$

(iii) Let $\overline{A}$ denote the closure of A (i.e., the smallest closed subset of $\mathbb{R}^m$ which contains A). Then
$$\overline{\dim}_B \overline{A} = \overline{\dim}_B A, \quad \underline{\dim}_B \overline{A} = \underline{\dim}_B A.$$

(iv) For any two sets $A_1, A_2 \subset \mathbb{R}^m$,
$$\overline{\dim}_B (A_1 \cup A_2) = \max\left\{\overline{\dim}_B A_1, \overline{\dim}_B A_2\right\}.$$
That is, $\overline{\dim}_B$ is finitely stable. On the other hand, $\underline{\dim}_B$ is not finitely stable.

(v) Neither $\overline{\dim}_B$ nor $\underline{\dim}_B$ is countably stable. That is, neither $\overline{\dim}_B$ nor $\underline{\dim}_B$ satisfies the analogue of property (iv) for a countable collection of subsets of $\mathbb{R}^m$.

A simple way to see why property (v) just above is satisfied is to consider the countable set $A = \{1, 1/2, 1/3, \ldots\}$ and note that $\dim_B A = 1/2$ whereas $\dim_B\{1/j\} = 0$ for each positive integer j.

The following proposition shows that one need only consider certain discrete sequences of scales which tend to zero in order to determine the box-counting dimension of a set.

Proposition 2.9. *Let $\lambda > 1$ and $A \subset \mathbb{R}^m$. Then*
$$\underline{\dim}_B A = \liminf_{k\to\infty} \frac{\log N_B(A, \lambda^k)}{\log \lambda^k},$$
$$\overline{\dim}_B A = \limsup_{k\to\infty} \frac{\log N_B(A, \lambda^k)}{\log \lambda^k}.$$

Proof. If $\lambda^k < x \leq \lambda^{k+1}$, then
$$\frac{\log N_B(A, x)}{\log x} \leq \frac{\log N_B(A, \lambda^{k+1})}{\log \lambda^k} = \frac{\log N_B(A, \lambda^{k+1})}{\log \lambda^{k+1} - \log \lambda}.$$
Therefore,
$$\limsup_{x\to\infty} \frac{\log N_B(A, x)}{\log x} \leq \limsup_{k\to\infty} \frac{\log N_B(A, \lambda^k)}{\log \lambda^k}.$$
The opposite inequality clearly holds and the case for the lower limits follows *mutatis mutandis.* □

Example 2.10 (Box-counting dimension of the Cantor set)**.** Let C be the Cantor set and $n \in \mathbb{N}$. Also, let $N_B(A, 3^n)$ denote the minimum number of disjoint closed intervals with length 3^{-n} required to cover C. Then $N_B(A, 3^n) = 2^n$, so by Proposition 2.9 we have

$$\dim_B C = \lim_{n\to\infty} \frac{\log 2^n}{\log 3^n} = \log_3 2. \tag{6}$$

In the next section, we discuss the Minkowski dimension, which is well known to be equivalent to the box-counting dimension.

2.3. Minkowski dimension. Minkowski content and Minkowski dimension require a specific notion of volume and can be stated concisely with the following notation.

Notation 2.11 (Distance, volume, and Big-O)**.** Let $\varepsilon > 0$ and $A \subset \mathbb{R}^m$. Let $d(x, A)$ denote the distance between a point $x \in \mathbb{R}^m$ and the set A given by

$$d(x, A) := \inf\{|x - a| : a \in A\},$$

where $|\cdot|$ denotes the m-dimensional Euclidean norm. The *ε-neighborhood* of A, denoted A_ε, is the set of points in $\mathbb{R}^m$ which are within ε of A. Specifically,

$$A_\varepsilon = \{x \in \mathbb{R}^m : d(x, A) < \varepsilon\}.$$

In the sequel, we fix the set A and are concerned with the m-dimensional Lebesgue measure (denoted vol_m) of its ε-neighborhood A_ε for a given $\varepsilon > 0$. Recall that the m-dimensional Lebesgue measure of a (measurable) set $A \subset \mathbb{R}^m$ is given by

$$\mathrm{vol}_m(A) := \inf \left\{ \sum_{n=1}^{\infty} \prod_{j=1}^{m} (b_{n,j} - a_{n,j}) : A \subset \bigcup_{n=1}^{\infty} \left(\prod_{j=1}^{m} [a_{n,j}, b_{n,j}] \right) \right\}.$$

In the case of an *ordinary fractal string* $\Omega \subset \mathbb{R}$ (see the latter part of Definition 3.1), we are interested in the 1-dimensional volume (i.e., length) of the *inner ε-neighborhood* of the boundary $\partial\Omega$. Specifically, given an ordinary fractal string Ω and $\varepsilon > 0$, the volume $V_{\mathrm{inner}}(\varepsilon)$ of the inner ε-neighborhood of $\partial\Omega$ is defined by

$$V_{\mathrm{inner}}(\varepsilon) := \mathrm{vol}_1\{x \in \Omega : d(x, \partial\Omega) < \varepsilon\}. \tag{7}$$

For two functions f and g, with g nonnegative, we write $f(x) = O(g(x))$ as $x \to \infty$ if there exists a positive real number c such that for all sufficiently large x, $|f(x)| \leq cg(x)$. In addition, if there exists C such that $|f(x)| \leq Cg(x)$ for all x sufficiently close to some value $a \in \mathbb{R} \cup \{\pm\infty\}$, then we write $f(x) = O(g(x))$ as $x \to a$. If both $f(x) = O(g(x))$ and $g(x) = O(|f(x)|)$ as $x \to a$, we write $f(x) \asymp g(x)$ as $x \to a$. Moreover, if $\lim_{x\to a} f(x)/g(x) = 1$ (or more generally, if $f(x) = g(x)(1 + o(1))$ as $x \to a$), then we write $f(x) \sim g(x)$ as $x \to a$. Analogous notation will be used for infinite sequences.

Definition 2.12 (Minkowski content)**.** Let r be a given nonnegative real number. The *upper* and *lower r-dimensional Minkowski contents* of a bounded set $A \subset \mathbb{R}^m$

are respectively given by

$$\mathscr{M}^{*r}(A) := \limsup_{\varepsilon\to 0^+} \frac{\mathrm{vol}_m(A_\varepsilon)}{\varepsilon^{m-r}},$$
$$\mathscr{M}^{r}_{*}(A) := \liminf_{\varepsilon\to 0^+} \frac{\mathrm{vol}_m(A_\varepsilon)}{\varepsilon^{m-r}}.$$

It is easy to see that if $\mathscr{M}^{*r}(A) < \infty$, then $\mathscr{M}^{*s}(A) = 0$ for each $s > r$. Furthermore, since A is bounded, then clearly $\mathscr{M}^{*r}(A) = 0$ for $r > m$. On the other hand, if $\mathscr{M}^{*r}(A) > 0$, then $\mathscr{M}^{*s}(A) = \infty$ for each $s < r$. Therefore, there exists a unique point in $[0, m]$ at which the function $r \mapsto \mathscr{M}^{*r}(A)$ jumps from the value of ∞ to zero. This unique point is called the *upper Minkowski dimension* of A. The *lower Minkowski dimension* of A is defined analogously by using the lower r-dimensional Minkowski content.

Definition 2.13 (Minkowski dimension)**.** The *upper* and *lower Minkowski dimensions* of a bounded set A are defined respectively by

$$\begin{aligned}\overline{\dim}_M A &:= \inf\{r \geq 0 : \mathscr{M}^{r*}(A) = 0\} = \sup\{r \geq 0 : \mathscr{M}^{r*}(A) = \infty\},\\ \underline{\dim}_M A &:= \inf\{r \geq 0 : \mathscr{M}^{r}_{*}(A) = 0\} = \sup\{r \geq 0 : \mathscr{M}^{r}_{*}(A) = \infty\}.\end{aligned} \tag{8}$$

When $\overline{\dim}_M A = \underline{\dim}_M A$, the common value is called the *Minkowski dimension* of A, denoted by $\dim_M A$.

When we write $\dim_M A$, we implicitly assume that the Minkowski dimension of A exists. In most applications we have that $\mathrm{vol}_m(A_\varepsilon) \asymp \varepsilon^\alpha$ as $\varepsilon \to 0^+$, where α is a number in $[0, m]$. Then $\dim_M A$ exists and is equal to $m - \alpha$ (in light of Definitions 2.12 and 2.13). Note that here

$$\alpha = \lim_{\varepsilon\to 0^+} \frac{\log \mathrm{vol}_m(A_\varepsilon)}{\log \varepsilon},$$

and hence,

$$\dim_M A = m - \lim_{\varepsilon\to 0^+} \frac{\log \mathrm{vol}_m(A_\varepsilon)}{\log \varepsilon}.$$

It is not difficult to show that the following more general result holds.

Proposition 2.14. *The upper and lower Minkowski dimensions of a bounded set $A \subset \mathbb{R}^m$ are respectively given by*

$$\overline{\dim}_M A = m - \liminf_{\varepsilon\to 0^+} \frac{\log \mathrm{vol}_m(A_\varepsilon)}{\log \varepsilon},$$
$$\underline{\dim}_M A = m - \limsup_{\varepsilon\to 0^+} \frac{\log \mathrm{vol}_m(A_\varepsilon)}{\log \varepsilon}.$$

Remark 2.15. The upper and lower Minkowski dimensions are, of course, independent of the ambient dimension m. The upper Minkowski dimension is equivalently given by

$$\overline{\dim}_M A = \inf\{\alpha \geq 0 : \mathrm{vol}_m(A_\varepsilon) = O(\varepsilon^{m-\alpha}) \text{ as } \varepsilon \to 0^+\}.$$

This equivalent form of the upper Minkowski dimension will prove to be useful in Section 4.

Remark 2.16. It is interesting that there exists a bounded set A in $\mathbb{R}^m$ such that the upper and lower box dimension are different (see, e.g., [**20**, p. 122]), and even such that $\overline{\dim}_M A = m$ and $\underline{\dim}_M A = 0$ (see [**22**, Theorem 1.2]).

