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Abstract: This past year has seen major progress in the study of tree-level supergravity amplitudes with zero cosmological constant using twistor methods. In this paper, we study amplitudes of conformal gravity and use these to deduce formulae for 'scattering amplitudes' on backgrounds with non-zero cosmological constant. Our approach is based firstly on the embedding of Einstein gravity into conformal gravity and secondly the twistor action for conformal gravity and its minimal $\mathcal{N} = 4$ supersymmetric extension. We derive conformal gravity amplitudes from the twistor action and show how they can then be restricted to Einstein states to give Einstein amplitudes. We employ a perturbative expansion to derive a new formula for the gravitational MHV amplitude with cosmological constant. We show that this formula is well-defined (i.e., is independent of certain gauge choices) and that it reproduces Hodges' formula for the MHV amplitude in the flat-space limit. We also discuss the possibility of a twistor-string origin for this formula, as well as more general properties of conformal (super-)gravity in twistor space. We also give a preliminary discussion of a possible MHV formalism for more general amplitudes obtained by reduction of one for conformal gravity obtained from the twistor action.

Contents

1 Introduction

Witten's twistor-string theory and related models [1–3] have inspired an extensive list of recent developments in our understanding of maximally supersymmectic ($\mathcal{N}=4$) super-Yang-Mills (SYM) theory, particularly with respect to scattering amplitudes in the planar sector. While these original twistor-string theories were limited in their applicability to perturbative gauge theory due to unwanted contributions from conformal gravity [4], twistor actions for Yang-Mills theory were discovered which isolated the gauge theoretic degrees of freedom [5, 6]. With the imposition of an axial gauge on twistor space, these twistor actions can be used to derive a particularly efficient Feynman diagram formalism, the MHV formalism, that had originally been suggested from twistor-string considerations [7]. In this formalism, the MHV amplitudes are extended off-shell to provide the vertices and can be used also for loop calculations [8]. These twistor actions have now been applied to study a wide variety of physical observables in $\mathcal{N} = 4$ SYM, including scattering amplitudes, null polygonal Wilson loops, and correlation functions (c.f., [9] for a review).

Currently, there is no twistor action for Einstein gravity, although there is one for *conformal gravity* [5]. This is the conformally invariant theory of gravity whose Lagrangian is the square of the Weyl tensor. It has fourth-order equations of motion so its quantum theory is non-unitary and is widely believed not to be suitable for a physical theory. Nevertheless, conformal gravity has many interesting mathematical properties: for instance, it can be extended to supersymmetric theories for $\mathcal{N} \leq 4$, and the maximally supersymmetric theory ($\mathcal{N} = 4$) comes in several variants which are finite and power-counting renormalizable (c.f., [10] for a review). Furthermore, solutions to Einstein gravity form a subsector of solutions to the field equations of conformal gravity. Maldacena has shown that evaluated on a de Sitter background, the tree-level S-matrix for conformal gravity reduces to that for Einstein gravity when Einstein states are inserted [11]. Thus we can hope to study the Einstein tree-level S-matrix using the conformal gravity twistor action, which is the goal of this paper.

Progress on understanding the scattering amplitudes of (super-)gravity in twistor space–even at tree-level–was elusive until Hodges' discovery of a manifestly permutation invariant and compact formula for the maximal-helicity-violating (MHV) tree amplitude of Einstein gravity [12]. This led to the development of the Cachazo-Skinner expression for the entire tree-level S-matrix of $\mathcal{N} = 8$ supergravity in terms of an integral over holomorphic maps from the Riemann sphere into twistor space [13, 14]. Perhaps most exciting of all is Skinner's development of a new twistor-string theory which produces this formula as a worldsheet correlation function of vertex operators; in other words, a twistor-string theory for $\mathcal{N} = 8$ Einstein supergravity [15].

Parallel work sought to derive Einstein amplitudes from Witten and Berkovits' original twistor-string formula for conformal gravity amplitudes by restricting to Einstein states [16, 17] and appealing to the Maldacena argument to obtain Einstein amplitudes. Although the correct amplitudes are obtained at three points, the relationship between Einstein and conformal gravity amplitudes requires *minimal* conformal supergravity rather than the non-minimal version arising from the Berkovits-Witten twistor-string. Nevertheless, in [17] it was shown that the correct Hodges formula is obtained at n-points when a tree ansatz is imposed on the worldsheet correlation function required in the Berkovits-Witten twistor-string formula. Although there is no clear motivation for the tree ansatz within the Berkovits-Witten twistor-string,[1] it is natural in the context of the Maldacena argument applied to the twistor action for conformal gravity [5], which does give the minimal conformal gravity. Part of the purpose of this paper is to give a complete presentation of that argument. It also allows us to provide a generalization of the Hodges formula for the MHV amplitude to the case of non-vanishing cosmological constant which is the regime in which the Maldacena argument is most straightforwardly applicable.

In this paper, we study conformal (super-)gravity on twistor space using the aforementioned twistor action and its generalization to $\mathcal{N} = 4$ supersymmetry. By exploiting the conformal/Einstein gravity correspondence, we obtain a twistor formula for the MHV amplitude on a background with non-vanishing cosmological constant. We check that this formula is independent of gauge choices made during its derivation, and also that it produces Hodges' formula in the flat-space limit.

For a cosmological constant $\Lambda \neq 0$, the traditional definition of a scattering amplitude for asymptotically flat space-times no longer applies. When $\Lambda > 0$, one can still define mathematical quantities corresponding to scattering from past infinity to future infinity, but these are not *physical* observables because no observer has access to the whole space-time. These mathematical analogues of scattering amplitudes have become known as meta-observables [18]: the theory allows them to be computed, even if no single physical observer can ever measure them. Actual physical observables can still be given in terms of the in-in formalism, where the observer only integrates over the portion of space-time containing his or her history. When $\Lambda < 0$, this situation is improved and the natural objects to compute are correlation functions in the conformal field theory on the boundary via the AdS/CFT correspondence (although mathematically the integration regions are not so dissimilar and indeed the formulae will be polynomial in Λ so that the analytic continuation from positive to negative Λ will be trivial).

For the remainder of this paper, we will refer only to 'scattering amplitudes' in (anti-) de Sitter space, trusting the reader to keep the implicit subtleties in mind. In the end, we will obtain a formula on twistor space, which is written in terms of arbitrary external states and a freely specified contour of integration in complexified space-time (see (1.1) below); the ambiguity in defining *what* observable we are computing can be absorbed into this choice of contour. Furthermore, although we will focus on the case of $\Lambda > 0$ de Sitter space in this paper, most of our arguments (and certainly the final formula) apply to anti-de Sitter space with trivial changes of sign and can therefore be applied to the AdS/CFT correspondence.

We begin in Section 2 with an exposition of the conformal/Einstein gravity correspondence. This includes a brief overview of different action principles for conformal gravity, as well as the relationship with general relativity on an asymptotically de Sitter background. From this, we derive a precise version of the correspondence for generating functionals of MHV amplitudes. Since we will be interested in scattering amplitudes, we also discuss the relationship between polarization states for conformal and Einstein gravity. Finally, we

[1] In Skinner's $\mathcal{N} = 8$ twistor-string the tree ansatz can be understood as arising from cancellation of the loops due to worldsheet supersymmetry.

consider maximally supersymmetric $\mathcal{N} = 4$ conformal supergravity; as we shall see, the conformal/Einstein correspondence can be extended only to the *minimal* version of this theory.

In Section 3, we study the twistor action for minimal $\mathcal{N} = 4$ conformal supergravity. After a brief review of some relevant aspects of twistor theory, we recall the definition of the twistor action for $\mathcal{N} = 0$ conformal gravity, and argue that its straightforward generalization to $\mathcal{N} = 4$ produces the minimal supersymmetric theory. By applying the conformal/Einstein gravity correspondence, we show how this twistor action can be restricted to Einstein states, thereby leading to a twistorial expression for the generating functional of MHV amplitudes in Einstein (super-)gravity.

We derive the new formula for the scattering amplitude with $\Lambda \neq 0$ in Section 4. This entails developing a Feynman diagram calculus on $\mathbb{CP}^1$ to operationalize the perturbative expansion of the generating functional on twistor space. As we shall see, this leads to a tree formalism for computing the Einstein amplitude within minimal $\mathcal{N} = 4$ conformal supergravity. A closely related formalism was recently used to extract Hodges' formula from the *non-minimal* conformal supergravity of the Berkovits-Witten twistor-string [17]; hence, we confirm that the tree formalism isolates the minimal sector in the twistor-string (at MHV).

By applying the diagram calculus we are able to derive an expression for the MHV amplitude in the presence of a cosmological constant. Explicitly, we will show that

$$\mathcal{M}_{n,0} = \frac{1}{\Lambda}\int \frac{\mathrm{d}^{8|8}X}{\mathrm{vol\ GL}(2,\mathbb{C})}\left[(X^2)^2\left|\mathbb{H}^{12}_{12}\right| + \sum_{i,j,k,l}\omega^1_{ij}\omega^2_{kl}\left|\mathbb{H}^{12ijkl}_{12ijkl}\right|\right]\prod_{m=1}^{n} h(Z(\sigma_m))\,\mathrm{D}\sigma_m\,, \tag{1.1}$$

where X^I_A are coordinates on the moduli space of degree one holomorphic maps $Z(\sigma)$ from $\mathbb{CP}^1$ (with homogeneous coordinates σ^A) to twistor space; $h(Z(\sigma_i))$ are twistor representatives for the external states; $\mathbb{H}$ is the Hodges matrix[2]

$$\mathbb{H}_{ij} = \begin{cases} \frac{1}{(ij)}\left[\frac{\partial}{\partial Z(\sigma_i)}, \frac{\partial}{\partial Z(\sigma_i)}\right] \text{ if } i \neq j \\ -\sum_{k\neq i}\mathbb{H}_{ik}\frac{(\xi k)^2}{(\xi i)^2} \text{ if } i = j \end{cases},$$

and the quantities ω^1_{ij} are given by

$$\omega^1_{ij} = -\Lambda\frac{(1\xi)^4(ij)}{(1i)^2(1j)^2(\xi i)^2(\xi j)^2}\left[\frac{\partial}{\partial Z(\sigma_i)}, \frac{\partial}{\partial Z(\sigma_j)}\right].$$

The notation $|\mathbb{H}^{12}_{12}|$ indicates the determinant of $\mathbb{H}$ with the row and columns corresponding to $h(Z(\sigma_1))$ and $h(Z(\sigma_2))$ removed, and $\xi \in \mathbb{CP}^1$ is an arbitrary reference spinor. We prove that (1.1) is independent of the choice of ξ, and limits smoothly onto Hodges' formula when

[2] Here, and throughout the paper, we denote the $\mathrm{SL}(2,\mathbb{C})$-invariant inner product on $\mathbb{CP}^1$ coordinates by $(ij) = \epsilon_{AB}\sigma^A_i\sigma^B_j$. The notation $[\ ,]$ stands for a contraction with a skew bi-twistor I^{IJ} called the *infinity twistor* which is introduced in Section 3. Similarly, $\langle\ ,\rangle$ denotes a contraction with the inverse infinity twistor I_{IJ}.

$\Lambda \to 0$. We also show how it can be manipulated into a form which is highly suggestive of a twistor-string origin and is the natural generalization of Hodges' formula to $\Lambda \neq 0$.

Section 5 concludes with a discussion of interesting future directions following on from this work. Most enticing is the possibility that the twistor action studied here could be used to define a MHV formalism [7] for conformal gravity, and in turn Einstein gravity. Indeed, the twistor action approach for $\mathcal{N} = 4$ SYM is one way of deriving this formalism in the gauge theory setting [19, 20] and other techniques such as Risager recursion fail in the gravitational context [21, 22]. We also discuss the possibility of defining twistor actions for non-minimal $\mathcal{N} = 4$ conformal supergravity, as well as how the twistor formula (1.1) could be converted into a meaningful physical observable in de Sitter space.

