The Nielsen and the Reidemeister Zeta Functions of maps on infra-solvmanifolds of type (R)

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ABSTRACT. We prove the rationality, the functional equations and calculate the radii of convergence of the Nielsen and the Reidemeister zeta functions of continuous maps on infra-solvmanifolds of type (R). We find a connection between the Reidemeister and Nielsen zeta functions and the Reidemeister torsions of the corresponding mapping tori. We show that if the Reidemeister zeta function is defined for a homeomorphism on an infra-solvmanifold of type (R), then this manifold is an infra-nilmanifold. We also prove that a map on an infra-solvmanifold of type (R) induced by an affine map minimizes the topological entropy in its homotopy class and it has a rational Artin-Mazur zeta function. Our main technical tool is the averaging formulas for the Lefschetz, the Nielsen and the Reidemeister numbers on infra-solvmanifolds of type (R).

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0. INTRODUCTION

We assume everywhere X to be a connected, compact polyhedron and $f : X \to X$ to be a continuous map. Let $p : \tilde{X} \to X$ be the universal cover of X and $\tilde{f} : \tilde{X} \to \tilde{X}$ a lifting of f, i.e., $p \circ \tilde{f} = f \circ p$. Two lifts $\tilde{f}$ and $\tilde{f}'$ are called *conjugate* if there is a $\gamma \in \Gamma \cong \pi_1(X)$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. The subset $p(\mathrm{Fix}(\tilde{f})) \subset \mathrm{Fix}(f)$ is called the *fixed point class* of f determined by the lifting class $[\tilde{f}]$. A fixed point class is called *essential* if its index is nonzero. The number of lifting classes of f (and hence the number of fixed point classes, empty or not) is called the *Reidemeister number* of f, denoted by $R(f)$. This is a positive integer or infinity. The number of essential fixed point classes is called the *Nielsen number* of f, denoted by $N(f)$ [31].

The Nielsen number is always finite. $R(f)$ and $N(f)$ are homotopy invariants. In the category of compact, connected polyhedra the Nielsen number of a map is, apart from in certain exceptional cases, equal to the least number of fixed points of maps with the same homotopy type as f.

Let G be a group and $\phi : G \to G$ an endomorphism. Two elements $\alpha, \alpha' \in G$ are said to be *ϕ-conjugate* if and only if there exists $\gamma \in G$ such that $\alpha' = \gamma \alpha \phi(\gamma)^{-1}$. The number of ϕ-conjugacy classes is called the *Reidemeister number* of ϕ, denoted by $R(\phi)$. This is a positive integer or infinity.

Taking a dynamical point of view, we consider the iterates of f and ϕ, and we may define following [14, 47, 15, 16] several zeta functions connected with the Nielsen fixed point theory. The Reidemeister zeta functions of f and ϕ and the Nielsen zeta function of f are defined as power series:

$$R_\phi(z) = \exp \left(\sum_{n=1}^{\infty} \frac{R(\phi^n)}{n} z^n \right),$$

$$R_f(z) = \exp \left(\sum_{n=1}^{\infty} \frac{R(f^n)}{n} z^n \right),$$

$$N_f(z) = \exp \left(\sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n \right).$$

Whenever we mention the Reidemeister zeta function $R_f(z)$, we shall assume that it is well-defined and so $R(f^n) < \infty$ and $R(\phi^n) < \infty$ for all $n > 0$. Hence $R_f(z) = N_f(z)$ on infra-nilmanifolds by Theorem 1.1 below and on infra-solvmanifolds of type (R) by Corollary 7.6. However, there are spaces and maps for which $R_f(z)$ is not defined. The zeta functions $R_f(z)$ and $N_f(z)$ are homotopy invariants. The function $N_f(z)$ has a positive radius of convergence for any continuous map f which has a sharp estimate in terms of the topological entropy of the map f [47]. The above zeta functions are

directly analogous to the Lefschetz zeta function

$$L_f(z) := \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} z^n\right),$$

where

$$L(f^n) := \sum_{k=0}^{\dim X} (-1)^k \mathrm{tr}\Big[f_{*k}^n : H_k(X; \mathbb{Q}) \to H_k(X; \mathbb{Q})\Big]$$

is the Lefschetz number of the iterate f^n of f. The Lefschetz zeta function is a rational function of z and is given by the formula:

$$L_f(z) = \prod_{k=0}^{\dim X} \det\left(I - f_{*k}.z\right)^{(-1)^{k+1}}.$$

A. Weil [51] introduced the function $L_f(z)$ in his study of the fixed points of the Frobenius map of an algebraic variety defined over a finite field. In the theory of discrete dynamical systems $L_f(z)$ was introduced by Smale [50] and in the singularity theory by Milnor [45].

The following problem was investigated: for which spaces and maps and for which groups and endomorphisms are the Nielsen and Reidemeister zeta functions rational functions? Are these functions algebraic functions?

The knowledge that a zeta function is a rational function is important because it shows that the infinite sequence of coefficients of the corresponding power series is closely interconnected, and is given by the finite set of zeros and poles of the zeta function.

In [16, 18, 19, 40, 17], the rationality of the Reidemeister zeta function $R_\phi(z)$ was proven in the following cases: the group is finitely generated and an endomorphism is eventually commutative; the group is finite; the group is a direct sum of a finite group and a finitely generated free Abelian group; the group is finitely generated, nilpotent and torsion free. In [52, Theorem 4] the rationality of the Reidemeister and Nielsen zeta functions was proven for infra-nilmanifold under some (rather technical) sufficient conditions. It is also known that the Reidemeister numbers of the iterates of an automorphism of an almost Bieberbach group satisfy remarkable Gauss congruences [20, 21].

In this paper we investigate the Reidemeister and the Nielsen zeta functions on infra-solvmanifolds of type (R). Our main technical tool is the averaging formulas for the Lefschetz numbers, the Nielsen numbers and the Reidemeister numbers on infra-nilmanifolds and on infra-solvmanifolds of type (R).

Recently, using these averaging formulas, K. Dekimpe and G.-J. Dugardein [10] calculated the Nielsen numbers via Lefschetz numbers and proved the rationality of the Nielsen zeta functions on infra-nilmanifolds.

We prove in this paper the rationality, the functional equations and calculate the radii of convergence of the Nielsen and the Reidemeister zeta functions of continuous maps on infra-solvmanifolds of type (R). We find

a connection between the Reidemeister and Nielsen zeta functions and the Reidemeister torsions of the corresponding mapping tori. We show that if the Reidemeister zeta function is defined for a homeomorphism on an infra-solvmanifold of type (R), then this manifold is an infra-nilmanifold. We also prove that a map on an infra-solvmanifold of type (R) induced by an affine map minimizes the topological entropy in its homotopy class and it has a rational Artin-Mazur zeta function.

Let us present the contents of the paper in more details. In Section 1 we describe the averaging formulas for the Lefschetz numbers, the Nielsen numbers and the Reidemeister numbers on infra-nilmanifolds and Dekimpe-Dugardein's formula for the Nielsen numbers. In Section 2, we obtain a partial generalization of K. Dekimpe and G.-J. Dugardein's formula from fixed points on infra-nilmanifolds to coincidences on infra-solvmanifolds of type (R) when the holonomy group is a cyclic group. The rationality and the functional equations for the Reidemeister and the Nielsen zeta functions on infra-solvmanifolds of type (R) are proven in Sections 3 and 7. After studying the asymptotic Nielsen numbers on infra-solvmanifolds of type (R) in Section 4, we discuss the relationship between the topological entropies, the asymptotic Nielsen numbers and the radius of convergence of the Nielsen and the Reidemeister zeta functions in Section 5. We also prove in Section 5 that a map on an infra-solvmanifold of type (R) induced by the affine map minimizes the topological entropy in its homotopy class . In Section 6, we find a connection between the Nielsen and the Reidemeister zeta functions and the Reidemeister torsions of the corresponding mapping tori. In Section 7, we obtain the averaging formula for the Reidemeister numbers on infra-solvmanifolds of type (R) and we are able to show that the Reidemeister zeta functions on infra-solvmanifolds of type (R) coincide with the Nielsen zeta functions. In Section 8, we show that if the Reidemeister zeta function is defined for a homeomorphism on an infra-solvmanifold of type (R), then this manifold is an infra-nilmanifold. In Section 9 we prove that the Artin- Mazur zeta function coincides with the Nielsen zeta function and is rational function with functional equation for a continuous map on an infra-solvmanifold of type (R) induced by an affine map.

1. Averaging formulas and Dekimpe-Dugardein's formula

We consider almost Bieberbach groups $\Pi \subset G \rtimes \mathrm{Aut}(G)$, where G is a connected, simply connected nilpotent Lie group, and infra-nilmanifolds $M = \Pi \backslash G$. It is known that these are exactly the class of almost flat Riemannian manifolds. It is L. Auslander's result (see, for example, [39])

that $\Gamma := \Pi \cap G$ is a lattice of G, and is the unique maximal normal nilpotent subgroup of Π. The group $\Phi = \Pi/\Gamma$ is the *holonomy group* of Π or M. Thus we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & G & \longrightarrow & G \rtimes \mathrm{Aut}(G) & \longrightarrow & \mathrm{Aut}(G) & \longrightarrow & 1 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \xrightarrow{p} & \Phi & \longrightarrow & 1
\end{array}
$$

Thus Φ sits naturally in $\mathrm{Aut}(G)$. Denote $\rho : \Phi \to \mathrm{Aut}(\mathfrak{G})$, $A \mapsto A_* =$ the differential of A.

Let $M = \Pi\backslash G$ be an infra-nilmanifold. Any continuous map $f : M \to M$ induces a homomorphism $\phi : \Pi \to \Pi$. Due to [38, Theorem 1.1], we can choose an affine element $(d, D) \in G \rtimes \mathrm{Endo}(G)$ such that

$$
(1) \qquad \phi(\alpha) \circ (d, D) = (d, D) \circ \alpha, \quad \forall \alpha \in \Pi.
$$

This implies that the affine map $(d, D) : G \to G$ induces a continuous map on the infra-nilmanifold $M = \Pi\backslash G$, which is homotopic to f. That is, f has an affine homotopy lift (d, D).

By [36, Lemma 3.1], we can choose a fully invariant subgroup $\Lambda \subset \Gamma$ of Π which is of finite index. Therefore $\phi(\Lambda) \subset \Lambda$ and so ϕ induces the following commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Lambda & \longrightarrow & \Pi & \longrightarrow & \Psi & \longrightarrow & 1 \\
& & \downarrow{\phi'} & & \downarrow{\phi} & & \downarrow{\bar\phi} & & \\
1 & \longrightarrow & \Lambda & \longrightarrow & \Pi & \longrightarrow & \Psi & \longrightarrow & 1
\end{array}
$$

where $\Psi = \Pi/\Lambda$ is finite. Applying (1) for $\lambda \in \Lambda \subset \Pi$, we see that

$$
\phi(\lambda) = dD(\lambda)d^{-1} = (\tau_d D)(\lambda)
$$

where τ_d is the conjugation by d. The homomorphism $\phi' : \Lambda \to \Lambda$ induces a unique Lie group homomorphism $F = \tau_d D : G \to G$, and hence a Lie algebra homomorphism $F_* : \mathfrak{G} \to \mathfrak{G}$. On the other hand, since $\phi(\Lambda) \subset \Lambda$, f has a lift $\bar{f} : N \to N$ on the nilmanifold $N := \Lambda\backslash G$ which finitely and regularly covers M and has Ψ as its group of covering transformations.

Theorem 1.1 (Averaging Formula [36, Theorem 3.4], [29, Theorem 6.11])**.** *Let f be a continuous map on an infra-nilmanifold $\Pi\backslash G$ with holonomy group Φ. Let f have an affine homotopy lift (d, D) and let $\phi : \Pi \to \Pi$ be the homomorphism induced by f. Then we have*

$$
L(f) = \frac{1}{|\Phi|} \sum_{A \in \Phi} \det(I - A_* F_*) = \frac{1}{|\Phi|} \sum_{A \in \Phi} \det(I - A_* D_*),
$$

$$
N(f) = \frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(I - A_* F_*)| = \frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(I - A_* D_*)|,
$$

$$
R(f) = R(\phi) = \frac{1}{|\Phi|} \sum_{A \in \Phi} \sigma\left(\det(A_* - F_*)\right) = \frac{1}{|\Phi|} \sum_{A \in \Phi} \sigma\left(\det(A_* - D_*)\right)
$$

where $\sigma : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is defined by $\sigma(0) = \infty$ and $\sigma(x) = |x|$ for all $x \neq 0$.

Recently, Dekimpe and Dugardein in [10] showed the following: Let $f : M \to M$ be a continuous map on an infra-nilmanifold M. Then the Nielsen number $N(f)$ is either equal to $|L(f)|$ or equal to the expression $|L(f) - L(f_+)|$, where f_+ is a lift of f to a 2-fold covering of M. By exploiting the exact nature of this relationship for all powers of f, they proved that the Nielsen zeta function $N_f(z)$ is always a rational function.

Let $M = \Pi \backslash G$ be an infra-nilmanifold with the holonomy group Φ and let $f : M \to M$ be a continuous map with an affine homotopy lift (d, D). Let $A \in \Phi$. Then we can choose $g \in G$ so that $\alpha = (g, A) \in \Pi$. Write $\phi(\alpha) = (g', A')$. By (1), we have $(g', A')(d, D) = (d, D)(g, A) \Rightarrow A'D = DA$. Thus ϕ induces a function $\hat{\phi} : \Phi \to \Phi$ given by $\hat{\phi}(A) = A'$ so that it satisfies that

$$(2) \qquad \hat{\phi}(A)D = DA, \quad \hat{\phi}(A)_* D_* = D_* A_*$$

for all $A \in \Phi$.

In what follows, we shall give a brief description of main results in [10]. We can choose a linear basis of $\mathfrak{G}$ so that $\rho(\Phi) = \Phi_* \subset \mathrm{Aut}(\mathfrak{G})$ can be expressed as diagonal block matrices

$$\begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{bmatrix} \subset \mathrm{GL}(n_1, \mathbb{R}) \times \mathrm{GL}(n_2, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{R})$$

and D_* can be written in block triangular form

$$\begin{bmatrix} D_1 & * \\ 0 & D_2 \end{bmatrix}$$

where D_1 and D_2 have eigenvalues of modulus ≤ 1 and > 1, respectively. We can assume $\Phi = \Phi_1 \times \Phi_2$. Every element $\alpha \in \Pi$ is of the form $(a, A) \in G \rtimes \mathrm{Aut}(G)$ and α is mapped to $A = (A_1, A_2)$. We define

$$\Pi_+ = \{\alpha \in \Pi \mid \det A_2 = 1\}.$$

Then Π_+ is a subgroup of Π of index at most 2. If $[\Pi : \Pi_+] = 2$, then Π_+ is also an almost Bieberbach group and the corresponding infra-nilmanifold $M_+ = \Pi_+ \backslash G$ is a double covering of $M = \Pi \backslash G$; the map f lifts to a map $f_+ : M_+ \to M_+$ which has the same affine homotopy lift (d, D) as f. If D_* has no eigenvalues of modulus > 1, then for any $A \in \Phi$, $A = A_1$ and in this case we take $\Pi_+ = \Pi$. Now, a main result, Theorem 4.4, of [10] is the following:

Theorem 1.2 ([10],Theorem 4.4, when $\Pi = \Pi_+$ see also proof of Theorem 3.5)**.** *Let f be a continuous map on an infra-nilmanifold $\Pi \backslash G$ with an affine homotopy lift (d, D). Then the Nielsen numbers of f^k are*

$$N(f^k) = \begin{cases} (-1)^{p+(k+1)n} L(f^k), & \text{when } \Pi = \Pi_+; \\ (-1)^{p+(k+1)n} \left(L(f_+^k) - L(f^k) \right), & \text{when } \Pi \neq \Pi_+, \end{cases}$$

where p be the number of real eigenvalues of D_* which are > 1 and n be the number of real eigenvalues of D_* which are < -1.