Remark 2.17. The upper Minkowski dimension of A is important in the study of the Lebesgue integrability of the distance function $d(x,A)^{-\gamma}$ in an ε-neighbourhood of A, where ε is a fixed positive number:

$$\text{if } \gamma < m - \overline{\dim}_M A, \text{ then } \int_{A_\varepsilon} d(x,A)^{-\gamma} dx < \infty. \tag{9}$$

This nice result is due to Harvey and Polking, and is implicitly stated in [**4**]; see also [**22**] for related results and references. This fact enabled the first and third authors, along with G. Radunović, to determine the abscissa of convergence of the so-called distance zeta function of A; see Definition 5.1 below along with Theorem 5.3 and [**12**] for details.

Definition 2.18 (Minkowski measurability)**.** Let $A \subset \mathbb{R}^m$ be such that $D_M = \dim_M A$ exists. The *upper* and *lower Minkowski content* of A are respectively defined as its D_M-dimensional upper and lower Minkowski contents, that is,

$$\mathscr{M}^* := \mathscr{M}^{*D_M}(A) = \limsup_{\varepsilon \to 0^+} \frac{\mathrm{vol}_m(A_\varepsilon)}{\varepsilon^{m-D_M}},$$
$$\mathscr{M}_* := \mathscr{M}_*^{D_M}(A) = \liminf_{\varepsilon \to 0^+} \frac{\mathrm{vol}_m(A_\varepsilon)}{\varepsilon^{m-D_M}}.$$

If the upper and lower Minkowski contents agree and lie in $(0,\infty)$, then A is said to be *Minkowski measurable* and the *Minkowski content of* A is given by

$$\mathscr{M} := \lim_{\varepsilon \to 0^+} \frac{\mathrm{vol}_m(A_\varepsilon)}{\varepsilon^{m-D_M}}.$$

For example, if A is such that $\mathrm{vol}_m(A_\varepsilon) \sim M\varepsilon^\alpha$ as $\varepsilon \to 0^+$, then $\dim_B A = m - \alpha$ and $\mathscr{M} = M$.

Open Problem 2.19. *If A and B are Minkowski measurable in $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, is their Cartesian product $A \times B$ Minkowski measurable in $\mathbf{R}^{m+n}$? See also Remark 5.8 below dealing with the so-called normalized Minkowski content, and its independence of the ambient dimension m.*

Remark 2.20. Another question to consider is whether or not the union $A \cup B$ of two Minkowski measurable sets is Minkowski measurable. If not, it would be interesting to find an explicit counter-example. (The answer is clearly affirmative if A and B are a positive distance apart.)

The Minkowski measurable sets on the real line have been characterized in [**11**]; see also Theorem 3.19. Some classes of Minkowski measurable sets are known in the plane in the case of smooth spirals, see [**23**], and in the case of discrete spirals, see [**16**]. It is interesting that in general, bilipschitz C^1 mappings do not preserve Minkowski measurability, even for subsets of the real line; see [**16**].

We close this section with the following (perhaps surprising) example.

Example 2.21. Let A be a bounded, Lebesgue non-measurable set in $\mathbb{R}^m$. Then $\dim_B A = m$. Indeed, the closure $\overline{A}$ cannot be of Lebesgue measure zero (i.e., we cannot have $\mathrm{vol}_m(\overline{A}) = 0$) since, in that case, A would also be of Lebesgue measure zero, implying that A is Lebesgue measurable. But then $\mathrm{vol}_m(\overline{A}) > 0$ immediately implies that $\underline{\dim}_B \overline{A} = m$, and therefore $\underline{\dim}_B A = m$. Since $\overline{\dim}_B A \le m$, this proves that $\dim_B A$ exists and $\dim_B A = m$.

3. Fractal strings and zeta functions

In this section we discuss a few of the many results on fractal strings presented in [**14**].

3.1. Fractal strings and ordinary fractal strings.

Definition 3.1. A *fractal string* $\mathcal{L}$ is a nonincreasing sequence of positive real numbers which tends to zero. Hence,

$$\mathcal{L} = (\ell_j)_{j\in\mathbb{N}} = \{l_n : l_n \text{ has multiplicity } m_n, n \in \mathbb{N}\},$$

where $(\ell_j)_{j\in\mathbb{N}}$ is nonincreasing, $\lim_{j\to\infty} \ell_j = 0$, $(l_n)_{n\in\mathbb{N}}$ is positive and strictly decreasing, and, for each $n \in \mathbb{N}$, m_n is the number of lengths ℓ_j such that $\ell_j = l_n$.

An *ordinary fractal string* Ω is a bounded open subset of the real line.

Remark 3.2. In [**14**], for instance, finite fractal strings (i.e., nonincreasing sequences of real numbers with a finite number of positive terms) are allowed. However, for reasons described in Remark 3.4, the finite case is not considered in this paper.

If an ordinary fractal string Ω is the union of a countably infinite collection of disjoint open intervals I_j (necessarily its connected components), then the lengths ℓ_j of the intervals I_j comprise a fractal string $\mathcal{L}$. Moreover, $\dim_M(\partial\Omega)$ is given by

$$\dim_M(\partial\Omega) = \inf\{\alpha \geq 0 : V_{\text{inner}}(\varepsilon) = O(\varepsilon^{1-\alpha}) \text{ as } \varepsilon \to 0^+\}, \tag{10}$$

where $V_{\text{inner}}(\varepsilon)$ is the 1-dimensional Lebesgue measure of the inner ε-neighborhood of Ω (see formula (7)) in Notation 2.11). In fact, Equation (10) is used to define the *Minkowski dimension* of (the boundary of) an ordinary fractal string in [**14**].[3] Moreover, it is shown in [**11**] that $V_{\text{inner}}(\varepsilon)$, and hence also $\dim_M(\partial\Omega)$, depends only on the fractal string $\mathcal{L}$ (but not on the particular rearrangement of the intervals I_j composing Ω).

Definition 3.3. Let $\mathcal{L}$ be a fractal string. The *abscissa of convergence* of the Dirichlet series $\sum_{j=1}^{\infty} \ell_j^s$ is defined by

$$\sigma = \inf\left\{\alpha \in \mathbb{R} : \sum_{j=1}^{\infty} \ell_j^{\alpha} < \infty\right\}. \tag{11}$$

Thus, $\{s \in \mathbb{C} : \text{Re}(s) > \sigma\}$ is the largest open half-plane on which this series converges; see, e.g., [**19**, §VI.2].

Remark 3.4. If $\mathcal{L}$ were allowed to be a finite sequence of positive real numbers (as in [**14**]), then we would have $\sigma = -\infty$ since the corresponding Dirichlet series would be an entire function. In the context of this paper, we always have that $\sigma \geq 0$ (since $\sum_{j=1}^{\infty} \ell_j^{\alpha}$ is clearly divergent when $\alpha = 0$). This explains why we consider only (bounded) infinite sets in the development of *box-counting fractal strings* in Section 4. Indeed, for clarity of exposition, we only consider fractal strings consisting of infinitely many positive lengths (or scales), and hence, ordinary fractal strings comprising infinitely many disjoint intervals.

[3] More specifically, $\dim_M(\partial\Omega)$ should really be denoted by $\dim_{M,\text{inner}}(\partial\Omega)$ and called the *inner Minkowski dimension* of $\partial\Omega$ (or of $\mathcal{L}$).

Remark 3.5. A key distinction between a fractal string $\mathcal{L}$ and an ordinary fractal string Ω lies in the sum of the corresponding lengths (or scales), denoted $(\ell_j)_{j\in\mathbb{N}}$ in either case. Specifically, since an ordinary fractal string Ω is bounded, $\sum_{j=1}^{\infty} \ell_j$ is necessarily convergent. On the other hand, for a fractal string $\mathcal{L}$, $\sum_{j=1}^{\infty} \ell_j$ may be divergent. See Example 4.6 for a bounded set in $\mathbb{R}^2$ whose *box-counting fractal string* is a fractal string whose lengths have an unbounded sum and yet contains pertinent information regarding the bounded set. (In a somewhat different setting, many other classes of examples are provided in [**14**, esp. §13.1 & §13.3] and in [**9**, **10**].)

Definition 3.6. Let $\mathcal{L}$ be a fractal string. The *geometric zeta function* of $\mathcal{L}$ is defined by

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s, \tag{12}$$

where $s \in \mathbb{C}$ and $\mathrm{Re}(s) > D_{\mathcal{L}} := \sigma$. The *dimension* of $\mathcal{L}$, denoted $D_{\mathcal{L}}$, is defined as the abscissa of convergence σ of the Dirichlet series which defines $\zeta_{\mathcal{L}}$.

In order to define the complex dimensions of a fractal string, as in [**14**], we assume there exists a meromorphic extension of the geometric zeta function $\zeta_{\mathcal{L}}$ to a suitable region. First, consider the *screen* S as the contour

$$S : S(t) + it \quad (t \in \mathbb{R}), \tag{13}$$

where $S(t)$ is a continuous function $S : \mathbb{R} \to [-\infty, D_{\mathcal{L}}]$. Next, consider the *window* W as the set

$$W = \{s \in \mathbb{C} : \mathrm{Re}(s) \geq S(\mathrm{Im}(s))\}. \tag{14}$$

By a mild abuse of notation, we denote by $\zeta_{\mathcal{L}}$ both the geometric zeta function of $\mathcal{L}$ and its meromorphic extension to some region.

Definition 3.7. Let $W \subset \mathbb{C}$ be a window on an open connected neighborhood of which $\zeta_{\mathcal{L}}$ has a meromorphic extension. The set of (*visible*) *complex dimensions* of $\mathcal{L}$ is the set $\mathcal{D}_{\mathcal{L}} = D_{\mathcal{L}}(W)$ given by

$$\mathcal{D}_{\mathcal{L}} = \{\omega \in W : \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}. \tag{15}$$

In the case where $\zeta_{\mathcal{L}}$ has a meromorphic extension to $W = \mathbb{C}$, the set $\mathcal{D}_{\mathcal{L}}$ is referred to as the *complex dimensions* of $\mathcal{L}$. Such is the case for the Cantor string Ω_{CS}.