It also worth noting that *a priori* one could hope to derive a formula for $\mathcal{M}_{n,0}$ from Skinner's $\mathcal{N} = 8$ twistor-string [15]. Unfortunately, although this twistor-string theory has been shown to give the correct tree-level amplitudes at $\Lambda = 0$, it has so far not been possible to make sense of the worldsheet correlations functions for the $\Lambda \neq 0$ regime. It is to be hoped that knowing the answer (1.1) will also allow us to understand how to make the Skinner twistor-string work for $\Lambda \neq 0$.

Notation

Throughout this paper, we use the following index conventions: space-time tensor indices are Greek letters from the middle of the alphabet ($\mu, \nu = 0, \cdots, 3$); positive and negative chirality Weyl spinor indices are primed and un-primed capital Roman letters respectively ($A, B = 0, 1$ or $A', B' = 0', 1'$); R-symmetry indices are lower-case Roman indices from the beginning of the alphabet ($a, b = 1, \ldots, \mathcal{N}$). We will also use bosonic twistor indices, denoted by Greek letters from the beginning of the alphabet (α, β), as well as supersymmetric twistor indices, denoted by capital Roman letters from the middle of the alphabet (I, J).

We denote the space of smooth n-forms on a manifold M by Ω^n_M; in the presence of a complex structure we denote the space of smooth (p, q) forms by $\Omega^{p,q}_M$. If we want to consider these spaces twisted by some sheaf V, then we write $\Omega^n_M(V)$ for 'the space of smooth n-forms on M with values in V,' and so forth. Dolbeault cohomology groups on M with values in V are denoted by $H^{p,q}(M, V)$. The complex line bundles $\mathcal{O}(k)$ denote the bundles of functions homogeneous of weight k on a (projective) manifold, and we make use of the abbreviation $\Omega^n_M(\mathcal{O}(k)) \equiv \Omega^n_M(k)$, and so on.

2 Einstein and Conformal Gravity on Asymptotically de Sitter Spaces

We will work on a 4-dimensional space-time M with metric g. Conformal gravity is the theory obtained from the action

$$S^{\mathrm{CG}}[g] = \frac{1}{\varepsilon^2} \int_M \mathrm{d}\mu \, C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} = \frac{1}{\varepsilon^2} \int_M \mathrm{d}\mu \left(\Psi^{ABCD} \Psi_{ABCD} + \widetilde{\Psi}^{A'B'C'D'} \widetilde{\Psi}_{A'B'C'D'} \right), \tag{2.1}$$

where ε^2 is a dimensionless coupling constant, $\mathrm{d}\mu = \mathrm{d}^4 x \sqrt{g}$ is the volume element, $C_{\mu\nu\rho\sigma}$ is the Weyl curvature tensor of g, and Ψ_{ABCD}, $\widetilde{\Psi}_{A'B'C'D'}$ are the anti-self-dual (ASD) and self-dual (SD) Weyl spinors respectively [23]. This theory is conformally invariant and hence only depends upon (and constrains) the conformal structure $[g]$ underlying g. The field equations are the vanishing of the Bach tensor, $B_{\mu\nu}$, which can be written in a variety of different forms thanks to the Bianchi identities:

$$
\begin{aligned}
B_{\mu\nu} &= 2\nabla^{\rho}\nabla^{\sigma}C_{\rho\mu\nu\sigma} + C_{\rho\mu\nu\sigma}R^{\rho\sigma} \\
&= \left(2\nabla_{\rho}\nabla_{(\mu}R^{\rho}_{\nu)} - \Box R_{\mu\nu} - \frac{2}{3}\nabla_{\mu}\nabla_{\nu}R - 2R_{\rho\mu}R^{\rho}_{\nu} + \frac{2}{3}R_{\mu\nu}R\right)_0 \\
&= 2(\nabla^{C}_{A'}\nabla^{D}_{B'} + \Phi^{CD}_{A'B'})\Psi_{ABCD} = 2(\nabla^{C'}_{A}\nabla^{D'}_{B} + \Phi^{C'D'}_{AB})\widetilde{\Psi}_{A'B'C'D'},
\end{aligned} \tag{2.2}
$$

where the subscript in the second line denotes 'trace-free part.' These imply that the field equations are satisfied whenever M is conformal to Einstein (i.e., $g_{\mu\nu} \propto R_{\mu\nu}$), or when its Weyl curvature is either self-dual or anti-self-dual.

The twistor actions for gauge theory start in space-time with the Yang-Mills Lagrangian in a format which explicitly gives a perturbative expansion around the self-dual sector. This is accomplished by using a 'BF,' or Chalmers-Siegel, action functional (c.f., [24, 25]). The field equations of conformal gravity can be understood as the Yang-Mills equations of the Cartan conformal connection (also known as the local twistor connection) on a $\mathrm{SU}(2,2)$ (or in the complex, $\mathrm{PSL}(4,\mathbb{C})$) bundle [26], so it is natural to expect analogous actions to exist for conformal gravity.

First, note that we can use a complex chiral action

$$
S^{\mathrm{CG}}[g] = \frac{2}{\varepsilon^2}\int_M \mathrm{d}\mu\; \Psi^{ABCD}\Psi_{ABCD}, \tag{2.3}
$$

which differs from (2.1) by

$$
\frac{1}{\varepsilon^2}\int_M \mathrm{d}\mu\left(\Psi^{ABCD}\Psi_{ABCD} - \widetilde{\Psi}^{A'B'C'D'}\widetilde{\Psi}_{A'B'C'D'}\right).
$$

This is a topological term, equal to $\frac{12\pi^2}{\varepsilon^2}(\tau(M) - \eta(\partial M))$, where $\tau(M)$ is the signature of M and $\eta(\partial M)$ is the η-invariant of the conformal boundary [27]. Hence, (2.3) is equivalent to the full action (2.1) up to terms which are irrelevant in perturbation theory.

To expand around the SD sector, we introduce the totally symmetric spinor field G_{ABCD} as a Lagrange multiplier, and write the action as [4]:

$$
S^{\mathrm{CG}}[g, G] = \int_M \mathrm{d}\mu\left(G^{ABCD}\Psi_{ABCD} - \varepsilon^2 G^{ABCD}G_{ABCD}\right). \tag{2.4}
$$

This has field equations [5]

$$
\Psi^{ABCD} = \varepsilon^2 G^{ABCD}, \qquad \left(\nabla^{C}_{A'}\nabla^{D}_{B'} + \Phi^{CD}_{A'B'}\right)G_{ABCD} = 0, \tag{2.5}
$$

so integrating out G returns (2.3). But now ε^2 becomes a parameter for expanding about the SD sector: when $\varepsilon = 0$, the field equations yield a SD solution and G_{ABCD} is a linear ASD solution propagating on the SD background.

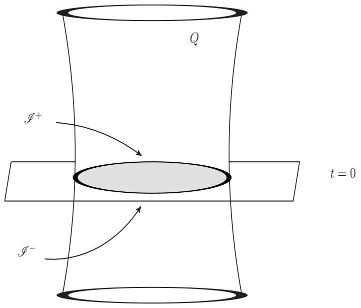

Figure 1. *De Sitter space as the quadric $Q \subset \mathbb{RP}^5$ and the identification of infinity.*

We now review the geometry of de Sitter space and the relationship between conformal gravity and Einstein gravity. We pay particular attention to how this relationship is manifested at the level of generating functionals for scattering amplitudes following an argument due to Maldacena [11]. Similar ideas hold for anti-de Sitter space with some sign changes and in that form these ideas can be applied to AdS/CFT duality.

2.1 The conformal geometry of de Sitter space

De Sitter, anti-de Sitter, and flat space-times in n-dimensions possess only scalar curvature and are hence conformally flat. Each is a dense open subset in the conformal compactification which is a projective quadric of signature $(2, n)$ in $\mathbb{RP}^{n+1}$ of topology $S^1 \times S^{n-1}/\mathbb{Z}_2$. The infinite points are respectively a space-like, time-like or null hypersurface (in fact a lightcone) in the conformal compactification obtained as the intersection of a hyperplane of appropriate signature in $\mathbb{RP}^{n+1}$. In four dimensions, de Sitter space (dS_4) is topologically $\mathbb{R} \times S^3$, and can be realized as the pseudosphere in $\mathbb{R}^{1,4}$ with coordinates (w, x^μ), $\mu = 0, \ldots, 3$ via the embedding [28]:

$$\eta_{\mu\nu} x^\mu x^\nu - w^2 = x^2 - w^2 = -\frac{3}{\Lambda}, \qquad \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)\,.$$

This makes manifest the isometry group $\mathrm{SO}(1, 4)$, the Lorentz group inherited from the embedding space.

The embedding as a projective quadric in $\mathbb{RP}^5$ can be realized with homogeneous coordinates (t, w, x^μ) as the $t \neq 0$ portion of:

$$2Q \equiv t^2 - w^2 + x^2 = 0,$$

with scale-invariant metric

$$\mathrm{d}s^2 = \frac{3}{\Lambda} \frac{\mathrm{d}t^2 - \mathrm{d}w^2 + \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu}{t^2}. \tag{2.6}$$

The intersection of Q with the plane $t = 0$ corresponds to the spatial S^3 at infinity, and is the identification of the past ($\mathscr{I}^-$) and future ($\mathscr{I}^+$) infinities (ordinarily, we will not make this identification); see Figure 1. The description of dS_4 as the pseudosphere in $\mathbb{R}^{1,4}$ is recovered by taking the patch $t = \sqrt{3/\Lambda}$.

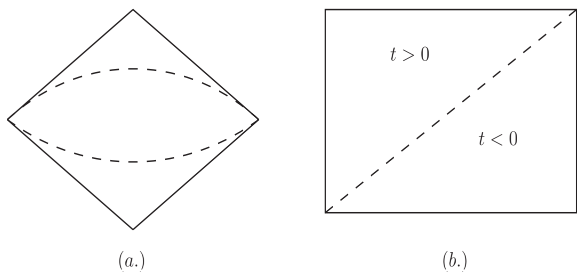

Figure 2. *De Sitter space on the affine Minkowski slicing* $(a.)$, *and the traditional Poincaré slicing* $(b.)$

There are two useful coordinate patches: the *affine* and *Poincaré* patches. The distinction between the two corresponds to choosing the point defining the light cone at infinity for some affine coordinates to be at a finite point of de Sitter space or at infinity, respectively. The former case corresponds to $t + w = 1$; after re-scaling the affine Minkowski coordinates x^μ the metric becomes

$$\mathrm{d}s^2 = \frac{\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}{(1 - \Lambda x^2)^2}. \tag{2.7}$$

Most of de Sitter infinity is then located at finite points in the affine space where $x^2 = \Lambda^{-1}$, although this obviously has an S^2 intersection with the affine (Minkowski) infinity. This has a straightforward $\Lambda \to 0$ limit whereupon (2.7) becomes the Minkowski metric (see Figure 2, $(a.)$).

The Poincaré patch, which is more familiar in the physics literature, corresponds to $x^0 + w = 1$, with metric:

$$\mathrm{d}s^2 = \frac{3}{\Lambda}\frac{\mathrm{d}t^2 - \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j}{t^2}. \tag{2.8}$$

The $t = 0$ slice is infinity minus a point whose light cone divides de Sitter space into two halves ($t > 0$ and $t < 0$), demonstrating that a physical observer at $\mathscr{I}^\pm$ has access to at most half of the space-time. The Poincaré patch manifests the three-dimensional rotation and translation symmetries of dS_4, but is not so well-behaved in the $\Lambda \to 0$ limit; see Figure 2, $(b.)$.

2.2 Einstein gravity amplitudes inside the conformal gravity S-matrix

We have seen from the definition of the Bach tensor that solutions to the Einstein gravity field equations are also solutions to those of conformal gravity. However, in order to show that Einstein tree amplitudes can be obtained from those of conformal gravity we need to relate the actions of the two theories. That is because we can define the tree-level S-matrix (or at least its phase) to be the value of the Einstein action evaluated on a classical solution to the Einstein equations that has been obtained perturbatively from the given fields involved in the scattering process. More formally, given n solutions g_i, $i = 1, \ldots, n$ to the linearized field equations and a classical background g^{cl}, we construct the solution g to the field equations whose asymptotic data is $\sum_i \epsilon_i g_i$. We can then–at least formally–define the amplitude to be

$$\mathcal{M}(1,\ldots,n) = \text{ coefficient of } \prod_{i=1}^{n} \epsilon_i \text{ in } S[g^{\text{cl}} + g]\,.$$

Thus, if the conformal gravity action of a solution to the Einstein equations yields the Einstein-Hilbert action of that same solution, then the tree-level conformal gravity S-matrix can be used to compute that for general relativity. We will see that this is the case up to a factor of Λ.