Remark 1.3. 1) In [10, Theorem 4.4] Nielsen numbers $N(f^n)$ are expressed in terms of Lefschetz numbers $L(f^n)$ and $L(f_+^n)$ via a table given by parity of n.
2) The proof of our Theorem 3.5 covers the case when $\Pi = \Pi_+$ in Theorem 1.2 above because in this case $N(f) = |L(f)|$.

2. COINCIDENCES ON INFRA-SOLVMANIFOLDS OF TYPE (R) WITH A CYCLIC HOLONOMY GROUP

In this section, we will be concerned with a generalization of Theorem 1.2 when $k = 1$ (that is, $N(f) = |L(f)|$ or $|L(f_+) - L(f)|$) from fixed points on infra-nilmanifolds to coincidences on infra-solvmanifolds of type (R). We obtain a partial result for coincidences on infra-solvmanifolds of type (R) when the holonomy group is a cyclic group.

Let S be a connected, simply connected solvable Lie group of type (R), and let C be a compact subgroup of $\mathrm{Aut}(S)$. Let $\Pi \subset S \rtimes C$ be torsion free and discrete which is a finite extension of the lattice $\Gamma = \Pi \cap S$ of S. Such a group Π is called an SB-*group* modeled on S. The quotient space $\Pi \backslash S$ is called an *infra-solvmanifold* of type (R) with holonomy group $\Phi = \Pi / \Gamma$. When $\Pi \subset S$, $\Pi \backslash S$ is a *special solvmanifold* of type (R). Thus the infra-solvmanifold $\Pi \backslash S$ is finitely and regularly covered by the special solvmanifold $\Gamma \backslash S$ with the group of covering transformations Φ. For more details, we refer to [37].

Let $M = \Pi \backslash S$ be an infra-solvmanifold of type (R) with the holonomy group Φ. Then Φ sits naturally in $\mathrm{Aut}(S)$. Write $\rho : \Phi \to \mathrm{Aut}(\mathfrak{S})$, $A \mapsto A_*$. Let $f, g : M \to M$ be maps with affine homotopy lifts $(d, D), (e, E) : S \to S$, respectively. Then f and g induce homomorphisms $\phi, \psi : \Pi \to \Pi$ by the following rules:

$$\phi(\alpha) \circ (d, D) = (d, D) \circ \alpha, \quad \psi(\alpha) \circ (e, E) = (e, E) \circ \alpha \quad \forall \alpha \in \Pi.$$

In turn, we obtain functions $\hat{\phi}, \hat{\psi} : \Phi \to \Phi$ satisfying

$$\hat{\phi}(A)D = DA \quad \text{and} \quad \hat{\psi}(A)E = EA \quad \forall A \in \Phi.$$

Thus

$$(3) \qquad \hat{\phi}(A)_* D_* = D_* A_* \quad \text{and} \quad \hat{\psi}(A)_* E_* = E_* A_* \quad \forall A \in \Phi.$$

Recall the following well-known facts from representation theory:

Theorem 2.1 (H. Maschke)**.** *Let $\rho : \Phi \to \mathrm{GL}(n, \mathbb{R})$ be a representation. Then there exist irreducible representations $\rho_i : \Phi \to \mathrm{GL}(n_i, \mathbb{R})$ such that ρ is similar to $\rho_1 \oplus \cdots \oplus \rho_s$.*

Theorem 2.2. *Let $\Phi = \langle A \rangle$ be a cyclic group of order n and let $\rho : \Phi \to$ $\mathrm{GL}(m, \mathbb{R})$ be a faithful $\mathbb{R}$-irreducible representation. If $n = 1$ then ρ is the trivial representation ρ_{triv}. If $n = 2$, then $m = 1$ and $\rho(A) = -1$. In this case, we denote ρ by τ. If $n > 2$, then there exists $k \in \mathbb{Z}$ such that $\gcd(n, k) = 1$ and ρ is similar to the irreducible rotation given by*

$$\Phi \longrightarrow \mathrm{GL}(2, \mathbb{R}), \ \ A \longmapsto \begin{bmatrix} \cos\frac{2k\pi}{n} & -\sin\frac{2k\pi}{n} \\ \sin\frac{2k\pi}{n} & \cos\frac{2k\pi}{n} \end{bmatrix}.$$

Consider the case where the infra-solvmanifold M of type (R) is orientable (for coincidences) with holonomy group Φ a cyclic group with a generator A_0. By Theorem 2.1, the natural representation $\rho : \Phi \to \mathrm{Aut}(S) \cong \mathrm{Aut}(\mathfrak{S})$ is similar to a sum of irreducible representations. If $\sigma : \Phi \to \mathrm{GL}(m, \mathbb{R})$ is irreducible, then the induced representation $\bar{\sigma} : \Phi/\ker\rho \to \mathrm{GL}(m, \mathbb{R})$ is faithful and irreducible. By Theorem 2.2, $\bar{\sigma}$ is similar to ρ_{triv}, τ or a rotation. Thus we may assume that $\rho = m\rho_{\mathrm{triv}} \oplus k\tau \oplus \rho_1 \oplus \cdots \oplus \rho_t$, where $\rho_i : \Phi \to \mathrm{GL}(2, \mathbb{R})$ is an irreducible rotation. That is, there is a linear basis of $\mathfrak{S}$ so that $\rho(A_0) \in \mathrm{Aut}(\mathfrak{S})$ can be represented as diagonal block matrices

$$\rho(A_0) = \begin{bmatrix} I_m & & & & \\ & -I_k & & & \\ & & \Phi_1 & & \\ & & & \ddots & \\ & & & & \Phi_t \end{bmatrix} \quad \text{where } \Phi_i = \rho_i(A_0) \in \mathrm{GL}(2, \mathbb{R}).$$

Remark that if $k > 0$ then the order of Φ is even, and $\det(\rho_i(A_0)) = 1$ for all i. Hence $\det(\rho(A_0)) = 1$ if and only if k is even. This is the only case when the infra-solvmanifold is orientable and hence k is even.

Using the identities (3), we can write D_* and E_* as block matrices

$$D_* = \begin{bmatrix} D_{\mathrm{triv}} & 0 & 0 \\ 0 & D_\tau & 0 \\ * & * & \hat{D} \end{bmatrix}, \quad E_* = \begin{bmatrix} E_{\mathrm{triv}} & 0 & 0 \\ 0 & E_\tau & 0 \\ * & * & \hat{E} \end{bmatrix}$$

where $D_{\mathrm{triv}}, E_{\mathrm{triv}}$ are $m \times m$, D_τ, E_τ are $k \times k$ and $\hat{D}, \hat{E}$ are $2t \times 2t$.

For $A \in \Phi$, $A = A_0^p$ for some p and

$$A_* = \begin{bmatrix} I_m & & \\ & (-1)^p I_k & \\ & & A_* \end{bmatrix}.$$

Write $\hat{\rho} = \rho_1 \oplus \cdots \oplus \rho_t : \Phi \to \mathrm{GL}(2t, \mathbb{R}), A \mapsto \hat{\rho}(A) = A_*$ (abusing the notation: $\rho(A) = A_*$). Then the identities (3) induce

$$\hat{\phi}(A)_* \hat{D} = \hat{D} A_*, \ \ \hat{\psi}(A)_* \hat{E} = \hat{E} A_*.$$

Hence, for all $A = A_0^p$ and $B = A_0^q \in \Phi$, we have that

$$(4) \quad \det(E_* - A_* D_*) \det(E_* - B_* D_*)$$
$$= \det(E_{\mathrm{triv}} - D_{\mathrm{triv}})^2 \det(E_\tau - (-1)^p D_\tau) \det(E_\tau - (-1)^q D_\tau)$$
$$\times \det(\hat{E} - A_* \hat{D}) \det(\hat{E} - B_* \hat{D}).$$

Note here that $\det(\hat{E} - A_* \hat{D}) \det(\hat{E} - B_* \hat{D}) \geq 0$, see [11, Lemma 6.3]. From [27, Theorem 4.5], immediately we have:

Theorem 2.3 (Compare with [11, Theorem 6.1])**.** *Let f and g be continuous maps on an orientable infra-solvmanifold of type* (R) *with cyclic holonomy group $\Phi = \langle A_0 \rangle$. If $\rho(A_0)$ has no eigenvalue -1, i.e., if $k = 0$, then $N(f, g) = |L(f, g)|$.*

Assume $k > 0$ (is even); then $\Phi = \langle A_0 \rangle$ is of even order. Let $\Phi_0 = \langle A_0^2 \rangle$ and let Π_0 be the subgroup of Π induced by the inclusion $\Phi_0 \hookrightarrow \Phi$. Remark also that if $D_\tau = 0$ or $E_\tau = 0$, then we still have $N(f, g) = |L(f, g)|$. We also assume that $D_\tau \neq 0$ and $E_\tau \neq 0$.

Theorem 2.4. *Then Π_0 is a subgroup of Π of index 2, Π_0 is also an SB-group and the corresponding infra-solvmanifold $M_0 = \Pi_0 \backslash S$ is a double covering of $M = \Pi \backslash S$; the maps f, g lift to map $f_0, g_0 : M_0 \to M_0$ which have the same affine homotopy lifts $(d, D), (e, E)$ as f and g.*

Proof. It is clear that $[\Pi : \Pi_0] = 2$ and that Π_0 is also an SB-group and the corresponding infra-solvmanifold $\Pi_0 \backslash S$ is a double covering of $\Pi \backslash S$.

To prove the last assertion, we may consider and assume that $(d, D) : S \to S$ induces f and that $\phi : \Pi \to \Pi$ is a homomorphism such that

$$\phi(\alpha)(d, D) = (d, D)\alpha, \quad \forall \alpha \in \Pi.$$

We need to show that (d, D) also induces a map on $\Pi_0 \backslash S$. For this purpose, it is enough to show that $\phi(\Pi_0) \subset \Pi_0$. For any $\beta = (a, A) \in \Pi_0$, let $\phi(\beta) = (b, \hat{\phi}(A))$. Since $(a, A) \in \Pi_0$, we have $A \in \Phi_0$. The above identity implies that

$$\hat{\phi}(A)_* D_* = D_* A_* \Rightarrow D_\tau = 0 \text{ or } \hat{\phi}(A) \in \Phi_0.$$

Since $D_\tau \neq 0$, this finishes the proof of the last assertion. $\qquad\square$

For any $A = A_0^p \in \Phi$, we recall from (4) that

$$\det(E_* - A_* D_*) = \det(E_{\mathrm{triv}} - D_{\mathrm{triv}}) \det(E_\tau - (-1)^p D_\tau) \det(\hat{E} - A_* \hat{D})$$

and

$$\det(\hat{E} - \hat{D}) \det(\hat{E} - A_* \hat{D}) \geq 0.$$

Let

$$\epsilon_o = \mathrm{sign}\, \det(E_\tau - D_\tau), \ \ \epsilon_e = \mathrm{sign}\, \det(E_\tau + D_\tau).$$

Then $\epsilon_o = \pm\epsilon_e$. Notice that the values ϵ_o and ϵ_e depend both on f and g. When $\epsilon_o = \epsilon_e$, we still have $N(f,g) = |L(f,g)|$. When $\epsilon_o = -\epsilon_e$, we have that

$$
\begin{aligned}
N(f,g) &= \frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(E_* - A_* D_*)| \\
&= \frac{1}{|\Phi|} \left(\sum_{A \in \Phi_0} |\det(E_* - A_* D_*)| + \sum_{A \notin \Phi_0} |\det(E_* - A_* D_*)| \right) \\
&= \frac{\epsilon_o}{|\Phi|} \left(\sum_{A \in \Phi_0} \det(E_* - A_* D_*) - \sum_{A \notin \Phi_0} \det(E_* - A_* D_*) \right) \\
&= \frac{\epsilon_o}{|\Phi|} \left(2 \sum_{A \in \Phi_0} \det(E_* - A_* D_*) - \sum_{A \in \Phi} \det(E_* - A_* D_*) \right) \\
&= \epsilon_o \left(\frac{1}{|\Phi_0|} \sum_{A \in \Phi_0} \det(E_* - A_* D_*) - \frac{1}{|\Phi|} \sum_{A \in \Phi} \det(E_* - A_* D_*) \right) \\
&= \epsilon_o (L(f_0, g_0) - L(f,g)).
\end{aligned}
$$

Therefore, we can summarize what we have observed as follows:

Theorem 2.5. *Let $M = \Pi \backslash S$ be an orientable infra-solvmanifold of type (R) with cyclic holonomy group $\Phi = \langle A_0 \rangle$. Let $\rho : \Phi \to \mathrm{Aut}(\mathfrak{G})$ be the natural presentation. Then ρ is similar to the sum of irreducible representations $m\rho_{\mathrm{triv}} \oplus k\tau \oplus \rho_1 \oplus \cdots \oplus \rho_t$, where $\rho_{\mathrm{triv}} : \Phi \to \mathrm{GL}(1,\mathbb{R})$ is the trivial representation, $\tau : \Phi \to \mathrm{GL}(1,\mathbb{R})$ is the representation given by $\tau(A_0) = -1$, and $\rho_i : \Phi \to \mathrm{GL}(2,\mathbb{R})$ is an irreducible rotation. Let $f, g : M \to M$ be continuous maps with affine homotopy lifts $(d, D), (e, E)$ respectively. Then D_* and E_* can be expressed as block matrices*

$$
D_* = \begin{bmatrix} D_{\mathrm{triv}} & 0 & 0 \\ 0 & D_\tau & 0 \\ * & * & \hat{D} \end{bmatrix}, \quad E_* = \begin{bmatrix} E_{\mathrm{triv}} & 0 & 0 \\ 0 & E_\tau & 0 \\ * & * & \hat{E} \end{bmatrix}
$$

where $D_{\mathrm{triv}}, E_{\mathrm{triv}}$ are $m \times m$, D_τ, E_τ are $k \times k$ and $\hat{D}, \hat{E}$ are $2t \times 2t$. Moreover, we have that:

(1) *If $k = 0$, then $N(f,g) = |L(f,g)|$.*
(2) *If $k > 0$ and $\det(E_\tau - D_\tau) \det(E_\tau + D_\tau) \geq 0$, then $N(f,g) = |L(f,g)|$.*
(3) *If $k > 0$ and $\det(E_\tau - D_\tau) \det(E_\tau + D_\tau) < 0$, then the maps f, g lift to maps $f_0, g_0 : M_0 \to M_0$ on a double covering M_0 of M which have the same homotopy lifts as f, g respectively so that the following formula holds*

$$
N(f,g) = |L(f_0, g_0) - L(f,g)|.
$$

Proof. We are left to notice only one thing: If $D_\tau = 0$ or $E_\tau = 0$, then $k > 0$ is even and so $\det(E_\tau - D_\tau) \det(E_\tau + D_\tau) \geq 0$. $\square$

3. THE RATIONALITY AND THE FUNCTIONAL EQUATION

We start with an example that shows how different can be the Nielsen, the Reidemeister and the Lefschetz zeta functions.