Example 3.8 (Complex dimensions of the Cantor string)**.** The Cantor string Ω_{CS} is the ordinary fractal string given by $\Omega_{CS} = [0,1] \setminus C$, where C is the Cantor set (see Example 2.10). The lengths of the Cantor string are given by the fractal string

$$\mathcal{L}_{CS} = \{3^{-n} : 3^{-n} \text{ has multiplicity } 2^{n-1}, n \in \mathbb{N}\}. \tag{16}$$

The geometric zeta function of the Cantor string, denoted ζ_{CS}, is given by

$$\zeta_{CS}(s) = \zeta_{\mathcal{L}_{CS}}(s) = \sum_{n=1}^{\infty} 2^{n-1} 3^{-ns} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}. \tag{17}$$

The closed form on the right-hand side of Equation (17) allows for the meromorphic continuation of ζ_{CS} to all of $\mathbb{C}$. Hence, $\zeta_{CS} = 3^{-s}(1 - 2 \cdot 3^{-s})^{-1}$ for all $s \in \mathbb{C}$. It follows that the complex dimensions of the Cantor string, denoted $\mathcal{D}_{CS}$, are the complex roots of the Moran equation $2 \cdot 3^{-s} = 1$. Thus,

$$\mathcal{D}_{CS} = \mathcal{D}_{\mathcal{L}_{CS}} = \left\{ \log_3 2 + i\frac{2\pi}{\log 3} z : z \in \mathbb{Z} \right\}. \tag{18}$$

Note that by Equation (6) the dimension $D_{CS} := D_{\mathcal{L}_{CS}} = \log_3 2$ of the Cantor string coincides with $\dim_B C = \dim_M C$, and this value is the unique real-valued complex dimension of the Cantor string.

3.2. Geometric counting function of a fractal string. The results in this section connect the counting function of the lengths of a fractal string to its dimension and geometric zeta function.

Definition 3.9. The *geometric counting function* of $\mathcal{L}$, or the *counting function of the reciprocal lengths* of $\mathcal{L}$, is given by

$$N_{\mathcal{L}}(x) := \#\{j \in \mathbb{N} : \ell_j^{-1} \leq x\} = \sum_{n \in \mathbb{N},\, l_n^{-1} \leq x} m_n$$

for $x > 0$.

The following easy proposition is identical to Proposition 1.1 of [**14**].

Proposition 3.10. *Let $\alpha \geq 0$ and $\mathcal{L}$ be a fractal string. Then $N_{\mathcal{L}}(x) = O(x^\alpha)$ as $x \to \infty$ if and only if $\ell_j = O(j^{-1/\alpha})$ as $j \to \infty$.*

PROOF. Suppose that for some $C > 0$ we have

$$N_{\mathcal{L}}(x) \leq Cx^\alpha.$$

Let $x = \ell_j^{-1}$, then $j \leq C\ell_j^{-\alpha}$, which implies that

$$\ell_j = O(j^{-1/\alpha}).$$

Conversely, if $\ell_j \leq cj^{-1/\alpha}$ for $j \in \mathbb{N}$ and some $c > 0$, then given $x > 0$, we have

$$\ell_j^{-1} > x \text{ for } j > (cx)^\alpha.$$

Therefore,

$$N_{\mathcal{L}}(x) \leq (cx)^\alpha.$$

□

Remark 3.11. Many additional (and harder) results connecting the asymptotic behavior of the geometric counting function, the spectral counting function, and the (upper and lower) Minkowski content(s) of a fractal string $\mathcal{L}$ are provided in [**11**]. The simplest one states that $N_{\mathcal{L}}(x) = O(x^\alpha)$ as $x \to \infty$ (i.e., $\ell_j = O(j^{-1/\alpha})$ as $j \to \infty$) if and only if $\mathscr{M}^{*\alpha}(\partial\Omega) < \infty$, where (consistent with our earlier comment) $\mathscr{M}^{*\alpha}(\partial\Omega)$ is given as in Definition 2.12 except with $\text{vol}_m(\cdot)$ replaced with $V_{\text{inner}}(\cdot)$.

Notation 3.12. The infimum of the nonnegative values of α which satisfy Proposition 3.10 plays a key role in our results. Hence, we let D_N denote that special value. That is,

$$D_N := \inf\{\alpha \geq 0 : N_{\mathcal{L}}(x) = O(x^\alpha) \text{ as } x \to \infty\}. \tag{19}$$

The following lemma is a restatement of a portion of Lemma 13.110 of [**14**].[4]

[4] In fact, a stronger result holds in the setting of generalized fractal strings (viewed as measures) in [**14**], but it is beyond the scope of this paper.

Lemma 3.13. *Let $\mathcal{L}$ be a fractal string. Then*

$$\zeta_{\mathcal{L}}(s) = s \int_0^\infty N_{\mathcal{L}}(x) x^{-s-1} dx \tag{20}$$

and, moreover, the integral converges (and hence, Equation (20) holds) if and only if $\sum_{j=1}^\infty \ell_j^s$ converges, i.e., if and only if $\mathrm{Re}(s) > D_{\mathcal{L}} = \sigma$.

Proof. For any given $n \in \mathbb{N}$, we have

$$s \int_0^{\ell_n^{-1}} N_{\mathcal{L}}(x) x^{-s-1} dx = \sum_{j=1}^{n-1} s \int_{\ell_j^{-1}}^{\ell_{j+1}^{-1}} N_{\mathcal{L}}(x) x^{-s-1} dx = \sum_{j=0}^{n-1} j(\ell_j^s - \ell_{j+1}^s)$$

since $N_{\mathcal{L}}(x) = 0$ for $x < \ell_1^{-1}$ and $N_{\mathcal{L}}(x) = j$ for $\ell_j^{-1} \le x < \ell_{j+1}^{-1}$. Furthermore,

$$s \int_0^{\ell_j^{-1}} N_{\mathcal{L}}(x) x^{-s-1} dx = \sum_{j=1}^{n-1} j\ell_j^s - \sum_{j=1}^{n} (j-1)\ell_j^s = \sum_{j=1}^{n} \ell_j^s - n\ell_n^s.$$

Now for $s \ge 0$, $n\ell_n^s \le 2\sum_{j=[n/2]}^n \ell_j^s$. Thus, Equation (20) holds if and only if $\sum_{j=1}^\infty \ell_j^s$ converges. Moreover,

$$\lim_{n\to\infty} s \int_0^{\ell_n^{-1}} N_{\mathcal{L}}(x) x^{-s-1} dx = \zeta_{\mathcal{L}}(s)$$

since the tail $\sum_{j=[n/2]}^\infty \ell_j^s$ converges to zero. (Here, $[y]$ denotes the integer part of the real number y.) □

The following proposition will be used to prove a portion of our main result, Theorem 6.1 (cf. Theorem 13.111 and Corollary 13.112 of [**14**], as well as [**6**, **7**], where this proposition is established in the context of p-adic fractal strings and also of ordinary (real) fractal strings).

Proposition 3.14. *Let $\mathcal{L}$ be a fractal string. Then*

$$D_{\mathcal{L}} = D_N,$$

where $D_{\mathcal{L}} = \sigma$ is the dimension of $\mathcal{L}$ given by Equation (11) (and Definition 3.6) and D_N is given by Equation (19).

Proof. Suppose $\mathrm{Re}(s) > D_N$. Denoting $t = \mathrm{Re}(s)$, we choose any fixed $\alpha \in (D_N, t)$. Using Lemma 3.13, for $x_1 = (\ell_1)^{-1}$ we have

$$|\zeta_{\mathcal{L}}(s)| \le |s| \int_{x_1}^\infty C x^\alpha x^{-t-1} dx = \left[\frac{|s| C x^{\alpha - t}}{\alpha - t} \right]_{x_1}^\infty = 0 - \frac{|s| C x_1^{\alpha - t}}{\alpha - t},$$

since $\alpha - t < 0$. Hence, $|\zeta_{\mathcal{L}}(s)| < \infty$. In other words, $t > D_{\mathcal{L}}$ for any $t > D_N$. Letting $t \searrow D_N$, we obtain that $D_{\mathcal{L}} \le D_N$.

The converse inequality $D_{\mathcal{L}} \ge D_N$ follows similarly as in the proof of Theorem 13.111 of [**14**]. □

For a given fractal string $\mathcal{L}$, Theorem 3.15 (cf. Theorem 5.10 and Theorem 5.18 in [**14**]) shows that under mild conditions the complex dimensions $\mathcal{D}_{\mathcal{L}}$ contain enough information to determine the geometric counting function $N_{\mathcal{L}}$.

Theorem 3.15. *Let $\mathcal{L}$ be a fractal string such that $\mathcal{D}_\mathcal{L}$ consists entirely of simple poles with respect to a window W. Then, under certain mild growth conditions on $\zeta_\mathcal{L}$,*[5] *we have*

$$N_\mathcal{L}(x) = \sum_{\omega \in \mathcal{D}_\mathcal{L}} \frac{x^\omega}{\omega} \mathrm{res}(\zeta_\mathcal{L}(s); \omega) + \{\zeta_\mathcal{L}(0)\} + R(x), \tag{21}$$

where $R(x)$ is an error term of small order and the term in braces is included only if $0 \in W \backslash \mathcal{D}_\mathcal{L}$.

Remark 3.16. If the poles are not simple, the explicit formula for $N_\mathcal{L}$ is slightly more complicated (see [**14**, Chs. 5,6]). If an ordinary fractal string Ω is *strongly languid* (see [**14**, Def. 5.3]), then by Theorem 5.14 and Theorem 5.22 of [**14**], Equation (21) holds with no error term (i.e., $W = \mathbb{C}$ and $R(x) \equiv 0$).

Remark 3.17. Similar (but harder to derive) explicit formulas called *fractal tube formulas* are obtained in [**14**, Ch. 8], which, as described therein, allows for the expression of $V_{\text{inner}}(\varepsilon)$ in terms of the underlying (visible) complex dimensions of $\mathcal{L}$. (Still in [**14**], they are used, in particular, to derive the equivalence of (i) and (iii) in Theorem 3.19 below.) We will return to this topic in Section 6 when discussing Open Problems 6.2 and 6.3.

Analogous results regarding connections between the structure of the complex dimensions $\mathcal{D}_\mathcal{L}$ of an ordinary fractal string Ω with lengths $\mathcal{L}$ and the (inner) Minkowski measurability of $\partial\Omega$ are presented in the next section.

3.3. Classic results.

The following theorem is precisely Theorem 1.10 of [**14**]. It is actually a classic theorem of Besicovitch and Taylor (see [**1**]) stated in terms of ordinary fractal strings.[6]

Theorem 3.18. *Suppose Ω is an ordinary fractal string with infinitely many lengths denoted by $\mathcal{L}$. Then the abscissa of convergence of $\zeta_\mathcal{L}$ coincides with the Minkowski dimension of $\partial\Omega$. That is, $D_\mathcal{L} = \dim_M(\partial\Omega)$.*

The following result is Theorem 8.15 of [**14**]. For complete details regarding connections between complex dimensions and Minkowski measurability, see [**14**, Ch. 8].