The Einstein-Hilbert action in the presence of a cosmological constant is

$$S^{\text{EH}}[g] = \frac{1}{\kappa^2}\int_M \mathrm{d}\mu(R - 2\Lambda),$$

where $\kappa^2 = 16\pi G_N$. On a de Sitter space, the field equations are $R_{\mu\nu} = \Lambda g_{\mu\nu}$, so the action reads

$$S^{\text{EH}}[dS_4] = \frac{2\Lambda}{\kappa^2}\int_{dS_4} \mathrm{d}\mu = \frac{2\Lambda}{\kappa^2} V(dS_4),$$

where $V(M)$ is the volume of M. For any asymptotically de Sitter manifold, this volume will be infinite so the action functional must be modified by the Gibbons-Hawking boundary term [29]. Additionally, we must include the holographic renormalization counter-terms (which also live on the boundary) in order to render the volume finite [30, 31]. After including these additions, one obtains the *renormalized* Einstein-Hilbert action [32], and if M is asymptotically de Sitter, we have:

$$S^{\text{EH}}_{\text{ren}}[M] = \frac{2\Lambda}{\kappa^2} V_{\text{ren}}(M), \tag{2.9}$$

where V_{ren} is the renormalized volume of the space-time (c.f., [33]).

On the other hand, if M was a Riemannian 4-manifold which was compact without boundary, the Chern-Gauss-Bonnet formula tells us that

$$\chi(M) = \frac{1}{8\pi^2}\int_M \mathrm{d}\mu \left(C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} - \frac{1}{2} R_{\mu\nu} R^{\mu\nu} + \frac{1}{6} R^2 \right).$$

If M were additionally Einstein ($R_{\mu\nu} = \Lambda g_{\mu\nu}$), then we would have

$$S^{\text{CG}}[M] = \frac{8\pi^2 \chi(M)}{\varepsilon^2} - \frac{2\Lambda^2}{3\varepsilon^2} V(M). \tag{2.10}$$

When M is (Lorentzian) asymptotically de Sitter, the Chern-Gauss-Bonnet formula requires a boundary term, and the volume is renormalized. However, a theorem[3] of Anderson tells us that (2.10) continues to hold even after boundary terms for the Euler characteristic are taken into account and the volume has been renormalized [34]. Furthermore, since M

[3]Note that Anderson's theorem is actually stated for asymptotically hyperbolic Riemannian four-manifolds; the extension to asymptotically de Sitter Lorentzian manifolds follows by analytic continuation.

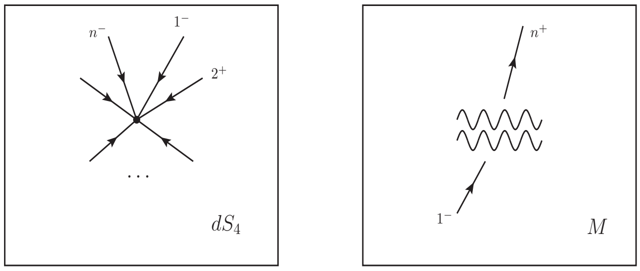

Figure 3. *Geometric picture of MHV graviton scattering*

is asymptotically de Sitter we can assume that we always perturb around the topologically trivial case (i.e., $\chi(M) = 0$), so comparing with (2.9) we find

$$S^{\mathrm{CG}}[M] = -\frac{\Lambda\,\kappa^2}{3\,\varepsilon^2} S^{\mathrm{EH}}_{\mathrm{ren}}[M]. \tag{2.11}$$

To relate the scattering amplitudes of the two theories, we need a way to single out Einstein scattering states inside conformal gravity. Maldacena has shown that this can be accomplished by employing boundary conditions on the metric [11]. We will use an equivalent explicit formula in twistor space to compute the tree-level scattering amplitudes of general relativity by using conformal gravity restricted to Einstein scattering states on a de Sitter background. We will refer to this as the *conformal/Einstein gravity correspondence.*

2.3 The MHV amplitude

We will focus on the tree-level amplitudes corresponding to the scattering of two negative helicity gravitons and $n-2$ positive helicity gravitons, the *MHV amplitudes* of general relativity. These are maximal because the positive and negative helicity states are dual to each other so that an 'all +' amplitude would correspond to a positive helicity particle picking up some negative helicity scattering on a positive helicity background. But this cannot happen by virtue of the consistency of the self-duality equations for general relativity. Similarly, the one minus and rest plus amplitude vanishes because the self-dual sector is integrable (it would correspond to the nontrivial scattering of a linear positive helicity particle on a positive helicity background). See Appendix A, lemma A.1 for more details.

Following [35], we absorb the $n-2$ SD gravitons of the MHV amplitude into a fully nonlinear SD background space-time M, which can subsequently be perturbatively expanded to recover the individual particle content. Reversing the momentum of one of the two negative helicity gravitons, the MHV amplitude is the probability for a pure ASD state at $\mathscr{I}^-$ to propagate across M and evolve into a SD state at $\mathscr{I}^+$ as illustrated in Figure 3.

In Appendix A, we derive the generating functional for these amplitudes in Einstein gravity by working with the chiral formalism in proposition A.1. We will denote this by I^{GR}, and its exact form can be found in (A.16). While we don't have a good off-shell expression and perturbation scheme for this Einstein generating function in twistor space, we do for the case of conformal gravity. The main point of this paper is that a perturbative expansion leading to the MHV amplitudes can be achieved by applying the conformal/Einstein gravity correspondence.

The generating functional for MHV amplitudes in *conformal gravity* is given by the second term in (2.4). The first term is precisely the action for the self-dual sector, so the second term is therefore the action for the first nontrivial deformation of the SD sector that is quadratic in the ASD part of the field. Evaluated on-shell with Einstein scattering states, the two ASD gravitons are given by Weyl spinor perturbations ψ_1, ψ_2 and the generating functional reads:

$$I^{\mathrm{CG}}[1^-, 2^-; M^+]\big|_{\mathrm{Ein}} = \frac{2i}{\varepsilon^2} \int_M \mathrm{d}\mu\, \psi_1^{ABCD} \psi_{2\,ABCD}, \tag{2.12}$$

where M is again the SD background which encodes the $n-2$ remaining gravitons. In proposition A.2, we prove that this is related to I^{GR} by

$$I^{\mathrm{GR}}[1^-, 2^-; M^+] = -\frac{3\varepsilon^2}{\Lambda\, \kappa^2}\, I^{\mathrm{CG}}[1^-, 2^-; M^+]\big|_{\mathrm{Ein}}\,, \tag{2.13}$$

in precise accordance with (2.11).[4] Note that although this correspondence appears to degenerate for $\Lambda \to 0$, the n-particle conformal gravity amplitude is a polynomial of degree $n-1$ in Λ with no $O(\Lambda^0)$ coefficient [17, 37]. This makes it possible to extract the flat-space amplitude for general relativity from (2.13) as well.

We shall see that I^{CG} has a very natural expression on twistor space which allows us to perform a perturbative expansion of the background M in terms of a diagram calculus on $\mathbb{CP}^1$. This will enable us to derive a twistorial expression for the MHV amplitudes with a cosmological constant in Section 4. Before proceeding we first discuss polarization states and the extension of the conformal/Einstein gravity correspondence to supersymmetric versions of conformal gravity.

2.4 Relations between Einstein and conformal gravity polarization states

The usual strategy for calculating scattering amplitudes is to express them in terms of a basis of momentum polarization states. We will in fact use a variety of different representations; however, we need some understanding of the relationship between linearized solutions to the Bach equations (2.2), spin-two fields ,and linearized Einstein solutions. Polarization states for conformal gravity were studied in [4, 38] and were found to contain twice as many states as for Einstein gravity.[5] We use a slightly different formulation here that allows us to retain Lorentz invariance (although not translation invariance), and will also tie in with our focus on de Sitter gravity.

[4] As discussed in the introduction, the 'scattering amplitudes' produced by this generating functional do not actually constitute *physical* observables, since the measurement is performed by integrating over all of $\mathscr{I}^+$. This is a space-like hypersurface, so no physical observer can perform this measurement. Hence, (2.13) generates a 'meta-observable' in the sense of the dS/CFT correspondence [18, 36], but limits nicely to the asymptotically flat definition of a scattering amplitude as $\Lambda \to 0$. We discuss how one might obtain physical observables in Section 5.

[5] On twistor space we will see three times as many conformal gravity states as for Einstein gravity and conceivably one has simply been missed in earlier treatments, but this will not materially alter our discussion.

Let $\{\psi_{ABCD}, \tilde{\psi}_{A'B'C'D'}\}$ be linearized spin-two fields and $\{\Psi_{ABCD}, \tilde{\Psi}_{A'B'C'D'}\}$ be the ASD and SD portions of the Weyl tensor. The key point in connecting conformal gravity to spin-two fields is that the Weyl tensor has conformal weight zero, whereas a linearized spin-two field has conformal weight -1 (c.f., [39]). Both fields satisfy

$$\nabla_A^{A'}\tilde{\Psi}_{A'B'C'D'} = \nabla_A^{A'}\tilde{\psi}_{A'B'C'D'} = 0 = \nabla_{A'}^{A}\Psi_{ABCD} = \nabla_{A'}^{A}\psi_{ABCD}\,, \tag{2.14}$$

in the Einstein conformal frame but the Weyl tensor only does so in its given Einstein conformal scale and no other. Einstein conformal scales can be specified as functions Ω of conformal weight $+1$ that satisfy the conformally invariant equation

$$(\nabla_\mu\nabla_\nu + \Phi_{\mu\nu})_0\Omega = 0, \tag{2.15}$$

where the subscript 0 denotes 'the trace-free part' and $\Phi_{\mu\nu}$ is half the trace-free part of the Ricci tensor.

In flat space, (2.15) has the general solution

$$\Omega = a + b_\mu x^\mu + cx^2\,. \tag{2.16}$$

It is clear in general that given such a solution Ω, rescaling so that $\Omega = 1$ gives a metric satisfying $\Phi_{\mu\nu} = 0$ from (2.15). This is the Einstein condition, and the solutions (2.16) give metrics with cosmological constant $\Lambda = 3(b_\mu b^\mu - ac)$. Upon setting

$$\Psi_{ABCD} = \Omega\psi_{ABCD}, \tag{2.17}$$

we see that the Weyl spinor Ψ_{ABCD} has conformal weight zero and satisfies the linearized vacuum Bianchi identity (2.14) for the conformal scale in which $\Omega = 1$. Since this is an Einstein scale and the Bach equations are simply another derivative of this equation, Ψ_{ABCD} so defined also satisfies the linearized Bach equations. But then, by conformal invariance of the Bach equations, it does so in any conformal scale.

We will not use momentum eigenstates much in what follows, but include the following in order to make contact with standard calculations. Standard momentum eigenstates for spin-two fields with 4-momentum $k_{AA'} = p_A\tilde{p}_{A'}$ are given by

$$\psi_{ABCD} = p_Ap_Bp_Cp_D\mathrm{e}^{ik\cdot x}\,,$$

where the polarization information is contained in the choice of scale of p_A.

