The role of the Jacobi Last Multiplier in Nonholonomic Systems and Almost Symplectic Structure

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Abstract

The relationship between Jacobi's last multiplier (JLM) and nonholonomic systems endowed with the almost symplectic structure is elucidated in this paper. In particular, we present an algorithmic way to describe how the two form and almost Poisson structure associated to nonholonomic system, studied by L. Bates and his coworkers, can be mapped to symplectic form and canonical Poisson structure using JLM. We demonstrate how JLM can be used to map an integrable nonholonomic system to a Liouville integrable system. We map the toral fibration defined by the common level sets of the integrals of a Liouville integrable Hamiltonian system with a toral fibration coming from a completely integrable nonholonomic system.

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1 Introduction

The Jacobi last multiplier (JLM) is a useful tool for deriving an additional first integral for a system of n first-order ODEs when $n-2$ first integrals of the system are known. Besides, the JLM allows us to determine the Lagrangian of a second-order ODE in many cases [13, 21, 25]. In his sixteenth lecture on dynamics Jacobi uses his method of the last multiplier [16, 17] to derive the components of the LaplaceRungeLenz vector for the two-dimensional Kepler problem. In recent years a number of articles have dealt with this particular aspect [21, 22, 10]. However, when a planar system of ODEs cannot be reduced to a second-order differential equation the question of interest arises whether the JLM can provide a mechanism for finding the Lagrangian of the system.

Let M be an even dimensional differentiable manifold endowed with a non-degenerate 2-form Ω, (M, Ω) is an almost symplectic manifold. An almost symplectic manifold (M, Ω) is called locally conformal symplectic manifold by Vaisman [24] if there is a global 1-form η, called the Lee form on M such that

$$d\Omega = \eta \wedge \Omega,$$

where $d\eta = 0$. (M, ω) is globally conformally symplectic if the Lee form η is exact and when $\eta = 0$, then (M, Ω) is a symplectic manifold. The notion of locally conformally symplectic forms is due to Lee and, in more modern form, to Vaisman. Chinea [8, 9] et al showed an extension of an observation made by I. Vaisman that locally conformal symplectic manifolds can be seen as a natural geometrical setting for the description of time-independent Hamiltonian systems. In a seminal paper Wojkowski and Liverani [26] studied the Lyapunov spectrum in locally conformal Hamiltonian systems. It was demonstrated that Gaussian isokinetic dynamics, Nośe-Hoovers dynamics and other systems can be studied through locally conformal Hamiltonian systems. It must be noted that the conformal Hamiltonian structure appears in various dissipative dynamics as well as in the activator-inhibitor model connected to Turing pattern formation. It has been shown by Haller and Rybicki [15] that the Poisson algebra of a locally conformally symplectic manifold is integrable by making use of a convenient setting in global analysis. In this paper we explore the role of the Jacobi last multiplier in nonholonomic free particle motion and nonholonomic oscillator. These systems were studied extensively studied by L. Bates and his coworkers [2, 3, 4, 5]. The two forms associated with these nonholonomic systems are not closed, in fact they satisfy l.c.s. condition. We apply JLM to such systems which guarantees that at least locally the symplectic form can be multiplied by a nonzero function to get a symplectic structure. In an interesting paper Bates and Cushman [4] compared the geometry of a toral fibration defined by the common level sets of the integrals of a Liouville integrable Hamiltonian system with a toral fibration coming from a completely integrable nonholonomic system. We apply JLM to study and compare these two toral fibrations. All the examples considered in this paper are taken from Bates et al. papers. Relatively very little has been done when the flow is not complete. A quarter of a century ago, Flaschka [12] raised a number of questions concerning a simple class of integrable Hamiltonian systems in $\mathbf{R}^4$ for which the orbits lie on surfaces.

This paper is organized as follows. The first section recalls the definitions of the locally conformal symplectic structure and the Jacobi last multiplier. We show the application of JLM in nonholonomic system in Section 3, a notion which, to our knowledge, does not appear explicitly in the literature. The paper ends with a list of remarks regarding the further applications of JLM in nonholonomic systems.