Theorem 3.19 (Criterion for Minkowski measurability)**.** *Let Ω be an ordinary fractal string whose geometric zeta function $\zeta_\mathcal{L}$ has a meromorphic extension which satisfies certain mild growth conditions.*[7] *Then the following are equivalent:*

(i) *$D_\mathcal{L}$ is the only complex dimension with real part $D_\mathcal{L}$, and it is simple.*
(ii) *$N_\mathcal{L}(x) = cx^{D_\mathcal{L}} + o(x^{D_\mathcal{L}})$ as $x \to \infty$, for some positive constant c.*[8]

---

[5]Namely, if $\zeta_\mathcal{L}$ is *languid* (see [**14**, Def. 5.2]) of a suitable order.

[6]There is, however, one significant difference with the setting of [**1**]. Namely, here, as in [**14**], we are assuming that we are working with the *inner* (rather than ordinary) Minkowski dimension and Minkowski content of $\partial\Omega$; see the statement and the proof of Theorem 1.10 in [**14**], along with Equation (7) above. By contrast, in the context of [**1**], one should assume that Ω is of full measure in its closed convex hull (i.e., in the smallest compact interval containing it).

[7]Specifically, $\zeta_\mathcal{L}$ is languid for a screen S passing strictly between the vertical line $\mathrm{Re}(s) = D_\mathcal{L}$ and all the complex dimensions (of the corresponding fractal string) $\mathcal{L}$ with real part strictly less than $D_\mathcal{L}$, and not passing through 0.

[8]In the spirit of Proposition 3.10, condition (ii) is easily seen to be equivalent to

$$\ell_j = Lj^{-1/D_\mathcal{L}} + o(j^{-1/D_\mathcal{L}}) \quad \text{as} \quad j \to \infty,$$

(iii) $\partial\Omega$ *is Minkowski measurable.*

Moreover, if any of these conditions is satisfied, then the Minkowski content $\mathscr{M}$ *of* $\partial\Omega$ *is given by*

$$\mathscr{M} = \frac{c2^{1-D_{\mathcal{L}}}}{1-D_{\mathcal{L}}} = 2^{1-D_{\mathcal{L}}} \frac{\operatorname{res}(\zeta_{\mathcal{L}}(s); D_{\mathcal{L}})}{D_{\mathcal{L}}(1-D_{\mathcal{L}})}.$$

Remark 3.20. We note that the equivalence of (ii) and (iii) in Theorem 3.19 was first established in [**11**] for any ordinary fractal string, without any hypothesis on the growth of the associated geometric zeta function. As was alluded to in Remark 3.17, however, the equivalence of (i) and (iii) in Theorem 3.19 was proved in [**14**] (and in earlier works of the authors of [**14**]) by using a suitable generalization of Riemann's explicit formula that is central to the theory of complex dimensions and is obtained in [**14**, Chs. 5 & 8].

Example 3.21 (The Cantor set is not Minkowski measurable)**.** By Equation (18) in Example 3.8, there is an infinite collection of complex dimensions $\omega \in \mathcal{D}_{CS}$ of the Cantor string with real part $D_{CS} = \log_3 2$. Hence, by Theorem 3.19, the Cantor set C is not Minkowski measurable. This fact was first established in [**11**] by using the equivalence of (ii) and (iii) and showing that (ii) does not hold. Actually, still in [**11**], for $\alpha = \dim_B C = D_{CS}$, both $\mathscr{M}^{\alpha *} = \mathscr{M}^*$ and $\mathscr{M}_*^\alpha = \mathscr{M}_*$ are explicitly computed and shown to be different (with $0 < \mathscr{M}_* < \mathscr{M}^* < \infty$). This result was significantly refined in [**14**, Ch. 10] in the broader context of generalized Cantor strings.

Remark 3.22. Example 3.21 is indicative of another result from [**14**] pertaining to a dichotomy in the properties of self-similar attractors of certain iterated function systems on compact intervals. Specifically, if an iterated function system on a compact interval I satisfies the open set condition with at least one gap and there is some unique $0 < r < 1$ and positive integers k_j such that the scaling ratios satisfy $r_j = r^{k_j}$ for each $j = 1, \dots, N$, then the complement $I \setminus A$ of the resulting attractor A is an ordinary fractal string known as a *lattice self-similar string*. For example, the Cantor string $\Omega_{CS} = [0,1] \setminus C$ is a lattice self-similar string. If no such r exists, then $I \setminus A$ is a *nonlattice self-similar string*. The complex dimensions of a self-similar string are given by (a subset of) the complex roots of the corresponding Moran equation (4). In the lattice case there are countably many complex dimensions with real part $D_{\mathcal{L}} = \dim_B A = \dim_M A$, so by Theorem 3.19, A is not Minkowski measurable. In the nonlattice case, Theorem 3.19 does not necessarily apply (because its hypotheses need not be satisfied, see [**14**, Example 5.32]), however the only complex dimension with real part $D_{\mathcal{L}}$ is $\dim_B A = \dim_M A$ and by Theorem 8.36 of [**14**] we have that A is Minkowski measurable. Therefore, the boundary of a self-similar string is Minkowski measurable if and only if it is nonlattice. See [**14**, Ch. 8] for details.

We conclude the section on classic results with the following definition and lemma which, in light of the expression for $V_{\text{inner}}(\varepsilon)$ obtained in [**11**] (see also [**14**, Eq. (8.1)]), can be be deduced from Lemma 1 in Section 1.4 in [**15**].[9] The lemma below provides yet another connection between counting functions and dimensions.

for some positive constant L. In that case, $c = L^{D_{\mathcal{L}}}$.

[9]For convenience, Definition 3.23 and Lemma 3.24 are stated in the language of fractal strings. A direct (and independent) proof of Lemma 3.24 can be found in [**14**].

Definition 3.23. Let $\mathcal{L}$ be a fractal string. The *order of the geometric counting function* $N_{\mathcal{L}}$ is given by

$$\rho_{\mathcal{L}} := \limsup_{x\to\infty} \frac{\log N_{\mathcal{L}}(x)}{\log x}. \tag{22}$$

Lemma 3.24. *For a fractal string* $\mathcal{L}$*,*

$$D_{\mathcal{L}} = \rho_{\mathcal{L}}.$$

Note that, for a given fractal string $\mathcal{L}$, the order of the counting function $\rho_{\mathcal{L}}$ given in Equation (22) and the value D_N given in Equation (19) provide essentially the same information regarding the geometric counting function $N_{\mathcal{L}}$. Indeed, it can be shown directly that $\rho_{\mathcal{L}} = D_N$, and hence Lemma 3.24 would follow from Proposition 3.14. This connection is examined further in [**13**].

In the next section, motivated by the box-counting function N_B and connections between the geometric counting function $N_{\mathcal{L}}$ and dimension $D_{\mathcal{L}}$ of a fractal string $\mathcal{L}$, we define and investigate the properties of box-counting fractal strings.

4. Box-counting fractal strings and zeta functions

In this section we develop the definition of and results pertaining to *box-counting fractal strings*. These fractal strings are defined in order to provide a framework in which one may, perhaps, extend the results on ordinary fractal strings via associated zeta functions and complex dimensions in [**14**] to bounded sets. Further exploration with box-counting fractal strings, such as Minkowski measurability of bounded sets, is central to the development of the authors' paper [**13**]. The box-counting fractal string and the box-counting zeta function for bounded sets in Euclidean spaces were introduced by the second author during the First International Meeting of the Permanent International Session of Research Seminars (PISRS) at the University of Messina, PISRS Conference 2011: Analysis, Fractal Geometry, Dynamical Systems, and Economics. The introduction took place after listening to a lecture of the third author about the results of [**12**] on distance and tube zeta functions for arbitrary compact subsets of $\mathbb{R}^m$. Some of these results are also discussed in Section 5 below.

4.1. Definition of box-counting fractal strings. If $A \subset \mathbb{R}^m$ is bounded, then the diameter of A, denoted $\operatorname{diam}(A)$, is finite. So for nonempty A and all x small enough, we have $N_B(A, x) = 1$ when $N_B(A, \cdot)$ is given as in Definition 2.6 or one of the options in Remark 2.8. Indeed, for a given bounded infinite set A, each such box-counting function uniquely defines a fractal string $\mathcal{L}_B$, which is introduced below and called the *box-counting fractal string*, by uniquely determining a sequence of distinct scales $(l_n)_{n\in\mathbb{N}}$ along with corresponding multiplicities $(m_n)_{n\in\mathbb{N}}$.

Given a fixed bounded infinite set A, the range of a chosen box-counting function $N_B(A, \cdot)$ can be thought of as a strictly increasing sequence of positive integers $(M_n)_{n\in\mathbb{N}}$. In this context, we can readily define a fractal string $\mathcal{L}_B$ whose geometric counting function $N_{\mathcal{L}_B}$ essentially coincides with $N_B(A, \cdot)$; see Lemma 4.7 below. To this end, the key idea is to make the distinct lengths (or rather, scales) l_n of the desired (box-counting) fractal string $\mathcal{L}_B$ correspond to the scales at which the box-counting function $N_B(A, \cdot)$ jumps. Furthermore, the multiplicities m_n are defined in order to have the resulting geometric counting function $N_{\mathcal{L}_B}$ (nearly) coincide with the chosen box-counting function $N_B(A, \cdot)$. Such *box-counting fractal strings* potentially allow for the development of a theory of complex dimensions of fractal strings, as presented in [**14**], by means of results in Section 3 similar to Theorem 3.15 above. These concepts are central to the development of the paper [**13**].

