Nonholonomic deformation of coupled and supersymmetric KdV equation and Euler-Poincaré-Suslov method

Partha GUHA

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Juin 2013

IHES/M/13/15

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Partha Guha*
Institut des Hautes Études Scientifiques
Le Bois-Marie 35, Route de Chartres
F-91440 Bures-sur-Yvette, France

S.N. Bose National Centre for Basic Sciences
JD Block, Sector-3, Salt Lake
Calcutta-700098, INDIA.

dedicated to Professor Victor Kac on his 70th birthday with great respect and admiration

Abstract

Recently Kupershmidt [36] presented a Lie algebraic derivation of a new sixth-order wave equation, which was proposed by Karasu-Kalkani et al [30]. In this paper we demonstrate that Kupershmidt's method can be interpreted as an infinite-dimensional analogue of the Euler-Poincaré-Suslov (EPS) formulation. In a finite-dimensional case we modify Kupershmidt's deformation of Euler top equation to obtain the standard EPS construction on $SO(3)$. We extend Kupershmidt's infinite-dimensional construction to construct nonholonomic deformation of a wide class of coupled KdV equations, all these equations follow from the Euler-Poincaré-Suslov flows of the right invariant L^2 metric on the semidirect product group $\widehat{Diff(S^1) \ltimes C^\infty}(S^1)$, where $Diff(S^1)$ is the group of orientation preserving diffeomorphisms on a circle. We generalize our construction to two component Camassa-Holm equation. We also give a derivation of a nonholonomic deformation of the $N = 1$ supersymmetric KdV equation, dubbed as sKdV6 equation and this method can be interpreted as an infinite-dimensional supersymmetric analogue of the Euler-Poincaré-Suslov (EPS) method.

Mathematics Subject Classifications (2000): 53A07, 53B50.

Key Words : diffeomorphism, geodesic flows, coupled KdV equations, Bott-Virasoro group, superconformal group, Kuper KdV equations, nonholonomic deformation.

*E-mail: partha@bose.res.in

Contents

1 Introduction

Recently Karasu-Kalkani et al. [30] applied Painlevé test to a class of sixth-order nonlinear wave equations and found three of these were previously known, but the 4^{th} one turned out to be new one

$$(\partial_x^2 + 8u_x\partial_x + 4u_{xx})(u_t + u_{xxx} + 6u_x^2) = 0. \tag{1}$$

One recognizes immediately the potential form of the KdV equation appearing in the second factor of the left-hand side of (1). The factored way of writing this equation has the advantage of showing immediately that all solutions of (potential) KdV are also solutions of the new sixth-order equation.

After a slight change of variables

$$v = u_x, \qquad w = u_t + u_{xxx} + 6u_x^2$$

this equation converts to the KdV equation with a source satisfying a third order ordinary differential equation

$$\begin{aligned} v_t + v_{xxx} + 12vv_x - w_x &= 0 , \\ w_{xxx} + 8vw_x + 4wv_x &= 0 , \end{aligned} \tag{2}$$

which Kupershmidt called it the KdV6 equation. The authors of [30] obtained the Lax pair and an auto-Bäcklund transformation for (2). They claimed that (2) is different from the KdV equation with self-consistent sources (KdVESCS) and posed an open problem to find higher symmetries and asked if higher conserved densities and a Hamiltonian formalism exist for (2). In a recent paper Ramani et al. [48] bilinearize the KdV6 equation and derive a new, simpler, auto-Bäcklund transformation, and starting from the solutions of the KdV equation they constructed those of the KdV6 in the form of M kinks and N poles, which indeed involve an arbitrary function of time. In spite of all these results Ramani et al. were unable to find higher symmetries of the KdV6 equation. In fact Kupershmidt described this equation as a nonholonomic perturbations of bi-Hamiltonian systems. Most important fact is that the eq. (2) had been written for the first time by Calogero [1] and it is contained in the book of Calogero and Degasperis [10]. As such it shares many of the properties of the equations associated to the Schrödinger spectral problem.

The soliton equation with self-consistent sources has many physical applications, for example, it describes the interaction of long and short capillary-gravity waves. In a recent paper Yao and Zeng [52] showed that the KdV6 equation is equivalent to the Rosochatius deformation of KdV equation with self-consistent sources. In our earlier paper [20] we extended Yao and Zeng result to construct many other equations equivalent to the KdV6 equation and we identified that the constraint equation of w is a stabilizer equation of the Virasoro orbit. We tacitly replaced this equation with an equivalent partner equation to obtain various new *avatars* of the KdV6 equation. Essentially Yao and Zeng [52] adopted this philosophy in a more adhoc style. We put it in a more systematic form using Kirillov's coadjoint orbit method. In [20], we extended Kuperschmidt's formalism [36] to extended Virasoro algebra $\widehat{Vir \ltimes C^\infty}(S^1)$ to construct the Ito6 equation. It is known [22, 23] that a wide class of coupled KdV equations can be manifested as geodesic flows of the right invariant L^2 metric on the semidirect product group $\widehat{Diff(S^1) \ltimes C^\infty}(S^1)$, where $Diff(S^1)$ is the group of orientation preserving diffeomorphisms on a circle. In this paper we construct nonholonomic deformation of *all* the coupled KdV systems, e.g., the Ito system, the modified dispersive water wave system, the Kaup-Boussinesq equation and

[1] Thanks to Professor Francesco Calogero for sharing this information

the Broer-Kaup system from the coadjoint representation of extended Virasoro algebra [22, 23].

The theory of the integration of nonholonomic mechanical systems is not as fully explored as that of holonomic systems. G. K. Suslov's [49] problem of the rotation of a rigid body about a fixed point with a constraint imposed on the angular velocity of the body, and S. A. Chaplygin's [11] problem of the rolling of a dynamically asymmetric ball on a rough horizontal plane are well-known examples of integrable mechanical systems with nonholonomic constraints. Authors generalize these two problems and present an entire family of new integrable systems with nonholonomic constraints by introducing additional gyroscopic forces and nonlinear potentials of sufficiently general form into the system. There is a large number of literature dedicated to the Chaplygin ball (see for example [8, 9, 16, 50]) as well as Suslov problem (for example [15, 19, 28]).

In an interesting paper Kersten et al [31] proved that the Kupershmidt deformation of a bi-Hamiltonian system is itself bi-Hamiltonian. Moreover, the (so called) Magri hierarchies of the initial system give rise to Magri hierarchies of Kupershmidt deformations as well. Note that the Magri hierarchy on a bi-Hamiltonian equation is an infinite sequence of conservation laws which satisfy Lenard recursion scheme, follows from the ingenious discovery of Magri [40, 41] that integrable Hamiltonian systems usually prove to be bi-Hamiltonian.

It is known that $N = 1$ superconformal (or super Virasoro) algebra can be related to fermionic extensions of the KdV equation. Using the superspace formalism one can construct *two* different fermionic extensions of the KdV equation. The first extension was proposed by Manin and Radul [44]. The Manin-Radul version of the super KdV is defined in terms of three independent variables ϑ, x, t, where ϑ is a Grassmann odd variable. In the supersymmeteric version of the KdV equation the variable x acquires a Grassmann partner ϑ, so $X \equiv (x, \vartheta)$ are coordinates in a one dimensional superspace. This $N = 1$ SUSY KdV is given by [44, 42]

$$\Phi_t + D^6\Phi + 3D^2(\Phi D\Phi) = 0, \tag{3}$$

where $\Phi(t, x, \vartheta) = \phi(t, x) + \vartheta\, u(t, x)$ stands for superfield. Thus equation (1.3) can be expressed as

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x - 3\phi\phi_{xx}\ , \\ \phi_t &= -\phi_{xxx} + 3(u\phi)_x. \end{aligned} \tag{4}$$

These equations are invariant under the infinitesimal supersymmetry transformation $\delta\phi = \epsilon u$, $\delta u = \epsilon\partial\phi$, where ϵ is a constant Grassmann parameter, which is equivalent to the superspace translation $\delta x = \vartheta\epsilon$, $\delta\vartheta = \epsilon$.

Mathieu [42] showed that this equation is associated to the (super) Lax operator

$$L = D^4 - \Phi D \qquad \text{or} \qquad L = D^4 - (D\Phi) + \Phi D,$$

where $D = \frac{\partial}{\partial\vartheta} + \vartheta\frac{\partial}{\partial x}$ denotes superderivative. After a suitable scaling Mathieu [42] showed that the Eqn. (4) is equivalent to the sKdV equation obtained by Manin and

Radul [44] from the reduction of the super Kadomtsev-Petviashvili (sKP) hierarchy. In fact, the second Hamiltonian structure of this system was shown by Mathieu [42] and it corresponds to the superconformal algebra of the superstring theories.

The *second class* of super KdV was proposed by Kupershmidt [35], given by

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x - 6\phi\phi_x \ , \\ \phi_t &= -\phi_{xxx} + 3u_x\phi + 6u\phi_x. \end{aligned} \tag{5}$$

This equation is associated to the Lax operator

$$L = \partial^2 - u + \phi\partial^{-1}\phi.$$

In fact Kupershmidt [35] demonstrated the bi-hamiltonian property of this equation. Ovsienko and Khesin [45] showed that this version of super KdV equation is the Euler-Poincaré flow corresponding to the inertia operator for the Neveu-Schwarz and Ramond superalgebras [33, 43], which are the simplest super analogues of the Virasoro algebra.

It should be noted that that the Kupershmidt's version of super KdV equation does not preserve SUSY transformation, although it yields bihamiltonian structures. Sometimes it is approapriate to call Eqn. (5) the *Kuper-KdV* equation. Moreover, a Painlevé analysis by Mathieu of possible supersymmetric extensions of the KdV equation seems to suggest that the only integrable extensions are the Manin-Radul KdV and the one of Kupershmidt.

1.1 Motivation, result and plan

This paper serves two purposes: it provides a (geometric) method to derive various nonholonomic deformations of coupled KdV and super KdV equations which are integrable, and it elaborates some techniques which promise to be useful in formulating new types of nonholonomic deformed bi-Hamiltonian systems. Also this method turns out to be an infinite-dimensional generalization of the celebrated Euler-Poincaré-Suslov method.