2 Preliminaries

We start with a brief review [19, 15, 24] of the locally conformal symplectic structure. A differentiable manifold M of dimension $2n$ endowed with a non-degenerate 2-form ω and a closed 1-form η is called a locally conformally symplectic (l.c.s.) manifold if

$$d\omega + \omega \wedge \eta = 0. \tag{2.1}$$

The 1-form η is called the Lee form of ω. This allows us to introduce the Lichnerowicz deformed differential operators

$$d_\eta : \Omega^*(M) \longrightarrow \Omega^{*+1}(M),$$

such that $d_\eta\theta = d\theta + \eta \wedge \theta$. clearly $d_\eta^2 = 0$ and $d_\eta\omega = 0$. It must be noted that l.c.s manifold is locally conformally equivalent to a symplectic manifold provided $\eta = df$ and $\omega = e^f\omega_0$, such that $d\omega_0 = 0$.

If (ω, η) is an l.c.s. structure on M and $f \in C^\infty(M, \mathbb{R})$, then $(e^f\omega, \eta - df) = (\omega', \eta')$ is again an l.c.s. structure on M then these two are conformally equivalent and two operators and two Lee forms are cohomologous: $\eta' = \eta - df$. Hence d_η and $d_{\eta'}$ are gauge equivalent

$$d_{\eta'}(\beta) = (d_\eta - df\wedge)\beta = e^f\, d(e^{-f}\beta).$$

The r.h.s is connected to Witten's differential. If $f \in C^\infty(M)$ and $t \geq 0$, Witten deformation of the usual differential $d_{tf} : \Omega^*(M) \longrightarrow \Omega^{*+1}(M)$ is defined by $d_{tf} = e^{tf}de^{-tf}$, which means $d_{tf}\beta = d\beta + t\beta\wedge df$. Since d_η and $d_{\eta'}$ are gauge equivalent hence the Lichnerowicz cohomology groups $H^*(\Omega^*(M), d_\eta)$ and $H^*(\Omega^*(M), d_\eta)$ are isomorphic and the isomorphism is given by the conformal transformation $[\beta] \longmapsto [e^f\beta]$.

It is clear from the definition that d_η does not satisfy the Leibniz property:

$$\begin{aligned} d_\eta(\theta \wedge \psi) &= (d + \eta\wedge)(\theta \wedge \psi) = d_\eta\theta \wedge \psi + (-1)^{deg\,\theta}\theta \wedge d\psi \\ &= d\theta \wedge \psi + (-1)^{deg\,\theta}\theta \wedge d_\eta\psi. \end{aligned}$$

For an l.c.s. manifold, we denote by

$$\text{Diff}\,{}^\infty_c(M, \omega, \eta) := \{f \in \text{ Diff}^\infty_c(M) | (f^*\omega, f^*\eta) \simeq (\omega, \eta)\}$$

the group of compactly supported diffeomorphisms preserving the conformal equivalence class of (ω, η). The corresponding Lie algebra of vector fields is

$$\chi_c(M, \omega, \eta) := \{X \in \chi_c(M) \mid \exists c \in \mathbb{R} : L^\eta_X\omega = c\omega\},$$

where $L^\eta_X\beta = L_X\beta + \eta(X)\beta$. The Cartan magic formula for L^η_X is given by

$$L^\eta_X = d_\eta \circ i_X + i_X \circ d_\eta.$$

Here we list some of the important properties of the Lie derivative.

1. $L^\eta_X L^\eta_Y - L^\eta_X L^\eta_X = L^\eta_{[X,Y]}$.
2. $L^\eta_X d_\eta - d_\eta L^\eta_X = 0$
3. $L^\eta_X i_Y - i_Y L^\eta_X = 0$.
4. Let η_1 and η_2 are two Lee forms then $L^{\eta_1+\eta_2}_X(\theta \wedge \psi) = (L^{\eta_1}_X\theta) \wedge \psi + \theta \wedge (L^{\eta_2}_X\psi)$.

Let X and Y be the two conformal vector fields then $[X, Y]$ becomes the symplectic vector field. The proof of this claim is very simple, can can easily show that $L^\eta_{[X,Y]}\omega = 0$.

2.1 Inverse problem and the Jacobi last multiplier

We start with a brief introduction [10, 13, 21, 22, 25] of the Jacobi last multiplier and inverse problem of calculus of variations [18]. Consider a system of second-order ordinary differential equations

$$y''_i = f_i(y_j, y'_j) \qquad \text{for} \quad 1 \leq i, j \leq n.$$

Geometrically these are the analytical expression of a second-order equation field Γ living on the first jet bundle $J^1\pi$ of a bundle $\pi : E \to \mathbb{R}$, so

$$\Gamma = y'_i \frac{\partial}{\partial y_i} + f_i(y_j, y'_j)\frac{\partial}{\partial y'_i}.$$

The local formulation of the general inverse problem is the question for the existence of a non-singular multiplier matrix $g_{ij}(y, y')$, such that

$$g_{ij}(y''_j - F_j) \equiv \frac{d}{dt}\Big(\frac{\partial L}{\partial y_i}\Big) - \frac{\partial L}{\partial y'_i},$$

for some Lagrangian L.

