Application of Jacobi's last multiplier for construction of singular Hamiltonian of the activator-inhibitor model and conformal Hamiltonian dynamics

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Abstract

The relationship between Jacobi's last multiplier and the Lagrangian of a second-order ordinary differential equation is quite well known. In this article we demonstrate the significance of the last multiplier in Hamiltonian theory by explicitly constructing the Hamiltonians of certain well known first-order systems of differential equations arising in the activator and inhibitor model and these are connected to conformal Hamiltonian structure.

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1 Introduction

Many second-order ordinary differential equations (ODEs) of the form $\ddot{x} = F(t, x, \dot{x})$ admit a Lagrangian description because of the existence of a Jacobi Last Multiplier (JLM), μ, which can be shown to be equal to $\partial^2 L/\partial \dot{x}^2$. In a recent article Nucci *et al* [16] have shown how one can obtain the Lagrangians for certain well known biological models described by planar systems of ODEs using Jacobi's last multiplier. Although in some cases the corresponding Lagrangians are known, it may not always be possible to reduce a given planar system of ODEs

$$\dot{x} = f(x, y, t), \tag{1.1}$$
$$\dot{y} = g(x, y, t), \tag{1.2}$$

to an equivalent second-order ODE. Therefore, the derivation of even a singular Lagrangian for such a system will be interesting from the mathematical point of view. A singular Lagrangian is one for which the Hessian matrix

$$\mathcal{H} = \left(\frac{\partial^2 L}{\partial u_i \partial u_j}\right)$$

is singular. In such cases it is not possible to express the equation of motion in the form $\ddot{x} = F(x, \dot{x}, t)$. For such singular Lagrangians the usual definition of the conjugate momentum turns out to be velocity independent and consequently one can not define a Hamiltonian by the usual process of a Legendre transformation.

The JLM is a useful tool for deriving an additional first integral for a system of n first-order ODEs when $n-2$ first integrals of the system are known. Besides, the JLM allows us to determine the Lagrangian of a second-order ODE in many cases [8, 9, 23, 4]. In recent years a number of articles have dealt with this particular aspect [11, 15, 2]. However, when a planar system of ODEs cannot be reduced to a second-order differential equation the question of interest arises whether the JLM can provide a mechanism for finding the Lagrangian of the system.

In an interesting paper Nucci and Tamizhmani [16] showed that the method used by Trubatch and Franco in [21] and Paine [17] for finding Lagrangians of certain representative biological models actually relies on the existence of a JLM. Nucci *et al* have re-derived the linear Lagrangians of these first-order systems using JLM. They have also obtained the Lagrangians of the corresponding single second-order equations which the earlier authors had failed to do, for example in the case of the host-parasite model.

In this article we apply JLM to find the Lagrangian and the Hamiltonian of certain systems of differential equations which appear in spatio-temporal studies and in biology. It may be noted that a variational problem with a Lagrangian L and configuration space Q may fail to satisfy the Legendre condition, i.e., the fibre derivative map $FL : TQ \to T^*Q$ may fail to be a local diffeomorphism. Therefore direct Hamiltonization of a nonlinear system based on the JLM [1, 7] offers a distinct advantage over the usual process using Legendre transformations. In fact it also yields the canonical coordinates in terms of which one can express the underlying system in canonical form. A similar though more restricted result is given by Lucey and Newman [10], who have shown that for a given system of autonomous ODEs there exists, locally at least a symplectic structure and a Hamiltonian function such that the given system of equations can be expressed in Hamiltonian form. In this article we apply the last multiplier to obtain the Hamiltonians of the Gierer-Meinhardt model [5], an activator-inhibitor model. In one of his seminal papers Turing [22] showed that differences in the diffusion constants of activator and inhibitor species can bring about destabilization of the uniform state and lead to spontaneous emergence of periodic spatial patterns. Turing patterns emerge in various areas of biological systems. The basic idea behind Gierer-Meinhardt system is the so called diffusion-driven instability, originally due to Turing, which asserts that different diffusion rates could lead to nonhomogeneous distributions of the reactants. Gierer and Meinhardt developed a model of two coupled reaction-diffusion equation for the production and diffusion of two different kinds of substances, called the activator and inhibitor (see for example, [18]). Let $u(t,x)$ and $v(t,x)$ stand for the concentration of the activator and inhibitor at (x,t) respectively, then the so called Gierer-Meinhardt model of morphogenesis is