Definition 4.1. Let A be a bounded infinite subset of $\mathbb{R}^m$ and let $N_B(A,\cdot)$ denote a box-counting function given by one of the options in Remark 2.8. Denote the range of $N_B(A,\cdot)$ as a strictly increasing sequence of positive integers $(M_n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, let l_n be the length (or scale) given by

$$l_n := (\sup\{x \in (0,\infty) : N_B(A,x) = M_n\})^{-1}. \tag{23}$$

Also, let $m_1 := M_2$, and for $n \geq 2$, let $m_n := M_{n+1} - M_n$. The *box-counting fractal string* of A, denoted $\mathcal{L}_B$, is given by

$$\mathcal{L}_B := \{l_n : l_n \text{ has multiplicity } m_n, n \in \mathbb{N}\}.$$

Remark 4.2. Note that the distinct lengths l_n and the multiplicities m_n are *uniquely defined by the box-counting function* $N_B(A,\cdot)$ since $N_B(A,x)$ is nondecreasing as $x \to \infty$. Also, each l_n is equivalently given by

$$l_n = \inf\{\varepsilon \in (0,\infty) : N_B(A,\varepsilon^{-1}) = M_n\}.$$

It remains to show that $\mathcal{L}_B$ is indeed a fractal string; see Definition 3.1. That is, since we want to use as many of the results from [**14**] as possible (some of which are presented in Section 3), we must verify that $\mathcal{L}_B = (\ell_j)_{j\in\mathbb{N}}$ is a nonincreasing sequence of positive real numbers which tends to zero. This is accomplished with the following proposition, in which other behaviors of $N_B(A,\cdot)$ are also determined.

For clarity of exposition and in order to ease the notation used in this section, in particular in the following proposition, take $N_B(A,\cdot)$ to be defined by option (i) of Remark 2.8 and let $N_B(A,0) := 0$. (Completely analogous results hold when $N_B(A,\cdot)$ is given by Definition 2.6 or one of the other options in Remark 2.8, *mutatis mutandis*.) Note that we have $N_B(A,x) \leq N_B(A,y)$ whenever $0 < x < y$. Furthermore, let $x_n := l_n^{-1}$ for each $n \in \mathbb{N}$, and note that we have $N_B(A,x_2) = m_1 = M_2$ and

$$N_B(A,x_{n+1}) - N_B(A,x_n) = m_n = M_{n+1} - M_n, \quad \text{for } n \geq 2.$$

Proposition 4.3. *Let A be a bounded infinite subset of $\mathbb{R}^m$ and let l_n be given by Equation* (23). *Then the sequence $(x_n)_{n\in\mathbb{N}} := (l_n^{-1})_{n\in\mathbb{N}}$ is a countably infinite, strictly increasing sequence of positive real numbers such that, for each $n \in \mathbb{N}$ and all x where $x_{n-1} < x \leq x_n$ (letting $x_0 = 0$), we have*

$$N_B(A,x_{n-1}) < N_B(A,x) = N_B(A,x_n). \tag{24}$$

Furthermore,

(i) $x_1 > 0$ *and* $N_B(A,x_1) = 1$,
(ii) $x_n \nearrow \infty$ *as* $n \to \infty$, *and*
(iii) $\bigcup_{n\in\mathbb{N}} N_B(A,x_n) = \text{range}\{N_B(A,\cdot)\}$.

Proof. We have that $N_B(A,x)$ is nondecreasing as $x \to \infty$. Further, the range of $N_B(A,\cdot)$, denoted range$N_B(A,\cdot)$ (also denoted by $(M_n)_{n\in\mathbb{N}}$ above), is at most countable since it is a subset of $\mathbb{N}$. In fact, range$N_B(A,\cdot)$ is countably infinite (otherwise A would be finite). Hence, $(x_n)_{n\in\mathbb{N}}$ is a unique, countably infinite, strictly increasing sequence of positive real numbers such that, for each $n \in \mathbb{N}$ and all x where $x_{n-1} < x \leq x_n$ (letting $x_0 = 0$), we have

$$N_B(A, x_{n-1}) < N_B(A, x) = N_B(A, x_n).$$

Since A is bounded and contains more than two elements, there exists a unique $x' \in (0, \infty)$ such that $N_B(A, x) = 1$ if $0 < x \leq x'$, and $N_B(A, x) > 1$ if $x > x'$. By the definition of the sequence $(x_n)_{n \in \mathbb{N}}$, we have $x' = x_1$.

Now, suppose $(x_n)_{n \in \mathbb{N}}$ has an accumulation point at some $x'' \in (0, \infty)$. Then $N_B(A, x'') = \infty$ since $N_B(A, \cdot)$ increases by some positive integer value at x_n for each $n \in \mathbb{N}$ and since range$N_B(A, \cdot) \subset \mathbb{N}$. However, this contradicts the boundedness of A. Further, assuming $N_B(A, \cdot)$ is bounded implies that A is finite. Hence, $x_n \nearrow \infty$ as $n \to \infty$.

Lastly, suppose there exists $k \in$ range$N_B(A, \cdot)$ such that we have $k \neq N_B(A, x_n)$ for all $n \in \mathbb{N}$. Since $x_n \nearrow \infty$ as $n \to \infty$ and $N_B(A, \cdot)$ is nondecreasing, there exists a unique $n_0 \in \mathbb{N}$ such that $x_{n_0 - 1} < y < x_{n_0}$ for all y such that $N_B(A, y) = k$. However, Equation (24) implies $N_B(A, y) = k = N_B(A, x_{n_0})$, which is a contradiction. Therefore, $\bigcup_{n \in \mathbb{N}} N_B(A, x_n) = \text{range} N_B(A, \cdot)$. □

Remark 4.4. By line (ii) of Proposition 4.3, $l_n \searrow 0$ as $n \to \infty$ and, hence, $\mathcal{L}_B$ is indeed a fractal string in the sense of Definition 3.1.

Example 4.5 (Box-counting fractal string of the Cantor set)**.** Consider the Cantor set C. For $x > 0$, let the box-counting function $N_B(C, x)$ be the minimum number of sets of diameter x^{-1} required to cover C (i.e., as in option (i) of Remark 2.8). Then the box-counting fractal string $\mathcal{L}_B$ of C is given by

$$\mathcal{L}_B = \{l_1 = 1 : m_1 = 2\} \cup \{l_n = 3^{-(n-1)} : m_n = 2^{n-1}, n \geq 2\}. \tag{25}$$

Indeed, for each $n \in \mathbb{N}$, exactly 2^n intervals of diameter 3^n are required to cover C. If $x^{-1} < 3^n$, then more than 2^n intervals of diameter x^{-1} are required to cover C.

Example 4.6 (Box-counting fractal string of a 1-dimensional fractal)**.** Consider the self-similar set F which is the attractor of the IFS $\mathbf{\Phi}_1 = \{\Phi_j\}_{j=1}^4$ on the unit square $[0,1]^2 \subset \mathbb{R}^2$ given by

$$\Phi_1(x) = \frac{1}{4}x, \quad \Phi_2(x) = \frac{1}{4}x + \left(\frac{3}{4}, 0\right), \quad \Phi_3(x) = \frac{1}{4}x + \left(\frac{3}{4}, \frac{3}{4}\right), \quad \text{and}$$
$$\Phi_4(x) = \frac{1}{4}x + \left(0, \frac{3}{4}\right).$$

The Moran equation of F is simply $4 \cdot 4^{-s} = 1$, hence $D_{\mathbf{\Phi}_1} = \dim_B F = \dim_M F = 1$ and F is a 1-dimensional self-similar set which is totally disconnected.

Let $N_B(F, x)$ be the minimum number of closed cubes with side x^{-1} required to cover F (as in option (iii) of Remark 2.8). Then $N_B(F, x) = 1$ when $0 < x \leq 1$ and $N_B(F, x) = 4^n$ when $4^n < x \leq 4^{n+1}$ for all $n \in \mathbb{N}$. Hence,

$$\text{range } N_B(F, \cdot) = \{4^n : n \in \mathbb{N} \cup \{0\}\}.$$

Thus, the box-counting fractal string of F is given by

$$\mathcal{L}_B = \{l_1 = 1 : m_1 = 4\} \cup \{l_n = 4^{-(n-1)} : m_n = 3 \cdot 4^{n-1}, n \geq 2\}. \tag{26}$$

Examples 4.5 and 4.6 will be revisited and expanded upon in the following subsection.

4.2. Box-counting zeta functions. Suppose A is a bounded infinite subset of $\mathbb{R}^m$. Each length (or rather, scale) $l_n \in \mathcal{L}_B$ is distinct and, for $n \geq 2$, counted according to the multiplicity $m_n := N_B(A, x_{n+1}) - N_B(A, x_n)$. It will help to note that we can also consider $\mathcal{L}_B$ to be given by the nonincreasing sequence $(\ell_j)_{j\in\mathbb{N}}$ where the distinct values among the ℓ_j repeat the l_n according to the multiplicities m_n. (The convention of distinguishing the notation ℓ_j and l_n in this way is established in [**14**] and its predecessors, where the distinction allows for various results therein to be obtained.) In this setting, we immediately have the following connection between $N_{\mathcal{L}_B}$, the counting function of the reciprocal lengths of $\mathcal{L}_B$, and the box-counting function $N_B(A, x)$.

Lemma 4.7. *For* $x \in (x_1, \infty) \setminus (x_n)_{n\in\mathbb{N}}$,

$$N_{\mathcal{L}_B}(x) = N_B(A, x).$$

Proof. The result follows immediately from Definitions 2.6 and 4.1. □

Moreover, for a bounded infinite set A, the geometric zeta function of the box-counting fractal string $\mathcal{L}_B$ is

$$\zeta_{\mathcal{L}_B}(s) = N_B(A, l_2^{-1}) l_1^s + \sum_{n=2}^{\infty} (N_B(A, l_{n+1}^{-1}) - N_B(A, l_n^{-1})) l_n^s = \sum_{j=1}^{\infty} \ell_j^s,$$

for $\mathrm{Re}(s) > D_{\mathcal{L}_B}$. We take this zeta function to be our *box-counting zeta function* for a bounded infinite set A in Definition 4.8.

Definition 4.8. Let A be a bounded infinite subset of $\mathbb{R}^m$. The *box-counting zeta function* of A, denoted ζ_B, is the geometric zeta function of the box-counting fractal string $\mathcal{L}_B$. That is,

$$\zeta_B(s) := \zeta_{\mathcal{L}_B}(s) = \sum_{n=1}^{\infty} m_n l_n^s,$$

for $Re(s) > D_B := D_{\mathcal{L}_B}$. The value D_B is the *abscissa of convergence* of ζ_B. The set of *box-counting complex dimensions* of A, denoted $\mathcal{D}_B$, is the set of complex dimensions $\mathcal{D}_{\mathcal{L}_B}$ of the box-counting fractal string $\mathcal{L}_B$.