Theorem 2.1 *(Douglas[11]) There exists a Lagrangian $L : TQ \to \mathbb{R}$ such that the equations are its Euler-Lagrange equations if and only if there exists a non-singular symmetric matrix g with entries g_{ij} satisfying the following three Helmholtz conditions:*

$$g_{ij} = g_{ji}, \qquad \Gamma(g_{ij}) = g_{ik}\Gamma^k_j + g_{jk}\Gamma^k_i,$$

$$g_{ik}\Phi^k_j = g_{jk}\Phi^k_i, \qquad \frac{\partial g_{ij}}{\partial y'_k} = \frac{\partial g_{ik}}{\partial y'_j},$$

where

$$\Gamma^k_j := -\frac{1}{2}\frac{\partial f_i}{\partial y'_j}.$$

When the system is one-dimensional we have $i = j = k = 1$ and then the three set of conditions become trivial and the fourth one reduces to one single P.D.E.

$$\Gamma(g) + g\frac{\partial f}{\partial v} \equiv v\frac{\partial g}{\partial x} + f\frac{\partial g}{\partial v} + g\frac{\partial f}{\partial v} = 0.$$

This is the equation defining the Jacobi multipliers, because $\text{div}\Gamma = \frac{\partial f}{\partial v}$. The main equation can also be expressed as

$$\frac{dg}{dt} + g \cdot div\, \Gamma = 0.$$

Then, the inverse problem reduces to find the function g (often denoted by μ) which is a Jacobi multiplier and L is obtained by integrating the function μ two times with respect to velocities.

An autonomous second-order differential equation $y'' = F(y, y')$ has associated a system of first-order differential equations

$$y' = v, \qquad v' = F(y, v) \tag{2.2}$$

whose solutions are the integral curves of the vector field in $\mathbb{R}^2$

$$\Gamma = v\frac{\partial}{\partial y} + F(y, v)\frac{\partial}{\partial v}. \tag{2.3}$$

A Jacobi multiplier μ for such a system must satisfy divergencefree condition

$$\frac{\partial}{\partial y}(\mu v) + \frac{\partial}{\partial v}(\mu F) = 0,$$

which implies μ must be such that

$$v\frac{\partial \mu}{\partial y} + \frac{\partial \mu}{\partial v}F + \mu\frac{\partial F}{\partial v} = 0.$$

which taking into account $\frac{dM}{dx} = v\frac{\partial M}{\partial y} + F\frac{\partial M}{\partial v}$ above equation can be written as

$$\frac{d \log \mu}{dx} + \frac{\partial F}{\partial v} = 0. \tag{2.4}$$

The normal form of the differential equation determining the solutions of the Euler-Lagrange equation defined by the Lagrangian function $L(y, v)$ admits as a Jacobi multiplier the function

$$\mu = \frac{\partial^2 L}{\partial v^2}. \tag{2.5}$$

Conversely, if $\mu(y, v)$ is a last multiplier function for a second-order differential equation in normal form, then there exists a Lagrangian L for the system related to μ by the above equation.

Let L be such that condition $M = \frac{\partial^2 L}{\partial v^2}$ be satisfied, then

$$\frac{\partial L}{\partial v} = \int^v M(y,\zeta)d\zeta + \phi_1(y)$$

which yields

$$L(y,v) = \int^v dv' \int^{v'} M(y,\zeta)d\zeta \; + \phi_1(y)v + \phi_2(y).$$

Geometrical Interpretation of JLM Let M be a smooth, real, n-dimensional orientable manifold with fixed volume form Ω. Let $\dot{x}_i(t) = \gamma_i(x_1(t), \cdots, x_n(t))$, $1 \leq i \leq n$ generated by the vector field Γ and we consider the $(n-1)$-form $\Omega_\gamma = i_\Gamma \Omega$. The function $\mu \in C^\infty(M)$ is called a JLM of the ODE system generated by Γ, if $\mu\omega$ is closed, i.e.,

$$d(\mu\Omega_\gamma) = d\mu \wedge \Omega_\gamma + \mu d\Omega_\gamma.$$

This is equivalent to $\Gamma(\mu) + \mu.$ div $\Gamma = 0$. Characterizations of the JLM can be obtained in terms of the deformed Lichnerowicz operator $d_\mu(\theta) = d\mu \wedge \theta + d\theta$, where the Lee form in terms of the last multiplier, i.e. $\eta = d\mu$. Hence, μ is a multiplier if and only if [6]

$$d(\mu\Omega_\gamma) \equiv d_\mu\Omega_\gamma + (m-1)d\Omega_\gamma = 0. \tag{2.6}$$