At first we consider Kupershmidt's derivation of the KdV6 equation or nonholonomic deformation of the KdV equation. We discuss the significance of his method and its connection to Euler-Poincaré-Suslov (EPS) method. this method can be considered as an infinite-dimensional analogue of the EPS scheme. We reformulate the derivation of Kupershmidt using coadjoint orbit method of the Virasoro algebra. Then we focus on to finite dimensional systems. We slightly modify Kupershmidt's deformation of the Euler top equation to obtain nonholonomic deformation of the top equation. Note that Kupershmidt's deformation leads to holonomic deformation of the top equation.

In this paper we construct nonholonomic deformation of all the coupled KdV systems, e.g., the Ito system, the modified dispersive water wave system, the Kaup-Boussinesq equation and the Broer-Kaup system from the coadjoint representation of extended Virasoro algebra [22, 23]. We give a geometrical construction the nonholonomic deformation of the two component Camassa-Holm equation. Given two Hamiltonian structures $\mathcal{O}_2$ and $\mathcal{O}_1$ for the KdV equation, it is customary to define constraint equation as $\mathcal{O}_2 w = 0$. Recently, Yao and Zeng [52, 53] considered another kind of generalized Kupershmidt deformation where the constraint equation is given by $(\mathcal{O}_2 - \lambda\mathcal{O}_1)w = 0$. In this paper we also discuss this nonholonomic deformation from the geometric point of view and compute the generalized Kupershmidt deformation of the two component Camassa-Holm equation. Finally we extend the Kupershmidt's programme to construct the nonholonomic deformation of the $N = 1$ supersymmetric KdV equation, dubbed as the $N = 1$ super KdV6 equation.

The **plan** of the paper is as follows. We give a comprehensive description of the Kupershmidt's construction of nonholonomic systems and its connection to the Euler-Poincaré-Suslov method in Section 2. We also show how Kupershmidt's nonholonomic deformation of Euler top can be mapped to EPS problem on $SO(3)$. Section 3 is devoted to Euler-Poincaré-Suslov flows on the coadjoint orbit of the extended Virasoro algebra. We divide our job in two steps, at first we compute the Hamiltonian structures of various coupled KdV systems, then using these we obtain the nonholonomic deformations of the coupled KdV equations. In Section 4, at first we discuss $N = 1$ superconformal algebra and superKdV equation and using this result we construct the nonholonomic deformation of the $N = 1$ super KdV equation. We finish our paper with a modest outlook.

2 Kupershmidt's construction and nonholonomic deformation

In an interesting paper Kupershmidt [36] described the new sixth-order equation as a *nonholonomic* deformation of the KdV equation and proved the integrability property. The novelty of his paper is far reaching, in fact he formulated a method which is closely associated to the Euler-Poincaré-Suslov (EPS) method. We describe his method both for finite and infinite dimensions separately. We start with the infinite-dimensional version of Kupershmidt's construction, in fact, this is the main aspect of his construction.

2.1 Kupershmidt's scheme for KdV6 equations and EPS formalism

Let us start with the sixth-order equation proposed by Karasu-Kalkani et al. By rescaling v and t of equation (2) Kupershmidt further modified this to

$$\begin{aligned} u_t - 6uu_x - u_{xxx} + w_x &= 0, \\ w_{xxx} + 4uw_x + 2u_x w &= 0, \end{aligned} \tag{6}$$

which he called the KdV6 equation. This can be converted into bi-Hamiltonian form

$$u_t = B^1\left(\frac{\delta H_{n+1}}{\delta u}\right) - B^1(w) = \quad B^2\left(\frac{\delta H_n}{\delta u}\right) - B^1(w), \quad B^2(w) = 0, \tag{7}$$

where

$$B^1 = \partial = \partial_x, \quad B^2 = \partial^3 + 2(u\partial + \partial u) \tag{8}$$

are the two standard Hamiltonian operators of the KdV hierarchy, $n = 2$, and

$$H_1 = u, \ H_2 = u^2/2, \ H_3 = u^3/3 - u_x^2/2, ... \tag{9}$$

are the conserved densities. Thus after simplifying the sixth-order equation Kupershmidt put it in a bihamiltonian form which indeed helps us to investigate the equation along the direction of coadjoint orbit method. We now present the reformulation the KdV6 equation using Virasoro orbit.

2.1.1 Virasoro orbit, infinite-dimensional EPS flow and KdV6

Let us consider the Lie algebra of vector fields $Vect(S^1)$ on a circle S^1. The dual of this algebra is identified with space of quadratic differential forms $\mathcal{F}_2$. The pairing between $f(x)\frac{d}{dx} \in Vect(S^1)$ and $u(x)dx^{\otimes 2} \in \mathcal{F}_2$ is defined as

$$< u(x)dx^2, f(x)\frac{d}{dx} >= \int_0^{2\pi} u(x)\ f(x)dx.$$

The Virasoro algebra Vir has a unique non-trivial central extension, described by the Gelfand-Fuchs cocycle

$$\omega_1(f,g) = \int_{S^1} f'g''dx.$$

The elements of Vir can be identified with the pairs (2π periodic function , real number). The commutator in Vir takes the form

$$[(f(x)\frac{d}{dx}, a), (g(x)\frac{d}{dx}, b)] = ((fg' - gf')\frac{d}{dx}, \int_{S^1} f'g''\ dx).$$

The dual space Vir^* can be identified to the set $\{(\mu, udx^2) \mid \mu \in \mathbf{R}\}$. A pairing between a point $(\lambda, f(x)\frac{d}{dx}) \in Vir$ and a point $(\mu, udx^2) \in Vir^*$ is given by

$$< (\mu, u(x)dx^2), (\lambda, f(x)\frac{d}{dx} >= \lambda\mu + \int_{S^1} f(x)u(x)\ dx.$$

Lemma 1 *The coadjoint action of the Virasoro algebra $(\lambda, f(x)\frac{d}{dx}) \in Vir$ on its dual $(\mu, udx^2) \in Vir^*$ is given by*

$$ad^*_{(\lambda, f(x)\frac{d}{dx})}(\mu, udx^2) \;=\; \mu f''' + 2f'u + fu'. \tag{10}$$

We fix the hyperplane $\mu = \frac{1}{2}$. The kernel of the ad^* yields the stabilier set of Virasoro orbit

$$f''' + 4u'f + 4uf' = 0. \tag{11}$$

The second Hamiltonian operator of the KdV equation can be easily derived from the coadjoint action of the Virasoro algebra, which is given by

$$\mathcal{O}^2_{KdV} = D^3 + 4uD + 2u_x, \qquad \text{where } D = \frac{d}{dx}. \tag{12}$$

It is known that the first Hamiltonian operator of the KdV equation can also be derived (please see next section for derivation of this operator when we deal with the coupled systems) from the coadjoint action using frozen Lie-Poisson structure. It is given by $\mathcal{O}^1_{KdV} = D$.

Proposition 1 *The KdV6 equation is constraint Hamiltonian flow on the Virasoro orbit*

$$u_t = ad^*_{\nabla H}(u) - w_x = \mathcal{O}^2_{KdV}\frac{\delta H}{\delta u} - \mathcal{O}^1_{KdV}(w) \qquad s.\ t. \qquad < \nabla H, w_x >= 0 \tag{13}$$

and

$$\mathcal{O}^2_{KdV}(w) = 0, \tag{14}$$

where $H = \frac{1}{2}\int_{S^1} u^2\, dx$.

Our task is to find w such that it satisfies $< \nabla H, w_x >= 0$. The workable approach to this problem is to choose Kupershmidt's scheme, i.e.,

$$w = \frac{\delta G}{\delta u}, \tag{15}$$

for some function G. This immediately leads to

$$< \nabla H, w_x >=< \frac{\delta H}{\delta u}, \partial(\frac{\delta G}{\delta u}) >= \{H, G\}_1 = 0.$$

Since $\mathcal{O}^2_{KdV}(w) = 0$, hence

$$\{H, G\}_2 = 0.$$

Thus G commutes with H with respect to both the Poisson structures. It is easy to generalize this construction to sequence of Hamiltonians H_n. Thus G commutes with H_n w.r.t. to both the brackets, i.e.,

$$\{H_n, G\}_1 = 0 = \{H_n, G\}_2. \tag{16}$$

Second type of nonholonomic deformation Let us define pencil of Hamiltonian structures

$$\mathcal{O}^\lambda_{KdV} = \mathcal{O}^2_{KdV} - \lambda\mathcal{O}^1_{KdV}. \tag{17}$$

Then equation (16) implies $\{H, G\}_\lambda = 0$, i.e. G commutes with H with respect to pencil of Hamiltonian structures. This immediately yields a slightly modified nonholonomic deformation of the KdV equation

$$u_t = ad^*_{\nabla H}(u) - w_x = \mathcal{O}^2_{KdV}\frac{\delta H}{\delta u} - \mathcal{O}^1_{KdV}(w) \qquad \mathcal{O}^\lambda_{KdV}(w) = 0, \tag{18}$$

which is given by

$$\begin{aligned} u_t &= 6uu_x + u_{xxx} - w_x\ , \\ w_{xxx} + 4uw_x + 2u_x w - \lambda w_x &= 0\ . \end{aligned} \tag{19}$$

This equation was derived by Yao and Zeng [52, 53].

Other avatars of KdV6 equation Suppose ψ_1 and ψ_2 are the solutions of the spectral equation

$$\Delta\psi \ = \ \psi_{xx} + (u - \lambda)\psi \ = \ 0, \tag{20}$$

then the product $w = \psi_i^2$ satisfies the constraint equation $w''' + 2u'w + 4uw' - \lambda w' = 0$, which yields

$$2\psi_1(\psi_{1xx} + (u-\lambda)\psi_1)_x + 6\psi_{1x}(\psi_{1xx} + (u-\lambda)\psi_1) = 0.$$

This gives rise to Ermakov-Pinney equation

$$\psi_{1xx} + (u-\lambda)\psi_1 = \frac{\mu}{\psi_1^3}. \tag{21}$$

If we start with the hierarchy and consider spectral problem with N distinct eigenvalues and follow the similar procedure we obtain the Rosochatius system. Essentially this is the observation of Yao and Zeng.