$$\begin{aligned} u_t &= D_u u_{xx} + \rho\rho_0 + \lambda\rho\frac{u^2}{v} - bu\ , \\ v_t &= D_v v_{xx} + \lambda'\rho' u^2 - cv\ , \end{aligned} \tag{1.3}$$

where D_u and D_v are the diffusion constants of the activator and inhibitor, $\rho\rho_0$ stands for the sourse concentration for the activator and ρ' is the one for the inhibitor, b and c are respectively denote the degradation coefficients of the activator and inhibitor, λ and λ' are related to activator and inhibitor productions.

Granero-Porati and Porati [6] considered the ODE version of the Gierer-Meinhardt model in the form

$$\begin{aligned} \dot{u} &= \rho\rho_0 + \lambda\rho\frac{u^2}{v} - bu\ , \\ \dot{v} &= \lambda'\rho' u^2 - cv\ . \end{aligned} \tag{1.4}$$

In the following we will deduce a Lagrangian and the corresponding Hamiltonian for this system of equations. It is well known that vector fields whose flows preserve a symplectic form up to a constant, such as simple mechanical systems with friction, are called conformal. The Duffing oscillator being a typical example of this class. In fact, interestingly enough, the system (1.4) also turns out to be a conformal Hamiltonian system [12, 14].

From the point of view of applications, the activator-inhibitor equations play a very important role in the study of Turing pattern formation which provides a credible theoretical explanation of animal coat patterns (see for example [13]). In an interesting paper Wojkowski and Liverani [24] have studied the Lyapunov spectrum in locally conformal Hamiltonian systems. Furthermore, it was demonstrated that Gaussian isokinetic dynamics, Nośe-Hoovers dynamics and other systems can also be studied through locally conformal Hamiltonian systems. In this paper we wish to focus on the dynamical aspects of these equations. We compute the Lagrangian and Hamiltonian of these equations. The paper is organized as follows. In Section 2 we describe the Jacobi Last Multiplier (JLM) formulation for singular Lagrangian equations illustrating our construction through examples. Section 3 is devoted to the Hamiltonian formulation of such singular activator-inhibitor type systems.

2 Preliminaries

Let us briefly recall the procedure described in [16] for finding Lagrangians for a planar system of ODEs from a knowledge of the last multiplier. We assume that the system (1.1) and (1.2) admits a Lagrangian and class of systems which we wish to deal in this paper are linear or affine in velocities, so that

$$L(t,x,y,\dot{x},\dot{y}) = F(t,x,y)\dot{x} + G(t,x,y)\dot{y} - V(t,x,y). \tag{2.1}$$

Then the Euler-Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) = \frac{\partial L}{\partial u}, \quad \text{with} \;\; u = x \text{ and } y$$

yield

$$\dot{y} = \left(\frac{F_t + V_x}{G_x - F_y}\right) = g(t,x,y), \tag{2.2}$$

and

$$\dot{x} = -\left(\frac{G_t + V_y}{G_x - F_y}\right) = f(t,x,y), \tag{2.3}$$

where the subscripts on F, G and V denote partial derivatives while the overdot represents derivative with respect to time. It is obvious that one must have $G_x \neq F_y$. In order to introduce the notion of Jacobi's last multiplier we assume that $G_x = -F_y$ and assign a common value,

$$\mu(t,x,y) := G_x = -F_y. \tag{2.4}$$

From (2.2) and (2.3) we have

$$2\mu f(t,x,y) = -(G_t + V_y) \tag{2.5}$$

$$2\mu g(t,x,y) = (F_t + V_x). \tag{2.6}$$

It is clear that the construction

$$\frac{\partial}{\partial x}(2\mu f) + \frac{\partial}{\partial y}(2\mu g)$$

leads to the following equation,

$$\frac{d}{dt}\log\mu + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0. \tag{2.7}$$