3 Nonholonomic systems, locally conformal symplectic structure and JLM

Let (x, y, z) be the configuration variables for the nonholonomic system. The results presented in this paper are quite general but in order to perform explicit calculation we stick to the example of the motion of a free particle with unit mass subjected to a constriant

$$\dot{z} = y\dot{x}, \tag{3.1}$$

and the Lagrangian is $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. The equations of motion are

$$\dot{x} = p_x, \;\; \dot{y} = p_y, \;\; \dot{z} = p_z, \;\; \dot{p_x} = -\lambda y, \;\; \dot{p_y} = 0 \;\; \dot{p_z} = \lambda, \tag{3.2}$$

where (p_x, p_y, p_z) are the momentum variables.

Using the constraint equation we eliminate the multiplier to obtain

$$\dot{x} = p_x, \quad \dot{y} = p_y, \quad \dot{p_x} = -y\frac{p_x p_y}{(1+y^2)}, \quad \dot{p_y} = 0. \tag{3.3}$$

The vector field corresponding to equation (3.3)

$$\Gamma = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} - \frac{y p_x p_y}{(1+y^2)}\frac{\partial}{\partial p_x} \tag{3.4}$$

satisfies

$$\Gamma \lrcorner \tilde{\omega}_{nh} = -dH, \quad \text{where} \quad H = \frac{1}{2}\big((1+y^2)p_x^2 + p_y^2\big) \tag{3.5}$$

and the two form is given by

$$\tilde{\omega}_{nh} = dp_x \wedge dx + dp_y \wedge dy - \frac{p_x y}{(1+y^2)} dy \wedge dx. \tag{3.6}$$

Here $\tilde{\omega}_{nh}$ is the nondenerate two form on phase space P, however it is not closed, i.e.,

$$d\tilde{\omega}_{nh} = y dx \wedge dp_x \wedge dy.$$

We now compute the JLM of these set of equations from

$$\frac{d}{dt}\log\mu + \big(-\frac{y\dot{y}}{1+y^2}\big) = 0,$$

thus we obtain

$$\mu = (1+y^2)^{1/2}. \tag{3.7}$$

The Lagrangian of the reduced system is $L_c = 1/2\big((1+y^2)\dot{x}^2 + \dot{y}^2\big)$. Let S be the configuration space and $Leg_c : TS \longrightarrow T^*S$ be the Legendre transformation of the reduced system. Then the momentum corresponding to the reduced equations are given by

$$m_x = \frac{\partial L}{\partial \dot{x}} = (1+y^2)\dot{x}, \qquad m_y = \frac{\partial L}{\partial \dot{y}}$$

and the corresponding Hamiltonian of the reduced system is given by

$$H_c = \frac{1}{2}\big(\frac{m_x^2}{1+y^2} + m_y^2\big). \tag{3.8}$$

The new set of equations is given by

$$\dot{x} = \frac{m_x}{1+y^2}, \quad \dot{x} = m_y, \quad \dot{m}_x = \frac{y m_x m_y}{1+y^2}, \quad \dot{m}_y = 0. \tag{3.9}$$

The vector field

$$\Gamma_h = \frac{m_x}{1+y^2}\partial_x + m_y\partial_y + \frac{y m_x m_y}{1+y^2}\partial_{m_x} \tag{3.10}$$

satisfies $\Gamma_h \lrcorner \omega_{nh} = -dH_c$, where

$$\omega_{nh} = dm_x \wedge dx + dm_y \wedge dy - \frac{m_x y}{1+y^2} dy \wedge dx. \tag{3.11}$$

This says that that we can still do Hamiltonian dynamics as long as we are willing to give up the existence of canonical coordinates and the Jacobi identies for the Poisson brackets. We will subsequently see that the Jacobi last multiplier plays a crucial role to obtain the canonical coordinates and Poisson structures.