As conformal gravity has fourth-order equations of motion, we need more polarization states and as mentioned above, it is usually thought that twice as many suffice [4, 38] although we will present three here to line up with the counting from twistor space. The first two arise from (2.17) as the pair

$$\Psi_{ABCD} = p_Ap_Bp_Cp_D\mathrm{e}^{ik\cdot x}\,, \qquad \Psi'_{ABCD} = x^2p_Ap_Bp_Cp_D\mathrm{e}^{ik\cdot x}\,, \tag{2.18}$$

and similarly for $\tilde{\Psi}$, $\tilde{\Psi}'$. This framework can also be used to characterize Einstein polarization states inside conformal gravity. In particular, on the affine patch of de Sitter space given by (2.7), we will have Einstein states

$$\Psi^\Lambda_{ABCD} = (1 - \Lambda x^2)p_Ap_Bp_Cp_D\mathrm{e}^{ik\cdot x}\,. \tag{2.19}$$

If instead we work on the Poincaré patch of de Sitter space (2.8), we would use:

$$\Psi^{\Lambda}_{ABCD} = \sqrt{\frac{\Lambda}{3}}\, t\, p_A p_B p_C p_D \mathrm{e}^{ik\cdot x}\,. \tag{2.20}$$

Clearly we can characterize the Einstein polarization state (2.19) as the linear combination of the conformal gravity polarization states (2.18) that vanishes at the hypersurface $\mathscr{I}_{\Lambda} = \{x|(1-\Lambda x^2)=0\}$ of de Sitter space.

Another linearized conformal gravity solution that is missed by the above is

$$\Psi_{ABCD} = \alpha_{(A} p_B p_C p_{D)} \mathrm{e}^{ik\cdot x}\,, \tag{2.21}$$

where α_A is an arbitrary constant spinor. The general solution for the spin two equation can be expressed by Fourier transform as

$$\psi_{ABCD}(x) = \int \mathrm{d}^4k\; \delta(k^2)\; \psi(k)\; p_A p_B p_C p_D \mathrm{e}^{ik\cdot x}\,.$$

Similarly, the general solution to the linearized Bach equations can be expressed as

$$\Psi_{ABCD}(x) = \int \mathrm{d}^4k\; \left(\delta(k^2)\Psi_0(k)_{(A} + p_{(A}\delta'(k^2)\Psi_1(k)\right)\; p_B p_C p_{D)} \mathrm{e}^{ik\cdot x}\,. \tag{2.22}$$

This can be seen by taking the Fourier transform of the Bach equations to yield

$$k^{AA'}k^{BB'}\Psi_{ABCD}(k) = 0.$$

Multiplying by k twice more we discover that $(k^2)^2\Psi_{ABCD}(k) = 0$ so that

$$\Psi_{ABCD}(k) = \Psi_{0\;ABCD}\;\delta(k^2) + \Psi_{1\;ABCD}\;\delta'(k^2).$$

Introduce $p_A = k_{AA'}o^{A'}$ so that $k^{AA'}p_A = \frac{k^2}{2}o^{A'}$. Then it is straightforward to see that the field equations are satisfied by (2.22) and that this is the general solution. Integrating by parts in (2.22), we can eliminate $\delta'(k^2)$ in favour of $\delta(k^2)$ but will then pick up explicit dependence on x^μ making contact with the polarization states (2.18).

2.5 Minimal and non-minimal conformal super-gravity

It is natural to ask if the classical correspondence between conformal and Einstein gravity persists in the presence of supersymmetry. Analogues of conformal gravity with extended supersymmetry were first constructed in [40], and it is believed that these theories are well-defined for $\mathcal{N} \leq 4$ (c.f., [41, 42]). In this paper, we will be concerned primarily with $\mathcal{N} = 4$ conformal supergravities (CSGs), since this is the degree of supersymmetry that arises most naturally in twistor theory. This $\mathcal{N} = 4$ CSG comes in two basic phenotypes: *minimal* and *non-minimal* based upon the presence of a certain global symmetry. The non-minimal type depends essentially on a free function of one variable. Einstein supergravity embeds into minimal CSG, but *not* into the non-minimal models.

The field content of $\mathcal{N} = 4$ CSG consists of the spin-2 conformal gravitons along with bosonic fields $V_{\mu\,b}^{a}$, anti-self-dual tensors $T^{ab}_{\mu\nu}$, scalars $\{E_{ab}, D^{ab}_{cd}, \varphi\}$ and fermions

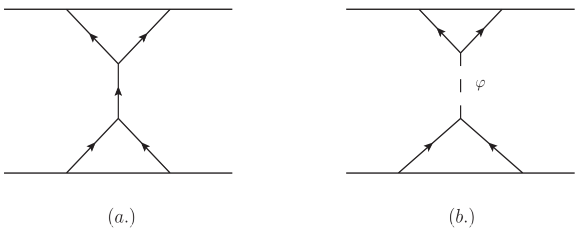

Figure 4. *In minimal $\mathcal{N} = 4$ CSG, external gravitons only couple to other gravitons in the bulk $(a.)$; in the non-minimal model they can couple to the scalar φ $(b.)$.*

$\{\psi^a_\mu, \chi^a_{bc}, \lambda_a\}$, where $a = 1, \ldots, 4$ is a SU(4) R-symmetry index. *Minimal* $\mathcal{N} = 4$ CSG is characterized by a global SU(1,1) symmetry acting non-linearly on the complex scalar φ (essentially the action of SU(1,1) on the upper-half plane) [40]. This relates to the presence of $\mathcal{N} = 4$ Poincaré supergravity sitting inside the CSG [43]. The minimal model also has a degenerate limit where SU(1,1) is replaced by a linear $\mathbb{E}_2$ (the Euclidean symmetries of the plane) action; once again this has an analogue in $\mathcal{N} = 4$ Einstein supergravity, and also arises in coupling $\mathcal{N} = 1$ supergravity to a scalar multiplet [43–45].

A general conformally invariant theory of gravity has a Lagrangian of the form

$$\mathcal{L} = f(\varphi)\Psi^2 + \varphi\Box^2\bar{\varphi} + c.c. + \ldots ,$$

where we just give two indicative terms of a rather extended Lagrangian. Because the field φ has conformal weight zero, we are allowed an arbitrary function of φ as a coefficient of the self-dual Weyl tensor squared Ψ^2. In a superfield formalism, it can bee seen that this will have a supersymmetric extension for arbitrary analytic f.

In the minimal $\mathcal{N} = 4$ case, the aforementioned SU(1,1) symmetry leads to a unique $\mathcal{N} = 4$ CSG Lagrangian. It follows from symmetry under the U(1) subgroup of SU(1,1) that we must have $f \equiv 1$, giving the Lagrangian:

$$\mathcal{L}^{\min} = C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + \varphi\Box^2\bar{\varphi} + \cdots .$$

Einstein supergravities at $\mathcal{N} = 4$ can be constructed from minimal CSG [46] and so restricting to Einstein scattering states, Maldacena's argument should still apply and we can extract the tree-level Einstein gravity scattering amplitudes (see Figure 4 (a)).

Without the global SU(1,1) symmetry, we can have an arbitrary $f(\varphi)$ and there are couplings between the complex scalar φ and the Weyl curvature. Such $\mathcal{N} = 4$ CSG theories are referred to as *non-minimal*, and were first conjectured to exist in [10, 47]. If $f' \neq 0$, the Weyl tensor will provide a source for the scalar field, and so even if it vanishes asymptotically it will become nontrivial in the interior. Since the scalar will then provide a source for the Weyl curvature, Einstein gravity will not be a subset of this theory and there will in general be no embedding of Einstein solutions into non-minimal CSG.

At the level of scattering amplitudes, conformal graviton scattering states in the non-minimal theory can interact with the scalar in the bulk via three-point vertices of the form $\varphi(\text{Weyl})^2$. This means that a tree-level scattering amplitude for conformal gravitons will include Feynman diagrams for which there is no analogue in Einstein supergravity, as illustrated in Figure 4 (b). Without a consistent algorithm for subtracting these diagrams, Maldacena's argument *can not* be applied to non-minimal CSG. The theory arising from the Berkovits-Witten twistor string is understood to be an example of non-minimal CSG, with $f(\varphi) = \mathrm{e}^{\varphi}$ [4]. And indeed, spurious amplitudes related to the non-minimal coupling between conformal gravitons and scalars were found explicitly in [16, 48].

While there is some doubt over whether non-minimal CSG is well-defined at the quantum level [49, 50], minimal conformal gravity maintains some independent interest. It has been shown that minimal $\mathcal{N} = 4$ CSG interacting with a $\mathrm{SU}(2) \times \mathrm{U}(1)$ $\mathcal{N} = 4$ SYM theory is ultraviolet finite and power-counting renormalizable [10, 51]. This theory can be obtained as a gauge theory of the superconformal group $\mathrm{SU}(2, 2|4)$. A weaker version of the minimal Lagrangian can also be obtained by coupling abelian $\mathcal{N} = 4$ SYM to a $\mathcal{N} = 4$ CSG background [46, 52] and extracting the UV divergent portion of the partition function [50, 53]. The theory has even been proposed as a basic model for quantum gravity (c.f., [38, 54]).

3 Twistor Action for Conformal (Super-)Gravity

In this section, we show how $\mathcal{N} = 4$ CSG can be formulated in terms of a classical action functional on twistor space. After first recalling some background material on twistor spaces for curved space-times, we define the twistor action for $\mathcal{N} = 0$ conformal gravity [5] and then consider its natural extension to $\mathcal{N} = 4$ supersymmetry.

3.1 Curved twistor theory

In flat Minkowski space $\mathbb{M}$, twistor space $\mathbb{PT}$ is an open subset of $\mathbb{CP}^3$, with homogeneous coordinates $Z^{\alpha} = (\lambda_A, \mu^{A'})$. The standard flat-space incidence relations

$$\mu^{A'} = i x^{AA'} \lambda_A,$$

represent a point $x \in \mathbb{M}$ by a linearly embedded $\mathbb{CP}^1 \subset \mathbb{PT}$. To study conformal gravity and the MHV generating functional (2.13), we need twistor theory adapted to curved space-times such as the self-dual background with cosmological constant, M.

The *non-linear graviton* construction is the basis for curved twistor theory. We state the theorem in the context of $\mathcal{N} = 0$, but its extension to the $\mathcal{N} = 4$ context is straightforward.

Theorem 1 (Penrose [55], Ward [56]) *There is a one-to-one correspondence between:* (a.) *Space-times M with self-dual conformal structure $[g]$, and* (b.) *twistor spaces $\mathbb{P}\mathscr{T}$ (a complex projective 3-manifold) obtained as a complex deformation of $\mathbb{PT}$ and containing at least one rational curve X_0 with normal bundle $N_{X_0} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$. Define the complex line bundle $\mathcal{O}(1) \to \mathbb{P}\mathscr{T}$ so that $\Omega^3_{\mathbb{P}\mathscr{T}} \cong \mathcal{O}(-4)$ (the appropriate 4^{th} root exists on the neighbourhood of X_0 from the previous assumption).*

There is a metric $g \in [g]$ with Ricci curvature $R_{\mu\nu} = \Lambda g_{\mu\nu}$ if and only if $\mathbb{P}\mathscr{T}$ is equipped with:

- *a non-degenerate holomorphic contact structure specified by* $\tau \in \Omega^{1,0}_{\mathbb{P}\mathscr{T}}(2)$*, and*
- *a holomorphic 3-form* $\mathrm{D}^3Z \in \Omega^{3,0}_{\mathbb{P}\mathscr{T}}(4)$ *obeying* $\tau \wedge \mathrm{d}\tau = \frac{\Lambda}{3}\mathrm{D}^3Z$.

Here D^3Z *is the tautologically defined section of* $\Omega^3_{\mathbb{P}\mathscr{T}}(4)$.

We define the non-projective twistor space $\mathscr{T}$ *to be the total space of the complex line bundle* $\mathcal{O}(-1)$.

Thus, points $x \in M$ (for M obeying the conditions of this theorem) correspond to rational, but no longer necessarily linearly embedded, curves $X \subset \mathbb{P}\mathscr{T}$ of degree 1. The conformal structure on M corresponds to requiring that if two of these curves X, Y intersect in $\mathbb{P}\mathscr{T}$, then the points $x, y \in M$ are null separated. Furthermore, $\mathbb{P}\mathscr{T}$ can be reconstructed from M as the space of totally null self-dual 2-planes in the complexification of M (c.f., [55, 57]).