The integrable Ermakov-Pinney deformation of the KdV6 equation follows from the usual KdV framework. If ψ_1 and ψ_2 satisfy Hill's equation then $\psi = \sqrt{A\psi_1^2 + 2B\psi_1\psi_2 + C\psi_2^2}$ satisfies Ermakov-Pinney equation

$$\psi'' + u(x)\psi = \frac{\sigma}{\psi^3}, \qquad \sigma = AC - B^2,$$

and $\{\psi_1^2, \psi_2^2, \psi_1\psi_2\}$ satisfy constraint equation $w''' + 4uw' + 2u'w = 0$.

2.2 Kupershmidt's construction for finite-dimensional system

We now consider the finite-dimensional case, this is relatively less studied by the integrable system community. In fact his method only leads to holonomic deformation, we modify it and establish a connection between his method and classical EPS method.

The constraint equation related to second Hamiltonian operator $B^2(w) = 0$[2] is, in general, nonholonomic only for systems which are either differential or difference on $\mathbb{Z}$. Kupershmidt showed that in Classical Mechanics with a finite number of degrees of freedom, the constrain $B_2(w) = 0$ becomes holonomic. Let us quickly recapitulate Kupershmidt formalism in finite-dimensional system.

Kupershmidt considered the Euler top equation

$$\dot{x}_i = \alpha_i x_j x_k, \qquad \text{with } \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3. \tag{22}$$

with $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ being arbitrary but fixed vector parameter. He considered Poisson brackets

$$\{x_i, x_j\}^\gamma = \gamma_k x_k, \qquad \text{with } \ \gamma \in \mathbb{R}^3, \tag{23}$$

and Hamiltonian $H_c = \frac{1}{2}\sum_{i=1}^3 c_i x_i^2$ to express top equation in a Hamiltonian form

$$\dot{x}_i = \{x_i, H_c\} = \mathbb{J}_\gamma \times \nabla H_c, \qquad (\text{ where } \ (\alpha, \mathbf{c}) = 0 \tag{24}$$

[2]Here we keep his notation, i.e., B_2 and B_1 stand for second and first Hamiltonian structures

where $\mathbb{J}_\gamma$ is the Poisson matrix related to (23). Kupershmidt considered this to be second Hamiltonian structure B^2 and first one B^1 is given another Poisson matrix $\mathbb{J}_\beta$ related to Poisson brackets $\{x_i, x_j\}^\beta = \beta_k x_k$. Therefore the constraint equation $B^2(w) = 0$ is

$$\mathbf{w} \times \mathbb{J}_\gamma = 0, \tag{25}$$

so that $\mathbf{w} = k_1 \mathbb{J}_\gamma$, where k_1 is a constant. Thus the deformation equation becomes

$$\dot{x}_i = \alpha_i x_j x_k - k(\mathbb{J}_\beta \times \mathbb{J}_\gamma)_i, \tag{26}$$

which can be easily reduced to $\dot{x}_i = K\alpha_i x_j x_k$, where K is another constant. Thus he showed that the deformation for Euler top is not nonholonomic anymore and overall effect of the perturbation amounts to the time rescaling of the original top. Hence Kupershmidt's formalism of deformation leads to only holonomic deformation of the Euler top equation. After a brief recapitulation of the finite-dimensional EPS method we will modify this construction.

2.3 Modified Kupershmidt's scheme for classical systems and the Euler-Poincaré-Suslov method

We wish to show that Kupershmidt's scheme for the Euler top is closely related to Euler-Poincaré-Suslov (EPS) formalism [6, 17, 18, 29]. The reduced dynamics of the constrained generalized rigid body is governed by the Euler-Poincaré-Suslov equations which in fact one of the best known demonstration of EPS formalism in integrable systems.

2.3.1 Recap of Euler-Poincaré-Suslov (EPS) method

We briefly recall the definition of EPS from [6, 18, 29]. Let Q be an n-dimensional manifold. The distribution can be defined by m independent 1-forms α_i via

$$\mathcal{D}_q = \{\gamma \in T_qQ,\ \alpha_i(\gamma) = 0,\ i = 1, \cdots, m\}.$$

The smooth path $c(t)$, $t \in \Delta$ is called admissible (or allowed by constraints) if the corresponding velocity $\dot{c}(t)$ belongs to $\mathcal{D}_{c(t)}$ for all $t \in \Delta$. The admissible path $c(t)$ is called a nonholonomic geodesic line if it satisfied d'AlambertLagrange equations

$$\pi(\nabla_{\dot{c}(t)}\dot{c}(t)) = 0,$$

where $\pi : T_qQ \to \mathcal{D}_q$ is the orthogonal projection.

Let $\{.,.\}$ be canonical Poisson brackets on T^*Q. A Hamiltonian vector field X_h satisfies $< dg, X_h >= \{g, h\}$ for all $g : T^*Q \to \mathbb{R}$. Let M_c be the constraint submanifold in the phase space T^*Q:

$$M_c = \{(p, q) \in T^*Q, p \in g_q(\mathcal{D}_q) \subset T_q^*Q\},$$

where $g_q : T_qQ \to T_q^*Q$. Let

$$h(p, q) = \frac{1}{2} p g_q^{-1} p, \qquad p \in T_q^*Q,$$

we can write the nonholonomic flow as

$$\dot{x} = X_h(x) + \sum_{i=1}^{m} \lambda_i \text{ vert } \alpha_i|_{\pi(x)}, \tag{27}$$

where x stands for the phase space coordinates. Here $\text{vert}\alpha_i|_{\pi(x)} \in T_x(T^*Q)$ is the vertical lift of $\alpha_i|_{\pi(x} \in T^*_{\pi(x)}Q, \pi : T^*Q \to Q, \pi : T^*Q \to Q$.

In canonical coordinates these becomes

$$\dot{p} = -\frac{\partial h}{\partial q} + \sum_{i=1}^{m} \lambda_i \alpha_i(q) \quad \dot{q} = \frac{\partial h}{\partial p}. \tag{28}$$

Nonholonomic systems on Lie groups We now replace the configuration space Q by a compact connected Lie group G and the system is characterized by a left-invariant Lagrangian

$$L = \frac{1}{2} < I\alpha, \alpha > \qquad \text{where} \quad \alpha = g^{-1}\dot{g} \in \mathfrak{g},$$

where $I : \mathfrak{g} \to \mathfrak{g}^*$ is the momentum map. Then distribution is given by

$$D = \{\alpha \in \mathfrak{g}, < \alpha, a_i >= 0, i = 1, \cdots m\} \subset \mathfrak{g}.$$

In other words, the nonholonomic constraints are expressed in terms of m linearly independent fixed covectors $a_i \in \mathfrak{g}^*$, $i = 1, \cdots, m$. We say a velocity vector satisfies constraints if $< a_i, \alpha >= 0$. Suppose $\mathcal{D} \subset \mathfrak{g}$ be the vector subspace of all velocities satisfying the constraints, we say the constraints are nonholonomic if $\mathcal{D}$ is not a Lie subalgebra of $\mathfrak{g}$.

Given a set of linearly independent vectors a^i and $H(\alpha) = 1/2 < I^{-1}\alpha, \alpha >$ this non-holonomic flow equation can be expressed in canonical coordinates $\alpha = (\alpha_1, \cdots, \alpha_n)$

$$\dot{\alpha} = ad^*_{\nabla H}\alpha + \sum_{i=1}^{m} \lambda_i a^i, \tag{29}$$

where the Lagrange multipliers are chosen such that

$$< \nabla H(\alpha), a^i >= 0, \qquad i = 1, \cdots, m.$$

Example : Nonholonomic rigid body dynamics The Euler equation of the angular velocity is given by $I\dot{\Omega} = I\Omega \times \Omega$. Let $M = I\Omega$ denotes angular momentum. The Euler equation for rigid body may be expressed equivalently in angular momentum vector $\dot{M} = M \times \frac{\partial H}{\partial M}$, where the Hamiltonian $H = 1/2 < M, I^{-1}M >$. The equations are Hamiltonian with the rigid body Poisson bracket [26, 44]

$$\{F, K\}(M) = -M \cdot \Big[\nabla F(M) \times \nabla K(M)\Big], \qquad \text{where} \quad \nabla F(M) = \frac{\partial F}{\partial M}.$$

The Euler-Poincaré-Suslov problem on $SO(3)$ can be formulated as the standard Euler equations subjected to the constraint $< \mathbf{a}, \Omega >= 0$, where $\mathbf{a} \in \mathfrak{g}^*$. The nonholonomic equations of motion are then given by

$$I\dot{\Omega} = I\Omega \times \Omega + \lambda\mathbf{a} \tag{30}$$

subject to the constraint. We can easily solve for λ:

$$\lambda = -\frac{I^{-1}\mathbf{a}\cdot(I\Omega\times\Omega)}{I^{-1}\mathbf{a}\cdot\mathbf{a}}. \tag{31}$$

The nonholonomic equation (30) can also be expressed in terms of coadjoint action

$$\dot{M} = M\times I^{-1}M - \frac{< M\times I^{-1}M, I^{-1}\mathbf{a} >}{< \mathbf{a}, I^{-1}\mathbf{a} >}\mathbf{a}. \tag{32}$$

This equation is a special case for $G = SO(3)$, for arbitrary Lie group this can be expressed as

$$\dot{M} = ad^*_{\nabla H(M)}M - \frac{< M, ad_{\nabla H(M)}I^{-1}\mathbf{a} >}{< \mathbf{a}, I^{-1}\mathbf{a} >}\mathbf{a}. \tag{33}$$

2.3.2 Kupershmidt's method and EPS formalism

Let us modify and update Kupershmidt's method for the Euler top. We show that a little modification of Kupershmidt's scheme yields nonholonomic deformation of the Euler top equation.