Proposition 3.1 *The nonholonomic two form* ω_{nh} *as well as* $\tilde{\omega}_{nh}$ *satisfy locally conformal symplectic structure and the Lee form is* $\eta = d\big(\log(1+y^2))^{1/2}\big) = d(\log\mu)$*, where* μ *is the Jacobi's last multiplier.*

Proof: It is straightforward to check

$$d\omega_{nh} = -\big(\frac{ydy}{1+y^2}\big) \wedge dm_x \wedge dx$$

$$= -d\big(\log(1+y^2)\big) \wedge \big(dm_x \wedge dx + dm_y \wedge dy - \frac{m_x y}{1+y^2} dy \wedge dx\big) = -\eta \wedge \omega_{nh}$$

and similarly for the other case. □

Corollary 3.1 *The (almost) Poisson structure corresponding to the locally conformal symplectic form* ω_{nh} *is given by*

$$\{x, m_x\} = 1, \quad \{y, m_y\} = 1, \quad \{m_x, m_y\} = \frac{ym_x}{1+y^2}. \tag{3.12}$$

The inverse multiplier play an important role for changing locally conformal symplectic form ω_{nh} to symplectic form. In this process we find new momemta which satisfy canonical Poisson structure.

Proposition 3.2 *Let* μ^{-1} *be the inverse multiplier, then* $\omega = \mu^{-1}\omega_{nh}$ *is a symplectic form, given by*

$$\tilde{\omega} = d\tilde{m}_x \wedge dx + d\tilde{m}_y \wedge dy, \tag{3.13}$$

where the new momenta are

$$\tilde{m}_x = \mu^{-1} m_x = \frac{m_x}{\sqrt{1+y^2}} \qquad \tilde{m}_y = \frac{m_x}{\sqrt{1+y^2}}. \tag{3.14}$$

Proof By direct computation one obtains

$$\mu^{-1}\omega_{nh} = \frac{1}{\sqrt{1+y^2}}\big(dm_x \wedge dx + dm_y \wedge dy - \frac{m_x y}{1+y^2} dy \wedge dx\big)$$

$$= \frac{dm_x}{\sqrt{1+y^2}} \wedge dx + \frac{dm_y}{\sqrt{1+y^2}} \wedge dy - \frac{m_x y}{(1+y^2)^3/2} dy \wedge dx \equiv d\tilde{m}_x \wedge dx + d\tilde{m}_y \wedge dy.$$

□

It is clear $d\tilde{\omega} = 0$ and the new momenta satisfy the canonical Poisson structure

$$\{x, \tilde{m}_x\} = 1, \qquad \{y, \tilde{m}_y\} = 1. \tag{3.15}$$

3.1 Role of Jacobi's multiplier and integrability of nonholonomic dynamical systems

We now address the question of integrability of the nonholonomic systems as posed by Bates and Cushman [4, 7]. In their papers, they explored to what extent nonholonomic systems behave like a integrable systems. The fundamental Liouville theorem states that it suffices to have n $\{f_1 = H, f_2, \cdots, f_n\}$ independent Poisson commuting functions to explicitly (i.e., by quadratures) integrate the equations of motion for generic initial conditions. Let $M_c = \{f_1 = c_1, \cdots, f_n = c_n\}$ be a common invarinat level set, which is regular (i.e., $df_1, \cdots df_n$ are independent), compact and connected, then it is diffeomorphic to n-dimensional tori $\mathbb{T}^n = \mathbb{R}^n/\Lambda$, where Λ is a lattice in $\mathbb{R}^n$. These tori are known as the Liouville tori [1, 7], In the neighborhood of M_c there exist canonical variables I, ϕ mod 2π, called action-angle variables which satisfy $\{\phi_i, I_j\} = \delta_{ij}$, $\{\phi_i, \phi_j\} = \{I_i, I_j\} = 0$, $i, j = 1, \cdots n$, such that the level sets of the actions $_I, \cdots, I_n$ are invariant tori and $H = H(I_1, \ldots, I_n)$.

The vector fields $X_{f_1}, \cdots, X_{f_n}$ corresponding to the n integrals of motion $f_1, \cdots f_n$ are independent (it follows from the independency of differentials) and span the tangent spaces of T_qM_c for all $q \in M_c$, since M_c is compact hence X_{f_i}s are complete. The Poisson commutativity implies the commutivity of vector fields. In other words, the so-called invariant manifolds, which are the (generic) submanifolds traced out by the n commuting vector fields X_{f_i} are Liouville tori, the flow of each of the vector fields X_{f_i} is linear, so that the solutions of Hamilton's equations are quasi-periodic. A proof in the case of a Liouville integrable system on a symplectic manifold was given by Arnold [1].