using the original system of ODEs $\dot{x} = f$ and $\dot{y} = g$. However, (2.7) is precisely the defining relation for JLM [23]. Thus we see that given the solution of this equation one can easily construct from (2.4) the coefficient functions F and G occurring in the expression for the Lagrangian since

$$F(t,x,y) = -\int \mu(t,x,y)dy \quad \text{and} \quad G(t,x,y) = \int \mu(t,x,y)dx. \tag{2.8}$$

Once these functions are determined one can obtain an expression for the partial derivatives of V from (2.2) and (2.3) as follows

$$\frac{\partial V}{\partial x} = 2\mu(t,x,y)g(t,x,y) + \frac{\partial}{\partial t}\left(\int \mu dy\right), \tag{2.9}$$

$$\frac{\partial V}{\partial y} = -2\mu(t,x,y)f(t,x,y) - \frac{\partial}{\partial t}\left(\int \mu dx\right). \tag{2.10}$$

In view of (2.7) it is easy to check the equality of the mixed derivatives,

$$\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x}.$$

2.1 Illustrations of finding Lagrangians using JLM

In this section we demonstrate the construction of the Lagrangians for spatio-temporal autocatalysis system and the celebrated Gierer-Meinhardt model, these are all autocatalytic systems. We apply Jacobi's last multiplier method to compute there Lagrangians.

2.1.1 Lagrangians for Spatio-temporal autocatalysis system

The following is an example of an auto-catalysis and activator inhibitor system.

$$\dot{x} = \frac{a}{b+y} - cx := f(x,y) \tag{2.11}$$

$$\dot{y} = dx - hy := g(x,y). \tag{2.12}$$

From the defining relation (2.7) for the last multiplier, we have

$$\frac{d}{dt}\log\mu - (c+h) = 0,$$

which gives $\mu = e^{(c+h)t}$. It must be noted that μ is not necessarily a purely temporal function, *μ is defined up to a factor that is a constant of motion.* It follows that the functions $G(x,y)$ and $F(x,y)$ are given by

$$G(x,y) = xe^{(c+h)t}, \quad F(x,y) = -ye^{(c+h)t}. \tag{2.13}$$

From (2.9) and (2.10) we have therefore

$$\frac{\partial V}{\partial x} = 2e^{(c+h)t}(dx - hy)$$

$$\frac{\partial V}{\partial y} = -2e^{(c+h)t}(\frac{a}{b+y} - cx)$$

whence from the equality of the mixed derivatives we obtain the condition $c + h = 0$. Using this condition the expression for the potential term becomes

$$V(x, y) = 2cxy - 2a\log(b + y) + dx^2$$

and hence a Lagrangian for the reduced system

$$\dot{x} = \frac{a}{b+y} - cx, \quad \dot{y} = dx + cy$$

is given by

$$L = x\dot{y} - y\dot{x} + 2a\log(b + y) - dx^2 - 2cxy. \tag{2.14}$$

Notice that the condition $c + h = 0$ causes the last multiplier to reduce to unity.

2.1.2 Lagrangians for the Gierer-Meinhardt model

Our second example is provided by the following system

$$\dot{u} = a - bu + \frac{u^2}{v}, \quad \dot{v} = u^2 - v, \tag{2.15}$$

which is known as the *Gierer-Meinhardt* model. Here u is a short-range autocatalytic substance, i.e., activator, and v is its long-range antagonist, i.e., inhibitor. In other words, this scheme considers autocatalytic activation of chemical u and self inhibitation of v. The model was formulated by Alfred Gierer and Hans Meinhardt in 1972 [5].