Later we will take the self-dual manifold M to correspond to the background of our MHV generating functional (2.12), encoding the $n-2$ positive helicity gravitons of the n-particle MHV amplitude and we will want to be very explicit about the presentation of the data and details of the construction. Theorem 1 tells us that M corresponds to a curved twistor space $\mathbb{P}\mathscr{T}$ which arises as a complex deformation of $\mathbb{PT}$. We will take M to be a finite but small perturbation away from flat space, so the deformed complex structure on $\mathbb{P}\mathscr{T}$ will be expressed as a small but finite deformation of the flat $\bar\partial$-operator:

$$\bar\partial_f = \bar\partial + f = \mathrm{d}\bar Z^{\bar\alpha}\frac{\partial}{\partial \bar Z^{\bar\alpha}} + f,$$

where $f \in \Omega^{0,1}_{\mathbb{PT}}(T_{\mathbb{PT}})$ and Z^α are homogeneous coordinates on $\mathbb{P}\mathscr{T}$. This induces a basis for $T^{0,1}_{\mathbb{P}\mathscr{T}}$ and $\Omega^{1,0}_{\mathbb{P}\mathscr{T}}$ with respect to the deformed complex structure:

$$T^{0,1}_{\mathbb{P}\mathscr{T}} = \mathrm{span}\left\{\frac{\partial}{\partial \bar Z^{\bar\alpha}} + f^\alpha_{\bar\alpha}\frac{\partial}{\partial Z^\alpha}\right\}, \tag{3.1}$$

$$\Omega^{1,0}_{\mathbb{P}\mathscr{T}} = \mathrm{span}\{\mathrm{D}Z^\alpha\} = \mathrm{span}\left\{\mathrm{d}Z^\alpha - f^\alpha\right\}, \tag{3.2}$$

where we have denoted $f = f^\alpha\partial_\alpha = f^\alpha_{\bar\alpha}\mathrm{d}\bar Z^{\bar\alpha}\partial_\alpha$. The forms f^α must descend from $\mathscr{T}$ to $\mathbb{P}\mathscr{T}$ which follows from

$$\bar Z^{\bar\alpha} f^\beta_{\bar\alpha} = 0\,, \qquad f^\alpha(\lambda Z) = \lambda f^\alpha(Z)\,, \quad \lambda \in \mathbb{C}^*. \tag{3.3}$$

Additionally, the vector field f on $\mathscr{T}$ is determined by one on $\mathbb{P}\mathscr{T}$ only up to multiples of the Euler vector field $Z^\alpha\partial_\alpha$, and this freedom can be fixed by imposing

$$\partial_\alpha f^\alpha = 0\,. \tag{3.4}$$

As it stands, $\bar\partial_f$ defines an almost complex structure. This is integrable if and only if

$$\bar\partial_f^2 = \bar\partial f^\alpha + [f,f]^\alpha = 0, \qquad [f,f]^\alpha = f^\beta \wedge \partial_\beta f^\alpha\,. \tag{3.5}$$

This integrability condition can be thought of as the twistor form of the field equations for self-dual conformal gravity. Kodaira theory implies the existence of a complex four parameter family of rational curves of degree one, and this family is identified with the complexification of space-time M. Thus to reconstruct M from $\mathbb{P}\mathscr{T}$ we must find a family of holomorphic maps

$$Z^\alpha(x^\mu, \sigma_A) : \mathbb{PS} \to \mathbb{P}\mathscr{T}, \qquad Z^\alpha(x, \sigma) = \left(\lambda_A(x, \sigma), \mu^{A'}(x, \sigma)\right),$$

where $\mathbb{PS} \cong M \times \mathbb{CP}^1$ is naturally identified with the un-primed projective spinor bundle of M and $Z(x, \sigma)$ is a map of degree one parametrized by $x \in M$. We will often denote the image of the map for $x \in M$ as X. The condition that these maps be holomorphic is

$$\bar{\partial}_\sigma Z^\alpha(x, \sigma) - f^\alpha(Z(x, \sigma)) = 0, \tag{3.6}$$

where $\bar{\partial}_\sigma = \mathrm{d}\bar{\sigma}\frac{\partial}{\partial\bar{\sigma}}$ is the $\bar{\partial}$-operator on $X \subset \mathbb{P}\mathscr{T}$ pulled back to $\mathbb{PS}$.

3.2 Twistor action

We construct a twistorial version of the chiral action (2.4) in twistor space in two parts. The first is an action for the self-dual sector of conformal gravity. By theorem 1, this is equivalent to a twistor space with almost complex structure $\bar{\partial}_f$ subject to the field equation that it be integrable. The integrability condition is the vanishing of

$$N = \left(\bar{\partial} f^\alpha + [f, f]^\alpha\right) \partial_\alpha \in \Omega^{0,2}_{\mathbb{P}\mathscr{T}}(T_{\mathbb{P}\mathscr{T}}). \tag{3.7}$$

This will follow as the field equations from the Lagrange multiplier action [4]:

$$S_1[g, f] = \int_{\mathbb{P}\mathscr{T}} \mathrm{D}^3 Z \wedge g_\alpha \wedge N^\alpha, \tag{3.8}$$

where $g := g_\alpha \mathrm{D}Z^\alpha \in \Omega^{0,1}_{\mathbb{P}\mathscr{T}}(\mathcal{O}(-4) \otimes \Omega^1)$ and is subject to $Z^\alpha g_\alpha = 0$ because f^α is defined modulo Z^α. (If we fix this freedom in f^α so that $\partial_\alpha f^\alpha = 0$, then we can allow a gauge freedom $g_\alpha \to g_\alpha + \partial_\alpha \chi$, although this makes less geometric sense as then g becomes non-projective.) The field equations for this action are

$$N^\alpha = 0, \qquad \bar{\partial}_f \left(g_\alpha \mathrm{D}Z^\alpha\right) = 0. \tag{3.9}$$

We additionally have the gauge freedom $g \to g + \bar{\partial}_f \alpha$ for $\alpha \in \Omega^1_{\mathbb{P}\mathscr{T}}(-4)$ because $\bar{\partial}_f N = 0$ follows from a Jacobi-like identity for the almost complex structure. Thus, on-shell at least, g defines a cohomology class in $H^{0,1}(\mathbb{P}\mathscr{T}, \Omega^1(-4))$. We can therefore apply the Penrose transform [58] to define a space-time field G_{ABCD} by:

$$G_{ABCD} = \int_X \lambda_A \lambda_B \lambda_C \lambda_D \, g(Z(x, \sigma)). \tag{3.10}$$

In Appendix B we show that G_{ABCD} satisfies the second field equation of (2.5). Thus g gives rise to a linear ASD conformal gravity field propagating on the SD background.

The action (3.8) is therefore equivalent to the first (self-dual) part of the chiral space-time action (2.4), i.e., with $\varepsilon^2 = 0$. To obtain the ASD interactions of the theory, we simply need to express the second term in (2.4) in twistor space. The Penrose transform (3.10) can be implemented off-shell to give:

$$S_2[g,f] = \int_{\mathbb{PS}\times_M \mathbb{PS}} \mathrm{d}\mu \wedge \langle\lambda_1\, \lambda_2\rangle^4\, g_1 \wedge g_2, \tag{3.11}$$

where $\mathbb{PS}\times_M \mathbb{PS} \cong M \times \mathbb{CP}^1 \times \mathbb{CP}^1$ is the fibre-wise product of $\mathbb{PS}$ with itself. In this expression for S_2, we implicitly assume that the SD background M is constructed via the non-linear graviton of theorem 1. This can be made explicit by introducing a Lagrange multiplier field $Y \in \Omega^{1,0}_{\mathbb{CP}^1}(T^*\mathbb{P}\mathscr{T})$ and re-writing the action as

$$S_2[g,f] = \int_M \mathrm{d}\mu \left[\int_{\mathbb{CP}^1} \left(Y_\alpha \bar{\partial}_\sigma Z^\alpha - f^\alpha Y_\alpha \right) + \int_{(\mathbb{CP}^1)^2} \langle\lambda_1\, \lambda_2\rangle^4\, g_1 \wedge g_2 \right]. \tag{3.12}$$

Integrating out the field Y_α produces the constraint $\bar{\partial}_\sigma Z^\alpha = f^\alpha$, matching (3.6) and returning (3.11). Note that the Lagrange multiplier Y appears in a similar fashion in the worldsheet action of the Berkovits-Witten twistor-string [2, 4].

This gives the twistor action for the full (i.e., non-self-dual) conformal gravity of the form:

$$S[g,f] = S_1[g,f] - \varepsilon^2 S_2[g,f]. \tag{3.13}$$

We should note that to define the action off shell, we must nevertheless solve (3.6) in order to define the integrals in S_2. This equation can be solved with the standard four complex dimensional family of solutions irrespective of whether the almost complex structure is integrable. However, the integral against $\mathrm{d}\mu$ in (3.11) is over a *real* four-dimensional contour and so we must also impose a reality condition on the data in order for the moduli space of solutions to have a real four-dimensional slice. This can be done by imposing a reality structure on the data that is adapted to either Euclidean or split signature. Thus for Euclidean signature we have an anti-linear involution $Z^\alpha \to \hat{Z}^\alpha$ that is quaternionic so that $\hat{\hat{Z}}^\alpha = -Z^\alpha$ and we require $\bar{f} = f(\hat{Z})$. This induces a conjugation on M whose fixed points are a real slice of Euclidean signature (an ordinary conjugation yields a real slice of split signature).

The following theorem confirms that this is equivalent to (2.4), as desired:

Theorem 2 (Mason [5]) *The twistor action $S[g,f]$ is classically equivalent to the conformal gravity action* (2.4) *in the sense that solutions to its Euler-Lagrange equations are in one-to-one correspondence with solutions to the field equations* (2.5) *up to space-time diffeomorphisms.*

3.3 The $\mathcal{N}=4$ minimal twistor action

The extension of the above construction to $\mathcal{N}=4$ supersymmetry is straightforward. The twistor space $\mathbb{P}\mathscr{T}$ becomes a projective (3|4)-dimensional supermanifold modelled on $\mathbb{CP}^{3|4}$ with homogeneous coordinates $Z^I = (Z^\alpha, \chi^a)$, $a = 1, \ldots, 4$. It is super-Calabi-Yau being equipped with a (canonical) holomorphic volume measure $\mathrm{D}^{3|4}Z$ (i.e., a canonical holomorphic section of the Berezinian). The data naturally extends to a deformed $\bar{\partial}$-operator and $(1,1)$-form on $\mathbb{P}\mathscr{T}$

$$\bar{\partial}_f = \bar{\partial} + f^I \frac{\partial}{\partial Z^I}\,, \qquad g := g_I \mathrm{D}Z^I \in \Omega^{1,1}_{\mathbb{P}\mathscr{T}}\,, \quad \mathrm{D}Z^I = \mathrm{d}Z^I - f^I\,.$$

With $\mathcal{N} = 4$ supersymmetry, the conditions $\partial_I f^I = 0$ and $Z^I g_I = 0$ no longer fix the gauge freedoms of adding a multiple of Z^I to f^I or ∂_I to g. Since $\partial_I f^I = 0$ on account of fermionic signs, $\partial_I f^I = 0$ is compatible with adding a multiple of Z^I to f^I, and $Z^I g_I = 0$ is compatible with adding $\partial_I \alpha$ to g, as α now has homogeneity zero rather than -4.