Remark 4.9. Note that we do not consider the case when A is finite. One may, of course, define the box-counting fractal string $\mathcal{L}_B$ for such a set as a finite sequence of positive real numbers. In that case, however, the box-counting zeta function would comprise a finite sum, which would yield an abscissa of convergence $-\infty$ and no complex dimensions; see Remark 3.4. That is, in the context of the theory of complex dimensions of fractal strings, the case of finite sets are not very interesting.

Example 4.10 (Box-counting zeta function of the Cantor set)**.** By Example 4.5, the box-counting fractal string $\mathcal{L}_B$ of the Cantor set C is given by Equation (25). It follows that for $\mathrm{Re}(s) > \log_3 2$, the box-counting zeta function of C is given by

$$\zeta_B(s) = 2 + \sum_{n=2}^{\infty} 2^{n-1} \cdot 3^{-(n-1)s} = 1 + \frac{1}{1 - 2 \cdot 3^{-s}}.$$

Thus, $D_B = \dim_B C = \dim_M C = \log_3 2$ and ζ_B has a meromorphic extension to all of $\mathbb{C}$ given by the last expression in the above equation. Moreover, we have

$$\mathcal{D}_B = \mathcal{D}_{CS} = \mathcal{D}_{\mathcal{L}_{CS}} = \left\{ \log_3 2 + i \frac{2\pi}{\log 3} z : z \in \mathbb{Z} \right\}.$$

Example 4.11. By Example 4.6, the box-counting fractal string of the 1-dimensional self-similar set F, the attractor of the IFS $\mathbf{\Phi}_1$, is given by Equation (26). Hence, the box-counting zeta function of F is given (for $\mathrm{Re}(s) > 1$) by

$$\zeta_B(s) = 4 + \sum_{n=2}^{\infty} 3 \cdot 4^{n-1} \cdot 4^{-(n-1)s} = 1 + \frac{3}{1 - 4 \cdot 4^{-s}}. \tag{27}$$

Thus, $D_B = \dim_B F = \dim_M F = 1$ and ζ_B has a meromorphic extension to all of $\mathbb{C}$ given by the last expression in the above equation. Moreover, we have

$$\mathcal{D}_B = \left\{ 1 + i\frac{2\pi}{\log 4} z : z \in \mathbb{Z} \right\}. \tag{28}$$

Note that the series corresponding to $\zeta_B(1)$ is divergent. Hence, the fractal string $\mathcal{L}_B$ does not correspond to an ordinary fractal string (which, by definition, requires $\zeta_{\mathcal{L}}(1) = \sum_{j=1}^{\infty} \ell_j$ to be convergent).

Remark 4.12. The Cantor set C and the 1-dimensional self-similar set F are each the attractor of a *lattice* iterated function system; see [**14**, §13.1] as well as [**8**–**10**]. Essentially, an IFS is lattice if there is a unique scaling ratio $0 < r < 1$ and positive integers k_j where $r_j = r^{k_j}$ for each $j = 1, \ldots, N$. Note that in each case, the box-counting complex dimensions comprise a set of complex numbers with a unique real part (equal to the box-counting dimension) and a vertical (and arithmetic) progression, in both directions, of imaginary parts.

In the case of the Cantor set C, the box-counting complex dimensions $\mathcal{D}_B$ coincide with the usual complex dimensions $\mathcal{D}_{CS}$. Moreover, the structure of $\mathcal{D}_{CS}$ allows for the application of Theorem 3.19 and, hence, we conclude that C is not Minkowski measurable.

In the case of the 1-dimensional self-similar set F of Examples 4.6 and 4.11, the set of complex dimensions $\mathcal{D}_B$ has no counterpart in the context of usual complex dimensions since F is not the complement of an ordinary fractal string. As such, Theorem 3.19 does not apply. Moreover, since $\dim_M F = 1$, the corresponding results in [**14**, §13.1] do not apply, either. (Fractals with nonnegative integer Minkowski dimension are not considered therein.) This provides motivation for developing a theory of complex dimensions which can take such examples, and many others, into account. The box-counting fractal strings defined in this paper, and investigated further in [**13**], provide a first step in developing one such theory. Analogous comments regarding the further development of a higher-dimensional theory of complex dimensions can be made about the results of [**12**] to be discussed in Section 5.

The following corollary follows readily from Lemma 4.7 and Proposition 3.14. It establishes the equivalence of the box-counting zeta function ζ_B and an integral transform of the (appropriately truncated) box-counting function $N_B(A, x)$.

Corollary 4.13. *Let A be a bounded set. Then*

$$\zeta_B(s) = \zeta_{\mathcal{L}_B}(s) = s \int_{x_1}^{\infty} x^{-s-1} N_B(A, x) dx,$$

for $Re(s) > D_B$.

We close this subsection with a theorem which is a partial statement of our main result, Theorem 6.1. Specifically, the upper box-counting dimension of a bounded infinite set is equal to the abscissa of convergence of the corresponding box-counting zeta function.

Theorem 4.14. *Let A be a bounded infinite subset of $\mathbb{R}^m$. Then $\overline{\dim}_B A = D_B$.*

Proof. The proof follows from a connection made through D_N, the asypmtotic growth rate of the geometric counting function $N_{\mathcal{L}}(x)$ given by Equation (19), where $\mathcal{L} := \mathcal{L}_B$. The equality $D_B = D_N$ follows from Proposition 3.14 and Corollary 4.13. The equality $\overline{\dim}_M A = D_B = D_N$ follows with the addition of Remark 2.15. □

4.3. Tessellation fractal strings and zeta functions. In this subsection, we loosely discuss another type of fractal string defined for a given bounded infinite subset A of $\mathbb{R}^m$. Unlike the box-counting fractal string $\mathcal{L}_B$, which is completely determined by the set A and the box-counting function $N_B(A,\cdot)$, the *tessellation fractal string* defined here depends on the set A, a chosen parameter, and a chosen family of tessellations of $\mathbb{R}^m$.

First, choose a scaling parameter $\lambda \in (0,1)$. For any $n \in \mathbb{N}$, consider the n-th tessellation of $\mathbb{R}^m$ defined by the family of cubes of length λ^n (obtained by taking translates of the cube $[0,\lambda^n]^m$ in $\mathbb{R}^m$). The number of cubes of the n-th tessellation that intersect A is denoted by $m_n(\lambda)$. Let the scale $l_n(\lambda) := \lambda^n$ be of multiplicity $m_n(\lambda)$. This defines the box-counting fractal string $\mathcal{L}(A,\lambda) = (\ell_j)_{j\in\mathbb{N}}$, where $(\ell_j)_{j\in\mathbb{N}}$ is the sequence starting with l_1 with multiplicity m_1, l_2 with multiplicity m_2, and so on. The geometric counting function $N_{\mathcal{L}(A,\lambda)}(x) = \#\{j \in \mathbb{N} : \ell_j^{-1} \geq x\}$ of the fractal string is then well defined.

A more general, but equivalent, definition of the *tessellation fractal string* starts again with a prescribed scaling factor $\lambda \in (0,1)$, and additionally with a *basic shape* U which contains A. Assume that U is a closed set of nonempty interior which can be used to tessellate the whole space $\mathbb{R}^m$ (using isometries from U into $\mathbb{R}^m$). In the applications, U usually contains A, so that it suffices to tessellate the set U only. For each fixed n, we perform tessellations of $\mathbb{R}^m$ (or U) with $\lambda^n U$ (n-th tessellation). Define $m_n(A,U,\lambda)$ analogously as above, by counting the number of elements of the n-th tessellation which intersect A. The *tessellation fractal string* $\mathcal{L}(A,U,\lambda)$ of the set A is then the fractal string defined by

$$\mathcal{L}(A,U,\lambda) := \{l_n(\lambda) = \lambda^n : \ l_n(\lambda) \text{ has multiplicity } m_n(\lambda),\, n \in \mathbb{N}\} = (\ell_j)_{j\in\mathbb{N}}.$$

The middle set is in fact a *multiset*, by that we mean that its elements repeat with prescribed multiplicity. The geometric zeta function of the tessellation fractal string $\mathcal{L}(A,U,\lambda)$, called the *tessellation fractal string*, is then given by

$$\zeta_{\mathcal{L}(A,U,\lambda)}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} m_n(\lambda)\lambda^{ns} \tag{29}$$

for $\mathrm{Re}(s)$ large enough. Also, when defined accordingly, the set of complex dimensions $\mathcal{D}_{\mathcal{L}(A,U,\lambda)}$ of the tessellation fractal string $\mathcal{L}(A,U,\lambda)$ is called the set of *tessellation complex dimensions*.

The main result regarding this zeta function is the following theorem.

Theorem 4.15. *Let U be a closed and bounded set which tessellates $\mathbb{R}^m$. The upper box-counting dimension of a bounded infinite set A in $\mathbb{R}^m$ is equal to the abscissa of convergence $D_{\mathcal{L}(A,U,\lambda)}$ of the geometric zeta function of its tessellation* *fractal string. That is,*

$$\overline{\dim}_B A = D_{\mathcal{L}(A,U,\lambda)}.$$

Proof. We provide the proof in the case of cubic tessellations, that is, when $U = [0,\lambda]^m$. (The proof is similar in the case of general tessellations.) We can assume without loss of generality that $A \subset U$. It suffices to exploit a version of Proposition 2.9 above (note that here $m_n(\lambda) = N_B(A, \lambda^{-n})$, where $N_B(A, x)$ is defined by (iv) in Remark 2.8). In the current setting, we have:

$$\overline{\dim}_B A = \limsup_{n\to\infty} \frac{\log m_n(\lambda)}{\log \lambda^{-n}}. \tag{30}$$

(Also, see [2, p. 41] or [20, p. 24]).