From the defining relation (2.7) for the last multiplier, we find that when the parameter $a = 0$ the multiplier is given by

$$\mu = \frac{1}{u^2}e^{(1-b)t}.$$

In this case proceeding in the same manner as above one finds that a consistent expression for the potential term exists provided the parameter $b = 1$ and the final expression for a singular Lagrangian may be given by

$$L = \frac{v}{u^2}\dot{u} + \frac{1}{u}\dot{v} + 2(u + \frac{v}{u} - \log v), \tag{2.16}$$

under the condition $a = 0$ and $b = 1$. The reduced system therefore has the appearance

$$\dot{u} = -u + \frac{u^2}{v}, \quad \dot{v} = u^2 - v. \tag{2.17}$$

3 Time dependent Hamiltonian systems and the Jacobi Last Multiplier

Let $\mathcal{M}$ denote a real two dimensional manifold with local coordinates x and y. Consider the following non autonomous system of differential equations:

$$\dot{x} = f(x, y, t), \quad \dot{y} = g(x, y, t), \tag{3.1}$$

where f and g are smooth real valued functions. We can associate with the system (3.1) the following:
(a) a vector field

$$X := \frac{\partial}{\partial t} + f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$$

defined on $\mathcal{M} \times \mathbb{R}$ whose integral curves are determined by the system (3.1), (b) alternatively we may consider the following set of one forms on $\mathcal{M} \times \mathbb{R}$

$$\alpha^{(1)} := dx - f(x, y, t)dt, \quad \alpha^{(2)} := dy - g(x, y, t)dt$$

and finally, (c) a two-form on $\mathcal{M} \times \mathbb{R}$ given by

$$\alpha^{(1)} \wedge \alpha^{(2)} = dx \wedge dy + (fdy - gdx) \wedge dt.$$

The classical Poincaré-Cartan form [19, 20] for a Hamiltonian H is given in the standard extended phase space coordinates $\{t, q^i, p^j\ 1 \geq i, j \geq n\}$, by

$$\Theta = p_i dq^i - Hdt. \tag{3.2}$$

The Poincaré-Cartan form consists of two terms - the standard "symplectic" 1-form $p_i dq^i$ and the Hamiltonian term. The duality between the Hamiltonian and Lagrangian formulations is well known by means of the Legendre transformation. Let $L : \mathbb{R} \times TQ \to \mathbb{R}$ be a non-autonomous Lagrangian. Let us recall that the Legendre transformation $FL : \mathbb{R} \times TQ \longrightarrow \mathbb{R} \times T^*Q$, is a fibre derivative. Given $f \in C^\infty(TQ, \mathbb{R})$ and a restriction $f_q := f|_{T_qQ}$ to the fibre over q, the fibre derivative of f is a mapping defined by

$$FL(f) : TQ \to T^*Q, \qquad FL\big(f(q, \dot{q})\big) := Df_q(\dot{q}).$$

The function f is said to be hyperregular if $FL(f)$ is a diffeomorphism. Therefore a Legendre transformation is locally given by

$$FL(t, q^i, \dot{q}^j) \equiv (t, q^i, FL(\dot{q}^j)) = (t, q^i, p_i = \partial L/\partial \dot{q}).$$

In terms of the jet coordinates $\{t, q^i, \dot{q}^j\}$ the one form Θ has a Lagrangian of the form

$$\Theta = Ldt + \frac{\partial L}{\partial \dot{q}^i}\big(dq^i - \dot{q}^i dt\big). \tag{3.3}$$

When L is hyperregular, i.e., FL is a diffeomorphism, then in such a situation both formulations are completely equivalent. Our focus here is on the Hamiltonian formulation. In Hamiltonian coordinates we have

$$d\Theta = \big(dp_i + \frac{\partial H}{\partial q^i}dt\big) \wedge \big(dq^i - \frac{\partial H}{\partial p_i}dt\big)$$

so the differential system takes the well-known Hamiltonian form

$$dp_i + \frac{\partial H}{\partial q^i}dt = 0, \qquad dq^i - \frac{\partial H}{\partial p_i}dt = 0.$$