This allows us to define (3.13) with respect to the new super-geometry by taking:

$$S_1[g,f] = \int_{\mathbb{P}\mathscr{T}} \mathrm{D}^{3|4}Z \wedge g_I \wedge N^I, \tag{3.14}$$

$$S_2[g,f] = \int_{\mathbb{PS}\times_M \mathbb{PS}} \mathrm{d}\mu \wedge g_1 \wedge g_2. \tag{3.15}$$

Here, as we will see later in §4, $\mathrm{d}\mu$ is a canonically defined measure on the $(4|8)$-dimensional chiral space-time M, the space of degree-one rational curves in $\mathbb{P}\mathscr{T}$. As in the $\mathcal{N} = 0$ setting, we can make the construction of the SD background M explicit by introducing the Lagrange multiplier Y and writing

$$S_2[g,f] = \int_M \mathrm{d}\mu \left[\int_{\mathbb{CP}^1} \left(Y_I \bar{\partial}_\sigma Z^I - f^I Y_I \right) + \int_{(\mathbb{CP}^1)^2} g_1 \wedge g_2 \right]. \tag{3.16}$$

In the supersymmetric setting, $g_I \mathrm{D}Z^I$ defines a chiral superfield on space-time:

$$\mathcal{G}(x,\theta) = \int_X g(Z(x,\theta,\sigma)), \tag{3.17}$$

where $\mathcal{G}$ has an expansion like:

$$\mathcal{G}(x,\theta) = \varphi + \cdots + \theta^{4\,ABCD}\Psi_{ABCD} + \cdots\,.$$

The Penrose transform can be used to show that the individual fields in $\mathcal{G}$ correspond to the chiral (ASD) half of the $\mathcal{N} = 4$ CSG field content, as desired. Heuristically, the space-time translation of our $\mathcal{N} = 4$ twistor action will look like

$$S[\mathcal{W},\mathcal{G}] = \int_M \mathrm{d}\mu \left(\mathcal{W}(x,\theta)\, \mathcal{G}(x,\theta) - \varepsilon^2 \mathcal{G}(x,\theta)^2 \right) \rightarrow \int_M \mathrm{d}\mu\, \mathcal{W}(x,\theta)^2, \tag{3.18}$$

where $\mathcal{W}(x,\theta)$ is the a chiral superfield which, on-shell, is a Lorentz scalar encoding the $\mathcal{N} = 4$ Weyl multiplet (c.f., [40]).

This action has the correct linear reduction for $\mathcal{N} = 4$ CSG [4], and must correspond to a *minimal* $\mathcal{N} = 4$ CSG since the functional form prohibits any cubic couplings between φ and the Weyl curvature. However, note that our twistor action only possesses the linearized $\mathbb{E}_2$ global symmetry of translating the scalar φ rather than the fully non-linear $\mathrm{SU}(1,1)$.[6] Nevertheless, since Einstein supergravity still forms a subsector of this degenerate theory [43], we are able to apply the conformal/Einstein gravity correspondence.

[6]The additional U(1)-symmetry of the minimal model can be seen as arising from $g \rightarrow \mathrm{e}^{4i\beta} g$ together with $\chi^a \rightarrow \mathrm{e}^{-i\beta}\chi^a$ which induces a similar phase rotation for θ^{aA}. This symmetry is the key for ruling out the φ (Weyl)^2 couplings and hence ensuring that the conformal/Einstein gravity correspondence applies.

3.4 The Einstein sector

We will apply Maldacena's argument to the minimal $\mathcal{N}=4$ CSG twistor action; this entails restricting $S[g,f]$ to Einstein polarization states. A conformal factor Ω from (2.16) relating spin-two and linearized Einstein fields can be specified on twistor space by introducing an *infinity twistor* I_{IJ}, a skew bi-twistor.[7]

Choose I_0 and I_1 to be of rank-two such that

$$I_{0\,IJ}Z^I \mathrm{d}Z^J = \langle\lambda \mathrm{d}\lambda\rangle, \qquad I_{1\,IJ}Z^I \mathrm{d}Z^J = [\mu\, \mathrm{d}\mu].$$

Then the infinity twistor appropriate to Einstein polarization states with cosmological constant Λ on the affine de Sitter patch is given by $I = I_0 + \Lambda I_1$. We can define an upstairs bosonic part by $I^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}I_{\gamma\delta}$ and we will have

$$I_{\alpha\beta}I^{\beta\gamma} = \Lambda\delta^{\gamma}_{\alpha}\,.$$

This relation can be extended supersymmetrically if we set:

$$I_{IJ} = \begin{pmatrix} \epsilon^{AB} & 0 & 0 \\ 0 & \Lambda\epsilon_{A'B'} & 0 \\ 0 & 0 & \sqrt{\Lambda}\delta_{ab} \end{pmatrix}, \qquad I^{IJ} = \begin{pmatrix} \Lambda\epsilon_{AB} & 0 & 0 \\ 0 & \epsilon^{A'B'} & 0 \\ 0 & 0 & \sqrt{\Lambda}\delta^{ab} \end{pmatrix}. \tag{3.19}$$

Geometrically, these are encoded into a weighted contact form τ and Poisson structure on $\mathbb{P}\mathscr{T}$:

$$\tau = I_{IJ}Z^I \mathrm{D}Z^J, \qquad \Pi = I^{IJ}\partial_I \wedge \partial_J, \qquad \{f,g\} = I^{IJ}\partial_I f\, \partial_J g. \tag{3.20}$$

We now require that the complex deformation $\bar{\partial}_f$ be Hamiltonian with respect to Π

$$\mathcal{L}_f \Pi = 0 \;\Rightarrow f = I^{IJ}\partial_I h \partial_J, \qquad h \in \Omega^{0,1}_{\mathbb{P}\mathscr{T}}(2).$$

Infinitesimal Hamiltonian diffeomorphisms are pure gauge and modulo such, h defines a cohomology class in $H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(2))$. The Penrose transform realizes this as a $\mathcal{N}=4$ graviton multiplet of helicity $+2$ via the integral formula

$$\tilde{\psi}(x,\theta)_{A'B'C'D'} = \int_X \frac{\partial^4 h}{\partial\mu^{A'}\cdots\partial\mu^{D'}} \wedge \tau.$$

Plugging this into (3.14), we get:

$$\begin{aligned} S_1[g,f] \to S_1[g,h] &= \int_{\mathbb{P}\mathscr{T}} \mathrm{D}^{3|4}Z \wedge g_I \wedge I^{IJ}\partial_J \left(\bar{\partial}h + \frac{1}{2}\{h,h\}\right) \\ &= \int_{\mathbb{P}\mathscr{T}} \mathrm{D}^{3|4}Z \wedge I^{IJ}\partial_I g_J \wedge \left(\bar{\partial}h + \frac{1}{2}\{h,h\}\right), \end{aligned} \tag{3.21}$$

with the second line following via integration by parts. On-shell $\tilde{h} := I^{IJ}\partial_I g_J$ defines an element of $H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-2))$. The Penrose transform identifies this with the $\mathcal{N}=4$ graviton multiplet of helicity -2 [37], this time starting with the scalar

$$\phi(x,\theta) = \int_X \tilde{h} \wedge \tau.$$

[7] In the supersymmetric case, the fermionic part of the infinity twistor corresponds to a gauging of the $\mathcal{N}=4$ R-symmetry [59]; this will not play an important role in this paper.

Given some $\tilde{h} \in H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-2))$ we can also write $g = \tilde{h} \wedge \tau$. With this, (3.21) becomes:

$$S_1[g,h] \to S_1[\tilde{h},h] = \int_{\mathbb{P}\mathscr{T}} \mathrm{D}^{3|4}Z \wedge I^{IJ}\partial_I \left(I_{JK}Z^K\tilde{h}\right) \wedge \left(\bar{\partial}h + \frac{1}{2}\{h,h\}\right)$$
$$= 2\Lambda \int_{\mathbb{P}\mathscr{T}} \mathrm{D}^{3|4}Z \wedge \tilde{h} \wedge \left(\bar{\partial}h + \frac{1}{2}\{h,h\}\right). \quad (3.22)$$

This is precisely the self-dual twistor action for Einstein gravity, up to the factor of Λ required by conformal/Einstein gravity correspondence [16, 60].

The Einstein reduction for the second term of the twistor action follows easily:

$$S_2[g,f] \to S_2[\tilde{h},h] = \int\limits_{\mathbb{PS}\times_M \mathbb{PS}} \mathrm{d}\mu \wedge \tilde{h}_1\, \tau_1 \wedge \tilde{h}_2\, \tau_2. \quad (3.23)$$

So the reduction of the conformal gravity twistor action to Einstein wavefunctions is simply

$$S[\tilde{h},h] = S_1[\tilde{h},h] - \varepsilon^2 S_2[\tilde{h},h]. \quad (3.24)$$

The remaining diffeomorphism freedom on $\mathbb{P}\mathscr{T}$ is captured by the transformations:

$$Z^\alpha \to Z^\alpha + \{Z^\alpha, \chi\}, \qquad h \to h + \bar{\partial}\chi + \{h,\chi\},$$

for χ a weight $+2$ function [60].

4 The MHV Amplitude with Cosmological Constant

We are now in a position to derive a twistorial formula for the MHV amplitude of Einstein supergravity with a cosmological constant. By (3.23) and proposition A.2, we know that the generating functional for these amplitudes (with $\mathcal{N}=4$ supersymmetry) is given by

$$\frac{1}{\Lambda} \int\limits_{\mathbb{PS}\times_M \mathbb{PS}} \mathrm{d}\mu \wedge \tilde{h}_1\, \tau_1 \wedge \tilde{h}_2\, \tau_2, \quad (4.1)$$

where the background space-time M is self dual and so can be obtained via the non-linear graviton construction by solving the equation

$$\bar{\partial}_\sigma Z^I(x,\sigma) = f^I(Z) = I^{IJ}\partial_J h(Z) \quad (4.2)$$

While we will focus on Einstein states, much of our calculation is easily applicable to conformal gravity, since polarization states for this theory can be expressed in terms of Einstein states. Given the permutation symmetry of the positive helicity and negative helicity fields amongst themselves, we can generate all conformal gravity amplitudes by considering Einstein states with one choice of infinity twistor for the positive helicity states (upstairs indices), and a different one for the negative helicity states (downstairs indices). Restricting to the Einstein subsector is then accomplished by requiring these infinity twistors be compatible as in (3.19).

Now, equation (4.2) has the four complex parameter family of solutions that defines (complexified) space-time [55, 61].[8] To obtain a formula for the n-point amplitude, we will obtain a perturbative expansion of the generating functional to $(n-2)^{\text{th}}$ order by solving equation (4.2) peturbatively. This will lead to Feynman diagrams on the $\mathbb{CP}^1$ factors for the integrand of the MHV generating function (4.1). These are then summed using the matrix-tree theorem to give a compact formula in terms of reduced determinants for the MHV amplitude analogous to that of Hodges [12]. This gives a clear explanation of the use of the matrix-tree theorem for this amplitude described in [17, 65].

4.1 The measure

To start with we will first define the measure $\mathrm{d}\mu$ used in (4.1). To this end, and for the the later perturbation expansion, we rewrite (4.2) as an integral equation

$$Z^I(x,\sigma) = X^I_A\sigma^A + \bar\partial_\sigma^{-1}\left(f^I(Z)\right)\,, \tag{4.3}$$

where $X^I_A\sigma^A$ solves the homogeneous equation and X^{IA} parametrizes its solutions. Since f^I has weight $+1$, there is an ambiguity in the choice of $\bar\partial_\sigma^{-1}$ and we can choose $\bar\partial^{-1}$ to vanishes at two points. For simplicity we will require that it vanishes at $\sigma_A = \xi_A$ to second order by setting

$$Z^I(x,\sigma) = X^I_A\sigma^A + \frac{1}{2\pi i}\int_{\mathbb{CP}^1}\frac{\mathrm{D}\sigma'}{(\sigma\sigma')}\frac{(\xi\sigma)^2}{(\xi\sigma')^2}f^I(Z(\sigma')). \tag{4.4}$$

Physical observables such as scattering amplitudes should be independent of ξ at the end of our calculations and we will check this explicitly.