Let us start with an explicit representation of the Poisson matrix $\mathbb{J}$ in $\mathbb{R}^3$. Any exact Poisson bivector in $\mathbb{R}^3$ corresponds to a certain function $\zeta(x,y,z)$ is given by

$$\Lambda^3_\zeta = \frac{\partial\zeta}{\partial x}\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} + \frac{\partial\zeta}{\partial y}\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} + \frac{\partial\zeta}{\partial z}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$$

This we can express in terms of a Poisson matrix

$$\mathbb{J}_\zeta = \begin{pmatrix} 0 & \frac{\partial\zeta}{\partial z} & -\frac{\partial\zeta}{\partial y} \\ -\frac{\partial\zeta}{\partial z} & 0 & \frac{\partial\zeta}{\partial x} \\ \frac{\partial\zeta}{\partial y} & -\frac{\partial\zeta}{\partial x} & 0 \end{pmatrix} \tag{34}$$

If we focus on the Jacobi identity, it is well known that in $\mathbb{R}^3$ the Jacobi equation for the Poisson structure, is a single scalar equation for the three components of the Poisson structure $\mathbb{J}$ [5, 46]. Note that if we stick to usual notation of Poisson matrix then the relevant properties of the matrix $\mathbb{J} = (J)_{ij}$ for the Poisson structure are

$$\begin{aligned} &(i) \qquad J_{ij} = -J_{ji} \quad i,j = 1,2,3 \quad \text{skew-symmetry} \\ &(ii)\ J^{li}\partial_l J^{jk} + J^{lj}\partial_l J^{ki} + J^{lk}\partial_l J^{ij} = 0 \quad i,j,k = 1,2,3, \end{aligned} \tag{35}$$

which is a consequence of the Jacobi identity.

A distinguished property of the Poisson structures in 3D is the invariant of the Jacobi identity under the multiplication of the Possion vector $\mathbb{J}(x)$ by an arbitrary but non-zero factor. In particular, under the transformation of Poisson vector

$$\mathbb{J}(x) \to h(x)\mathbb{J}(x)$$

the Jacobi identity transforms as

$$\mathbb{J}\cdot(\nabla\times\mathbb{J}) \to h(x)^2\mathbb{J}\cdot(\nabla\times\mathbb{J}). \tag{36}$$

Our *ansätz* is to choose *trivial* structure, as Kupershmidt prescribed for infinite-dimensional case B^1 operator. Then the deformed Euler top equation becomes

$$\dot{\mathbf{x}} = \mathbb{J}_\gamma \times \nabla H(x) + \sum_{i=1}^{3} \lambda_i w^i, \tag{37}$$

and the system is subject to the constraint

$$< \nabla H, w^i >= 0, \qquad i = 1, 2, 3. \tag{38}$$

Thus replacing $\mathbb{J}_\beta$ by a constant matrix we can reduce the Kupershmidt construction to Euler-Poincaré-Suslov problem on $SO(3)$.

The nonholonomic description on the semidirect space $SO(3) \ltimes \mathbf{R}^3$ is closely related to Veselova system [51] on the motion of a rigid body about a fixed point under the action of the nonholonomic constraint $(\Omega, \Gamma) = 0$.

3 Euler-Poincaré formalism on extended Virasoro algebra and coupled KdV equations

We start with some basic definitions collected from [25, 44, 43]. Let $\rho : G \to Aut(V)$ denotes a Lie group (left) representation of G in the vector space V, and $\tilde{\rho} : \mathfrak{g} \to End(V)$ is the induced Lie algebra representation. Let us denote $G \ltimes V$ the semidirect product group of G with V by ρ with multiplication [12, 44]

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_1 + \rho(g_1) v_2).$$

Let $\mathfrak{g} \ltimes V$ be the Lie algebra of $G \ltimes V$. The Lie bracket on $\mathfrak{g} \ltimes V$ is given by

$$[(\xi_1, u_1), (\xi_2, u_2)] = ([\xi_1, \xi_1], \tilde{\rho}(\xi_1) u_2 - \tilde{\rho}(\xi_2) u_1).$$

We have already seen that a prototypical example of a semidirect product structure is when $\mathfrak{g}$ is the Lie algebra $so(3)$ associated with the rotation group $SO(3)$ and u is $\mathbf{R}^3$. Their semidirect product is the algebra of the 6-parameter Galilean group of rotations and translations.

We can build the Lie-Poisson brackets from these algebras. The $\pm$ Lie-Poisson bracket of $f, g : (\mathfrak{g} \ltimes V)^* \to \mathbf{R}$ is given as

$$\{f, g\}_{\pm}(\mu, a) = \pm\Big\langle \mu, [\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}] \Big\rangle \pm \Big\langle a, \tilde{\rho}(\frac{\delta f}{\delta \mu}) \cdot \frac{\delta g}{\delta a} \Big\rangle \mp \Big\langle a, \tilde{\rho}(\frac{\delta g}{\delta \mu}) \cdot \frac{\delta f}{\delta a} \Big\rangle,$$

where $\frac{\delta f}{\delta \mu} \in \mathfrak{g}$ and $\frac{\delta f}{\delta a} \in V$, dual of μ under the pairing $<,>: \mathfrak{g}^* \times \mathfrak{g} \to \mathbf{R}$.

3.1 Extended Bott-Virasoro group

Let $Diff(S^1)$ be the group of orientation preserving diffeomorphisms of a circle. It is known that the group $Diff(S^1)$ as well as its algebra, i.e., the Lie algebra of vector fields on S^1, $T_{id}Diff(S^1) = Vect(S^1)$, have non-trivial one-dimensional central extensions, the Bott-Virasoro group $\widehat{Diff(S^1)}$ and the Virasoro algebra Vir respectively [33].

The Lie algebra $Vect(S^1)$ is the algebra of smooth vector fields on S^1. This satisfies the commutation relations

$$[f\frac{d}{dx}, g\frac{d}{dx}] := (f(x)g'(x) - f'(x)g(x))\frac{d}{dx}. \tag{39}$$

One parameter family of $Vect(S^1)$ acts on the space of smooth functions $C^\infty(S^1)$ by

$$L^{(\mu)}_{f(x)\frac{d}{dx}} a(x) = f(x)a'(x) + \mu f'(x)a(x), \tag{40}$$

where

$$L^{(\mu)}_{f(x)\frac{d}{dx}} = f(x)\frac{d}{dx} + \mu f'(x)$$

is the Lie derivative with respect to the vector field $f(x)\frac{d}{dx}$ on the tensor density $a(x)dx^\mu$.

The Lie algebra of $Diff(S^1) \ltimes C^\infty(S^1)$ is the semidirect product Lie algebra

$$\mathfrak{g} = Vect(S^1) \ltimes C^\infty(S^1).$$

An element of $\mathfrak{g}$ is a pair $(f(x)\frac{d}{dx}, a(x))$, where $f(x)\frac{d}{dx} \in Vect(S^1)$ and $a(x) \in C^\infty(S^1)$.

It is known that this algebra has a three dimensional central extension given by the following non-trivial cocycles [3, 43]

$$\omega_1((f\frac{d}{dx}, a), (g\frac{d}{dx}, b)) = \int_{S^1} f'(x)g''(x)dx \tag{41}$$

$$\omega_2((f\frac{d}{dx}, a), (g\frac{d}{dx}, b)) = \int_{S^1} f''(x)b(x) - g''a(x))dx \tag{42}$$

$$\omega_3((f\frac{d}{dx}, a), (g\frac{d}{dx}, b)) = 2\int_{S^1} a(x)b'(x)dx. \tag{43}$$

The first cocycle ω_1 is the well known Gelfand-Fuchs cocycle and we have seen earlier that the Virasoro algebra is the unique non-trivial central extension of $Vect(S^1)$ via this ω_1 cocycle.

3.1.1 Modified Gelfand-Fuchs cocycle

Consider the following "modified" Gelfand-Fuchs cocycle [33] on $Vect(S^1)$:

$$\omega_{mGF}(f(x)\frac{d}{dx}, g(x)\frac{d}{dx}) = \int_{S^1} (af'g'' + bf'g)dx. \tag{44}$$

This cocycle is cohomologues to the Gelfand-Fuchs cocycle, hence, the corresponding central-extension is isomorphic to the Virasoro algebra. The additional term in (44) is a coboundary term. It is easy to check that the functional

$$\int_{S^1} f'g\ dx = \frac{1}{2}\int_{S^1}(f'g - fg')dx$$

depends on the commutator of $f\frac{d}{dx}$ and $g\frac{d}{dx}$.

The Gelfand-Fuchs theorem states that $H^2(Vect(S^1)) = \mathbf{R}$, and therefore, every nontrivial cocycle is proportional to the Gelfand-Fuchs cocycle upto a coboundary. Thus one has

$$\tilde{\omega}_1 = \lambda\omega_1 + b,$$

where b is a coboundary

$$b(f\frac{d}{dx}, g\frac{d}{dx})(u) =< udx^2, [f,g]\frac{d}{dx} >$$

for some $udx^2 \in Vir^*$ is the element of the dual space of Virasoro algebra.

3.1.2 Coadjoint orbit and Hamiltonian structure

Let us consider the extension of $\mathfrak{g} = Vect(S^1) \ltimes C^\infty(S^1)$. This extended algebra is given by

$$\widehat{\mathfrak{g}} = Vect(S^1) \ltimes C^\infty(S^1) \oplus \mathbf{R}^3. \tag{45}$$

Definition 1 *The commutation relation in $\widehat{\mathfrak{g}}$ is given by*

$$[(f\frac{d}{dx}, a, \alpha), (g\frac{d}{dx}, b, \beta)] := ((fg' - f'g)\frac{d}{dx}, fb' - ga', \omega) \tag{46}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbf{R}^3$, $\omega = (\omega_1, \omega_2, \omega_3)$ are the two cocycles.

The dual space of smooth functions $C^\infty(S^1)$ is the space of distributions (generalized functions) on S^1. Of particular interest are the orbits in $\widehat{\mathfrak{g}}^*_{reg}$. In the case of current group, Gelfand, Vershik and Graev have constructed some of the corresponding representations [43].

Definition 2 *Let $\widehat{\mathfrak{g}}^*_{reg}$ be the regular part of the dual of the extended Virasoro algebra $\widehat{\mathfrak{g}}$, defined as*

$$\mathfrak{g}^*_{reg} = C^\infty(S^1) \oplus C^\infty(S^1) \oplus \mathbf{R}^3.$$

The $L^2 2$ pairing between this space and the extended Virasoro algebra $\widehat{\mathfrak{g}}$ $< \cdot, \cdot >_{L^2}$: $\mathfrak{g}^*_{\mathfrak{reg}} \otimes \mathfrak{g} \to \mathbf{R}$ is given by

$$< \hat{u}, \hat{f} >_{L^2} = \int_{S^1} f(x)u(x)dx + \int_{S^1} a(x)v(x)dx + \alpha \cdot \gamma, \tag{47}$$

where $\hat{u} = (u(x), v, \gamma)$, $\hat{f} = (f\frac{d}{dx}, a, \alpha)$.