Example 3.1 ([17]). Let $f : S^2 \vee S^4 \to S^2 \vee S^4$ to be a continuous map of the bouquet of spheres such that the restriction $f|_{S^4} = id_{S^4}$ and the degree of the restriction $f|_{S^2} : S^2 \to S^2$ equal to -2. Then $L(f) = 0$, hence $N(f) = 0$ since $S^2 \vee S^4$ is simply connected. For $k > 1$ we have $L(f^k) = 2 + (-2)^k \neq 0$, therefore $N(f^k) = 1$. $R(f^k) = 1$ for all $k \geq 1$ since $S^2 \vee S^4$ is simply connected. From this we have by direct calculation that

$$N_f(z) = \exp(-z) \cdot \frac{1}{1-z}; \ R_f(z) = \frac{1}{1-z}; \ L_f(z) = \frac{1}{(1-z)^2(1+2z)}.$$

Hence $N_f(z)$ is a meromorphic function, and $R_f(z)$ and $L_f(z)$ are rational and different.

We give now some other examples of the Nielsen and the Reidemeister zeta functions on infra-nilmanifolds. For the explicit computation of the zeta functions, the following is useful.

Proposition 3.2. *Let f be a continuous map on an infra-nilmanifold $\Pi \backslash G$ with holonomy group Φ. Let f have an affine homotopy lift (d, D) and let $\phi : \Pi \to \Pi$ be the homomorphism induced by f. Then*

$$N_f(z) = \prod_{A \in \Phi} {}^{|\Phi|}\sqrt{\exp\left(\sum_{n=1}^{\infty} \frac{|\det(A_* - D_*^n)|}{n} z^n\right)}.$$

When $R_f(z)$ is defined, $R_f(z) = R_\phi(z) = N_f(z)$.

Proof. By Theorem 1.1, we have that

$$N_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n\right)$$

$$= \exp\left(\sum_{n=1}^{\infty} \frac{\frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(A_* - F_*^n)|}{n} z^n\right)$$

$$= \prod_{A \in \Phi} \left(\exp\left(\sum_{n=1}^{\infty} \frac{|\det(A_* - F_*^n)|}{n} z^n\right)\right)^{\frac{1}{|\Phi|}}$$

$$= \prod_{A \in \Phi} {}^{|\Phi|}\sqrt{\exp\left(\sum_{n=1}^{\infty} \frac{|\det(A_* - F_*^n)|}{n} z^n\right)}. \quad \square$$

Example 3.3. This is an example used by Anosov to show that the Anosov relation does not hold when the manifold is not a nilmanifold [1].

Let $\alpha = (a, A)$ and $t_i = (e_i, I_2)$ be elements of $\mathbb{R}^2 \rtimes \text{Aut}(\mathbb{R}^2)$, where

$$a = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then A has period 2, $(a, A)^2 = (a + Aa, I_2) = (e_1, I_2)$, and $t_2\alpha = \alpha t_2^{-1}$. Let Γ be the subgroup generated by t_1 and t_2. Then it forms a lattice in $\mathbb{R}^2$ and the quotient space $\Gamma \backslash \mathbb{R}^2$ is the 2-torus. It is easy to check that the subgroup

$$\Pi = \langle \Gamma, (a, A) \rangle \subset \mathbb{R}^2 \rtimes \text{Aut}(\mathbb{R}^2)$$

generated by the lattice Γ and the element (a, A) is discrete and torsion free. Furthermore, Γ is a normal subgroup of Π of index 2. Thus Π is an (almost) Bieberbach group, which is the Klein bottle group, and the quotient space $\Pi \backslash \mathbb{R}^2$ is the Klein bottle. Thus $\Gamma \backslash \mathbb{R}^2 \to \Pi \backslash \mathbb{R}^2$ is a double covering projection.

Let $K : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear automorphism given by

$$K = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

It is not difficult to check that K induces $\bar{f} : \Gamma \backslash \mathbb{R}^2 \to \Gamma \backslash \mathbb{R}^2$ and $f : \Pi \backslash \mathbb{R}^2 \to \Pi \backslash \mathbb{R}^2$ so that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathbb{R}^2 & \xrightarrow{\ K\ } & \mathbb{R}^2 \\
\downarrow & & \downarrow \\
\Gamma \backslash \mathbb{R}^2 & \xrightarrow{\ \bar{f}\ } & \Gamma \backslash \mathbb{R}^2 \\
\downarrow & & \downarrow \\
\Pi \backslash \mathbb{R}^2 & \xrightarrow{\ f\ } & \Pi \backslash \mathbb{R}^2
\end{array}
$$

Note that all the vertical maps are the natural covering maps. In particular, $\Gamma \backslash \mathbb{R}^2 \to \Pi \backslash \mathbb{R}^2$ is a double covering by the holonomy group of Π/Γ, which is $\Phi = \{I, A\} \cong \mathbb{Z}_2$. By Theorem 1.1, we have

$$L(f^n) = \frac{1}{2} \left(\det(I - K^n) + \det(I - AK^n) \right) = 1 - (-1)^n,$$
$$N(f^n) = 2^n (1 - (-1)^n).$$

In particular, $R(f^n) = 2^{n+1}$ when n is odd; otherwise, $R(f^n) = \infty$. Therefore, the Reidemeister zeta function $R_f(z)$ is not defined, and

$$L_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{2}{2n-1} z^{2n-1}\right) = \frac{1+z}{1-z},$$

$$N_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{2^{2n}}{2n-1} z^{2n-1}\right)$$

$$= \exp\left(\sum_{n=1}^{\infty} \frac{2}{2n-1} (2z)^{2n-1}\right) = \frac{1+2z}{1-2z}.$$

Example 3.4. Consider Example 3.5 of [36] in which an infra-nilmanifold M modeled on the 3-dimensional Heisenberg group Nil has the holonomy group of order 2 generated by A and a self-map f on M is induced by the automorphism $D : \text{Nil} \to \text{Nil}$ given by

$$D : \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & -4x-y & z' \\ 0 & 1 & 6x+2y \\ 0 & 0 & 1 \end{bmatrix}$$

where $z' = -2z - (12x^2 + 10xy + y^2)$. Then with respect to the ordered (linear) basis for the Lie algebra of Nil

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{e}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

the differentials of A and D are

$$A_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \ D_* = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & -1 \\ 0 & 6 & 2 \end{bmatrix}.$$

By Proposition 3.2, we have

$$N_f(z) = \sqrt{\exp\left(\sum_{n=1}^{\infty} \frac{|\det(I - D_*^n)|}{n} z^n\right)} \sqrt{\exp\left(\sum_{n=1}^{\infty} \frac{|\det(A_* - D_*^n)|}{n} z^n\right)}.$$

Remark that A_* is a block diagonal matrix with 1×1 block I_1 and 2×2 block $-I_2$. We have

$$\begin{aligned}
|\det(A_* - D_*^n)| &= |\det(I_1 - D_1^n) \det(-I_2 - D_2^n)| \\
&= |\det(I_1 - D_1^n)| |\det(I_2 + D_2^n)| \\
&= |(1 - (-2)^n)| (-1)^n \det(I_2 + D_2^n) \\
&= (2^n - (-1)^n)(-1)^n \sum_i \text{tr}(\bigwedge^i D_2^n).
\end{aligned}$$

Consequently, we obtain

$$\exp\left(\sum_{n=1}^{\infty}\frac{|\det(A_* - D_*^n)|}{n}z^n\right)$$

$$= \exp\left(\sum_{n=1}^{\infty}\frac{(2^n - (-1)^n)(-1)^n\sum_i \operatorname{tr}(\bigwedge^i D_2^n)}{n}z^n\right)$$

$$= \exp\left(\sum_{n=1}^{\infty}\frac{\sum_i \operatorname{tr}(\bigwedge^i D_2^n)}{n}(-2z)^n - \sum_{n=1}^{\infty}\frac{\sum_i \operatorname{tr}(\bigwedge^i D_2^n)}{n}z^n\right)$$

$$= \prod_i \frac{\det(I - z\bigwedge^i D_2)}{\det(I + 2z\bigwedge^i D_2)}$$

$$= \frac{1-z}{1+2z}\cdot\frac{1+2z-2z^2}{1-4z-8z^2}\cdot\frac{1+2z}{1-4z}.$$

In a similar fashion, we compute

$$\exp\left(\sum_{n=1}^{\infty}\frac{|\det(I - D_*^n)|}{n}z^n\right)$$

$$= \exp\left(\sum_{n=1}^{\infty}\frac{|\det(I_1 - D_1^n)\det(I_2 - D_2^n)|}{n}z^n\right)$$

$$= \exp\left(\sum_{n=1}^{\infty}\frac{(2^n - (-1)^n)(-1)^{n+1}\sum_i(-1)^i\operatorname{tr}(\bigwedge^i D_2^n)}{n}z^n\right)$$

$$= \prod_i \left(\frac{\det(I + 2z\bigwedge^i D_2)}{\det(I - z\bigwedge^i D_2)}\right)^{(-1)^i}$$

$$= \frac{1+2z}{1-z}\cdot\frac{1+2z-2z^2}{1-4z-8z^2}\cdot\frac{1-4z}{1+2z}.$$

The last identity of the above computations follows from the definition of $\bigwedge^i D_2$ (see [28, Lemma 3.2]). Namely, we have

$$\bigwedge^0 D_2 = 1,\ \bigwedge^1 D_2 = D_2,\ \bigwedge^2 D_2 = \det(D_2) = -2.$$

In all, we obtain that

$$N_f(z) = R_f(z) = \frac{1+2z-2z^2}{1-4z-8z^2}.$$

Theorem 3.5. *Let f be a continuous map on an infra-nilmanifold with an affine homotopy lift (d, D). Assume $N(f) = |L(f)|$. Then the Nielsen zeta function $N_f(z)$ is a rational function and is equal to*

$$N_f(z) = L_f((-1)^q z)^{(-1)^r}$$

where q is the number of real eigenvalues of D_* which are < -1 and r is the number of real eigenvalues of D_* of modulus > 1. When the Reidemeister

zeta function $R_f(z)$ is defined, we have

$$R_f(z) = R_\phi(z) = N_f(z)$$

.

Proof. By [46, Theorem 8.2.2] $N(f) = |L(f)|$ implies $N(f^n) = |L(f^n)|$ for all n. Let ϵ_n be the sign of $\det(I - D_*^n)$. Let q be the number of real eigenvalues of D_* which are less than -1 and r be the number of real eigenvalues of D_* of modulus > 1. Then $\epsilon_n = (-1)^{r+qn}$. By Theorem 1.1, we have that $\epsilon_1 \det(I - A_* D_*) \geq 0$ for all $A \in \Phi$. In particular, we have

$$\det(I - A_* D_*) \det(I - B_* D_*) \geq 0 \quad \text{for all } A, B \in \Phi.$$

Choose arbitrary $n > 0$. By [46, Lemma 8.2.1],

$$\det(I - A_* D_*^n) \det(I - D_*^n) \geq 0 \quad \text{for all } A \in \Phi.$$

Hence we have $N(f^n) = \epsilon_n L(f^n) = (-1)^{r+qn} L(f^n)$. Consequently,

$$
\begin{aligned}
N_f(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n\right) \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{r+qn} L(f^n)}{n} z^n\right) \\
&= \left(\exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n}((-1)^q z)^n\right)\right)^{(-1)^r} \\
&= L_f((-1)^q z)^{(-1)^r}
\end{aligned}
$$

is a rational function.

Assume $R_f(z)$ is defined. So, $R(f^n) = R(\phi^n) < \infty$ for all $n > 0$. On infranilmanifolds, by Theorem 1.1, it is equivalent to saying that $\det(A_* - D_*^n) \neq 0$ for all $A \in \Phi$ and all n, and hence $\sigma\left(\det(A_* - D_*^n)\right) = |\det(A_* - D_*^n)|$. Thus

$$
\begin{aligned}
R(f^n) = R(\phi^n) &= \frac{1}{|\Phi|} \sum_{A \in \Phi} \sigma\left(\det(A_* - D_*^n)\right) \\
&= \frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(A_* - D_*^n)| = N(f^n).
\end{aligned}
$$

This implies that $R_f(z) = R_\phi(z) = N_f(z)$. $\qquad\square$

Therefore, for those classes of maps on infra-nilmanifolds for which Anosov relation $N(f) = |L(f)|$ holds [35, 41, 7] and for those classes of infranilmanifolds for which Anosov relation $N(f) = |L(f)|$ holds for ALL maps [1, 6, 7, 8], the Nielsen zeta functions and the Reidemeister zeta functions are rational functions.

In general case, using the results of Theorem 1.2, Dekimpe and Dugardein described the Nielsen zeta function of f as follows:

Theorem 3.6 ([10, Theorem 4.5])**.** *Let f be a continuous map on an infra-nilmanifold $\Pi\backslash G$ with an affine homotopy lift (d, D). Then the Nielsen zeta function is a rational function and is equal to*

$$N_f(z) = \begin{cases} L_f((-1)^n z)^{(-1)^{p+n}} & \text{when } \Pi = \Pi_+; \\ \left(\dfrac{L_{f_+}((-1)^n z)}{L_f((-1)^n z)}\right)^{(-1)^{p+n}} & \text{when } \Pi \neq \Pi_+, \end{cases}$$

where p is the number of real eigenvalues of D_* which are > 1 and n is the number of real eigenvalues of D_* which are < -1.

When the Reidemeister zeta function $R_f(z)$ is defined, we have

$$R_f(z) = R_\phi(z) = N_f(z)$$

.

Remark 3.7. In [10, Theorem 4.5] the Nielsen zeta function is expressed in terms of Lefschetz zeta functions $L_f(z)$ and $L_{f_+}(z)$ via a table given by parity of p and n. The class of infra-solvmanifolds of type (R) contains and shares a lot of properties of the class of infra-nilmanifolds such as the averaging formula for Nielsen numbers, see [28, 37]. Therefore, Theorem 1.2 and the statement about $N_f(z)$ in Theorem 3.6 can be generalized directly to the class of infra-solvmanifolds of type (R), see Remark in [10, Sec. 4].

To write down a functional equation for the Reidemeister and the Nielsen zeta functionc, we recall the following functional equation for the Lefschetz zeta function:

Lemma 3.8 ([22, Proposition 8], see also [13])**.** *Let M be a closed orientable manifold of dimension m and let $f : M \to M$ be a continuous map of degree d. Then*

$$L_f\left(\frac{\alpha}{dz}\right) = \epsilon\,(-\alpha dz)^{(-1)^m \chi(M)}\,L_f(\alpha z)^{(-1)^m}$$

where $\alpha = \pm 1$ and $\epsilon \in \mathbb{C}$ is a non-zero constant such that if $|d| = 1$ then $\epsilon = \pm 1$.

Proof. In the Lefschetz zeta function formula (0), we may replace f_* by $f^* : H^*(M; \mathbb{Q}) \to H^*(M; \mathbb{Q})$. Let $\beta_k = \dim H_k(M; \mathbb{Q})$ be the kth Betti number of M. Let $\lambda_{k,j}$ be the (complex and distinct) eigenvalues of $f_{*k} : H_k(M; \mathbb{Q}) \to H_k(M; \mathbb{Q})$

Via the natural non-singular pairing in the cohomology $H^k(M; \mathbb{Q}) \otimes H^{m-k}(M; \mathbb{Q}) \to \mathbb{Q}$, the operators f^*_{m-k} and $d(f^*_k)$ are adjoint to each other. Hence since $\lambda_{k,j}$ is an eigenvalue of f^*_k, $\mu_{\ell,j} = d/\lambda_{k,j}$ is an eigenvalue of $f^*_{m-k} = f^*_\ell$. Furthermore, $\beta_k = \beta_{m-k} = \beta_\ell$.