We now write $Z^I(x,\sigma) = \mathcal{X}^I_A\sigma^A$ defining

$$\mathcal{X}^{IA}(x,\sigma) = X^{IA} + \frac{\xi^A}{2\pi i}\int_{\mathbb{CP}^1}\frac{\mathrm{D}\sigma'}{(\sigma\sigma')}\frac{(\xi\sigma)}{(\xi\sigma')^2}f^I(Z(\sigma'))\,. \tag{4.5}$$

which solves

$$\bar\partial_\sigma\mathcal{X}^{IA}(x,\sigma) = \frac{\xi^A f^I}{(\xi\sigma)}. \tag{4.6}$$

This enables us to take the exterior derivative of $\mathcal{X}$ with respect to the space-time coordinate x, finding

$$\bar\partial_\sigma\left(\mathrm{d}_x\mathcal{X}^{IA}(x,\sigma)\right) = \partial_J f^I\frac{\xi^A\sigma_B}{(\xi\sigma)}\mathrm{d}_x\mathcal{X}^{JB}(x,\sigma). \tag{4.7}$$

Since $\partial_I f^I = 0$, this means that the top-degree form $\mathrm{d}^{8|8}\mathcal{X}$ is holomorphic in σ and of weight zero; by Liouville's theorem, it is therefore independent of σ. But this means that

$$\mathrm{d}\mu = \frac{\mathrm{d}^{8|8}\mathcal{X}}{\mathrm{vol}\ \mathrm{GL}(2,\mathbb{C})} = \frac{\mathrm{d}^{8|8}X}{\mathrm{vol}\ \mathrm{GL}(2,\mathbb{C})},$$

is an invariant volume form on the space-time M itself.[9]

[8] When $\Lambda = 0$, it is a twistorial formulation of the 'good cut equation' [62–64].

[9] Here $\mathrm{GL}(2,\mathbb{C})$ is the choice of homogeneous coordinates σ_A on $X \cong \mathbb{CP}^1$. The division by vol $\mathrm{GL}(2,\mathbb{C})$ is understood in the Fadeev-Popov sense: one chooses a section of the group action, and multiplies by the appropriate Jacobian factor to obtain a well-define volume form on the $(4|8)$-dimensional quotient. One can also define this form to be that obtained by contracting a basis set of the generators of $\mathrm{GL}(2,\mathbb{C})$ into the volume form in the numerator and observing that the form is one pulled-back from the quotient.

4.2 A Feynman diagram calculus for the perturbation theory

We now introduce a Feynman diagram calculus on $\mathbb{CP}^1$ for the perturbative evaluation of the generating functional (4.1). We compute the n-point functional as a sum of diagrams defined as follows:

- Draw a black vertex for each of $\tilde{h}_1$, $\tilde{h}_2$.
- Draw a grey vertex for each contact structure τ_1, τ_2.
- Draw a white vertex for each of the $n-2$ fields h_i.
- Draw an oriented edge out from each white vertex to some other vertex such that the resulting diagram is a forest of trees rooted at the black or grey vertices.

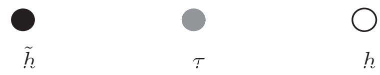

Figure 5. *Building blocks for Feynman diagrams*

The computational dictionary associated to these diagrams comes directly from the generating functional (4.1). Recall that we can make the SD background space-time M explicit by introducing a Lagrange multiplier field Y_I as in (3.16); at the level of the MHV generating functional, this takes the form:

$$\int_M \mathrm{d}\mu \left[\int_{\mathbb{CP}^1} \left(Y_I \bar{\partial}_\sigma Z^I - f^I Y_I\right) + \frac{1}{\Lambda}\int_{(\mathbb{CP}^1)^2} \tilde{h}_1\, \tau_1 \wedge \tilde{h}_2\, \tau_2\right]. \tag{4.8}$$

Our diagrams arise by considering the tree-level (since the curve $X \subset \mathbb{P}\mathscr{T}$ is built by the classical solution to (4.2)) Feynman rules on $\mathbb{CP}^1$.

Each diagram corresponds to an integrand to be integrated over the n-fold product of the $\mathbb{CP}^1$ factor in (4.1) and then over $\mathbb{M}$. The vertices are each associated to a point σ_i on the i^{th} $\mathbb{CP}^1$ factor. For $i = 1, 2$ we have a wavefunction $\tilde{h}(Z(\sigma_i))$ for the black vertices or $\tau_i = I_{IJ} Z^I(\sigma_i)\partial Z^J(\sigma_i)$ for the grey vertices. Writing

$$f^I = \sum_{j=3}^{n} f^I(Z(\sigma_j)) = \sum_{j=3}^{n} I^{IJ}\frac{\partial h(Z(\sigma_j))}{\partial Z^J(\sigma_j)},$$

we obtain the $n-2$ white vertices of the form $[Y(Z(\sigma_j)), \partial h(Z(\sigma_j))]$. The kinetic term $Y_I \bar{\partial}_\sigma Z^I$ defines a propagator in accordance with (4.4), so an edge from a white node $j > 2$ to a black or white node i corresponds to the differential operator

$$\frac{(\xi\sigma_i)^2 \mathrm{D}\sigma_j}{(\xi\sigma_j)^2(\sigma_j\sigma_i)} I^{IJ} \frac{\partial h(Z(\sigma_j))}{\partial Z^J(\sigma_j)} \frac{\partial}{\partial Z^I(\sigma_i)} \tag{4.9}$$

acting on the wave function at the i^{th} node of the diagram. We will give the formulae for the action on the τ associated to the grey vertices below, as they require a more subtle treatment.

Since there is a single Y_I in each white vertex, there are $n-2$ total edges in each diagram, and the fact that they are forests of trees requires them to be rooted at black or grey vertices. Additionally, since the wavefunctions $\tilde{h}$, h depend non-polynomially on Z, the white and black vertices can have have an arbitrary number of incoming edges. Since $\tau = \langle Z(\sigma), \partial Z(\sigma)\rangle$ is of order two in Z, the grey vertices can absorb at most two edges.

To summarize, we represent the perturbative expansion of the MHV generating functional (4.1) by using a $\mathbb{CP}^1$-Feynman diagram calculus, which follows naturally from the 'explicit' form of the generating functional (4.8). Since we work classically, each diagram corresponds to a forest of trees on $n+2$ ($2\ \tau$s $+\ 2\ \tilde{h}$s $+\ n-2\ h$s) vertices, rooted at a black or grey vertex. Restricting to Einstein states, this perturbative expansion acts on $Z(x,\sigma)$ as

$$Z^I(x,\sigma) \to \int_{\mathbb{CP}^1} \frac{\mathrm{D}\sigma'}{(\sigma\sigma')} \frac{(\xi\sigma)^2}{(\xi\sigma')^2} I^{IJ} \frac{\partial h(Z(\sigma'))}{\partial Z^J(\sigma')}, \tag{4.10}$$

while its action on a wavefunction h or $\tilde{h}$ is:

$$h(Z(\sigma)) \to \int_{\mathbb{CP}^1} \frac{\mathrm{D}\sigma'}{(\sigma\sigma')} \frac{(\xi\sigma)^2}{(\xi\sigma')^2} \left[\frac{\partial}{\partial Z(\sigma)}, \frac{\partial}{\partial Z(\sigma')}\right] h(Z(\sigma))\, h(Z(\sigma')). \tag{4.11}$$

Note that the diagram calculus of the perturbative iteration is identical to the semi-classical connected tree formalism which arose in the context of twistor-string theory in [17]. There, the trees emerged in order to extract Einstein amplitudes from the Berkovits-Witten twistor-string at degree one. Here, the trees arise naturally from the twistor action of minimal $\mathcal{N}=4$ CSG: this proves that they are isolating the minimal content of BW-CSG for a degree-one instanton.

4.3 The role of the contact structure

All the diagrams have two grey vertices corresponding to the contact structures $\tau_i = \langle Z(\sigma_i), \partial Z(\sigma_i)\rangle$, $i=1,2$. These are quadratic in Z and so can have at most two incoming arrows; higher numbers of incoming arrows will vanish. In fact, if the upstairs infinity twistor is the inverse of the downstairs one (as in the Einstein case), other contributions vanish as follows.

Lemma 4.1 *If a Feynman diagram has a disconnected piece with just one white vertex connected to a grey vertex we will refer to it as* isolated *if the corresponding white vertex in the perturbative expansion has no incoming arrows as in* (a.) *of Figure 6. Such isolated deformations of the contact structure give a vanishing contribution to the vertex generating functional.*

Figure 6. *An isolated* (a.) *and un-isolated* (b.) *deformation of τ.*

Proof: Without loss of generality, consider perturbative expansions with a single *isolated* arrow from (white) vertex i to τ_1. This corresponds to a contribution

$$\begin{aligned}\langle Z(\sigma_1), \partial Z(\sigma_1)\rangle \rightarrow I_{IJ}\int_{\mathbb{CP}^1} \frac{\mathrm{D}\sigma_i}{(1i)^2}\frac{(\xi 1)}{(\xi i)^2}\big[&(\xi 1)(1i)\partial Z^J(\sigma_1)f^I(\sigma_i) + (\xi 1)(i\mathrm{d}\sigma_1)Z^I(\sigma_1)f^J(\sigma_i)\\ &+2(1i)(\xi \mathrm{d}\sigma_1)Z^I(\sigma_1)f^J(\sigma_i)\big]\\ = I_{IJ}\int_{\mathbb{CP}^1} \frac{\mathrm{D}\sigma_i}{(1i)^2}\frac{(\xi 1)}{(\xi i)^2}\big[&(\xi 1)(1i)\partial Z^J(\sigma_1)f^I(\sigma_i) + \mathrm{D}\sigma_1(\xi i)Z^I(\sigma_1)f^J(\sigma_i)\\ &+(1i)(\xi \mathrm{d}\sigma_1)Z^I(\sigma_1)f^J(\sigma_i)\big]\,,\end{aligned}$$

with the second expression following by the Schouten identity.

Now recall that the map to twistor space takes the form $Z^I(\sigma) = X^I_A\sigma^A = (X\sigma)^I$, so the Schouten identity gives

$$\partial Z^J(\sigma_1)\;(1i) = Z^J(\sigma_i)\;\mathrm{D}\sigma_1 - Z^J(\sigma_1)\;(i\mathrm{d}\sigma_1),$$

and feeding this into the above expression leaves us with

$$I_{IJ}\mathrm{D}\sigma_1\int_{\mathbb{CP}^1}\frac{\mathrm{D}\sigma_i}{(1i)^2}\frac{(\xi 1)}{(\xi i)^2}\big(2(\xi i)\;Z^I(\sigma_1) - (\xi 1)\;Z^I(\sigma_i)\big)\,f^J(Z(\sigma_i)).$$

Using $f^I = I^{IJ}\partial_J h$, we obtain a contraction between two infinity twistors which gives $I_{IJ}I^{JK} = \Lambda\delta_I^K$ and we have

$$\begin{aligned}&\Lambda\mathrm{D}\sigma_1\int_{\mathbb{CP}^1}\frac{\mathrm{D}\sigma_i\,(\xi 1)}{(1i)^2(\xi i)^2}\big(2(\xi i)Z^I(\sigma_1) - (\xi 1)Z^I(\sigma_i)\big)\,\partial_I h(\sigma_i)\\ &\qquad= 2\Lambda\mathrm{D}\sigma_1\int_{\mathbb{CP}^1}\frac{\mathrm{D}\sigma_i\,(\xi 1)}{(1i)^2(\xi i)^2}\left((\xi i)\sigma_1\cdot\partial_i h(\sigma_i) - (\xi 1)h(\sigma_i)\right)\\ &\qquad\qquad= 2\Lambda\mathrm{D}\sigma_1\sigma_{1A}\int_{\mathbb{CP}^1}\frac{\partial}{\partial\sigma_{iA}}\left(\frac{\mathrm{D}\sigma_i\,(\xi 1)h(\sigma_i)}{(1i)^2(\xi i)}\right)\\ &\qquad\qquad\qquad= 2\Lambda\mathrm{D}\sigma_1\sigma_{1A}\int_{\mathbb{CP}^1}\partial_i\left(\frac{\sigma_i^A(\xi 1)h(\sigma_i)}{(1i)^2(\xi i)}\right).\end{aligned}$$

In the second line we have used the homogeneity relation, chain rule, and the linearity of $Z(\sigma_i)$ in σ_i to deduce that $\sigma_1\cdot\partial_i h(Z(\sigma_i)) = Z^I(\sigma_1)\partial_I h(Z(\sigma_i))$.