Extend this to a right invariant metric on the semi-direct product group $Diff(\widehat{S^1) \ltimes C^\infty}(S^1)$ by setting

$$< \hat{u}, \hat{f} >_{\hat{\xi}} = < d_{\hat{\xi}} R_{\hat{\xi}^{-1}} \hat{u}, d_{\hat{\xi}} R_{\hat{\xi}^{-1}} \hat{f} >_{L^2} \tag{48}$$

for any $\hat{\xi} \in \widehat{\mathfrak{g}}$ and $\hat{u}, \hat{f} \in T_{\hat{\xi}}\widehat{G}$, where $R_{\hat{\xi}} : \widehat{\mathfrak{g}} \longrightarrow \widehat{\mathfrak{g}}$ is the right translation by $\hat{\xi}$.

Given any two elements $\hat{f} = (f\frac{d}{dx}, a, \alpha), \hat{g} = (g\frac{d}{dx}, b, \beta) \in \widehat{\mathfrak{g}}$, the coadjoint action of $\hat{f}$ on its dual element $(udx^2, v(x), c) \in \widehat{\mathfrak{g}}^*_{reg}$ is given below.

Lemma 2

$$ad^*_{\hat{f}}\hat{u} =$$
$$\begin{pmatrix} (2f'(x)u(x) + f(x)u'(x) + a'v(x) - c_1(af''' + bf') + c_2 a'' \\ f'v(x) + f(x)v'(x) - c_2 f''(x) + 2c_3 a'(x) \\ 0 \end{pmatrix}$$

Proof: This follows from

$$< ad^*_{\hat{f}}\hat{u}, \hat{g} >_{L^2} = < \hat{u}, [\hat{f}, \hat{g}] >_{L^2}$$

$$= < (u(x)\frac{d}{dx}, v(x), c), [(fg' - f'g)\frac{d}{dx}, fb' - ga', \omega) >_{L^2}$$

$$= -\int_{S^1} (fg' - f'g)u(x)dx - \int_{S^1} (fb' - ga')vdx - c_1 \int_{S^1} f'(x)g''(x)dx -$$
$$c_2 \int_{S^1} (f''(x)b(x) - g''(x)a(x))dx - 2c_3 \int_{S^1} a(x)b'(x)dx.$$

Since f, g, u are periodic functions, hence integrating by parts we obtain

$$R.H.S. = < (2f'(x)u(x) + f(x)u'(x) + a'(x)v(x) - c_1 f'''(x) +$$
$$c_2 a''(x), f'(x)v(x) + f(x)v'(x) - c_2 f'' b(x) + 2c_3 a'(x), 0)$$

□

The Hamiltonian structure associated with the coadjoint action is given by

$$\mathcal{O} = \begin{pmatrix} -c_1(aD^3 + bD) + 2uD + u_x & vD + c_2 D^2 \\ v_x + vD - c_2 D^2 & 2c_3 D \end{pmatrix}. \tag{49}$$

This is most general Hamiltonian structure for the Antonowicz-Fordy system. So all other Hamiltonian structures follow from this.

The Euler-Poincaré equation is the Hamiltonian flow on the coadjoint orbit in $\mathfrak{g}^*$, generated by the Hamiltonian

$$H(u, v) = < (u(x), v(x)), (u(x), v(x)) > = \frac{1}{2} \int_{S^1} (u^2 + v^2)\, dx, \tag{50}$$

given by

$$\frac{d\hat{u}}{dt} = ad^*_{\nabla H}\hat{u}(t). \tag{51}$$

Let V be a vector space and assume that the Lie group G acts on the left by linear maps on V, thus G acts on the left on its dual space V^*. The Euler-Poincaré flows on the coadjoint orbits of dual space of the extended Virasoro algebra can be identified with the geodesic flow the extended Bott-Virasoro group from the following proposition.

Proposition 2 *Let $G \ltimes V$ be a semidirect product space (possibly infinite dimensional), equipped with a metric $< \cdot, \cdot >$ which is right translation. A curve $t \to c(t)$ in $G \ltimes V$ is a geodesic of this metric if and only if $\hat{u}(t) = d_{c(t)}R_{c(t)^{-1}}\dot{c}(t)$ satisfies the Euler-Poincaré equation.*

We now consider various examples of the Euler-Poincaré flows.

3.2 Examples of Euler-Poincaré Flows on Semidirect product spaces and Hamiltonian structures

We give a series of examples of coupled KdV equations and their corresponding second Hamiltonian structures. All these flows are restricted to certain hyperplanes on the dual space of the extended Virasoro algebra.

1. **Ito system**

   We choose the *hyperplane* in the dual space. The coadjoint action leaves the parameter space invariant. Let us consider a hyperplane $c_1 = -1, a = 1, b = 0, c_2 = c_3 = 0$. The Hamiltonian structure of the well known Ito system

   $$u_t = u_{xxx} + 6uu_x + 2vv_x, \qquad v_t = 2(uv)_x \tag{52}$$

   is given by

   $$\mathcal{O}_{Ito} = \begin{pmatrix} D^3 + 4uD + 2u_x & 2vD \\ 2v_x + 2vD & 0 \end{pmatrix},$$

   where $\frac{\delta H}{\delta u} = u$, $\frac{\delta H}{\delta v} = v$.

2. **Modified dispersive Water wave equation**

   When we restrict to a hyperplane $c_1 = 0, c_2 = 1, c_3 = 0$, we obtain the modified dispersive water wave equation

   $$\begin{aligned} u_t &= 6uu_x + 2vv_x + v_{xx} \ , \\ v_t &= 2(vu)_x - u_{xx}. \end{aligned} \tag{53}$$

Thus the Hamiltonian structure of the modified dispersive water wave is

$$\mathcal{O}_2 = \begin{pmatrix} 4uD + 2u_x & 2vD + D^2 \\ 2v_x + 2vD - D^2 & 0 \end{pmatrix},$$

with Hamiltonian $H = \frac{1}{2}\int_{S^1}(u^2 + v^2)dx$.

3. **<u>The Kaup-Boussinesq system</u>**

   The Kaup-Boussinesq equation

$$u_t = (uv)_x + \frac{1}{4}v_{xxx}, \qquad v_t = vv_x + u_x. \tag{54}$$

   This equation is also related to a hyperplane $c_1 = -\frac{1}{2}$, $a = 1$, $b = 0$, $c_2 = 0$ and $c_3 = 1$ in the coadjoint orbit of the extension of the Bott-Virasoro group. Its Hamiltonian structure is

$$\mathcal{O}_2 = \begin{pmatrix} 2uD + u_x + \frac{1}{2}D^3 & vD \\ Dv & 2D \end{pmatrix},$$

   with

$$\frac{\delta H}{\delta v} = \frac{1}{2}u, \qquad \frac{\delta H}{\delta u} = \frac{1}{2}v.$$

4. **<u>The Broer-Kaup system</u>**

   The Broer-Kaup system

$$u_t = -u_{xx} + 2(uv)_x, \qquad v_t = v_{xx} + 2vv_x - 2u_x \tag{55}$$

   is a geodesic flow associated to the hyperplane $c_1 = 0$, $c_2 = -1$ and $c_3 = -1$. Hence the Hamiltonian structure is

$$\mathcal{O}_{BK} = \begin{pmatrix} uD + Du & -D^2 + vD \\ D^2 + Dv & -2D \end{pmatrix}, \qquad \text{with } H = \int_{S^1} uv.$$

We will use all these second Hamiltonian structures to compute nonholonomic deformations of various coupled KdV systems in the next section.

3.3 Frozen Lie-Poisson bracket and First Hamiltonian structures

We also consider the dual of the Lie algebra of Vir^* with a Poisson structure given by the "frozen" Lie-Poisson structure. In otherwords, we fix some poinr $\mu_0 \in \mathfrak{g}^*$ and define a Poisson structure given by

$$\{f, g\}_0(\mu) := < [df(\mu), dg(\mu)], \mu_0 >$$

It was shown by Khesin and Misiolek [32] that

Proposition 3 *The brackets* $\{\cdot,\cdot\}_{LP}$ *and* $\{\cdot,\cdot\}_0$ *are compatible for every "freezing" point* μ_0.

Proof: Let us take any linear combination

$$\{\cdot,\cdot\}_\lambda := \{\cdot,\cdot\}_{LP} + \lambda\{\cdot,\cdot\}_0$$

is again a Poisson bracket, it is just the translation of the Lie-Poisson bracket from the origin to the point $-\lambda\mu_0$.
□

Let us proceed to compute frozen brackets. In general, given

$$(u_0, v_0, c) \in \widehat{Vect(S^1) \ltimes C^\infty}(S^1)^* \simeq C^\infty(S^1) \oplus C^\infty(S^1) \oplus \mathbf{R}^3,$$

the frozen bracket is given by

$$\{f,g\}(u,v,c) =< (u_0, v_0, c), [\frac{\delta f}{\delta(u,v,c)}, \frac{\delta g}{\delta(u,v,c)}] >,$$

$$=< -ad^*_{\frac{\delta f}{\delta(u,v,c)}}(u_0, v_0, c), \frac{\delta f}{\delta(u,v,c)} > .$$

Furthermore, recall the corresponding equations of motions are given by

$$\frac{d}{dt}(u,v,c) = -ad^*_{\frac{\delta f}{\delta(u,v,c)}}(u_0, v_0, c).$$

We would like find the frozen bracket at $(u(x), v(x), c) \equiv (0, 0, c)$ and this gives rise to the first Hamiltonian structure of the coupled KdV type systems

$$\mathcal{O}_1 = \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix}, \tag{56}$$

where $c = (-1, 0, \frac{1}{2})$ and $a = 0, b = 1$.

Remark on coboundary operator and frozen structure Every 2-cocycle Γ defines a Lie-Poisson structure on $\mathfrak{g}^*$. The vanishing of Schouten-Nijenhuis bracket for Poisson bivector can be recast as a cocycle condition $\partial\Gamma = 0$, where $\partial : \wedge^k\mathfrak{g}^* \to \wedge^{k+1}\mathfrak{g}^*$. A special case of Lie-Poisson structure is given by a 2-cocycle Γ which is a coboundary [13, 14]. If $\Gamma = \partial\mu_0$ for some $\mu_0 \in \mathfrak{g}^*$, the expression

$$\{f,g\}_0(\mu) = \mu_0([d_\mu f, d_\mu g])$$

considered to be Lie-Poisson bracket which has been "frozen" at a point $\mu_0 \in \mathfrak{g}^*$.