We will soon figure out that the (reduced) nonholonomic problem which we are considered in this paper has two constants of motion H (Hamiltonian) and K, these are Poisson commuting. However, because the nonholonomic system does not satisfy the Jacobi identity, the associated vector fields X_H and X_K do not commute, i.e. $[X_H, X_K] \neq 0$, on the torus. So Bates and Cushman asked if such system is integrable in some sense or how can it be converted to integrable systems.

3.1.1 JLM and commuting of vector fields

It has been observed the reduced Hamiltonian equation of motion lies on the invariant manifold given by

$$K = \frac{m_x}{\sqrt{1+y^2}}, \tag{3.16}$$

where K satisfies $\frac{dK}{dt} = 0$. The Hamiltonian vector field

$$X_K = \frac{1}{\sqrt{1=y^2}} \tag{3.17}$$

satisfies $X_k \lrcorner \omega_{nh} = -dK$.

The Hamiltonian vector field X_H satisfies

$$L_{X_H}K = X_H(K) = 0, \tag{3.18}$$

which implies

$$\omega_{nh}(X_H, X_K) = X_K \lrcorner X_H \lrcorner \omega_{nh} = X_K \lrcorner \Big(\frac{m_x}{1+y^2}dm_x + m_y dm_y - \frac{mx^2y}{(1+y^2)^2}dy\Big) = 0.$$

Next observe that the Lie bracket between vector fields X_H and X_K

$$[X_H, X_k] = -\frac{ym_x}{1+y^2}X_K. \tag{3.19}$$

This has been demonstrated by Bates and Cushman the vector fields X_H and X_K do not commute on the torus, because the two form ω_{nh} is not closed. They try to seek an integrating factor g such that $[gX_K, X_H] = 0$. Next propositon addresses the value of g.

Proposition 3.3 *Let μ be the Jacobi last multiplier, then the modified vector field $\mu^{-1}X_K$ commutes with the Hamiltonian vector field X_H, i.e.,*

$$[\mu^{-1}X_K, X_h] = 0. \tag{3.20}$$

Proof We know that the JLM $\mu = \sqrt{1+y^2}$, so that $\mu^{-1}X_K = \partial_x$. Hence we obtain $[\mu^{-1}X_K, X_h] = 0$. □

4 Final comments and outlook

Our formalism can be easily extended to nonholonomic oscillator. In this case, Lagrangian is given by $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}y^2$, subject to the nonholonomic constraint $\dot{z} = y\dot{x}$. The reduced system of equations are given by

$$\dot{x} = p_x, \quad \dot{y} = p_y, \quad \dot{p_x} = -\frac{y}{1+y^2}p_x p_y, \quad \dot{p_y} = -y.$$

One can easily check that the last multiplier is $\mu = (1+y^2)^{1/2}$. The two form associated to the reduced nonholonomic oscillator equation

$$\omega_{as} = (1+y^2)dp_x \wedge dx + dp_y \wedge dy + yp_x dy \wedge dx$$

satisfies locally conformal symplectic structure, $d\omega_{as} + \eta \wedge \omega_{as} = 0$, where the Lee form $\eta = d\big(\log(1+y^2)^{1/2}\big)$. Hence the inverse Jacobi's last multiplier transforms ω_{as} into a symplectic form

$$\mu^{-1}\omega_{as} \equiv \tilde{\omega} = d\tilde{p}_x \wedge dx + d\tilde{p}_y \wedge dy,$$

where the modified momenta are given by $\tilde{p}_x = \sqrt{1+y^2}p_x$ and $p_y = \frac{p_y}{\sqrt{1+y^2}}$. Thus everything can be repeated here.

The application of the Jacobi Last Multiplier (JLM) for finding Lagrangians of any second-order differential equation has been extensively studied. It is known that the ratio of any two multipliers is a first integral of the system, in fact, it plays a role similar to the integrating factor for system of first-order differential equations. But so far, it has not been applied to nonholonomic systems. In this paper we have studied nonholonomic system endowed with a two form, which is closely related to locally conformal symplectic structure. We have applied JLM to map it to symplectic frame work. Also, we have shown how a toral fibration defined by the common level sets of integrable nonholonomic system, studied by Bates and Cushman, can be mapped to toral fibration defined of the integrals of a Liouville integrable Hamiltonian system.

There are some open problems popped up from this article. Firstly, it would be nice to study the time-dependent nonholonomic systems using JLM. Secondly, we have considered examples from the integrable domain, hence it would be great to apply JLM in nonintegrable domain.

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