In other words, if (3.1) admits a Hamiltonian description then

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{3.4}$$

An alternative description is given in terms of the Euler vector field

$$X_E = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}, \tag{3.5}$$

defined by

$$i_{X_E} d\Theta = 0 \qquad \text{and} \qquad i_{X_E} dt = 1. \tag{3.6}$$

Thus $X_E = \frac{\partial}{\partial t} + X_H$, where X_H is the standard Hamiltonian vector field defined by the canonical symplectic form $\omega = dp \wedge dq$. In fact the first equation yields the Hamiltonian equation in the following form

$$dH + i_{X_H}\omega = 0.$$

3.1 Jacobi's last multiplier and Hamiltonians

Our basic aim is to study the generalization of Hamiltonian mechanics and to construct an exact expression for dH using Jacobi's last multiplier. The algorithm to be employed for this purpose is based on an application of the exterior algebra and is described below. Let

$$\beta^{(1)} := dq - \frac{\partial H}{\partial p}dt, \quad \beta^{(2)} := dp + \frac{\partial H}{\partial q}dt. \tag{3.7}$$

Clearly the two-form

$$\Omega := \beta^{(1)} \wedge \beta^{(2)} = dq \wedge dp + (dH - \frac{\partial H}{\partial t}dt) \wedge dt = dq \wedge dp + dH \wedge dt, \tag{3.8}$$

and is *closed*. It is obvious that when (3.1) is expressible in the form of (3.4) then the two-form $\alpha^{(1)} \wedge \alpha^{(2)}$ must be proportional to the closed two-form $\beta^{(1)} \wedge \beta^{(2)}$, so that there exists a function $\sigma(x, y, t)$ such that

$$\sigma^{-1}[dx \wedge dy + (f dy - g dx) \wedge dt] = [dq \wedge dp + dH \wedge dt]. \tag{3.9}$$

Since the two-form on the rhs is necessarily closed it follows that we must have

$$\sigma^{-1}\left(\frac{\partial\sigma}{\partial t}+f\frac{\partial\sigma}{\partial x}+g\frac{\partial\sigma}{\partial y}\right)=\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right). \tag{3.10}$$

The last equation may be written as

$$\frac{d\log\sigma}{dt}=\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right),$$

and a comparison with (2.7), the determining equation for the JLM, shows that $\sigma^{-1}=\mu$. Hence from (3.9) we see that the existence of σ satisfying (3.10) implies that

$$\sigma^{-1}(fdy-gdx)=dH+\text{terms proportional to }\ dt, \tag{3.11}$$

provided $\frac{\partial\sigma}{\partial t}=0$. The latter condition being true for autonomous differential equations.

For nonautonomous cases satisfying (3.9) form one must modify (3.11) by introducing two auxiliary functions ψ and ϕ [1] such that

$$\sigma^{-1}((f-\psi)dy-(g-\phi)dx)=dH+\text{terms proportional to } dt, \tag{3.12}$$

when $\frac{\partial\sigma}{\partial t}\neq 0$. This essentially removes the explicit time-dependent terms and allows for the construction of a Hamiltonian for the remaining autonomous part. However, the time dependence is not altogether lost; it being manifested in the the data of the new coordinates.

Note that (3.12) implies

$$\frac{\partial}{\partial x}\left(\sigma^{-1}(f-\psi)\right)+\frac{\partial}{\partial y}\left(\sigma^{-1}(g-\phi)\right)=0. \tag{3.13}$$

It will now be observed that one may write (3.9) in the following manner

$$\sigma^{-1}[dx\wedge dy+(fdy-gdx)\wedge dt]=\sigma^{-1}[(dx-\psi dt)\wedge(dy-\phi dt)+(f-\psi)dy\wedge dt-(g-\phi)dx\wedge dt],$$

which in view of (3.12) becomes

$$\sigma^{-1}[dx\wedge dy+(fdy-gdx)\wedge dt]=\sigma^{-1}(dx-\psi dt)\wedge(dy-\phi dt)+dH\wedge dt. \tag{3.14}$$