On the other hand, using Cauchy's criterion for convergence, we obtain that the series (29) converges for all $s \in \mathbb{C}$ such that

$$\limsup_{n\to\infty} m_n(\lambda)^{1/n} \lambda^{\mathrm{Re}(s)} < 1,$$

that is,

$$\mathrm{Re}(s) > \frac{\log(\limsup_{n\to\infty} m_n(\lambda)^{1/n})}{\log \lambda^{-1}}.$$

The series (29) diverges if we have the opposite inequality. Therefore, the abscissa of convergence of (29) is

$$D_{\mathcal{L}(A,U,\lambda)} = \frac{\log(\limsup_{n\to\infty} m_n(\lambda)^{1/n})}{\log \lambda^{-1}} = \limsup_{n\to\infty} \frac{\log m_n(\lambda)}{\log \lambda^{-n}}. \tag{31}$$

From (30) and (31) we see that $\overline{\dim}_B A = D_{\mathcal{L}(A,U,\lambda)}$. □

Example 4.16. Let F be the 1-dimensional self-similar set from Examples 4.6 and 4.11. We define U as the unit square $[0,1]^2$ and $\lambda = 1/4$. Here, the scale $l_n(1/4) = 1/4^n$ occurs with multiplicity $m_n(1/4) = 4^n$, defining the corresponding tessellation fractal string $\mathcal{L}(F,U,1/4)$. For $\mathrm{Re}(s) > \log_4 4 = 1$, the tessellation zeta function is given by $\zeta_{\mathcal{L}(F,U,1/4)}(s) = \sum_{n=1}^{\infty} 4^n \cdot 4^{-ns} = 4(4^s - 4)^{-1}$ (cf. Equation (27) in Example 4.11). The dimension is $D_{\mathcal{L}(F,U,1/4)} = \log_4 4 = 1$ which (by Theorem 4.15) is equal to the box-counting dimension of F. It follows that $\zeta_{\mathcal{L}(F,U,1/4)}(s)$ has a meromorphic extension to all of $\mathbb{C}$ given by $4(4^s - 4)^{-1}$. Furthermore, the set of tessellation complex dimensions is equal to the set $\mathcal{D}_B$ of box-counting complex dimensions given in Equation (28). That is,

$$\mathcal{D}_{\mathcal{L}(F,U,1/4)} = \mathcal{D}_B = \left\{ 1 + i\frac{2\pi}{\log 4} z : z \in \mathbb{Z} \right\}.$$

Note that the tessellation fractal string $\mathcal{L}(F,U,1/4)$ is unbounded in the sense that the series given by $\zeta_{\mathcal{L}(F,U,1/4)}(1)$ is divergent.

Analogous results hold regarding the Cantor set C and its (classical and box-counting) fractal strings, zeta functions, and complex dimensions. Further (higher-dimensional) examples will be studied in [**13**].

5. Distance and tube zeta functions

In this section we deal with a class of zeta functions introduced by the first author during the 2009 ISAAC Conference at the University of Catania in Sicily, Italy. More generally, the main results of this section are obtained in the forthcoming paper [**12**], written by the first and third author, along with Goran Radunović. We state here only some of the basic results, without attempting to work at the greatest level of generality. We refer to [**12**] for more general statements and additional results and illustrative examples.

The following definition can be found in [**12**].

Definition 5.1. Let $A \subset \mathbb{R}^m$. The *distance zeta function* of A, denoted ζ_d, is defined by

$$\zeta_d(s) := \int_{A_\varepsilon} d(x, A)^{s-m} dx \tag{32}$$

for $\mathrm{Re}(s) > D_d$, where D_d denotes the abscissa of convergence of the distance zeta function ζ_d and ε is a fixed positive number.

Remark 5.2. It is shown in [**12**] that changing the value of ε modifies the distance zeta function by adding an entire function to ζ_d. Hence, the main properties of ζ_d do not depend on the choice of $\varepsilon > 0$. Such is the case for D_d, the abscissa of covergence of ζ_d (cf. Theorem 5.3), and $\mathrm{res}(\zeta_d; D_d)$, the residue of ζ_d at $s = D_d$ (cf. Theorem 5.5).

The distance zeta function can be used as an effective tool in the computation of the box-counting dimensions of various subsets A of some Euclidean space; see [**12**]. Indeed, one of the basic results concerning the distance zeta function is given in the following theorem, which is Theorem 1 in [**12**]. Note: unlike in Theorem 4.14 above, we allow A to be finite here.

Theorem 5.3. *Let A be a bounded subset of $\mathbb{R}^m$. Then $D_d = \overline{\dim}_B A$.*

Remark 5.4. We do not know if the value of the *lower* box-counting dimension $\underline{\dim}_B A$ can be computed from the distance zeta function ζ_d.

It is shown in [**12**] that the distance zeta function represents a natural extension of the geometric zeta function $\zeta_{\mathcal{L}}$ of a bounded (i.e., summable) fractal string $\mathcal{L} = (\ell_j)_{j\in\mathbb{N}}$. Indeed, we can identify the string with an ordinary fractal string of the form $\Omega = \cup_{j=1}^{\infty} I_j$, where $I_j := (a_{j+1}, a_j)$ and $a_j := \sum_{k\geq j} \ell_k$. Note that $|I_j| = \ell_j$. Defining $A = \{a_j\}_{j=1}^{\infty}$, it is easy to see that $\zeta_d(s) = a(s)\zeta_{\mathcal{L}}(s) + b(s)$, where $a(s)$ vanishes nowhere and $a(s)$ and $b(s)$ are explicit meromorphic functions in the complex plane with poles at the origin. Hence, both zeta functions have the same abscissa of convergence.

5.1. Minkowski content and residue of the distance zeta function.

A remarkable property of the distance zeta function is that its residue computed at $s = D_d$ is closely related to the D_d-dimensional Minkowski content of A; see [**12**].

Theorem 5.5. *Let A be a nonempty bounded set in $\mathbb{R}^m$. Assuming that the distance zeta function can be meromorphically extended to a neighborhood of $s = D_d$ and $D_d < m$, then for its residue at $s = D_d$ we have that*

$$(m - D_d)\mathscr{M}_*^{D_d} \leq \mathrm{res}(\zeta_d(s); D_d) \leq (m - D_d)\mathscr{M}^{*D_d}. \tag{33}$$

If, in addition, A is Minkowski measurable, it then follows that

$$\operatorname{res}(\zeta_d(s); D_d) = (m - D_d)\mathscr{M}^{D_d}. \tag{34}$$

The last part of this result (namely, Equation (34)) generalizes the corresponding one obtained in [**14**] in the context of ordinary fractal strings to the case of arbitrary bounded sets in Euclidean spaces; see [**12**].

Example 5.6. It can be shown that, in the case of the Cantor set C, we have strict inequalities in Equation (33). Indeed, in this case $m = 1$, $D_d = \log_3 2$, and

$$\operatorname{res}(\zeta_d(s); D_d) = \frac{2}{\log 2}\, 6^{-D_d},$$

whereas the values of the lower and upper D_d-dimensional Minkowski contents have been computed in [**14**, Theorem 2.16] (as well as earlier in [**11**]):

$$\mathscr{M}_*^{D_d}(A) = \frac{1}{D_d}\left(\frac{2D_d}{1-D_d}\right)^{1-D_d}, \quad \mathscr{M}^{*D_d}(A) = 2^{2-D_d}.$$

This is a special case of an example in [**12**] dealing with generalized Cantor sets. Generalized Cantor strings, which are a certain type of generalized fractal strings, and their (geometric and spectral) oscillations are studied in [**14**, Ch. 10].

Remark 5.7. An open problem is to determine whether there exists a set A such that one of the inequalities in Equation (33) is strict and the other is an equality.

Remark 5.8. According to a recent result due to Maja Resman in [**18**], we know that if A is Minkowski measurable, then the value of the *normalized D_d-dimensional Minkowski content* of a bounded set $A \subset \mathbb{R}^m$,[10] defined by

$$\frac{\mathscr{M}^{D_d}(A)}{\omega(m - D_d)}, \tag{35}$$

is independent of the ambient dimension m. Here, for $t > 0$, we let

$$\omega(t) := 2\pi^{t/2}t^{-1}\Gamma(t/2)^{-1},$$

where Γ is the classic Gamma function. For any positive integer k, $\omega(k)$ is equal to the k-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^k$. In other words, the value given in Equation (35) is intrinsic to the set A and hence independent of the embedding of A in $\mathbb{R}^k$. Therefore, we may ask if the value of the normalized residue,

$$\frac{\operatorname{res}(\zeta_d(s); D_d)}{(m - D_d)\,\omega(m - D_d)},$$

is also independent of m. Combining the preceding two results (namely, Theorems 5.3 and 5.5), we immediately obtain that if A is Minkowski measurable, then the answer is positive.

[10]This choice of normalized Minkowski content is well known in the literature; see, e.g., [**3**].

5.2. Tube zeta function. Given $\varepsilon > 0$, it is also natural to introduce the following zeta function of a bounded set A in $\mathbb{R}^m$, involving the tube around A (which we view as the mapping $t \mapsto \mathrm{vol}_m(A_t)$, for $0 \le t \le \varepsilon$):

$$\tilde{\zeta}_A(s) = \int_0^\varepsilon t^{s-m-1}\mathrm{vol}_m(A_t)\,dt, \tag{36}$$

for $\mathrm{Re}(s)$ sufficiently large, where ε is a fixed positive number. Hence, $\tilde{\zeta}_A$ is called the *tube zeta function* of A. Assuming $\overline{\dim}_B A < m$, its abscissa of convergence is equal to $\overline{\dim}_B A$, which follows immediately from Theorems 5.3 above and 5.9 below. Tube zeta functions are closely related to distance zeta functions, as shown by the following result; see [**12**].

Theorem 5.9. *If* $\overline{\dim}_B A < m$, *then for any* $\varepsilon > 0$,

$$\zeta_d(s) = \varepsilon^{s-m}\mathrm{vol}_m(A_\varepsilon) + (m-s)\tilde{\zeta}_A(s). \tag{37}$$

It follows from Equation (37) that the abscissae of convergence of the zeta functions ζ_d and $\tilde{\zeta}_A$ are the same. This identity extends the analogous one obtained in [**12**] in the case of fractal strings. Using this result and Theorem 5.5, it is easy to derive the following consequence; see [**12**].

Corollary 5.10. *If* $D = \dim_B A$ *exists,* $D < m$, *and there exists a meromorphic extension of* $\tilde{\zeta}_A(s)$ *to a neighborhood of* $s = D$, *then*

$$\mathscr{M}_*^D \le \mathrm{res}(\tilde{\zeta}_A(s); D) \le \mathscr{M}^{*D}.$$

In particular, if A *is Minkowski measurable, then*

$$\mathrm{res}(\tilde{\zeta}_A(s); D) = \mathscr{M}^D.$$

As we can see, the tube zeta function is ideally suited to study the Minkowski content.