3.4 Nonholonomic deformation of coupled systems

In this section we present one of the main result of the paper, i.e., the nonholonomic deformations of the coupled KdV systems using Kupershmidt's prescription. These are all dubbed as the Ito6, the Modified dispersive water wave 6 (or MDWW6), the Kaup-Boussinesq 6 (KB6) and the Broer-Kaup 6 (or BK6) equatiions. These are all considered to be multi-component generalization of the KdV6 equation.

Proposition 4 *1. The Ito6 equation is a constraint flow on the dual space of semi-direct algebra* $\mathfrak{g}$ *restricted to hyperplane* $c_1 = -1, a = 1, b = 0, c_2 = c_3$

$$u_t = u_{xxx} + 6uu_x + 2vv_x - w_{1x} \qquad v_t = 2(uv)_x - w_{2x}, \tag{57}$$

where $\mathbf{w} = (w_1, w_2)$ *satisfies*

$$\begin{aligned} w_{1xxx} + 4uw_{1x} + 2u_x w_1 + 2vw_{2x} &= 0\,, \\ 2(w_1 v)_x &= 0. \end{aligned} \tag{58}$$

(59)

2. The MDWW6 equation is a constraint flow on the dual space of $\mathfrak{g}$ *restricted to hyperplane* $c_1 = 0, c_2 = 1, c_3 = 0$

$$u_t = 6uu_x + 2vv_x + v_{xx} - w_{1x} \qquad v_t = 2(vu)_x - u_{xx} - w_{2x}, \tag{60}$$

where the constraint equations satisfy

$$2uw_{1x} + 2(uw_1)_x + 2vw_{2x} + w_{2xx} = 0 \qquad 2(vw_1)_x - w_{1xx} = 0. \tag{61}$$

3. The Kaup-Boussinesq 6 (KB6) equation is a constraint flow on the dual space of $\mathfrak{g}$ *restricted to hyperplane* $c_1 = 1/2, c_2 = 0, c_3 = 1, c_4 = 0$

$$u_t = (uv)_x + \frac{1}{4}v_{xxx} - w_{1x}, \qquad v_t = vv_x + u_x - w_{2x}, \qquad v_t = (uv)_x + \frac{1}{4}u_{xxx} - w_{2x}, \tag{62}$$

where $\mathbf{w}$ *equations satisfy the following constraint equations*

$$2uw_{1x} + u_x w_1 + \frac{1}{2}w_{1xxx} + vw_{2x} = 0, \qquad (vw_1)_x + 2w_{2x} = 0. \tag{63}$$

4. The Broer-Kaup 6 (BK6) equation is a constraint flow on the dual space of $\mathfrak{g}$ *restricted to hyperplane* $c_1 = 0, c_2 = -1, c_3 = -1, c_4 = 0$

$$u_t = -u_{xx} + 2(uv)_x - w_{1x}, \qquad v_t = v_{xx} + 2vv_x - 2u_x - w_{2x} \tag{64}$$

where constriant equations satisfy

$$uw_{1x} + (uw_1)_x - w_{2xx} + vw_{2x} = 0, \qquad w_{1xx} + (vw_1)_x - 2w_{2x} = 0. \tag{65}$$

Proof: We use Kupershmidt formulation or Euler-Poincaré-Suslov type equation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{O}^2 \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix} - \mathcal{O}^1 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

and constraint equation

$$\mathcal{O}^2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$$

to produce our result. □

Following Kupershmidt we can show the the existence of infinite number of conserved densities

$$\frac{dH_m}{dt} = \nabla H_m(\mathbf{u})\Big[\mathcal{O}^2(\nabla H_n(\mathbf{u}) - \mathcal{O}^1(\mathbf{w})\Big]$$
$$= \nabla H_m(\mathbf{u})\mathcal{O}^2(\nabla H_n \mathbf{u}) - \nabla H_n(\mathcal{O}^1)\mathbf{w} = 0.$$

where all the operations are defined upto exact differential and $\nabla H_m(\mathbf{u}) = \begin{pmatrix} \frac{\delta H_m}{\delta u} \\ \frac{\delta H_m}{\delta v} \end{pmatrix}$ and the operators $\mathcal{O}^2$ and $\mathcal{O}^1$ are identified with the Kupershmidt's B_2 and B_1 operators. Like Kupershmidt's case, if we proceed to develop the variational calculus in the (u, v, w)-variables for general system, we would be obstructed since the calculus works only when the factor $\Omega^1/\partial(\Omega^1)$ is free module, where Ω^1 is the module of differential forms. But it can be performed well for the KdV type systems. We demonstrate this for the coupled KdV equations.

Remark By changing the constriant equation from $\mathcal{O}^2\mathbf{w} = 0$ to pencil of Poisson structure $\mathcal{O}^\lambda\mathbf{w} = (\mathcal{O}^2 - \lambda\mathcal{O}^1)\mathbf{w} = 0$ we may construct second type of nonholonomic deformed coupled KdV systems.

By changing the norm from L^2-norm to H^1 we may study the Euler-Poincaré-Suslov flows on the extended Virasoro group. Since these are bihamiltonian in nature so we can apply Kupershmidt scheme and this will yield nonholonomic deformation of the coupled Camassa-Holm type equations. In the next section we present this construction.

3.5 The two component Camassa-Holm equation and nonholonomic deformation

Let us consider extended Virasoro algebra $\hat{\mathfrak{g}} = Vect(S^1) \ltimes C^\infty(S^1) \oplus \mathbf{R}^3$. In fact it is known [24] that the two component Camassa-Holm (CH) equation also follows from the geodesic flow on the extended Bott-Virasoro with respect to H^1 inner product.

Let us introduce H^1 inner product on the algebra $\widehat{\mathfrak{g}}$

$$\langle \widehat{f}, \widehat{g} \rangle_{H^1} = \int_{S^1} [\, f(x)g(x) + a(x)b(x) + \partial_x f(x)\partial_x g(x)\,]\, dx + \alpha \cdot \beta, \tag{66}$$

where

$$\widehat{f} = \left(f\frac{d}{dx}, a, \alpha\right), \qquad \widehat{g} = \left(g\frac{d}{dx}, b, \beta\right).$$

Now we compute :

Lemma 3 *The coadjoint operator with respect to the H^1 inner product is given by*

$$ad^*_{\widehat{f}}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (1-\partial^2)^{-1}[2f'(x)(1-\partial_x^2)u(x) + f(x)(1-\partial_x^2)u'(x) + a'v(x)] - c_1(af''' + bf') + c_2 a'' \\ f'v(x) + f(x)v'(x) - c_2 f''(x) + 2c_3 a'(x) \end{pmatrix} \tag{67}$$

Proof: Since we have identified $\mathfrak{g}$ with $\mathfrak{g}^*$, it follows from the definition that

$$\langle\, ad^*_{\widehat{f}}\widehat{u}\,,\, \widehat{g}\,\rangle_{H^1} = \langle\, \widehat{u}\,,\, [\widehat{f}, \widehat{g}]\,\rangle_{H^1} = -\int_{S^1}[\,(fg' - f'g)u - (fb' - ga')v - \partial_x(fg' - f'g)\partial_x u\,]dx.$$

After computing all the terms by integrating by parts and using the fact that the functions $f(x), g(x), u(x)$ and $a(x), b(x), v(x)$ are periodic, the right hand side can be expressed as above.

Let us compute now the left hand side:

$$ad^*_{\widehat{f}}\begin{pmatrix} u \\ v \end{pmatrix} = \int_{S^1}[\,(ad^*_{\widehat{f}}u)g + (ad^*_{\widehat{f}}u)'g' + (ad^*_{\widehat{f}}v)b\,]\; dx$$

$$= \int_{S^1}[\,[(1-\partial^2)ad^*_{\widehat{f}}u]g + (ad^*_{\widehat{f}}v)b\,]\; dx = \Big\langle\, ((1-\partial^2)ad^*_{\widehat{f}}u\,,\, (ad^*_{\widehat{f}}v)), (g, b)\,\Big\rangle$$

Thus by equating the the right and left hand sides, we obtain the desired formula. $\square$

We conclude that the Hamiltonian operator arising from the induced Lie–Poisson structure is

$$\widehat{J} = \begin{pmatrix} D\rho + \rho D - c_1(aD^3 + bD) & vD + c_2D^2 \\ Dv - c_2D^2 & 2c_3D \end{pmatrix}, \tag{68}$$

where $\rho = (1-\partial_x^2)u$.

If we restrict it to hyperplane $c_1 = c_2 = c_3 = 0$, then this yields the two-component Camassa-Holm equation

$$\begin{aligned} u_t - u_{xxt} = u_{xxx} + 3uu_x + vv_x - \left(uu_{xx} + \frac{1}{2}u_x^2\right)_x \\ v_t = 2(uv)_x \end{aligned} \tag{69}$$

and second Hamiltonian structure

$$J_2 = \begin{pmatrix} D\rho + \rho D & vD \\ Dv & 0 \end{pmatrix}. \tag{70}$$

It is a protypical example of a two-component integrable system.

3.5.1 Nonholonomic deformation of two component Camassa-Holm equation

The two-component Camassa-Holm equation can also be expressed in bihamiltonian form and the first Hamiltonian structure of the two-component Camassa-Holm equation is obtained from the frozen Lie-Poisson structure. Consider hyperplane $c_1 = 1$, $c_3 = \frac{1}{2}$ and

$c_2 = 0$, furthermore we set $a = 1 = -b$. This immediately yields the first Hamiltonian operator

$$J_1 = \begin{pmatrix} D - D^3 & 0 \\ 0 & D \end{pmatrix}. \tag{71}$$

One can put the two-component Camassa-Holm equation as a bihamiltonian system using the two compatible Hamiltonian operators.