But a comparison of (3.14) with (3.9) shows that

$$\sigma^{-1}(dx-\psi dt)\wedge(dy-\phi dt)=dq\wedge dp. \tag{3.15}$$

In view of (3.10) and (3.13) it is clear that the lhs of (3.15) is indeed closed. Thus the problem of recasting (3.1) into the form of Hamilton's equations reduces to a determination of the auxiliary functions ϕ and ψ such that H is identified from (3.12). As for the canonical variables q and p these are to be identified from (3.15) once ϕ and ψ are known and σ has been obtained by solving (3.10). We illustrate the application of the above procedure with a few examples.

4 Hamiltonians of some systems of ODEs appearing in biology

Let (M, ω) be a symplectic manifold, where M is a differentiable manifold endowed with a symplectic form ω. Consider a diffeomorphism ϕ such that $\phi^*(\omega) = k\omega$, where $k \in \mathbb{R}$. Let the vector field Γ be the generator of a one-parameter group of canonical transformations $\phi^*_\epsilon(\omega) = k(\epsilon)\omega$. Then, there exists a real number $a \neq 0$ such that Lie derivative of the dynamical vector field Γ satisfies $L_\Gamma \omega = a\omega$, with k and a related by $k(\epsilon) = exp(a\epsilon)$. In this section we analyse Hamiltonian structure of both the activator-inhibitor systems. In the first case we map it to conformal Hamiltonian dynamics, but the Gierer-Meinhardt case yields almost conformal Hamiltonian structure.

4.1 Autocatalysis system and conformal Hamiltonian Systems

Let us start our analysis with the autocatalysis system. One can check directly that Lie derivative of the dynamical vector field of the autocatalysis system satisfies $L_\Gamma \omega = (c+h)\omega$, hence Γ is said to be conformal vector field with parameter $c+h$. When we impose $c+h=0$, Γ becomes symplectic or Hamiltonian vector field. When we restricted to the symplectic case, i.e., $c = -h$, the associated Hamiltonian is given by

$$H = a\log(b+y) - d\frac{x^2}{2} - cxy, \tag{4.1}$$

with standard Poisson Brackets $\{x, y\} = 1$ with the equations of motion given by

$$\dot{x} = \{x, H\} = \frac{\partial H}{\partial y} = \frac{a}{b+y} - cx$$

$$\dot{y} = \{y, H\} = -\frac{\partial H}{\partial x} = dx + cy.$$

When the manifold is $\mathbb{R}^2$ with coordinates (x, y) and $\omega = dx \wedge dy$, the conformal vector fieldsthose whose flow is conformalhave the form:

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x} + \kappa y$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is the Hamiltonian. Their flow has the property $\phi^*\omega = e^{\kappa t}\omega$, so the symplectic inner product of any two tangent vectors contracts exponentially if $\kappa < 0$. If ω is an locally symplectic structure (l.c.s.) then two l.c.s. ω and $\omega' = e^{\kappa t}\omega$ are conformally equivalent.

Thus general activator-inhibitor system can be expressed

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x} - (c+h)y.$$

Given $H \in C^\infty(M)$, the vector field Γ (or X_H^c- usual notation) satisfies

$$i_{X_H^c}\omega = dH - (c+h)\theta, \quad \theta = ydx$$

is conformal. The conformal vector field is given by $X_H + (c+h)Z$, where Z is defined by $i_Z\omega = -\theta$. Here it turns out $Z = y\partial_y$.