Consequently, we have

$$
\begin{aligned}
L_f\left(\frac{\alpha}{dz}\right) &= \prod_{k=0}^{m}\prod_{j=1}^{\beta_k}\left(1-\lambda_{k,j}\frac{\alpha}{dz}\right)^{(-1)^{k+1}} \\
&= \prod_{k=0}^{m}\prod_{j=1}^{\beta_k}\left(1-\frac{d}{\lambda_{k,j}}\alpha z\right)^{(-1)^{k+1}}\left(-\frac{\alpha dz}{\lambda_{k,j}}\right)^{(-1)^{k}} \\
&= \prod_{\ell=0}^{m}\prod_{j=1}^{\beta_{m-\ell}}(1-\mu_{\ell,j}\alpha z)^{(-1)^{m-\ell+1}}\prod_{k=0}^{m}\prod_{j=1}^{\beta_k}\left(-\frac{\alpha dz}{\lambda_{k,j}}\right)^{(-1)^{m-\ell}} \\
&= \left(\prod_{\ell=0}^{m}\prod_{j=1}^{\beta_\ell}(1-\mu_{\ell,j}\alpha z)^{(-1)^{\ell+1}}\prod_{k=0}^{m}\prod_{j=1}^{\beta_k}\left(-\frac{\alpha dz}{\lambda_{k,j}}\right)^{(-1)^{\ell}}\right)^{(-1)^{m}} \\
&= L_f(\alpha z)^{(-1)^m}\cdot(-\alpha dz)^{\sum_{\ell=0}^{m}(-1)^{\ell}\beta_\ell}\cdot\prod_{k=0}^{m}\prod_{j=1}^{\beta_k}\lambda_{k,j}^{(-1)^{k+1}} \\
&= L_f(\alpha z)^{(-1)^m}\,\epsilon(-\alpha dz)^{(-1)^m\chi(M)}.
\end{aligned}
$$

Here,

$$
\epsilon = \prod_{k=0}^{m}\prod_{j=1}^{\beta_k}\lambda_{k,j}^{(-1)^{k+1}} = \pm\prod_{k=0}^{m}\det(f_k^*). \qquad \square
$$

We obtain:

Theorem 3.9 (Functional Equation)**.** *Let f be a continuous map on an orientable infra-nilmanifold $M = \Pi\backslash G$ with an affine homotopy lift (d, D). Then the Reidemeister zeta function, whenever it is defined, and the Nielsen zeta function have the following functional equations:*

$$
R_f\left(\frac{1}{dz}\right) = \begin{cases} R_f(z)^{(-1)^m}\epsilon^{(-1)^{p+n}} & \text{when } \Pi = \Pi_+; \\ R_f(z)^{(-1)^m}\epsilon^{-1} & \text{when } \Pi \neq \Pi_+ \end{cases}
$$

and

$$
N_f\left(\frac{1}{dz}\right) = \begin{cases} N_f(z)^{(-1)^m}\epsilon^{(-1)^{p+n}} & \text{when } \Pi = \Pi_+; \\ N_f(z)^{(-1)^m}\epsilon^{-1} & \text{when } \Pi \neq \Pi_+ \end{cases}
$$

where d is a degree f, $m = \dim M$, ϵ is a constant in $\mathbb{C}^\times$, $\sigma = (-1)^n$, p is the number of real eigenvalues of D_* which are > 1 and n is the number of real eigenvalues of D_* which are < -1. If $|d| = 1$ then $\epsilon = \pm 1$.

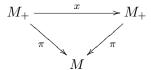*Proof.* Assume $\Pi = \Pi_+$. Then $R_f(z) = N_f(z) = L_f(\sigma z)^{(-1)^{p+n}}$. By Lemma 3.8, we have

$$R_f\left(\frac{1}{dz}\right) = N_f\left(\frac{1}{dz}\right) = L_f\left(\frac{\sigma}{dz}\right)^{(-1)^{p+n}}$$

$$= \left(\epsilon(-\sigma dz)^{(-1)^m\chi(M)}L_f(\sigma z)^{(-1)^m}\right)^{(-1)^{p+n}}$$

$$= N_f(z)^{(-1)^m}\epsilon^{(-1)^{p+n}}(-\sigma dz)^{(-1)^{m+p+n}\chi(M)}$$

$$= R_f(z)^{(-1)^m}\epsilon^{(-1)^{p+n}}(-\sigma dz)^{(-1)^{m+p+n}\chi(M)}.$$

Assume now that $\Pi \neq \Pi_+$. First we claim that f and f_+ have the same degree. Let $\pi : M_+ \to M$ be the natural double covering projection. Then $\Pi/\Pi_+ \cong \mathbb{Z}_2$ is the group of covering transformations of π. By [3, III.2], the homomorphism $\pi^* : H^m(M;\mathbb{Q}) \to H^m(M_+;\mathbb{Q})$ induces an isomorphism $\pi^* : H^m(M;\mathbb{Q}) \to H^m(M_+;\mathbb{Q})^{\Pi/\Pi_+}$. In particular, π^* is injective. If x is the nontrivial covering transformation, we have the commutative diagram

$$M_+ \xrightarrow{\quad x \quad} M_+$$
$$\pi \searrow \quad \swarrow \pi$$
$$M$$

This induces the following commutative diagram

$$H^m(M_+;\mathbb{Q}) \xrightarrow{\quad x^* \quad} H^m(M_+;\mathbb{Q})$$
$$\pi^* \nwarrow \quad \nearrow \pi^*$$
$$H^m(M;\mathbb{Q})$$

We denote generators of $H^m(M;\mathbb{Q})$ and $H^m(M_+;\mathbb{Q})$ by $[M]$ and $[M_+]$, respectively. The above diagram shows that $x^*(\pi^*([M])) = \pi^*([M])$, which induces that $x^*([M_+]) = [M_+]$ as π^* is injective, and hence x acts on $H^m(M_+;\mathbb{Q})$ trivially. In other words, $H^m(M_+;\mathbb{Q}) = H^m(M_+;\mathbb{Q})^{\Pi/\Pi_+}$ and $\pi^* : H^m(M;\mathbb{Q}) \to H^m(M_+;\mathbb{Q})$ is an isomorphism. This implies that f and f_+ have the same degree.

By Theorem 3.6 and Lemma 3.8, we have

$$R_f\left(\frac{1}{dz}\right) = N_f\left(\frac{1}{dz}\right) = L_{f_+}\left(\frac{\sigma}{dz}\right)^{(-1)^{p+n}} \cdot L_f\left(\frac{\sigma}{dz}\right)^{(-1)^{p+n+1}}$$

$$= \left(\epsilon(-\sigma dz)^{(-1)^m\chi(M)}L_{f_+}(\sigma z)^{(-1)^m}\right)^{(-1)^{p+n}}$$

$$\times \left(\epsilon(-\sigma dz)^{(-1)^m\chi(M)}L_f(\sigma z)^{(-1)^m}\right)^{(-1)^{p+n+1}}$$

$$= N_f(z)^{(-1)^m}\left(\epsilon(-\sigma dz)^{(-1)^m\chi(M)}\right)^{-1}$$

$$= R_f(z)^{(-1)^m}\left(\epsilon(-\sigma dz)^{(-1)^m\chi(M)}\right)^{-1}.$$

On the other hand, it is known that $\chi(M) = 0$, e.g. see the remark below, which finishes our proof. $\qquad\square$

Remark 3.10. Let G be a torsion-free polycylic group. Then $\chi(G) = 0$. For, by induction, we may assume that G is an extension of $\mathbb{Z}^m$ by $\mathbb{Z}^n$; then as $\chi(\mathbb{Z}) = 0$, we have $\chi(G) = \chi(\mathbb{Z}^m)\chi(\mathbb{Z}^n) = 0$, [5, Theorem 6.4.2]. Another proof: A solvmanifold is aspherical and its fundamental group contains a nontrivial Abelian normal subgroup. By Gottlieb's theorem, its Euler characteristic is zero. If S is a torsion-free extension of G by a finite group of order k, then $k \cdot \chi(S) = \chi(G) = 0 \Rightarrow \chi(S) = 0$.

Remark 3.11. As it is mentioned above, since Theorem 3.6 is true for the Nielsen zeta functions on infra-solvmanifolds of type (R), the functional equation for the Nielsen zeta functions in Theorem 3.9 is true on infra-solvmanifolds of type (R) (see Theorem 7.9 for the Reidemeister zeta functions).

4. Asymptotic Nielsen numbers

The growth rate of a sequence a_n of complex numbers is defined by

$$\text{Growth}(a_n) := \max\left\{1, \limsup_{n\to\infty} |a_n|^{1/n}\right\}.$$

We define the asymptotic Nielsen number [30] and the asymptotic Reidemeister number to be the growth rate $N^\infty(f) := \text{Growth}(N(f^n))$ and $R^\infty(f) := \text{Growth}(R(f^n))$ correspondingly. These asymptotic numbers are homotopy type invariants. We denote by $\text{sp}(A)$ the spectral radius of the matrix or the operator A, $\text{sp}(A) = \lim_n \sqrt[n]{\|A^n\|}$ which coincide with the largest modulus of an eigenvalue of A. We denote by $\bigwedge F_* := \bigoplus_{\ell=0}^m \bigwedge^\ell F_*$ a linear operator induced in the exterior algebra $\bigwedge^* \mathbb{R}^m := \bigoplus_{\ell=0}^m \bigwedge^\ell \mathbb{R}^m$ of $\mathfrak{G}$ considered as the linear space $\mathbb{R}^m$.

Theorem 4.1 (see also the proof of [42, Theorem 1.5]). *Let $M = \Gamma\backslash S$ be a special solvmanifold of type* (R) *and let f be a continuous map on M with a Lie group homomorphism $D : S \to S$ as a homotopy lift. Then we have*

$$N^\infty(f) = \text{sp}(\bigwedge D_*)$$

provided that 1 is not in the spectrum of D_* and $\text{sp}(D_*) > 1$.

Proof. We give a *very elementary proof* of this theorem. Compare with the proof of [42, Theorem 1.5]. By [37, Theorem 2.2], we may assume that f is induced by a Lie group homomorphism $D : S \to S$. Let $\{\lambda_1, \cdots, \lambda_m\}$ be the eigenvalues of D_*, counted with multiplicities. First we note from definition that

$$\text{sp}(\bigwedge D_*) = \begin{cases} \prod_{|\lambda_j|>1} |\lambda_j| & \text{when } \text{sp}(D_*) > 1 \\ \text{sp}(D_*) & \text{when } \text{sp}(D_*) \leq 1. \end{cases}$$

Recalling $N(f^n) = |\det(I - D_*^n)| = \prod_{j=1}^m |1 - \lambda_j^n|$, we first consider the case where $N(f^n) \neq 0$ for all n. Then we have

$$\log \limsup_{n \to \infty} N(f^n)^{1/n} = \limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^m \log |1 - \lambda_j^n|$$

$$= \sum_{j=1}^m \limsup_{n \to \infty} \frac{1}{n} \log |1 - \lambda_j^n|.$$

If $|\lambda| \leq 1$ then $\limsup_n \frac{1}{n} \log |1 - \lambda^n| = 0$. For, $\log |1 - \lambda^n| \leq \log 2$. If $|\lambda| > 1$ then using L'Hôpital's rule, we have

$$|\lambda|^n - 1 \leq |1 - \lambda^n| \leq |\lambda|^n + 1 \Rightarrow \lim_{n \to \infty} \frac{1}{n} \log |1 - \lambda^n| = \log |\lambda|.$$

Hence

$$N^\infty(f) = \max \left\{ 1, \limsup_{n \to \infty} N(f^n)^{1/n} \right\}$$

$$= \max \left\{ 1, \prod_{|\lambda| > 1} |\lambda| \right\} = \prod_{|\lambda| > 1} |\lambda| = \mathrm{sp}(\bigwedge D_*).$$

Now consider the case where $N(f^n) = 0$ for some n. Then some λ_j is an nth root of unity. For each such λ_j, consider all k's for which $|1 - \lambda_j^k| \neq 0$. Since by the assumption $\lambda_j \neq 1$, there are infinitely many such k's. Furthermore, there are infinitely many k's for which $|1 - \lambda_j^k| \neq 0$ for all such (finitely many) λ_j. Therefore, we have

$$\log \limsup_{n \to \infty} N(f^n)^{1/n} = \limsup_{k \to \infty} \frac{1}{k} \log N(f^k)$$

$$= \limsup_{k \to \infty} \frac{1}{k} \sum_{j=1}^m \log |1 - \lambda_j^k|$$

$$= \sum_{|\lambda| > 1} \limsup_{k \to \infty} \frac{1}{k} \log |1 - \lambda^k|$$

$$= \log \left(\prod_{|\lambda| > 1} |\lambda| \right).$$

This completes the proof. $\qquad \square$

In fact, what we have shown in the above proof is the following:

Corollary 4.2. *Let D be an integer matrix with eigenvalues $\lambda_1, \cdots, \lambda_m$, counted with multiplicities. Let $L(D^n) = \det(I - D^n)$. If 1 is not in the spectrum of D and $\mathrm{sp}(D) > 1$, then*

$$\mathrm{Growth}\,(L(D^n)) = \mathrm{sp}(\bigwedge D).$$

Recall that if $f : M \to M$ is a continuous map on an infra-nilmanifold $M = \Pi \backslash G$ with the holonomy group Φ and if f has an affine homotopy lift (d, D), then f induces a homomorphism $\phi : \Pi \to \Pi$ defined by the rule:

$$\forall \alpha \in \Pi, \ \phi(\alpha) \circ (d, D) = (d, D) \circ \alpha.$$

Furthermore, the homomorphism ϕ induces a function $\hat{\phi} : \Phi \to \Phi$ satisfying the identity (2):

$$\forall A \in \Phi, \ \hat{\phi}(A)D = DA.$$

For any $n \geq 1$, we can observe that:

(1) f^n has an affine homotopy lift $(d, D)^n = (*, D^n)$,
(2) f^n induces a homomorphism $\phi^n : \Pi \to \Pi$,
(3) the homomorphism ϕ^n induces a function $\widehat{\phi^n} = \hat{\phi}^n : \Phi \to \Phi$.