Example 5.11 (Minkowski measurable discrete spiral in the plane)**.** We consider a discrete spiral A constructed in [**16**]. Let p be a fixed positive number, and define A to be the union of the vertices of the sequence of regular polygons P_n of circumradius n^{-p}, with common center at the origin, $n \ge 3$, such that the distance between any two consecutive vertices of P_n is asymptotically the same as $n^{-p} - (n+1)^{-p}$; see [**16**, p. 462]. (Note that this implies that the number of vertices on P_n is asymptotically equal to n,[11] so that we can assume that P_n is the regular n-gon of circumradius n^{-p}.) Then A is Minkowski measurable, with box dimension $D = 2/(p+1)$; see [**16**, Theorem 2]. The corresponding Minkowski content is equal to

$$\mathscr{M}^D(A) = \frac{\pi^2}{p}\left(\frac{p}{2}\right)^{\frac{2}{p+1}} + p\left(\frac{\sqrt{2}}{p}\right)^{\frac{2p}{p+1}} + C_p\frac{2\pi p^{\frac{2}{p+1}-1}}{p+1}$$

(see [**16**, Lemmas 1 to 4]), where the constant C_p is given in [**16**, §2.1.1 and §2.1.2]. According to Corollary 5.10, assuming the existence of a meromorphic extension of the tube zeta function $\tilde{\zeta}_A(s)$ to a neighborhood of $s = D$ (which is expected, but

[11]Indeed, since the circumradius of P_n is asymptotically n^{-p} as $n \to \infty$, its circumference is asymptotically equal to $2\pi n^{-p}$. On the other hand, if N_n is the number of vertices of P_n, then since the length of each side is $n^{-p} - (n+1)^{-p} \sim p n^{-p-1}$ as $n \to \infty$, we conclude that the circumference of P_n is asymptotically equal to $N_p p n^{-p-1}$. From $2\pi n^{-p} \sim N_p p n^{-p-1}$, we deduce that $N_n \sim (2\pi/p)\, n$ as $n \to \infty$.

not proved, at this stage), it follows that the residue of $\tilde{\zeta}_A(s)$ at $s = D$ is equal to the indicated value of $\mathscr{M}^D(A)$.

The question of the existence of a meromorphic extension of $\tilde{\zeta}_A$ (for the above example and especially for more general bounded sets $A \subset \mathbb{R}^m$) will be the subject of a sequel to [**12**]. It is interesting to note that the discrete spiral A has the same Minkowski dimension as the union B of the sequence of circles of radii n^{-p} with a common center at the origin, $n \in \mathbb{N}$; see [**23**, Corollary 2]. However, the corresponding Minkowski contents are different, and clearly, $\mathscr{M}^D(A) < \mathscr{M}^D(B)$.

Remark 5.12. The box-counting zeta function ζ_B of a set $A \subset \mathbb{R}^m$ given by Definition 4.8 is closely related to the tube zeta function $\tilde{\zeta}_A$. To see this, it suffices to perform the change of variables $x = t^{-1}$ in Equation (36) and compare with Corollary 4.13. Note that for $x > 0$, we have (under suitable hypotheses) $\mathrm{vol}_m(A_{1/x}) \asymp x^{-m} N_B(A, x)$ as $x \to \infty$. Here, $N_B(A, x)$ is defined as the number of x^{-1}-mesh cubes that intersect A; see (4) in Remark 2.8. It is clear, however, that these two zeta functions are in general not equal to each other. Moreover, we do not know if the related two sets of complex dimensions of A, corresponding to these two zeta functions, coincide.

Various generalizations of the notion of distance zeta function are possible. One of them, which is especially interesting, deals with zeta functions associated to relative fractal drums. By a *relative fractal drum*, introduced in [**12**], we mean an ordered pair (A, Ω), where A is an arbitrary nonempty subset of $\mathbb{R}^m$, and Ω an open subset such that A_ε contains Ω for some positive ε and the m-dimensional Lebesgue measure of Ω is finite. The corresponding *relative zeta function* (or the distance zeta function of the relative fractal drum), also introduced in [**12**], is defined much as in Equation (32):

$$\zeta_d(s; A, \Omega) := \int_\Omega d(x, A)^{s-m} dx.$$

It is possible to show that the abscissa of convergence of the relative zeta function is equal to the relative box dimension $\overline{\dim}_B(A, \Omega)$; see [**12**] for details and illustrative examples. Note that the sets A and Ω may even be unbounded.

Remark 5.13. It is easy to see that the notion of relative fractal drum (A, Ω) is a natural extension of the notion of fractal string $\mathcal{L} = \{\ell_j\}$. Indeed, for a given (standard) fractal string $\mathcal{L} = \{\ell_j\}$, it suffices to define $A = \{a_j\}$, where $a_j := \sum_{k \geq j} \ell_k$ and Ω is the ε-neighborhood of $\cup_{k \geq 1}(a_{k+1}, a_k)$, with ε being any fixed positive number. We warn the reader that the notion of generalized fractal string already exists and does not coincide with the notion of relative fractal drum. Specifically, in [**14**, Ch. 4], a generalized fractal string is defined to be a locally positive or locally complex measure on $(0, \infty)$ supported on a subset of (x_0, ∞), for some positive real number x_0.

6. Summary of results and open problems

For a bounded infinite set A, recall that $\overline{\dim}_B A$ denotes the upper box-counting dimension of A given by Equation (5), $\overline{\dim}_M A$ denotes the upper Minkowski dimension of A given by Equation (8), D_B denotes the abscissa of convergence of the box-counting zeta function ζ_B of A given in Definition 4.8, $\rho_{\mathcal{L}}$ denotes the order of the geometric counting function $N_{\mathcal{L}}$ given by Equation (22) where $\mathcal{L} = \mathcal{L}_B$, D_N

denotes the value corresponding to the (asymptotic) growth rate of $N_{\mathcal{L}}$ given by Equation (19), and D_d is the abscissa of convergence of the distance zeta function ζ_d given in Definition 5.1.

The following theorem summarizes our main result (as stated in Theorem 1.1 of the introduction), which pertains to the determination of the box-counting dimension of a bounded infinite set. (Recall that the equality $\overline{\dim}_B A = D_d$ is established in [**12**]; see Theorem 5.3 above.)

Theorem 6.1. *Let A be a bounded infinite subset of $\mathbb{R}^m$ and let $\mathcal{L} = \mathcal{L}_B$ be the corresponding box-counting fractal string. Then the following equalities hold:*

$$\overline{\dim}_B A = \overline{\dim}_M A = D_B = \rho_{\mathcal{L}} = D_N = D_d.$$

Proof. The classic equality $\overline{\dim}_B A = \overline{\dim}_M A$ is established in [**2**]. The equality $\overline{\dim}_M A = D_B = D_N$ follows from Theorem 4.14. The equality $\rho_{\mathcal{L}} = D_N$ is then the result of Lemma 3.24. Finally, as was recalled just above, the equality $\overline{\dim}_M A = D_d$ is established in [**12**]; see Theorem 5.3. □

Recall that, as stated in Definition 2.6, $N_B(A, x)$ denotes the maximum number of disjoint balls of radius x^{-1} centered in A. In this setting and for $\varepsilon > 0$ we have

$$B_m \varepsilon^m N_B(A, \varepsilon^{-1}) \leq \mathrm{vol}_m(A_\varepsilon), \tag{38}$$

where A_ε is the ε-neighborhood of A, B_m is the m-dimensional volume of a ball in $\mathbb{R}^m$ with unit radius, and $0 < \varepsilon < x_1^{-1}$, where x_1^{-1} is given by Proposition 4.3.

Motivated by Equation (38) and Theorem 3.15, we propose the following open problem (which is stated rather roughly here).

Open Problem 6.2. *Let A be a bounded infinite subset of $\mathbb{R}^m$ with box-counting fractal string $\mathcal{L}_B$. Assume suitable growth conditions on ζ_B (such as the languidity of ζ_B on an appropriate window, see [**14**, Chs. 5 & 8]) and assume for simplicity that all of the complex dimensions are simple (i.e., are simple poles of ζ_B). Then, as $\varepsilon \to 0^+$, compare the quantities*

$$\mathrm{vol}_m(A_\varepsilon), \quad \varepsilon^m N_{\mathcal{L}_B}(\varepsilon^{-1}), \quad \text{and } \varepsilon^m \left(\sum_{\omega \in \mathcal{D}_B} \frac{\varepsilon^{-\omega}}{\omega} \mathrm{res}(\zeta_B(s); \omega) + R(\varepsilon^{-1}) \right), \tag{39}$$

where $R(\varepsilon^{-1})$ is an error term of small order.

If one were to provide a more precise version of the above open problem and solve it, one might consider pursuing a generalization of Theorem 3.19 in the spirit of the theory of complex dimensions of fractal strings, as described in [**14**], and of its higher-dimensional counterpart in [**8**–**10**]. Naturally, the clarified version of this open problem would consist of replacing the implicit 'approximate equalities' in Equation (39) with true equalities, modulo suitable modifications and under appropriate hypotheses.

Analogously (but possibly more accurately), in light of the results from [**12**] discussed in Section 5, as well as from the results about fractal tube formulas obtained in [**14**, Ch. 8] for fractal strings and in [**8**, **9**] and especially [**10**] in the higher-dimensional case (for fractal sprays and self-similar tilings),[12] we propose the following open problem. (A similar problem can be posed for the tube zeta function $\tilde{\zeta}_A$ discussed in Section 5.2.)

[12]A survey of the results of [**8**–**10**] can be found in [**14**, §13.1].

Open Problem 6.3. *Let A be a bounded subset of $\mathbb{R}^m$ with distance zeta function ζ_d. Under suitable growth assumptions on ζ_d (such as the languidity of ζ_B on an appropriate window, see* [**14**, Chs. 5 & 8]*), and assuming for simplicity that all of the corresponding complex dimensions are simple, calculate the volume of the tubular neighborhood of A in terms of the complex dimensions of A (defined here as the poles of the meromorphic continuation of ζ_d union the 'integer dimensions' $\{0,1,\ldots,m\}$) and the associated residues.*

Moreover, even without assuming that the complex dimensions are simple, express the resulting fractal tube formula as a sum of residues of an appropriately defined 'tubular zeta function' (in the sense of [**8**–**10**]*).*

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