4.2 The Gierer-Meinhardt model and almost conformal Hamiltonian structure

At first we study $b = 1$ case, then the integrating factor becomes $\mu = u^{-2}$. Consider planar Hamiltonian system

$$\dot{u} = J(u,v)H_v, \qquad \dot{v} = -J(u,v)H_u,$$

where J is associated with the symplectic structure. Using last multiplier equation we obtain

$$\frac{d}{dt}\log\mu + \frac{\dot{J}}{J} = 0,$$

which yields $\mu = \frac{1}{J}$. Thus symplectic matrix is given by

$$\mathbb{J} = \begin{pmatrix} 0 & \mu^{-1} \\ -\mu^{-1} & 0 \end{pmatrix}.$$

Thus only multiplying with the inverse integrating factor $\mu^{-1} = u^2$ admits a Hamiltonian structure

$$H = \log v - u - \frac{v}{u}, \tag{4.2}$$

with fundamental Poisson Brackets given by

$$\{u,u\} = \{v,v\} = 0, \;\; \{u,v\} = \mu^{-1} = u^2. \tag{4.3}$$

In general for any arbitrary nonsingular function $f(u)$ it is possible to change the nonstandard Poisson structure to standard Poisson structure by substituting $v \longmapsto f^{-1}(u)v$. Then the "new" Poisson bracket satisfies

$$\{u,v\}_f = f^{-1}(u)\{u,v\} = 1.$$

It is now easy to verify that the Hamiltonian (for nonstandard Poisson structure) equations

$$\dot{u} = \{u,H\} = u^2\frac{\partial H}{\partial v} = -u + \frac{u^2}{v}, \tag{4.4}$$

$$\dot{v} = \{v,H\} = -u^2\frac{\partial H}{\partial u} = u^2 - v, \tag{4.5}$$

reproduce the equations of the reduced system (2.17). Furthermore as $dH/dt = 0$ the Hamiltonian H is also a first integral of the reduced system (2.17). The Poisson structure associated with (4.3) is called nonstandard Poisson structure. It is clear that the equations (4.4) and (4.5) are not divergence free. The multiplication by the JLM μ yields volume preserving condition

$$\sum_{i=1}^{2} \frac{\partial}{\partial w_i} \mu \frac{dw_i}{dt} = 0, \qquad \text{where} \quad w_i = u, v,$$

so the phase space volume is preserved.

Let μ be the multiplier and the dynamical vector field associated to $b = 1$ Gierer-Meinhardt equation is given by

$$\Gamma = (-u + \frac{u^2}{v}) \frac{\partial}{\partial u} + (u^2 - v) \frac{\partial}{\partial v}. \tag{4.6}$$

We consider transformation of vector field Γ corresponding to the transformation of Poisson bracket, $\Gamma \longmapsto \mu\Gamma$, which satisfies

$$i_{\mu\Gamma}\Omega = dH, \qquad \text{where} \qquad \Omega = du \wedge dv. \tag{4.7}$$

Here μ can be identified with the inverse integrating factor. The terminology "integrating factor" for the function V comes from the fact that $1/V$ is an integrating factor for the vector field Γ, i.e., that $V\Gamma$ is divergence-free. These allow the use of techniques from the theory of (local) Hamiltonian differential equations.

The vector field $\tilde{\Gamma}$ associated to the general equation satisfies

$$i_{\mu\tilde{\Gamma}}\Omega = dH - (b-1)\frac{dv}{u}. \tag{4.8}$$

Hence the conformal vector field can be decomposed into

$$\mu\tilde{\Gamma} = X_H - (b-1)u^{-1}\frac{\partial}{\partial u}, \tag{4.9}$$

where

$$X_H = (-\frac{1}{u} + \frac{1}{v})\frac{\partial}{\partial u} + (1 - \frac{v}{u^2})\frac{\partial}{\partial v}. \tag{4.10}$$

Thus we give a Hamiltonian description of the Gierer-Meinhardt model.

5 Conclusion

In this paper we have studied the Hamiltonization of systems of equations appearing in activator and inhibitor model. The Hamiltonians described here are true in the sense that they allow us to reproduce the original equations through the standard Hamilton equations. This method described here is specifically suited for systems which are intrinsically described by singular Lagrangians. There are some open problems popped up from this paper, it would be nice to study this formulation for other activator-inhibitor models. In particular, we wish to study a model commonly referred to as the Thomas model [13], proposed in 1975, it is an empirical model based on a specific reaction involving uric acid and oxygen.

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