Recall from Theorem 1.1 the averaging formula:

$$N(f^n) = \frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(I - A_* D_*^n)|.$$

Since

$$\frac{1}{|\Phi|} |\det(I - D_*^n)| \leq N(f^n),$$

we have

$$\frac{1}{n} \log N(f^n) \geq \frac{1}{n} \left(\log |\det(I - D_*^n)| - \log |\Phi| \right)$$
$$\Rightarrow \limsup \frac{1}{n} \log N(f^n) \geq \limsup \frac{1}{n} \left(\log |\det(I - D_*^n)| \right).$$

This induces from Corollary 4.2 that

$$\mathrm{sp}(\bigwedge D_*) = \mathrm{Growth}(L(D_*^n)) \leq N^\infty(f).$$

Next we recall [7, Lemma 3.1]: Give $A \in \Phi$, we can choose a sequence $(B_i)_{i \in \mathbb{N}}$ of elements in Φ be taking $B_1 = A$ and such that $B_{i+1} = \hat{\phi}(B_i)$, associated to f. Since Φ is finite, this sequence will become periodic from a certain point onwards. Namely, there exist $j, k \geq 1$ such that $B_{j+k} = B_j$. It is shown in [7, Lemma 3.1] that

(1) $\forall i \in \mathbb{N}, \ \det(I - \hat{\phi}(B_i)_* D_*) = \det(I - \hat{\phi}(B_{i+1})_* D_*)$,
(2) $\exists \ell \in \mathbb{N}$ such that $(\hat{\phi}(B_j)_* D_*)^\ell = D_*^\ell$,

Since A is of finite order, $\det A_* = \pm 1$.

Let $\lambda_1, \cdots, \lambda_m$ be the eigenvalues of D_* counted with multiplicities and let $\mu_1, \cdots, \mu_m$ be the eigenvalues of $\hat{\phi}(B_j)_* D_*$ counted with multiplicities. Since $(\hat{\phi}(B_j)_* D_*)^\ell = D_*^\ell$, $(\hat{\phi}(B_j)_* D_*)^\ell$ has the eigenvalues

$$\{\lambda_1^\ell, \cdots, \lambda_m^\ell\} = \{\mu_1^\ell, \cdots, \mu_m^\ell\}.$$

We may assume that $\lambda_i^\ell = \mu_i^\ell$ for all $i = 1, \cdots, m$. Thus $|\mu_i| = |\lambda_i|$. Now,

$$
\begin{aligned}
|\det(I - A_* D_*)| &= |\det A_* \det(A_*^{-1} - D_*)| = |\det(I - D_* A_*)| \\
&= |\det(I - \hat{\phi}(B_j)_* D_*)| \quad \text{(by (1))} \\
&= \prod_{i=1}^{m} |1 - \mu_i| \leq \prod_{i=1}^{m} (1 + |\mu_i|) \quad \text{(by triangle inequality)} \\
&= \prod_{i=1}^{m} (1 + |\lambda_i|)
\end{aligned}
$$

Applying the above argument to D^n, we obtain that

$$
|\det(I - A_* D_*^n)| \leq \prod_{i=1}^{m} (1 + |\lambda_i|^n).
$$

By the averaging formula, we have

$$
\begin{aligned}
N(f^n) &= \frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(I - A_* D_*^n)| \\
&\leq \frac{1}{|\Phi|} \sum_{A \in \Phi} \prod_{i=1}^{m} (1 + |\lambda_i|^n) = \prod_{i=1}^{m} (1 + |\lambda_i|^n),
\end{aligned}
$$

which induces

$$
\begin{aligned}
\limsup \frac{1}{n} \log N(f^n) &\leq \sum_{i=1}^{m} \limsup \frac{1}{n} \log (1 + |\lambda_i|^n) \\
&= \sum_{|\lambda|>1} \log |\lambda| = \log \left(\prod_{|\lambda|>1} |\lambda| \right).
\end{aligned}
$$

Hence it follows that

$$
N^\infty(f) \leq \mathrm{sp}(\bigwedge D_*).
$$

Because the above (algebraic) properties [7] and the averaging formula for the Nielsen number [37] on infra-nilmanifolds can be generalized to infra-solvmanifolds of type (R), we have proven in all that:

Theorem 4.3. *Let f be a continuous map on an infra-solvmanifold of type* (R) *with an affine homotopy lift (d, D). Then we have*

$$
N^\infty(f) = \mathrm{sp}(\bigwedge D_*)
$$

provided that 1 is not in the spectrum of D_* and $\mathrm{sp}(D_*) > 1$.

Remark 4.4. The above theorem was proved when M is a special solvmanifold of type (R), see Theorem 4.1 and the proof of [42, Theorem 1.5]. In the paper [42], it is assumed that $\mathrm{sp}(D_*) > 1$. Since $N(f) = |\det(I - D_*)|$, 1 is not in the spectrum of D_* if and only if f is not homotopic to a fixed point free map.

Now, we provide an example of the asymptotic Nielsen numbers.

Example 4.5. Let $f : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ be any continuous map on the Klein bottle $\Pi\backslash\mathbb{R}^2$ of type (r, ℓ, q). Recall from [33, Theorem 2.3] and its proof that r is odd or $q = 0$, and

$$N(f^n) = \begin{cases} |q^n(1 - r^n)| & \text{when } r \text{ is odd and } q \neq 0; \\ |1 - r^n| & \text{when } q = 0, \end{cases}$$

$$D_* = \begin{cases} \begin{bmatrix} r & 0 \\ 0 & q \end{bmatrix} & \text{when } r \text{ is odd and } q \neq 0; \\ \begin{bmatrix} r & 0 \\ 2\ell & 0 \end{bmatrix} & \text{when } q = 0. \end{cases}$$

Assume $q = 0$. If $|r| \leq 1$, then $N(f^n) \leq 2$ and so $N^\infty(f) = 1$; if $|r| > 1$ then

$$\log \limsup_{n\to\infty} N(f^n)^{1/n} = \limsup_{n\to\infty} \frac{1}{n} \log|1 - r^n| = \log|r|.$$

Thus $N^\infty(f) = \max\{1, |r|\}$.

Assume $q \neq 0$ and r is odd. If $r = 1$ then $N(f^n) = 0 \Rightarrow N^\infty(f) = 1$. If $r \neq 1$ is odd, then

$$\log \limsup_{n\to\infty} N(f^n)^{1/n} = \limsup_{n\to\infty} \left(\log|q| + \frac{1}{n} \log|1 - r^n| \right)$$

$$= \begin{cases} \log|q| & \text{when } |r| \leq 1, \text{ i.e., } r = -1; \\ \log|qr| & \text{when } |r| > 1. \end{cases}$$

Thus

$$N^\infty(f) = \begin{cases} 1 & q \neq 0 \text{ and } r = 1 \\ \max\{1, |q|, |qr|\} & q \neq 0 \text{ and } r \neq 1 \text{ is odd.} \end{cases}$$

On the other hand, since $\mathrm{sp}(\bigwedge D_*)$ is the largest modulus of an eigenvalue of $\bigwedge D_*$, it follows that

$$\mathrm{sp}(\bigwedge D_*) = \max\{|r|, |q|, |qr|\}.$$

Hence:

(1) If $r = 1$, then $N^\infty(f) = 1$ and $\mathrm{sp}(\bigwedge D_*) = |q| \geq 1$ (since $r = 1$ is odd and so $q \neq 0$).
(2) If $r = 0$ (even), then q must be 0 and so $N^\infty(f) = 1$ but $\mathrm{sp}(\bigwedge D_*) = 0$.
(3) Otherwise, $N^\infty(f) = \mathrm{sp}(\bigwedge D_*)$.

We observe explicitly in this example that the conditions that 1 is not in the spectrum of D_* and $\mathrm{sp}(D_*) > 1$ induce the identity $N^\infty(f) = \mathrm{sp}(\bigwedge D_*)$. If $q = 0$ then $\mathrm{sp}(D_*) = |r| > 1$ and so $N^\infty(f) = |r| = \mathrm{sp}(\bigwedge D_*)$. If $q \neq 0$

then r is odd and $\mathrm{sp}(D_*) = \max\{|r|, |q|\} > 1$; if $\mathrm{sp}(D_*) = |r| \geq |q|$ then $|r| > 1$ and so $N^\infty(f) = |qr| = \mathrm{sp}(\bigwedge D_*)$; if $\mathrm{sp}(D_*) = |q| \geq |r|$ then $|q| > 1$ and $|r| > 1$ or $r = -1$ (because r cannot be 1) so $N^\infty(f) = |qr| = \mathrm{sp}(\bigwedge D_*)$.

5. TOPOLOGICAL ENTROPY AND THE RADIUS OF CONVERGENCE

The most widely used measure for the complexity of a dynamical system is the topological entropy. For the convenience of the reader, we include its definition. Let $f : X \to X$ be a self-map of a compact metric space. For given $\epsilon > 0$ and $n \in \mathbb{N}$, a subset $E \subset X$ is said to be (n, ϵ)-separated under f if for each pair $x \neq y$ in E there is $0 \leq i < n$ such that $d(f^i(x), f^i(y)) > \epsilon$. Let $s_n(\epsilon, f)$ denote the largest cardinality of any (n, ϵ)-separated subset E under f. Thus $s_n(\epsilon, f)$ is the greatest number of orbit segments $x, f(x), \cdots, f^{n-1}(x)$ of length n that can be distinguished one from another provided we can only distinguish between points of X that are at least ϵ apart. Now let

$$h(f, \epsilon) := \limsup_n \frac{1}{n} \log s_n(\epsilon, f)$$

$$h(f) := \limsup_{\epsilon \to 0} h(f, \epsilon).$$

The number $0 \leq h(f) \leq \infty$, which to be independent of the metric d used, is called the topological entropy of f. If $h(f, \epsilon) > 0$ then, up to resolution $\epsilon > 0$, the number $s_n(\epsilon, f)$ of distinguishable orbit segments of length n grows exponentially with n. So $h(f)$ measures the growth rate in n of the number of orbit segments of length n with arbitrarily fine resolution.

A basic relation between topological entropy $h(f)$ and Nielsen numbers was found by N. Ivanov [30]. We present here a very short proof by B. Jiang of the Ivanov's inequality.

Lemma 5.1 ([30])**.** *Let f be a continuous map on a compact connected polyhedron X. Then*

$$h(f) \geq \log N^\infty(f)$$

Proof. Let δ be such that every loop in X of diameter $< 2\delta$ is contractible. Let $\epsilon > 0$ be a smaller number such that $d(f(x), f(y)) < \delta$ whenever $d(x, y) < 2\epsilon$. Let $E_n \subset X$ be a set consisting of one point from each essential fixed point class of f^n. Thus $|E_n| = N(f^n)$. By the definition of $h(f)$, it suffices to show that E_n is (n, ϵ)-separated. Suppose it is not so. Then there would be two points $x \neq y \in E_n$ such that $d(f^i(x), f^i(y)) \leq \epsilon$ for $o \leq i < n$ hence for all $i \geq 0$. Pick a path c_i from $f^i(x)$ to $f^i(y)$ of diameter $< 2\epsilon$ for $0 \leq i < n$ and let $c_n = c_0$. By the choice of δ and ϵ, $f \circ c_i \simeq c_{i+1}$ for all i, so $f^n \circ c_0 \simeq c_n = c_0$. This means x, y in the same fixed point class of f^n, contradicting the construction of E_n. $\square$

This inequality is remarkable in that it does not require smoothness of the map and provides a common lower bound for the topological entropy of all maps in a homotopy class.

Let $H^*(f) : H^*(M; \mathbb{R}) \to H^*(M; \mathbb{R})$ be a linear map induced by f on the total cohomology $H^*(M; \mathbb{R})$ of M with real coefficients. By $\mathrm{sp}(f)$ we denote the spectral radius of $H^*(f)$, which is a homotopy invariant. In 1974 Michael Shub asked, [49], the extent to which the inequality

$$h(f) \geq \log(\mathrm{sp}(f))$$

holds. From this time this inequality has been usually called the Entropy Conjecture. Later A. Katok conjectured [32] that Entropy Conjecture holds for all continuous map for M being a manifold with the universal cover homeomorphic to $\mathbb{R}^m$. In [43], this was confirmed for every continuous map on an infra-nilmanifold.

Theorem 5.2. *Let f be a continuous map on an infra-solvmanifold M of type* (R) *with an affine homotopy lift (d, D). If 1 is not in the spectrum of D_* and $\mathrm{sp}(D_*) > 1$, then*

$$h(f) \geq \log(\mathrm{sp}(f)).$$

If $\bar{f}$ is the map on M induced by the affine map (d, D), then

$$h(f) \geq h(\bar{f}) \geq \log \mathrm{sp}(f),$$
$$h(\bar{f}) = \log \mathrm{sp}(\bigwedge D_*) = \log N^\infty(\bar{f}) = \log N^\infty(f).$$

Hence $\bar{f}$ minimizes the entropy in the homotopy class of f.

Proof. Let $\bar{f}$ be the map on M induced by the affine map (d, D). Thus f is homotopic to $\bar{f}$. By [37, Lemma 2.1], there is a special solvmanifold which regularly and finitely covers the infra-solvmanifold M so that f can be lifted to $\hat{f}$ on the solvmanifold. We also remark that the Lie group homomorphism $\tau_d D$ induces a map $\hat{\bar{f}}$ on the solvmanifold so that $\bar{f}$ lifts to $\hat{\bar{f}}$, $\hat{f}$ is homotopic to ϕ_f, the linearization D_* of f is also a linearization of the lift $\hat{f}$, and the topological entropies of $f, \bar{f}$ and their lifts $\hat{f}, \hat{\bar{f}}$ are the same, i.e., $h(f) = h(\hat{f})$ and $h(\bar{f}) = h(\hat{\bar{f}})$. Moreover, since the spectral radius is a homotopy invariant, $\mathrm{sp}(f) = \mathrm{sp}(\bar{f})$ and $\mathrm{sp}(\hat{f}) = \mathrm{sp}(\hat{\bar{f}})$. It is also known that $\mathrm{sp}(f) \leq \mathrm{sp}(\hat{f})$. See, for example, [43, Proposition 2].

Now observe that

$$\log \mathrm{sp}(\bigwedge D_*) = \log N^\infty(f) \quad \text{(Theorem 4.3)}$$
$$= \log N^\infty(\bar{f}) \quad \text{(homotopy invariance of } N^\infty(\cdot))$$
$$\leq h(\bar{f}) \quad \text{(Lemma 5.1)}$$
$$= h(\hat{\bar{f}}) \quad \text{(lift under a finite regular cover)}$$
$$\leq \log \mathrm{sp}(\bigwedge D_*)$$

$$\text{(first paragraph of the proof of [42, Theorem 4.13]).}$$

This implies that

$$\log \mathrm{sp}(\bigwedge D_*) = \log N^\infty(f) = \log N^\infty(\bar{f}) = h(\bar{f}).$$

Furthermore,

$$\log \mathrm{sp}(\bigwedge D_*) \geq \log \mathrm{sp}(\bar{\hat{f}}) \quad ([42, \text{ Theorem 2.1}])$$
$$= \log \mathrm{sp}(\hat{f}) \quad (\text{homotopy invariance of } \mathrm{sp}(\cdot))$$
$$\geq \log \mathrm{sp}(f) \quad (\text{lift under a finite regular cover}).$$

Thus we have

$$\log \mathrm{sp}(f) \leq \log \mathrm{sp}(\bigwedge D_*) = \log N^\infty(f) = \log N^\infty(\bar{f}) = h(\bar{f})$$
$$\leq h(f).$$

The last inequality follows from Ivanov's inequality, Lemma 5.1.   □

Remark 5.3. If $\mathrm{sp}(D_*) \leq 1$, then $\mathrm{sp}(\bigwedge D_*) = \mathrm{sp}(D_*) \leq 1$ and so we have the obvious inequality

$$h(f) \geq 0 \geq \log \mathrm{sp}(\bigwedge D_*) \geq \log \mathrm{sp}(f).$$

Remark 5.4. The first inequality $h(f) \geq \log \mathrm{sp}(f)$ in Theorem 5.2, i.e. Entropy Conjecture, also follows from [42, Proposition 4.2 and Theorem 1.5] and by taking a regular finite covering to a special solvmanifold of type (R). The second inequality in Theorem 5.2 generalizes the same results [40, Theorem 4.13] and [41, Theorem B] on nilmanifolds and infra-nilmanifolds.