Now we follow the recipe of Kupershmidt to derive the nonholonomic deformation of the two-component Camassa-Holm equation, which yields

$$\begin{aligned} u_t - u_{xxt} = 3uu_x + vv_x - \left(uu_{xx} + \frac{1}{2}u_x^2 \right)_x - (w_{1xxx} - w_{1x}), \\ v_t = 2(uv)_x - w_{2x} \end{aligned} \tag{72}$$

and the constraint equation is given by

$$\begin{aligned} 2w_{1x}\rho + \rho_x w_1 + vw_{2x} = 0, \\ (vw_2)_x = 0. \end{aligned} \tag{73}$$

Note that these Hamiltonian operators appeared in tri-hamiltonian system [47].

Second type of nonholonomic deformation of 2-component CH equation
Compatibility of J_1 and J_2 suggest us to define pencil of Hamiltonian structures

$$J_\lambda = J_2 - \lambda J_1. \tag{74}$$

This allows us to modify the nonholonomic deformation of the two-component Camassa-Holm equation, where the constraint equation is given by

$$\begin{aligned} 2w_{1x}\rho + \rho_x w_1 + vw_{2x} - \lambda(w_{1xxx} - w_{1x}) = 0, \\ (vw_2)_x - \lambda w_{2x} = 0. \end{aligned} \tag{75}$$

Thus we extend the Kupershmidt method to construct the nonholonomic deformation of the two component Camassa-Holm (CH) equation. In fact the same method can be applied to other bihamiltonian coupled CH equations. We now move to another class of coupled KdV type equation coming from superconformal algebra.

4 Neveu-Schwarz algebra and Kuper-KdV flows

We now concentrate on the nonholonomic deformations of the supersymmetric KdV equation. This is also yield coupled KdV equation and we formulate this nonholonomic deformation using superconformal algebra. We restrict ourselves to only $N = 1$ superconformal algebra but this can be generalized also to $N = 2$ superconformal algebra.

The first characteristic special property of a super algebra is that all the additive groups of its basic and derived structures are $\mathbb{Z}_2$. In other words replace vector spaces by $\mathbb{Z}_2$-graded vector spaces and invoke the "sign rule" when commuting homogeneous elements. A graded vector space $V = V_0 \oplus V_1$ is simply a vector space which is presented as the direct sum of two subspaces. Elements in V_0 are termed *even*, and elements in V_1 are termed *odd*.

An ordinary Lie algebra is a vector space $\mathfrak{g}$ with a bracket $[.,.] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which is skew-symmetric and satisfies Jacobi identity. A Lie superalgebra is a Z_2- graded algebra

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

together with a bracket. An element v of $\mathfrak{g}_0$ (resp. $\mathfrak{g}_1$) is said to be even (resp. odd). The axioms are equivalent to the following: (a) $\mathfrak{g}_0$ is an ordinary Lie algebra, (b) $\mathfrak{g}_1$ is a representation of $\mathfrak{g}_0$, and (c) there is an anticommutator given by a symmetric bilinear map

$$[.,.]_+ : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0.$$

The supercommutator of a pair of elements $v, w \in \mathfrak{g}$ is defined by

$$[v, w] = vw - (-1)^{\tilde{v}\tilde{w}} wv,$$

where $\tilde{v}$ and $\tilde{w}$ are the gradings of v and w respectively. It also satisfies the super Jacobi identity

$$(-1)^{\tilde{v}\tilde{u}}[v, [w, u]] + (-1)^{\tilde{v}\tilde{w}}[w, [u, v]] + (-1)^{\tilde{w}\tilde{u}}[u, [v, w]] = 0.$$

Let Ω be the cotangent bundle of S^1. The cotangent bundle of S^1 has two representation; a trivial one is given by cylinder and the nontrivial one by Möbius strip. We denote by $\Omega^{\pm 1/2}$ the square root of the tangent and cotangent bundle of S^1, respectively.

4.1 Kirillov's construction and Lie superalgebra

Consider the space $\mathcal{F}_{-1/2}$ of $-1/2$-tensor densities on S^1, where elements are given by $\xi = \xi(x)(dx)^{-1/2}$. Here $-1/2$ is the degree (or weight), x is a local coordinate on S^1. Once again as a vector space $\mathcal{F}_{-1/2}$ is isomorphic to $C^\infty(S^1)$.

Let us define a super algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ from the vector fields on a circle. We denote $\mathfrak{g}_0 \equiv Vect(S^1)$ and $\mathfrak{g}_1 \equiv \Omega^{-1/2}(S^1)$. Then $\mathfrak{g}$ forms a Lie superalgebra on $S^{1,1}$ [33], where $\mathfrak{g}_1$ is the super-partner of $\mathfrak{g}_0$. The anticommutator bracket is defined as

$$\left[\xi(x)\sqrt{\frac{d}{dx}}, \phi(x)\sqrt{\frac{d}{dx}}\right]_+ = \xi\phi\frac{d}{dx}. \tag{76}$$

Since $\mathfrak{g}_1$ is the super-partner of $\mathfrak{g}_0$ hence $\mathfrak{g}_1$ is the $\mathfrak{g}_0$ module and it is compatible with the structure of $\mathfrak{g}_0$ module and satisfies $\mathfrak{g}_1 \otimes \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$. The action of $Vect(S^1)$ on $\mathcal{F}_{-\frac{1}{2}}$ is defined as

$$L^{-1/2}_{f(x)\frac{d}{dx}}\xi = (f\xi' - \frac{1}{2}f'\xi). \tag{77}$$

In this realization, $(f(x)\frac{d}{dx} + \xi(x)\sqrt{\frac{d}{dx}})$ i.e. $(f(x), \xi(x))$ forms a super Lie algebra and the pair $(f(x), \xi(x))$ satisfies

$$f(x + 2\pi) = f(x)$$

$$\xi(x+2\pi) = \pm\xi(x)$$

When it is in the '+ ' sector, it is called the Ramond sector super Lie algebra, and the '$-$' sector is known as the Neveu-Schwarz sector. All these spaces are $Vect(S^1)$-module.

The Gelfand-Fuchs cocycle may be extended to this superalgebra via

$$\omega_F(\xi_1, \xi_2) = \int_{S^1} \xi_1' \xi_2' \, dx. \tag{78}$$

This centrally extended Lie superalgebra is denoted by $\hat{\mathfrak{g}}$.

Definition 3 *There exists a unique nontrivial central extension of the Lie superalgebra* $\mathfrak{g}$ *is defined by the following two cocycles*

$$\tilde{\omega}\Big((f(x)\frac{d}{dx}, \xi(x)\sqrt{\frac{d}{dx}}), (g(x)\frac{d}{dx}, \phi(x)\sqrt{\frac{d}{dx}})\Big) = \int_{S^1} (f'g'' + \xi'\phi')\, dx \tag{79}$$

Proposition 5 *The supervector fields* $(\xi_1\sqrt{\frac{d}{dx}}, \xi_2\sqrt{\frac{d}{dx}}, \xi_3\sqrt{\frac{d}{dx}})$ *satisfy the Jacobi identity*

$$\Big[[\xi_1\sqrt{\frac{d}{dx}}, \xi_2\sqrt{\frac{d}{dx}}], \xi_3\sqrt{\frac{d}{dx}}\Big] + \ \textit{cyclic} \quad \textit{terms} \ = 0$$

Proof: We tacitly use two different commutation relations, i.e.,

$$\Big[\xi_1\sqrt{\frac{d}{dx}}, \xi_2\sqrt{\frac{d}{dx}}\Big] = \xi_1\xi_2\frac{d}{dx} \text{ and } \Big[\xi_1\xi_2\frac{d}{dx}, \xi_3\sqrt{\frac{d}{dx}}\Big] = (\xi_3'\xi_1\xi_2 - \frac{1}{2}\xi_3(\xi_1\xi_2)').$$

The second relation follows from the basic definition of the action of $Vect(S^1)$ on $\mathcal{F}_{-\frac{1}{2}}$

$$L^{-1/2}_{f(x)\frac{d}{dx}}\xi = (f\xi' - \frac{1}{2}f'\xi).$$

Therefore we obtain

$$[[\xi_1, \xi_2], \xi_3] \ + \ \ \text{cyclic} \quad \text{terms} \ = 0.$$

□

Definition 4 *A Lie superalgebra is defined on a superspace is given by*

$$\hat{\mathfrak{g}} = Vect(S^1) \oplus \mathcal{F}_{-1/2} \oplus \mathbf{C}. \tag{80}$$

This Lie superalgebra is called the Neveu-Schwarz algebra.

The $N = 1$ Neveu-Schwarz algebra $\hat{\mathfrak{g}}$ has commutation relation

$$\begin{aligned}
&\Big[(f(x)\frac{d}{dx}, \phi(x)(dx)^{-1/2}, \lambda), (g(x)\frac{d}{dx}, \psi(x)(dx)^{-1/2}, \mu)\Big] \\
&= \Big[(fg' - f'g + \frac{1}{2}\psi\phi)\frac{d}{dx}, ((f\psi' - \frac{1}{2}f'\psi) - (g\phi' - \frac{1}{2}g'\phi))(dx)^{-1/2}, \\
&a\int_{S^1} f'g''\, dx + b\int_{S^1} \phi'\psi' dx\Big]
\end{aligned}$$

4.1.1 Expression of the $N=1$ Neveu-Schwarz algebra in terms of super bracket

We define the supercircle $S^{1|1}$ in terms of its superalgebras of functions, denoted by $C^{\infty}_{\mathbb{C}}(S^{1|1})$, consisting of elements of the form $F(x,\vartheta) = f(x) + \vartheta\phi(x)$, where x is an arbitrary parameter on S^1 and ϑ is an formal Grassmann coordinate such that $\vartheta^2 = 0$. A vector field on $S^{1|1}$ is a superderivation of $C^{\infty}_{\mathbb{C}}(S^{1|1})$. There is an isomorphism

$$Vect(S^{1|1}) \simeq \mathcal{F}_{-1/2} \oplus \mathcal{F}_{-1}. \tag{81}$$

Using super derivative we can express this commutation relation

$$[(F,a)\,,\,(G,b)] = \left(FD^2G - D^2FG + (-)^{p(F)}\frac{1}{2}DFDG\ ,\ \mathrm{Res}\ FDG_{xx}\right) \tag{82}$$

where $D := \frac{\partial}{\partial\vartheta} + \vartheta\frac{\partial}{\partial x}$ is the superderivate on supercircle $S^{1|1}$ and $p(F)$ is a parity function.