We denote by R the radius of convergence of the zeta functions $N_f(z)$ or $R_f(z)$.

Theorem 5.5. *Let f be a continuous map on an infra-nilmanifold with an affine homotopy lift (d, D). Then the Nielsen zeta function $N_f(z)$ and the Reidemeister zeta function $R_f(z)$, whenever it is defined, have the same positive radius of convergence R which admits following estimation*

$$R \geq \exp(-h) > 0,$$

where $h = \inf\{h(g) \mid g \simeq f\}$.
If 1 is not in the spectrum of D_* and $\mathrm{sp}(D_*) > 1$, the radius of convergence R is

$$R = \frac{1}{N^\infty(f)} = \frac{1}{\exp h(\bar{f})} = \frac{1}{\mathrm{sp}(\bigwedge D_*)}.$$

Proof. When $R_f(z)$ is defined, as it was observed before, $R(f^n) < \infty$ and so $R(f^n) = N(f^n) > 0$ for all $n > 0$ on infra-nilmanifolds. In particular, $R_f(z) = N_f(z)$. By the Cauchy-Hadamard formula,

$$\frac{1}{R} = \limsup_{n\to\infty} \left(\frac{N(f^n)}{n}\right)^{1/n} = \limsup_{n\to\infty} N(f^n)^{1/n}.$$

Since $N(f^n) \geq 1$ for all $n > 0$, it follows that $\limsup_{n \to \infty} N(f^n)^{1/n} \geq 1$. Thus

$$\frac{1}{R} = N^\infty(f) \leq \exp h(f).$$

This induces the inequality $R \geq \exp(-h)$ by the homotopy invariance of the radius R of the Reidemeister zeta function $R_f(z)$. We consider a smooth map $g : M \to M$ which is homotopic to f. As it is known in [48], the entropy $h(g)$ is finite. Thus $\exp(-h) \geq \exp(-h(g)) > 0$. Now the identities in our theorem follow from Theorem 5.2.

Consider next the Nielsen zeta function $N_f(z)$. If $\limsup_{n \to \infty} N(f^n)^{1/n} \geq 1$, then we obtain the same inequality for R as for $R_f(z)$. Thus, we assume $\limsup_{n \to \infty} N(f^n)^{1/n} < 1$. This happens only when $N(f^n) = 0$ for all but finitely many n. In this case, $1/R = \limsup_{n \to \infty} N(f^n)^{1/n} = 0$ and so $R = \infty$ and $N^\infty(f) = 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6. Zeta functions and the Reidemeister torsion of the mapping torus

The Reidemeister torsion is a graded version of the absolute value of the determinant of an isomorphism of vector spaces.

Let $d^i : C^i \to C^{i+1}$ be a cochain complex C^* of finite dimensional vector spaces over $\mathbb{C}$ with $C^i = 0$ for $i < 0$ and large i. If the cohomology $H^i = 0$ for all i we say that C^* is *acyclic*. If one is given positive densities Δ_i on C^i then the Reidemeister torsion $\tau(C^*, \Delta_i) \in (0, \infty)$ for acyclic C^* is defined as follows:

Definition 6.1. Consider a chain contraction $\delta^i : C^i \to C^{i-1}$, i.e., a linear map such that $d \circ \delta + \delta \circ d = \mathrm{id}$. Then $d + \delta$ determines a map $(d + \delta)_+ : C^+ := \oplus C^{2i} \to C^- := \oplus C^{2i+1}$ and a map $(d + \delta)_- : C^- \to C^+$. Since the map $(d + \delta)^2 = id + \delta^2$ is unipotent, $(d + \delta)_+$ must be an isomorphism. One defines $\tau(C^*, \Delta_i) := |\det(d + \delta)_+|$.

Reidemeister torsion is defined in the following geometric setting. Suppose K is a finite complex and E is a flat, finite dimensional, complex vector bundle with base K. We recall that a flat vector bundle over K is essentially the same thing as a representation of $\pi_1(K)$ when K is connected. If $p \in K$ is a base point then one may move the fibre at p in a locally constant way around a loop in K. This defines an action of $\pi_1(K)$ on the fibre E_p of E above p. We call this action the holonomy representation $\rho : \pi \to GL(E_p)$.

Conversely, given a representation $\rho : \pi \to GL(V)$ of π on a finite dimensional complex vector space V, one may define a bundle $E = E_\rho = (\tilde{K} \times V)/\pi$. Here $\tilde{K}$ is the universal cover of K, and π acts on $\tilde{K}$ by covering transformations and on V by ρ. The holonomy of E_ρ is ρ, so the two constructions give an equivalence of flat bundles and representations of π.

If K is not connected then it is simpler to work with flat bundles. One then defines the holonomy as a representation of the direct sum of π_1 of

the components of K. In this way, the equivalence of flat bundles and representations is recovered.

Suppose now that one has on each fibre of E a positive density which is locally constant on K. In terms of ρ_E this assumption just means $|\det \rho_E| = 1$. Let V denote the fibre of E. Then the cochain complex $C^i(K; E)$ with coefficients in E can be identified with the direct sum of copies of V associated to each i-cell σ of K. The identification is achieved by choosing a basepoint in each component of K and a basepoint from each i-cell. By choosing a flat density on E we obtain a preferred density Δ_i on $C^i(K, E)$. A case of particular interest is when E is an acyclic bundle, meaning that the twisted cohomology of E is zero ($H^i(K; E) = 0$). In this case one defines the R-torsion of (K, E) to be $\tau(K; E) = \tau(C^*(K; E), \Delta_i) \in (0, \infty)$. It does not depend on the choice of flat density on E.

The Reidemeister torsion of an acyclic bundle E on K has many nice properties. Suppose that A and B are subcomplexes of K. Then we have a multiplicative law:

$$(5) \qquad \tau(A \cup B; E) \cdot \tau(A \cap B; E) = \tau(A; E) \cdot \tau(B; E)$$

that is interpreted as follows. If three of the bundles $E|A \cup B$, $E|A \cap B$, $E|A$, $E|B$ are acyclic then so is the fourth and the equation (5) holds.

Another property is the simple homotopy invariance of the Reidemeister torsion. In particular τ is invariant under subdivision. This implies that for a smooth manifold, one can unambiguously define $\tau(K; E)$ to be the torsion of any smooth triangulation of K.

In the case $K = S^1$ is a circle, let A be the holonomy of a generator of the fundamental group $\pi_1(S^1)$. One has that E is acyclic if and only if $I - A$ is invertible and then

$$\tau(S^1; E) = |\det(I - A)|$$

Note that the choice of generator is irrelevant as $I - A^{-1} = (-A^{-1})(I - A)$ and $|\det(-A^{-1})| = 1$.

These three properties of the Reidemeister torsion are the analogues of the properties of Euler characteristic (cardinality law, homotopy invariance and normalization on a point), but there are differences. Since a point has no acyclic representations ($H^0 \neq 0$) one cannot normalize τ on a point as we do for the Euler characteristic, and so one must use S^1 instead. The multiplicative cardinality law for the Reidemeister torsion can be made additive just by using $\log \tau$, so the difference here is inessential. More important for some purposes is that the Reidemeister torsion is not an invariant under a general homotopy equivalence: as mentioned earlier this is in fact why it was first invented.

It might be expected that the Reidemeister torsion counts something geometric (like the Euler characteristic). D. Fried [23] showed that it counts the periodic orbits of a flow and the periodic points of a map. We will show that the Reidemeister torsion counts the periodic point classes of a map (fixed point classes of the iterations of the map).

Some further properties of τ describe its behavior under bundles. Let $p : X \to B$ be a simplicial bundle with fiber F where F, B, X are finite complexes and p^{-1} sends subcomplexes of B to subcomplexes of X over the circle S^1. We assume here that E is a flat, complex vector bundle over B . We form its pullback p^*E over X. Note that the vector spaces $H^i(p^{-1}(b), \mathbb{C})$ with $b \in B$ form a flat vector bundle over B, which we denote H^iF. The integral lattice in $H^i(p^{-1}(b), \mathbb{R})$ determines a flat density by the condition that the covolume of the lattice is 1. We suppose that the bundle $E \otimes H^iF$ is acyclic for all i. Under these conditions D. Fried [23] has shown that the bundle p^*E is acyclic, and

$$\tau(X; p^*E) = \prod_i \tau(B; E \otimes H^iF)^{(-1)^i}.$$

Let $f : X \to X$ be a homeomorphism of a compact polyhedron X. Let $T_f := (X \times I)/(x, 0) \sim (f(x), 1)$ be the mapping torus of f.

We shall consider the bundle $p : T_f \to S^1$ over the circle S^1. We assume here that E is a flat, complex vector bundle with finite dimensional fibre and base S^1. We form its pullback p^*E over T_f. Note that the vector spaces $H^i(p^{-1}(b), \mathbb{C})$ with $b \in S^1$ form a flat vector bundle over S^1, which we denote H^iF. The integral lattice in $H^i(p^{-1}(b), \mathbb{R})$ determines a flat density by the condition that the covolume of the lattice is 1. We suppose that the bundle $E \otimes H^iF$ is acyclic for all i. Under these conditions D. Fried [23] has shown that the bundle p^*E is acyclic, and we have

$$(6) \qquad \tau(T_f; p^*E) = \prod_i \tau(S^1; E \otimes H^iF)^{(-1)^i}.$$

Let g be the preferred generator of the group $\pi_1(S^1)$ and let $A = \rho(g)$ where $\rho : \pi_1(S^1) \to GL(V)$. Then the holonomy around g of the bundle $E \otimes H^iF$ is $A \otimes (f^*)^i$. Since $\tau(S^1; E) = |\det(I - A)|$ it follows from (6) that

$$\tau(T_f; p^*E) = \prod_i |\det(I - A \otimes (f^*)^i)|^{(-1)^i} .$$

We now consider the special case in which E is one-dimensional, so A is just a complex scalar λ of modulus one. Then in terms of the rational function $L_f(z)$ we have :

$$(7) \qquad \tau(T_f; p^*E) = \prod_i |\det(I - \lambda(f^*)^i)|^{(-1)^i} = |L_f(\lambda)|^{-1}$$

This means that the special value of the Lefschetz zeta function is given by the Reidemeister torsion of the corresponding mapping torus. Let us consider an infra-nilmanifold $M = \Pi \backslash G$ and a continuous map f on M. As in Section 1, we consider the subgroup Π_+ of Π of index at most 2. Then Π_+ is also an almost Bieberbach group as Π itself and the corresponding infra-nilmanifold $M_+ = \Pi_+ \backslash G$ is a double covering of the infra-nilmanifold $M = \Pi \backslash G$; the map f lifts to a map $f_+ : M_+ \to M_+$ which has the same affine homotopy lift (d, D) as f. Let T_f and T_{f_+} be the mapping torus

of f and f_+ correspondingly. We shall consider two bundles $p : T_f \to S^1$ and $p_+ : T_{f_+} \to S^1$ over the circle S^1. We assume here that E is a flat, complex vector bundle with one dimensional fibre and base S^1. We form its pullback p^*E over T_f and pullback p_+^*E over T_{f_+}. We suppose that the bundles $E \otimes H^i M$ and $E \otimes H^i M_+$ are acyclic for all i. Then Theorem 3.6 and the formula (7) imply the following result about special values of the Reidemeister and Nielsen zeta functions

Theorem 6.2. *Let f be a homeomorphism on an infra-nilmanifold $\Pi \backslash G$ with an affine homotopy lift (d, D). Then*

$$|R_f((-1)^n \lambda)^{(-1)^{p+n}}| = |R_\phi((-1)^n \lambda)^{(-1)^{p+n}}| = |N_f((-1)^n \lambda)^{(-1)^{p+n}}|$$

$$= \begin{cases} |L_f(\lambda)| = \tau(T_f; p^*E)^{-1} & when \ \Pi = \Pi_+; \\ |L_{f_+}(\lambda) L_f(\lambda)^{-1}| = \tau(T_f; p^*E)\tau(T_{f_+}; p_+^*E)^{-1} & when \ \Pi \neq \Pi_+, \end{cases}$$

where p is the number of real eigenvalues of D_* which are > 1 and n is the number of real eigenvalues of D_* which are < -1.

7. Jiang-type spaces and averaging formula for the Reidemeister numbers on infra-solvmanifolds of type (R)

A closed manifold M is called a *Jiang-type space* if for all continuous maps $f : M \to M$,

$$L(f) = 0 \Rightarrow N(f) = 0;$$
$$L(f) \neq 0 \Rightarrow N(f) = R(f).$$

A closed orientable manifold M is called a *Jiang-type space for coincidences* ([25]) if for any continuous maps $f, g : N \to M$ where N is any closed orientable manifold of equal dimension,

$$L(f, g) = 0 \Rightarrow N(f, g) = 0;$$
$$L(f, g) \neq 0 \Rightarrow N(f, g) = R(f, g).$$

It is well-known that Jiang spaces are of Jiang-type for coincidences. When $N = M$ is a nilmanifold and ϕ, ψ are homomorphisms on the group of covering transformations induced by self-maps f, g on N, it is proven in [24, Theorem 2.3] that

$$N(f, g) > 0 \Leftrightarrow \mathrm{coin}(\phi, \psi) = 1 \Leftrightarrow R(f, g) < \infty$$

Further if one of the above holds then

$$R(f, g) = N(f, g) = |L(f, g)|.$$

Furthermore, nilmanifolds are Jiang-type spaces for coincidences, see [25]. Recall that if N is a finite connected complex and M is a nilmanifold then $N(f, g) \neq 0 \Rightarrow R(f, g) < \infty$; if both N and M are nilmanifolds of equal dimension, then two conditions are equivalent and in that case we have $N(f, g) = R(f, g)$.

Recall what C. McCord proved in [44, Sec. 2]. Let S_i be simply connected solvable Lie groups of type (E) with equal dimension, and let Γ_i be lattices of S_i. Let $D_i : S_1 \to S_2$ be Lie group homomorphisms such that $D_i(\Gamma_1) \subset \Gamma_2$. Write $\phi_i = D_i|_{\Gamma_1} : \Gamma_1 \to \Gamma_2$. Thus D_i induce maps f_i between orbit spaces $M_i = \Gamma_i \backslash S_i$, special solvmanifolds of type (E). When S_i are of type (R), we can always assume that any f_i is induced from a Lie group homomorphism D_i, see [37, Theorem 2.2] or [27, Theorem 4.2].

Denote $C_\gamma := \mathrm{coin}(\gamma \circ D_1, D_2)$ and $\mathbb{S}_\gamma = p_1 \left(\mathrm{coin}(\gamma \circ D_1, D_2) \right)$ for each $\gamma \in \Gamma_2$. We also consider the map $D : S_1 \to S_2$ defined by $D(s) = D_1(s)^{-1} D_2(s)$ for $s \in S_1$.

Lemma 7.1 ([44, Lemmas 2.6 and 2.7, and Theorem 2.1]). *The following are equivalent:*

(1) $\mathrm{coin}(\phi_1, \phi_2) = 1$.
(2) $\dim(C_1) = 0$.
(3) D *is injective.*
(4) $C_1 = \mathbb{S}_1$.
(5) $\mathrm{ind}(\mathbb{S}_1) = \pm 1$.
(6) $\mathrm{ind}(\mathbb{S}_1) \neq 0$.

These statements are also valid for any other coincidence class $\mathbb{S}_\gamma$, and all $\mathrm{ind}(\mathbb{S}_\gamma)$ have the same sign. Hence $N(f_1, f_2) = |L(f_1, f_2)|$.

We generalize [24, Theorem 2.3] from nilmanifolds to special solvmanifolds of type (R).

Theorem 7.2. *Let f_1 and f_2 be maps on a special solvmanifold $\Gamma \backslash S$ of type (R). Let $\phi_1, \phi_2 : \Gamma \to \Gamma$ be homomorphisms induced by f_1, f_2 respectively. Then the following are equivalent:*

(a) $N(f_1, f_2) > 0$.
(b) $\mathrm{coin}(\phi_1, \phi_2) = 1$.
(c) $R(f_1, f_2) < \infty$.

Further if one of the above holds then

$$R(f_1, f_2) = N(f_1, f_2) = |L(f_1, f_2)|.$$

Proof. By Lemma 7.1, (a) $\Leftrightarrow$ (b). Now we will show (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a) together with the identity $R(f_1, f_2) = N(f_1, f_2)$.

Let S be a simply connected solvable Lie group of type (R). Let $N = [S, S]$ and $\Lambda = S/N$. Then N is nilpotent and $\Lambda \cong \mathbb{R}^k$ for some $k > 0$. A lattice Γ of S yields a lattice $N \cap \Gamma$ of N. Moreover, the lattice Γ induces a short exact sequence $1 \to N \cap \Gamma \to \Gamma \to \Gamma/N \cap \Gamma \cong \Gamma \cdot N/N \to 1$ so that the following diagram is commutative

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & S & \longrightarrow & \Lambda = S/N & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & N \cap \Gamma & \longrightarrow & \Gamma & \longrightarrow & \Gamma \cdot N/N & \longrightarrow & 0
\end{array}
$$

This gives rise to the fibration, called a *Mostow fibration*,

$$N \cap \Gamma \backslash N \longrightarrow M = \Gamma \backslash S \longrightarrow \Gamma \cdot N \backslash S$$

over a torus base $\Gamma \cdot N \backslash S$ with compact nilmanifold fiber $N \cap \Gamma \backslash N$. It is known that this fibration is orientable if and only if the solvmanifold M is a nilmanifold.

Let $E : S \to S$ be a homomorphism. Then E induces a homomorphism $E' : N \to N$ and hence a homomorphism $\bar{E} : \Lambda \to \Lambda$ so that the following diagram is commutative

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & S & \longrightarrow & \Lambda & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle E'} & & \downarrow{\scriptstyle E} & & \downarrow{\scriptstyle \bar{E}} & & \\
1 & \longrightarrow & N & \longrightarrow & S & \longrightarrow & \Lambda & \longrightarrow & 0
\end{array}
$$

Hence we have the following diagram is commutative

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & N \cap \Gamma & \longrightarrow & \Gamma & \longrightarrow & \Gamma \cdot N / N & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \phi_i'} & & \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \bar{\phi}_i} & & \\
1 & \longrightarrow & N \cap \Gamma & \longrightarrow & \Gamma & \longrightarrow & \Gamma \cdot N / N & \longrightarrow & 0
\end{array}
$$

Denote $\Gamma' = N \cap \Gamma$ and let $\bar{\Gamma} = \Gamma \cdot N / N$.

By [37, Theorem 2.2] or [27, Theorem 4.2], we may assume that f_1, f_2 are induced by Lie group homomorphisms $D_1, D_2 : S \to S$ respectively. Then

$$\varphi_i(\gamma) \circ D_i = D_i \circ \gamma \quad \forall \gamma \in \Gamma.$$

Evaluating at the identity of S, we obtain that $\phi_i(\gamma) = D_i(\gamma)$ for all $\gamma \in \Gamma$. So, ϕ_i is the restriction of D_i on Γ.

Assume (b): $\operatorname{coin}(\phi_1, \phi_2) = 1$. Then $\operatorname{coin}(D_1, D_2) = 1$ by Lemma 7.1. By taking differential, we see that $\operatorname{coin}(D_{1*}, D_{2*}) = 0$, or $D_{2*} - D_{1*}$ is a linear isomorphism. We can write $D_{2*} - D_{1*}$ as

$$D_{2*} - D_{1*} = \begin{bmatrix} \bar{D}_{2*} - \bar{D}_{1*} & 0 \\ * & D_{2*}' - D_{1*}' \end{bmatrix}$$

with respect to some linear basis of the Lie algebra of S. This implies that $\bar{D}_{2*} - \bar{D}_{1*}$ is an isomorphism and so $\operatorname{coin}(\bar{D}_{2*}, \bar{D}_{1*}) = 0$ or $\operatorname{coin}(\bar{D}_1, \bar{D}_2) = \bar{1} = \operatorname{coin}(\bar{\varphi}_1, \bar{\varphi}_2)$. This happens on $\Lambda \cong \mathbb{R}^k$ with the lattice Γ' and so on the torus $\Gamma \cdot N \backslash S = \Gamma' \backslash \Lambda$. Hence $\operatorname{coin}(\bar{\phi}_1, \bar{\phi}_2) = \bar{1}$ implies $R(\bar{\phi}_1, \bar{\phi}_2) < \infty$.

On the other hand, since $\operatorname{coin}(\phi_1', \phi_2') = 1$ from $\operatorname{coin}(\phi_1, \phi_2) = 1$, by [24, Theorem 2.3], $R(\phi_1', \phi_2') < \infty$. Now the above commutative diagram induces a short exact sequence of the sets of Reidemeister classes

$$\mathcal{R}(\phi_1', \phi_2') \longrightarrow \mathcal{R}(\phi_1, \phi_2) \longrightarrow \mathcal{R}(\bar{\phi}_1, \bar{\phi}_2) \longrightarrow 1.$$

Because both sets $\mathcal{R}(\phi_1', \phi_2')$ and $\mathcal{R}(\bar{\phi}_1, \bar{\phi}_2)$ are finite, it follows that the middle set $\mathcal{R}(\phi_1, \phi_2)$ is also finite. Hence $R(\phi_1, \phi_2) < \infty$.

Assume (c): $R(\phi_1, \phi_2) < \infty$. Then $R(\bar{\phi}_1, \bar{\phi}_2) < \infty$ on the torus $\Gamma' \backslash \Lambda$. We already know that this implies $0 < N(\bar{f}_1, \bar{f}_2) = R(\bar{\phi}_1, \bar{\phi}_2)$ and $\operatorname{coin}(\bar{\phi}_1, \bar{\phi}_2) = \bar{1}$. Assume that $R(\phi_1', \phi_2') = \infty$. By [24, Theorem 2.3], $\operatorname{coin}(\phi_1', \phi_2') \neq 1$ and

then by Lemma 7.1, $\mathrm{coin}(D_1', D_2') \neq 1$ and hence $D_{2_*}' - D_{1_*}'$ is singular, which implies $D_{2_*} - D_{1_*}$ is also singular and so contradicts $\mathrm{coin}(\phi_1, \phi_2) = 1$. Hence $R(\phi_1', \phi_2') < \infty$ on the nilmanifold $\Gamma' \backslash N$. This implies that $0 < N(f_1', f_2') = R(\phi_1', \phi_2')$. Hence we have

$$
\begin{aligned}
N(f_1, f_2) &= |L(f_1, f_2)| \text{ ([44, Theorem 2.1])} \\
&= |\det(D_{2_*} - D_{1_*})| \text{ ([28, Theorem 3.1])} \\
&= |\det(\bar{D}_{2_*} - \bar{D}_{1_*})||\det(D_{2_*}' - D_{1_*}')| \\
&= N(\bar{f}_1, \bar{f}_2)N(f_1', f_2') = R(\bar{\phi}_1, \bar{\phi}_2)R(\phi_1', \phi_2') \\
&\geq R(\phi_1, \phi_2) \text{ (exactness and finiteness of each Reidemeister set).}
\end{aligned}
$$

Consequently, sine it is always true that $N(f_1, f_2) \leq R(\phi_1, \phi_2)$, we have the identity $N(f_1, f_2) = R(\phi_1, \phi_2)$. $\qquad\square$

Immediately, from Theorem 7.2 we obtain the following: for any maps $f_1, f_2 : M \to M$ on a special solvmanifold M of type (R), we have

$$
\begin{aligned}
L(f_1, f_2) = 0 &\Rightarrow N(f_1, f_2) = 0; \\
L(f_1, f_2) \neq 0 &\Rightarrow N(f_1, f_2) = R(f_1, f_2).
\end{aligned}
$$

Example 7.3. Consider the closed 3-manifolds with Sol-geometry. We refer to [27, Sec. 6] for details about the Reidemeister numbers on these manifolds. These are infra-solvmanifolds $\Pi \backslash \mathrm{Sol}$ of type (R). When $\Pi = \Pi_0$ or $\Pi_2^{\pm}$, the corresponding manifold is a torus bundle over S^1, and when $\Pi = \Pi_3$ or Π_6, the manifold is a sapphire space. Only $\Pi_0 \backslash \mathrm{Sol}$ is the special solvmanifold and the remaining manifolds are non-special, infra-solvmanifolds. For any *homeomorphism* $f : \Pi \backslash \mathrm{Sol} \to \Pi \backslash \mathrm{Sol}$, let F_* be its linearization. Then the following can be found in [27, Sec. 6]:

(1) When $\Pi = \Pi_0$ or Π_2^+, $L(f) = N(f) = R(f) = 4$ only when F_* is of type (II) with $\det F_* = -1$; otherwise, $L(f) = N(f) = 0$ and $R(f) = \infty$.
(2) When $\Pi = \Pi_2^-$, F_* is always of type (I) and $L(f) = N(f) = 0$, but $R(f) = \infty$.
(3) When $\Pi = \Pi_3$, $L(f) = N(f) = 0$, but $R(f) = \infty$.
(4) When $\Pi = \Pi_6$, $L(f) = N(f)$, which is 0 or 2 according as $\det F_* = 1$ or -1, but $R(f) = \infty$.

These results show that Theorem 7.2 (i.e., $N(f) > 0 \Leftrightarrow R(f) < \infty$; in this case, $N(f) = R(f)$) is true for the special solvmanifold $\Pi_0 \backslash \mathrm{Sol}$ and infra-solvmanifolds $\Pi_2^{\pm} \backslash \mathrm{Sol}$ and $\Pi_3 \backslash \mathrm{Sol}$, but not true anymore for the infra-solvmanifold $\Pi_6 \backslash \mathrm{Sol}$.

Now we can state a practical formula for the Reidemeister number of a pair of continuous maps on an infra-solvmanifold of type (R). This is a straightforward generalization of [29, Theorem 6.11] and its proof from infra-nilmanifolds.

Theorem 7.4. *Let $M = \Pi \backslash S$ be an infra-solvmanifold of type* (R) *with holonomy group* Φ. *Let $f, g : M \to M$ be continuous maps with affine homotopy lifts $(d, D), (e, E)$ respectively. Then*

$$R(f, g) = \frac{1}{|\Phi|} \sum_{A \in \Phi} \sigma \left(\det(E_* - A_* D_*) \right),$$

where A_*, D_* and E_* induced by A, D and E are expressed with respect to a preferred basis of $\Pi \cap S$ and where $\sigma : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is given by $\sigma(0) = \infty$ and $\sigma(x) = |x|$ for all $x \neq 0$.

Proof. Choose a fully invariant subgroup $\Lambda \subset \Gamma := \Pi \cap S$ of Π with finite index ([37, Lemma 2.1]). Then f, g lift to maps $\bar{f}, \bar{g}$ on the special solvmanifold $\Lambda \backslash S$ of type (R) and by [26, Corollary 1.3] we have

$$R(f, g) = \frac{1}{[\Pi : \Lambda]} \sum_{\bar{\alpha} \in \Pi / \Lambda} R(\bar{\alpha}\bar{f}, \bar{g}).$$

By Theorem 7.2, $R(\bar{\alpha}\bar{f}, \bar{g}) = \sigma \left(N(\bar{\alpha}\bar{f}, \bar{g}) \right)$ for all $\bar{\alpha} \in \Pi / \Lambda$.

On the other hand, we may assume that f, g are induced by the affine maps $(d, D), (e, E)$ respectively. This induces that $\bar{f}, \bar{g}$ are induced by the Lie group homomorphisms $\mu(d) \circ D, \mu(e) \circ E : S \to S$, where $\mu(\cdot)$ is conjugation. If $(a, A) \in \Pi$ is a preimage of $\bar{\alpha} \in \Pi / \Lambda$, then the transformation $\bar{\alpha}$ on $\Lambda \backslash S$ is induced by the Lie group automorphism $\mu(a) \circ A$. By [28, Theorem 3.1] and Lemma 7.1, we have that

$$N(\bar{\alpha}\bar{f}, \bar{g}) = |\det(\mathrm{Ad}(e)E_* - \mathrm{Ad}(a)A_*\mathrm{Ad}(d)D_*)| = |\det_\Lambda(E_* - A_*D_*)|$$

with respect to any preferred basis of Λ. If we regard this as a basis of Γ, then we can see that

$$[\Gamma : \Lambda] \det_\Lambda(E_* - A_*D_*) = \det_\Gamma(E_* - A_*D_*),$$

for example see the proof of [29, Theorem 6.11]. Hence

$$R(f, g) = \frac{1}{[\Pi : \Lambda]} \sum_{\bar{\alpha} \in \Pi / \Lambda} \sigma \left(N(\bar{\alpha}\bar{f}, \bar{g}) \right)$$

$$= \frac{1}{[\Pi : \Lambda]} \sum_{A \in \Phi} [\Gamma : \Lambda] \, \sigma \left(\det_\Lambda(E_* - A_*D_*) \right)$$

$$= \frac{1}{|\Phi|} \sum_{A \in \Phi} \sigma \left(\det_\Gamma(E_* - A_*D_*) \right). \qquad \square$$

The following corollaries generalize [12, Theorems 5.1 and 5.2] from infra-nilmanifolds to infra-solvmanifolds of type (R).

Corollary 7.5. *Let $M = \Pi \backslash S$ be an orientable infra-solvmanifold of type* (R). *Let $f, g : M \to M$ be continuous maps. If $R(f, g) < \infty$, then $R(f, g) = N(f, g)$.*

Proof. Because M is orientable, the Nielsen number $N(f, g)$ is defined and is equal to, by [27, Theorem 4.5],

$$N(f, g) = \frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(E_* - A_* D_*)|.$$

Since $R(f, g) < \infty$, by Theorem 7.4, $\sigma(\det(E_* - A_* D_*))$ is finite for all $A \in \Phi$. By the definition of σ, we have $\sigma(\det(E_* - A_* D_*)) = |\det(E_* - A_* D_*)|$ for all $A \in \Phi$. This finishes the proof. $\square$

Corollary 7.6. *Let $M = \Pi \backslash S$ be an infra-solvmanifold of type* (R) *with holonomy group Φ. Let $f : M \to M$ be a continuous map with an affine homotopy lift (d, D). Then*

$$R(f) = \frac{1}{|\Phi|} \sum_{A \in \Phi} \sigma\left(\det(I - A_* D_*)\right),$$

and if $R(f) < \infty$ then $R(f) = N(f)$.