4.2 Dual of the Neveu-Schwarz algebra and the Kuper-KdV equation

A typical element of $\hat{\mathfrak{g}}$ would be

$$(f(x)\frac{d}{dx}, \xi(x)\sqrt{\frac{d}{dx}}, a) \qquad \text{where } a \in \mathbb{C}$$

and the super Lie bracket is given by

$$\left[\begin{pmatrix} f_1 \\ \xi_1 \\ a_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ \xi_2 \\ a_2 \end{pmatrix}\right] = \begin{pmatrix} [f_1, f_2] + \xi_1\xi_2 \\ \{f_1, \xi_2\} + \{\xi_1, f_2\} \\ 0 \end{pmatrix},$$

where $[f_1, f_2] = f_1 f_2' - f_1' f_2$ and $\{f_1, \xi_2\} = f_1\xi_2' - \frac{1}{2}f_1'\xi_2$.

Since the topological dual of the superalgebra $\mathfrak{g}$ is too big, once again we restrict our attention to the regular part of the dual of Kirillov's superalgebra. The regularized dual space to the superalgebra is naturally isomorphic to

$$\mathfrak{g}^* = \mathcal{F}_2 \oplus \mathcal{F}_{3/2} \oplus \mathbb{C}.$$

It is clear that the module $\mathcal{F}_{3/2}$ is dual to the " Fermionic" part of the superalgebra. Therefore, the regular dual space of superalgebra consists of elements $(u,\phi,c) = (u(x)dx^2, \phi(x)dx^{3/2}, c)$.

Proposition 6 *The coadjoint representation of the superalgebra is given by*

$$ad^*_{(f(x)\frac{d}{dx}, \xi\sqrt{\frac{d}{dx}}, a)} \begin{pmatrix} u(dx)^2 \\ \phi(dx)^{3/2} \\ c \end{pmatrix} = \begin{pmatrix} (fu' + 2f'u + cf''' + \frac{1}{2}\xi\phi' + \frac{3}{2}\xi'\phi)dx^2 \\ f\phi' + \frac{3}{2}f'\phi - \frac{1}{2}u\xi + c\xi'')(dx)^{3/2} \\ 0 \end{pmatrix}.$$

Proof: We obtain the above result by direct computation. □

The Hamiltonian operator (for $c = \frac{1}{2}$) is given by

$$\mathcal{O}^2_{sKdV} = \left(\begin{array}{c|c} \frac{1}{2}\partial^3 + \partial u + u\partial & \frac{1}{2}\partial\xi + \xi\partial \\ \hline \partial\xi + \frac{1}{2}\xi\partial & \frac{1}{2}\partial^2 - \frac{1}{2}u \end{array} \right). \tag{83}$$

We consider Hamiltonian

$$H = \int_{S^1} (u^2 + \frac{1}{2}\xi\xi_x)\, dx$$

such that

$$\frac{\delta H}{\delta u} = 2u \qquad \frac{\delta H}{\delta \xi} = \xi_x.$$

Using the Euler-Poincaré (or Hamiltonian) equation

$$\mathbf{u}_t = -\mathcal{O}_{sKdV}\nabla H, \qquad \mathbf{u} = (u, \xi),$$

we obtain the Kuper-KdV equation

$$u_t + u_{xxx} + 6uu_x + \frac{3}{2}\xi\xi_{xx} = 0 \qquad \xi_t + \xi_{xxx} + 3u\xi_x + \frac{3}{2}u_x\xi = 0. \tag{84}$$

4.3 Frozen structure and first Hamiltonian structure of sKdV equation

Using modified cocycle the coadjoint orbit yields following Hamiltonian operator

$$\mathcal{O}^{gen}_{sKdV} = \left(\begin{array}{cc} c_1\partial^3 + c_2\partial + \partial u + u\partial & \frac{1}{2}\partial\xi + \xi\partial \\ \partial\xi + \frac{1}{2}\xi\partial & c_3\partial^2 - \frac{1}{2}u \end{array} \right). \tag{85}$$

Using frozen Lie-Poisson structure we can derive the first Poisson structure The frozen Hamiltonian operator at $(u, \xi, c_1, c_2, c_3) = (u_0, 0, \kappa, 0)$ is given by

$$\mathcal{O}^{frozen}_{sKdV} = \left(\begin{array}{cc} (2u_0 + \kappa)\partial & 0 \\ 0 & -\frac{1}{2}u_0 \end{array} \right). \tag{86}$$

We choose u_0 and κ such a way so that the *first Hamiltonian structure* is given by

$$\mathcal{O}^1_{sKdV} = \left(\begin{array}{cc} \partial & 0 \\ 0 & 1 \end{array} \right) \tag{87}$$

with

$$H_2 = \int_{S^1} (u^3 - \frac{1}{2}u_x^2 + \frac{3}{2}u\xi\xi_x + \frac{1}{2}\xi\xi_{xxx})dx.$$

4.4 Nonholonomic deformation of $N = 1$ Super KdV equation

In this section we propose the $N = 1$ supersymmetric KdV6 equation. It is a nonholonomic deformation of the $N = 1$ supersymmetric KdV equation.

Proposition 7 *The sKdV6 equation is a constraint flow on the dual space of superconformal algebra $\mathfrak{g}$, confined to a hyperplane $c_1 = 1$, $c_2 = 0$ and $c_3 = 1$, is given by*

$$\begin{aligned} u_t &= u_{xxx} + 6u_x u + \frac{3}{2}\xi_{xx}\xi - w_{1x} \\ \xi_t &= \xi_{xxx} + 3u\xi_x + \frac{3}{2}u_x\xi - \nu_1. \end{aligned} \tag{88}$$

where $\mathbf{w} = \begin{pmatrix} w_1 \\ \nu_1 \end{pmatrix}$ *satisfies*

$$w_{1xxx} + 2uw_{1x} + u_x w_1 + \frac{1}{2}(\xi\nu_1)_x + \xi\nu_{1x} = 0$$

$$(\xi w_1)_x + \frac{1}{2}\xi w_{1x} + \nu_{1xx} - \frac{1}{2}u\nu_1 = 0.$$

Proof: We use Kupershmidt's scheme or Euler-Poincaré-Suslov type equation

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \mathcal{O}^2_{sKdV}\begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta \xi} \end{pmatrix} - \mathcal{O}^1_{sKdV}\begin{pmatrix} w_1 \\ \nu_1 \end{pmatrix},$$

and constraint equation

$$\mathcal{O}^2_{sKdV}\begin{pmatrix} w_1 \\ \nu_1 \end{pmatrix} = 0$$

to produce our result. □

Let us denote $\mathbf{u} = (u, \xi)$. Following Kupershmidt we can show the the existence of infinite number of conserved densities

$$\begin{aligned} \frac{dH_m}{dt} &= \nabla H_m(\mathbf{u})\Big[\mathcal{O}^2_{sKdV}(\nabla H_n(\mathbf{u}) - \mathcal{O}^1_{sKdV}(\mathbf{w})\Big] \\ &= \nabla H_m(\mathbf{u})\mathcal{O}^2_{sKdV}(\nabla H_n\mathbf{u}) - \nabla H_n(\mathcal{O}^1_{sKdV})\mathbf{w} = 0. \end{aligned}$$

where all the operations are defined upto exact differential and $\nabla H_m(\mathbf{u}) = \begin{pmatrix} \frac{\delta H_m}{\delta u} \\ \frac{\delta H_m}{\delta v} \end{pmatrix}$.

In this case also we can introduce the nonholonomic deformation of second kind, which is given by the constraint equation

$$(\mathcal{O}^2_{sKdV} - \lambda\mathcal{O}^{frozen}_{sKdV})\begin{pmatrix} w_1 \\ \nu_1 \end{pmatrix} = 0. \tag{89}$$

Thus we derive another version of the supersymmetric KdV6 equation.

5 Conclusion

In this article using the idea of Kupershmidt's programme on nonholonomic deformation method of KdV equation we have proposed an algorithmic method to derive nonholonomic deformations of the entire KdV family. Our method is deeply connected to the the Euler-Poincaré-Suslov (EPS) method, in fact, we have argued in the paper that our method can be manifested as an infinite-dimensional analogue of the EPS method. The classical EPS method yields Hamiltonian formulation of the finite-dimensional nonholonomic systems. There are numerous classical works dedicated to finite-dimensional cases that develop the equations of motion for mechanical systems with non-holonomic constraints, but infinite-dimensional systems are rare.

In particular, we have derived several coupled nonholonomic deformed KdV equations whose flows are defined on the semidirect products of the Bott-Virasoro group. Our method is useful tool for deriving new types of coupled constraint integrable systems. This method also explained various other generalization of Kupershmidt's programme, for example, we have discussed the generalized Kupershmidt deformation proposed by Yao and Zeng [52] and many other KdV6 avatars follows from method.

We have extended our method to supersymmetric KdV equation. Starting from the coadjoint action of the $N = 1$ superconformal algebra we derived explicit representation of the nonholonomic deformation of the Kupershmidt version of $N = 1$ supersymmetric KdV equation, also known as the Kuper-KdV6 equation. This method can be regarded as the infinite-dimensional supersymmetric Euler-Poincaré-Suslov method.

It would be interesting to study the integrability aspect of the Kupershmidt deformation method in general. It would be pertinent to study the solutions of these new sets of equations. The $N = 2$ supersymmetric integrable hierarchies have attracted much attention in integrable systems. It is commonly believed that all these super hierarchies will help us to study superconformal field theory. This work opens up various generalization of nonholonomic deformed integrable systems. Extension of the construction and integrability properties of the nonholonomic deformation of the $N = 2$ Labelle-Mathieu type super KdV and super coupled KdV type systems is under investigation. We also wish to study the super coupled KdV type systems using the extended superconformal algebra. All these programmes would lead to a infinite-dimensional supersymmetric extension of the Euler-Poincaré-Suslov method.

Acknowledgment

The author is indebted to Tony Bloch, Tudor Ratiu, Darryl Holm, Chand Devchand, Basil Grammaticos, Alfred Ramani and Andy Hone for their encouragement and valuable suggestions in various stages of the work. The final part of this work was done while the author was visiting IHES. He would like to express his gratitude to the members of IHES for their warm hospitality.

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