By Remarks 3.7 and 3.11, since the averaging formulas for the Lefschetz number and the Nielsen number are generalized from infra-nilmanifolds to infra-solvmanifolds of type (R) (see [28, 37]), all results and proofs concerning the Nielsen number and the Nielsen zeta function in this article directly generalize to the class of infra-solvmanifolds of type (R).

By Corollary 7.6 and [37, Theorem 4.3], the averaging formulas for the Reidemeister number and the Nielsen number on infra-solvmanifolds of type (R), we can generalize all results and proofs concerning the Reidemeister zeta function, whenever it is defined, to the class of infra-solvmanifolds of type (R). If $R_f(z)$ is defined, then $R(f^n) < \infty$ and so by Corollary 7.6 $R(f^n) = N(f^n) > 0$ for all $n > 0$ and thus $R_f(z) = N_f(z)$. For example, we can generalize Theorems 3.5, 3.6, 3.9, and 6.2, and their proofs from infra-nilmanifolds to infra-solvmanifolds of type (R) to obtain the following:

Theorem 7.7. *Let f be a continuous map on an infra-solvmanifold of type* (R) *with an affine homotopy lift (d, D). Assume $N(f) = |L(f)|$. Then the Nielsen zeta function $N_f(z)$ is a rational function and is equal to*

$$N_f(z) = L_f((-1)^q z)^{(-1)^r}$$

where q is the number of real eigenvalues of D_* which are < -1 and r is the number of real eigenvalues of D_* of modulus > 1. When the Reidemeister zeta function $R_f(z)$ is defined, we have $R_f(z) = R_\phi(z) = N_f(z)$.

Theorem 7.8. *Let f be a continuous map on an infra-solvmanifold $\Pi \backslash S$ of type* (R) *with an affine homotopy lift (d, D). Then the Reidemeister zeta function, whenever it is defined, is a rational function and is equal to*

$$R_f(z) = N_f(z) = \begin{cases} L_f((-1)^n z)^{(-1)^{p+n}} & \text{when } \Pi = \Pi_+; \\ \left(\frac{L_{f_+}((-1)^n z)}{L_f((-1)^n z)}\right)^{(-1)^{p+n}} & \text{when } \Pi \neq \Pi_+, \end{cases}$$

Theorem 7.9 (Functional Equation)**.** *Let f be a continuous map on an orientable infra-solvmanifold $M = \Pi \backslash S$ of type* (R) *with an affine homotopy lift* (d, D)*. Then the Reidemeister zeta function, whenever it is defined, and the Nielsen zeta function have the following functional equations:*

$$R_f \left(\frac{1}{dz} \right) = \begin{cases} R_f(z)^{(-1)^m} \epsilon^{(-1)^{p+n}} & when\ \Pi = \Pi_+; \\ R_f(z)^{(-1)^m} \epsilon^{-1} & when\ \Pi \neq \Pi_+ \end{cases}$$

and

$$N_f \left(\frac{1}{dz} \right) = \begin{cases} N_f(z)^{(-1)^m} \epsilon^{(-1)^{p+n}} & when\ \Pi = \Pi_+; \\ N_f(z)^{(-1)^m} \epsilon^{-1} & when\ \Pi \neq \Pi_+ \end{cases}$$

where d is a degree f, $m = \dim M$, ϵ is a constant in $\mathbb{C}^\times$, $\sigma = (-1)^n$, p is the number of real eigenvalues of D_* which are > 1 and n is the number of real eigenvalues of D_* which are < -1. If $|d| = 1$ then $\epsilon = \pm 1$.

Theorem 7.10. *Let f be a continuous map on an infra-solvmanifold of type* (R) *with an affine homotopy lift* (d, D)*. Then the Nielsen zeta function $N_f(z)$ and the Reidemeister zeta function $R_f(z)$, whenever it is defined, have the same positive radius of convergence R which admits following estimation*

$$R \geq \exp(-h) > 0,$$

where $h = \inf\{h(g) \mid g \simeq f\}$.

   If 1 is not in the spectrum of D_* and $\mathrm{sp}(D_*) > 1$, the radius of convergence R is

$$R = \frac{1}{N^\infty(f)} = \frac{1}{\exp h(\bar{f})} = \frac{1}{\mathrm{sp}(\bigwedge D_*)}.$$

Theorem 7.11. *Let f be a homeomorphism on an infra-solvmanifold $\Pi \backslash S$ of type* (R) *with an affine homotopy lift* (d, D)*. Then*

$$|N_f((-1)^n \lambda)^{(-1)^{p+n}}|$$
$$= \begin{cases} |L_f(\lambda)| = \tau(T_f; p^* E)^{-1} & when\ \Pi = \Pi_+; \\ |L_{f_+}(\lambda) L_f(\lambda)^{-1}| = \tau(T_f; p^* E) \tau(T_{f_+}; p_+^* E)^{-1} & when\ \Pi \neq \Pi_+, \end{cases}$$

where p is the number of real eigenvalues of D_* which are > 1 and n is the number of real eigenvalues of D_* which are < -1.

Remark 7.12. One may formulate the above theorem also for the Reidemeister zeta function of a homeomorphism f on an infra-solvmanifold of type (R) . However it will be seen in Theorem 8.2 that in the case of $R_f(z)$ such a manifold must be an infra-nilmanifold.

8. The Reidemeister zeta function is never defined for any homeomorphism of infra-solvmanifold of type (R), not an infra-nilmanifold

Consider now as an example closed 3-manifolds with Sol-geometry. We refer to [27, Sec. 6] for details about the Reidemeister numbers on these

manifolds. These are infra-solvmanifolds $\Pi\backslash\mathrm{Sol}$ of type (R). Let Π_1 be a lattice of Sol:

$$\Pi_1 = \Gamma_A = \langle a_1, a_2, \tau \mid [a_1, a_2] = 1, \tau a_i \tau^{-1} = A(a_i)\rangle,$$

where A is a 2×2-integer matrix of determinant 1 and trace > 2. Consider a homomorphism ϕ on Π_1 of type (III), i.e., ϕ is given by the form

$$\phi(a_1) = \phi(a_2) = 1, \phi(\tau) = a_1^p a_2^q \tau^r, \ r \neq \pm 1.$$

Then it is shown in [27, Theorem 6.1] that $R(\phi) = |1 - r|$. We can observe easily that ϕ^n is also of type (III) and $R(\phi^n) = |1 - r^n|$ for all $n > 0$. Hence

$$R_\phi(z) = \exp\left(\sum_{n=1}^{\infty} \frac{|1 - r^n|}{n} z^n\right) = \begin{cases} \frac{1}{1-z}, & \text{when } r = 0; \\ \frac{1-\frac{r}{|r|}z}{1-|r|z} & \text{when } |r| > 1. \end{cases}$$

It can be seen also that if ϕ is not of type (III), then $R(\phi) = \infty$ or $R(\phi^2) = \infty$. Thus the associated Reidemeister zeta function is not defined. A similar phenomenon happens for the infra-solvmanifold $\Pi^{\pm}\backslash\mathrm{Sol}$. For the remaining infra-solvmanifolds $\Pi_3\backslash\mathrm{Sol}$ and $\Pi_6\backslash\mathrm{Sol}$, it is shown that only trivial map has a finite Reidemeister number, which is 1. That is, only for the trivial map the Reidemeister zeta function is well-defined. The homomorphisms above are eventually commutative, and in fact, for every eventually commutative homomorphism the Reidemeister zeta function, whenever it is defined, is a rational function, see Theorem 9 and Theorem 10 in [17].

We will show now that if the Reidemeister zeta function is defined for a homeomorphism on an infra-solvmanifold of type (R), then this manifold must be an infra-nilmanifold.

Recall the following

Proposition 8.1 ([2, Ex. 21(b), p. 97], [50, Proposition 3.6]). *Let σ be a Lie algebra automorphism. If none of the eigenvalues of σ is a root of unity, then the Lie algebra must be nilpotent.*

Theorem 8.2. *If the Reidemeister zeta function $R_f(z)$ is defined for a homeomorphism f on an infra-solvmanifold M of type (R), then M is an infra-nilmanifold.*

Proof. Let f be a homeomorphism on an infra-solvmanifold $M = \Pi\backslash S$ of type (R). By [37, Theorem 2.2], we may assume that f has an affine map as a homotopy lift. By [37, Lemma 2.1], there is a special solvmanifold $N = \Lambda\backslash S$ which covers M finitely and on which f has a lift $\bar{f}$, which is induced by a Lie group automorphism D on the solvable Lie group S.

From [26, Corollary 1.3], we have an averaging formula for Reidemeister numbers:

$$R(f^n) = \frac{1}{[\Pi : \Lambda]} \sum_{\bar{\alpha} \in \Pi/\Lambda} R(\bar{\alpha}\bar{f}^n).$$

Assume now that f defines the Reidemeister zeta function. Then $R(f^n) < \infty$ for all $n > 0$. The above averaging formula implies that $R(\bar{f}^n) < \infty$ for

all n. By Theorem 7.2, we must have

$$R(\bar{f}^n) = N(\bar{f}^n) = |L(\bar{f}^n)| > 0.$$

Since $L(\bar{f}^n) = \det(I - D_*^n) \neq 0$ for all $n > 0$ by [28, Theorem 3.1], this would imply that the differential D_* of D has no roots of unity. By Proposition 8.1, S must be nilpotent. $\qquad\square$

Remark 8.3. Let A be an Anosov diffeomorphism on an infra-nilmanifold. Then an iteration A^n will be also an Anosov diffeomorphism for every $n \geq 1$. The Reidemeister number of an Anosov diffeomorphism is always finite [9]. Hence the Reidemeister zeta $R_A(z)$ is well-defined. From Theorem 3.6 and Theorem 3.9 it follows that the Reidemeister zeta function $R_A(z)$ of an Anosov diffeomorphism on an infra-nilmanifold is a rational function with functional equation. It is known that a nilmanifold modelled on a free c-step nilpotent Lie group on r generators admits an Anosov diffeomorphism if and only if $r > c$ [4]. Hence the Reidemeister zeta function of an Anosov diffeomorphism on such nilmanifold is well-defined if $r > c$ and is a rational function with functional equation.

9. The Artin-Mazur zeta functions on infra-solvmanifolds of type (R)

Let f be a continuous map on a topological space X. Then the *Artin-Mazur zeta function* of f is defined as follows:

$$AM_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{F(f^n)}{n} z^n\right)$$

where $F(f)$ is the number of isolated fixed points of f.

Proposition 9.1 ([35, Proposition 1]). *Let f be a continuous map on an infra-solvmanifold $\Pi \backslash S$ of type (E) induced by an affine map $F : S \to S$. For any $\alpha \in \Pi$, $\mathrm{Fix}(\alpha \circ F)$ is an empty set or path connected. Hence every nonempty fixed point class of f is path connected, and every isolated fixed point class forms an essential fixed point class.*

Proof. Let $x, y \in \mathrm{Fix}(\alpha \circ F)$. So, the affine map αF fixes x and y. Writing $\alpha F = (d, D) \in S \rtimes \mathrm{Endo}(S)$, we see that

- $(d, D)(x) = x \Rightarrow D(x) = d^{-1}x$,
- $(d, D)(y) = y \Rightarrow D(y) = d^{-1}y$,
- $(x, I)^{-1}(\alpha \circ F)(x, I) = (x, I)^{-1}(d, D)(x, I) = (x^{-1}dD(x), D) = (1, D)$, and D fixes 1 and $x^{-1}y$.

Since S is of type (E), $\exp : \mathfrak{S} \to S$ is a diffeomorphism with inverse log. Let $X = \log(x^{-1}y) \in \mathfrak{S}$. Then the 1-parameter subgroup $\{\exp(tX) \mid t \in \mathbb{R}\}$ of S is fixed by the endomorphism D. Consequently, the affine map $\alpha \circ F$ fixes the 'line' connecting the points x and y. $\qquad\square$

Remark 9.2. The above proposition generalizes [35, Proposition 1] from infra-nilmanifolds to infra-solvmanifolds of type (E). Further, the linear part of the affine map F need not be an automorphism.

Proposition 9.1 is proved when the manifold is a special solvmanifold of type (E) and the map is induced by a homomorphism in Lemma 7.1, [44]. In fact, the converse is also proved. That is, every essential fixed point class consists of a single element. We will prove the converse of the proposition on infra-solvmanifolds of type (R).

Proposition 9.3. *Let f be a continuous map on an infra-solvmanifold of type* (R) *induced by an affine map. Then every essential fixed point class of f consists of a single element.*

Proof. Let $\tilde{f} = (d, D)$ be the affine map on the connected, simply connected solvable Lie group S of type (R) which induces $f : \Pi \backslash S \to \Pi \backslash S$. Then f induces a homomorphism $\phi : \Pi \to \Pi$.

By [37, Lemma 2.1], we can choose a fully invariant subgroup $\Lambda \subset \Pi \cap S$ of Π with finite index. Hence $\phi(\Lambda) \subset \Lambda$. This implies that $\tilde{f}$ induces a map $\bar{f}$ on $\Lambda \backslash S$.

Then we have an averaging formula, [37, Theorem 4.2],

$$N(f) = \frac{1}{[\Pi : \Lambda]} \sum_{\bar{\alpha} \in \Pi/\Lambda} N(\bar{\alpha} \circ \tilde{f}).$$

Assume that f has an essential fixed point class. The averaging formula tells that this essential fixed point class of f is lifted to an essential fixed point class of some $\bar{\alpha} \circ \bar{f}$. That is, there is $\alpha = (a, A) \in \Pi$ such that the fixed point class $p'(\mathrm{Fix}(\alpha \circ \tilde{f}))$ of $\bar{\alpha} \circ \bar{f}$ is essential (and so $p(\mathrm{Fix}(\alpha \circ \tilde{f}))$ is an essential fixed point class of f). It suffices to show that the fixed point class $p'(\mathrm{Fix}(\alpha \circ \tilde{f}))$ consists of only one point.

Let $F = \alpha \circ \tilde{f} = (a, A)(d, D) := (e, E)$ be the affine map on S, and let $\bar{F} = \bar{\alpha} \circ \bar{f}$. Then $p'(\mathrm{Fix}(F))$ is essential and $N(\bar{F}) = |\det(I - E_*)| \neq 0$. Choose $x \in \mathrm{Fix}(F) = \mathrm{Fix}((e, E))$. Then the left multiplication by x^{-1},

$$\ell_{x^{-1}} : y \in \mathrm{Fix}((e, E)) \mapsto x^{-1}y \in \mathrm{Fix}(E),$$

is a bijection. Further, since $\exp : \mathfrak{S} \to S$ is a diffeomorphism, it follows that $\mathrm{Fix}(E) \leftrightarrow \mathrm{fix}(E_*) = \ker(I - E_*)$. Since $I - E_*$ is invertible, we see that $\mathrm{Fix}(F)$ and hence $p'(\mathrm{Fix}(F))$ and $p(\mathrm{Fix}(F))$ consist of a single element. $\square$

In Propositions 9.1 and 9.3, we have shown that for any continuous map on an infra-solvmanifold of type (R) induced by an affine map the isolated fixed points of f are the essential fixed point classes of f. That is, $F(f) = N(f)$. Similarly $F(f^n) = N(f^n)$ for all n. Therefore, by Theorem 7.8 and Theorem 7.9 we have

Theorem 9.4. *Let f be a continuous map on an infra-solvmanifold of type* (R) *induced by an affine map. Then $AM_f(z) = N_f(z)$, i.e. $AM_f(z)$ is a rational function with functional equation.*

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