The integrand of this expression has potential poles at $\sigma_i = \sigma_1, \xi$ which could lead to boundary contributions when we apply Stokes theorem. If we take $\sigma_i = \sigma_1 + z\xi$, then the integral takes the form:

$$\int_{\mathbb{CP}^1}\partial_i\left(\frac{g(z)\mathrm{d}\bar{z}}{z}\right) = \oint_{r=\infty}\frac{g(z)\mathrm{d}\bar{z}}{z} - \oint_{r=0}\frac{g(z)\mathrm{d}\bar{z}}{z},$$

where $g(z)$ is a smooth weighted holomorphic function. Writing $z = re^{i\theta}$, we are left with

$$-i\oint_{r=\infty} g(z)e^{-2i\theta}\mathrm{d}\theta + i\oint_{r=0} g(z)e^{-2i\theta}\mathrm{d}\theta = 0,$$

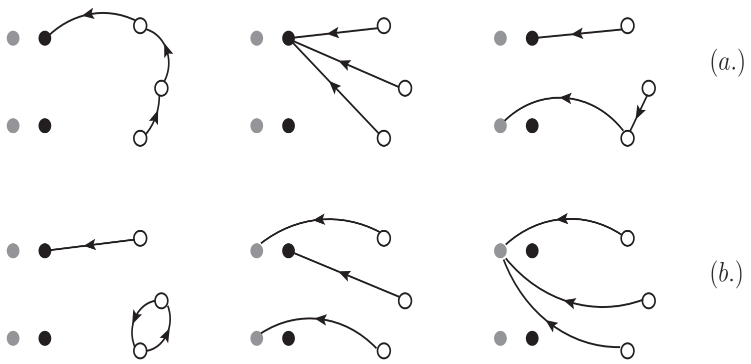

Figure 7. *Some diagrams for the 5-point amplitude which have a non-vanishing* (a.)*, or excluded/vanishing* (b.) *contribution.*

so any potential boundary terms do indeed vanish. The case with two isolated contractions into τ_1 from vertices i and j follows similarly. □

While this lemma ensures that we can neglect any *isolated* arrows to the contact structures in our diagrams, it does not rule out un-isolated contributions. Indeed, if τ_1 is connected to vertex i which is in turn connected to vertex j, then additional σ_i-dependence is introduced by the propagator and we do not obtain the exact derivative that was the key to the proof of lemma 4.1. Hence, we will still need propagators for when one or two deformations act at τ.

Since both the contact structure and the perturbative iteration involve the infinity twistor, it is clear that these propagators will be $O(\Lambda)$. After a bit of algebra, we find that the propagator for a single deformation of a contact structure (say, τ_1) is given by:

$$\psi_i^1 = \Lambda \frac{(\xi 1)}{(1i)^2(\xi i)^2} \left[(\xi i)\ Z^I(\sigma_1) + (1i)\ Z^I(\xi) \right] \frac{\partial}{\partial Z^I(\sigma_i)}. \tag{4.12}$$

Similarly, the propagator for two deformations of the same contact structure is given by:

$$\omega_{ij}^1 = -\Lambda \frac{(1\xi)^4(ij)}{(1i)^2(1j)^2(\xi i)^2(\xi j)^2} \left[\frac{\partial}{\partial Z(\sigma_i)}, \frac{\partial}{\partial Z(\sigma_j)} \right]. \tag{4.13}$$

Note that there are many equivalent formulae for these propagators following from the Schouten identity; the two we have presented here are the most useful for our following calculations.

Clearly, at any order in n there are many diagrams which can be drawn on the $n+2$ vertices which are either excluded from our diagram calculus or give a vanishing contribution to the generating functional. In Figure 7, we illustrate several examples for the case of the 5-point amplitude. All the diagrams in *(a.)* give a non-vanishing contribution, while all those in *(b.)* are either excluded or give a vanishing contribution. In the latter case, the first diagram of *(b.)* is excluded because of the loop; the second vanishes because there are isolated deformations of the contact structure so lemma 4.1 applies; and the third vanishes because there are more than two deformations of a contact structure.

4.4 The MHV Amplitude

At this point, we are ready to implement our diagram calculus by perturbatively expanding the generating functional $S_2[h, \tilde{h}]$ and recovering the MHV amplitude. For n-points, this involves summing all the associated $\mathbb{CP}^1$ Feynman diagrams; as explained before, each of these diagrams will be a forest on $n+2$ vertices ($2\tilde{h}+(n-2)h+2\tau$). Using some basic facts from algebraic combinatorics, we can perform this sum of Feynman diagrams to obtain the MHV amplitude. We then verify that this result is independent of $\xi \in \mathbb{CP}^1$ and that as $\Lambda \to 0$ it limits onto the Hodges formula for the MHV amplitude [12].

Denote the set of all Feynman diagrams contributing to the n-point amplitude as $\mathcal{F}^n$. This set has a natural disjoint-union splitting based upon the number of arrows which are incoming at each of the two contact structures τ_1, τ_2. Explicitly, we have

$$\mathcal{F}^n = \bigsqcup_{k=0}^{4} \mathcal{F}^n_k,$$

where each diagram $\Gamma \in \mathcal{F}^n_k$ is a forest on $n+2$ vertices which has k arrows into the contact structures (for $k > 0$ all the diagrams have a vanishing contribution).

The simplest case involves no deformations of the contact structures; its contribution to the n-point vertex can be written heuristically as:

$$\sum_{\Gamma \in \mathcal{F}^n_0} \int_{\mathscr{M}_{n,1}} \mathrm{d}\mu \; (X^2)^2 \; F_\Gamma \; \prod_{i=1}^{n} h(Z(\sigma_i)) \; \mathrm{D}\sigma_i,$$

where $\mathscr{M}_{n,1}$ is the moduli space of n-pointed holomorphic maps $Z^I : \mathbb{CP}^1 \to \mathbb{PT}$ of degree one [66], $X^2 \equiv I_{IJ} X^I_A X^{JA}$, and F_Γ encodes the contribution from diagram Γ built out of the propagators (5.2). Since there are no arrows into either of the contact structures, we have simply written $\tau_{1,2} = X^2 \; \mathrm{D}\sigma_{1,2}$.

Each term in this sum corresponds to a forest of trees rooted at the two black vertices corresponding to $\tilde{h}_1$ and $\tilde{h}_2$. As was first illustrated in [17, 65] the sum of such forests can be accomplished using the Matrix-Tree theorem, an analogue of Kirchoff's theorem for directed graphs (c.f., [67–69]). This results in the contribution

$$\int_{\mathscr{M}_{n,1}} \mathrm{d}\mu \; (X^2)^2 \; \left|\mathbb{H}^{12}_{12}\right| \; \prod_{i=1}^{n} h(Z(\sigma_i)) \; \mathrm{D}\sigma_i, \tag{4.14}$$

where $\mathbb{H}$ is (up to an irrelevant conjugation) the weighted $n \times n$ Laplacian matrix for the master graph on all black and white vertices whose entries are given by

$$\mathbb{H}_{ij} = \begin{cases} \frac{1}{(ij)} \left[\frac{\partial}{\partial Z(\sigma_i)}, \frac{\partial}{\partial Z(\sigma_i)} \right] \text{ if } i \neq j \\ -\sum_{j \neq i} \mathbb{H}_{ij} \frac{(\xi j)^2}{(\xi i)^2} \text{ if } i = j \end{cases}. \tag{4.15}$$

The notation $|\mathbb{H}^{12}_{12}|$ indicates the determinant of $\mathbb{H}$ with the row and columns corresponding to $\tilde{h}_1$ and $\tilde{h}_2$ removed.

We can now apply the Matrix-Tree theorem in a similar fashion to the other subsets of Feynman graphs $\mathscr{F}^n_{k>0}$. For instance, consider graphs in $\mathscr{F}^n_1$. The single deformation of the contact structure may come from any white vertex $i = 3, \ldots, n$, and results in a propagator ψ^1_i or ψ^2_i from (4.12). All the remaining arrows in the graph will correspond to propagators of the form (4.15), so once we factor out the propagator to τ we are in the business of counting forests of trees rooted at vertices 1, 2, or i. Via the Matrix-Tree theorem, we then have:

$$\sum_{\Gamma\in\mathscr{F}^n_1} \int_{\mathscr{M}_{n,1}} \mathrm{d}\mu\, X^2\, F_\Gamma\, \prod_{i=1}^n h(Z(\sigma_i))\, \mathrm{D}\sigma_i \\ = \int_{\mathscr{M}_{n,1}} \mathrm{d}\mu\, X^2 \sum_{i=3}^n \psi^1_i\, \left|\mathbb{H}^{12i}_{12i}\right| \prod_{j=1}^n h(Z(\sigma_j))\, \mathrm{D}\sigma_j + (1 \leftrightarrow 2). \quad (4.16)$$

A similar pattern follows for the remaining subsets in $\mathscr{F}^n$. Adding all of them together and including the required factor of Λ^{-1} from the conformal/Einstein gravity correspondence gives us the following formula for the MHV amplitude:

$$\mathcal{M}_{n,0} = \frac{1}{\Lambda}\int_{\mathscr{M}_{n,1}} \mathrm{d}\mu\, \Bigg[(X^2)^2 \left|\mathbb{H}^{12}_{12}\right| + X^2 \sum_i \psi^1_i \left|\mathbb{H}^{12i}_{12i}\right| \\ + X^2 \sum_{i,j} \omega^1_{ij} \left|\mathbb{H}^{12ij}_{12ij}\right| + \sum_{i,j} \psi^1_i \psi^2_j \left|\mathbb{H}^{12ij}_{12ij}\right| + \sum_{i,j,k} \psi^1_i \omega^2_{jk} \left|\mathbb{H}^{12ijk}_{12ijk}\right| \\ + \sum_{i,j,k,l} \omega^1_{ij}\omega^2_{kl} \left|\mathbb{H}^{12ijkl}_{12ijkl}\right| \Bigg] \prod_{m=1}^n h(Z(\sigma_m))\, \mathrm{D}\sigma_m + (1 \leftrightarrow 2). \quad (4.17)$$

In this expression, the sums are understood to run over all indices which are not excluded from the determinant, and also to symmetrize on those indices. For instance, in the first term of the second line $\sum_{i,j}$ runs over all $i, j = 3, \ldots n$ with $i \neq j$.

This formula is a perfectly valid representation of the MHV amplitude with cosmological constant; it can be simplified substantially if we investigate its properties a bit further, however. Each term in (4.17) takes the form of a differential operator acting on the wavefunctions. With momentum eigenstates and a generic infinity twistor these operators become rather complicated, involving derivatives of delta-functions. Our manipulations would be considerably simpler if we could treat these terms algebraically. This can be accomplished by working with *dual twistor* wavefunctions:

$$h(Z(\sigma_i)) = \int_{\mathbb{C}} \frac{\mathrm{d}t_i}{t_i^{1+w_i}} \exp\left(i t_i W_i \cdot Z(\sigma_i)\right), \qquad w_i = \begin{cases} -2 & \text{if } i = 1,2 \\ 2 & \text{otherwise} \end{cases}. \quad (4.18)$$

Here $W_{i\,I} = (\tilde{\mu}^A, \tilde{\lambda}^{A'}_i)$ are coordinates on n copies of dual twistor space, $\mathbb{PT}^\vee$. These wavefunctions have been used before in other contexts [14, 70], and can be paired with momentum eigenstates in an appropriate manner to obtain functionals of momenta at the end of any calculation. Furthermore, the scaling parameters t_i can be absorbed into the worldsheet coordinates by defining a new set of non-homogeneous coordinates: $\sigma_i t_i \to \sigma_i$, $\mathrm{d}t_i \mathrm{D}\sigma_i \to \mathrm{d}^2\sigma_i$.

With (4.18), all the propagators of the Feynman diagram calculus become purely algebraic. In particular, we now have:

$$\mathbb{H}_{ii} = \sum_{j\neq i} \frac{[W_i, W_j]}{(ij)} \frac{(\xi j)^2}{(\xi i)^2},$$

$$\psi_i^1 = \Lambda \, i \frac{(\xi 1) \, W_{i\,I}}{(1i)^2(\xi i)^2} \left[(\xi i) \, Z^I(\sigma_1) + (1i) \, Z^I(\xi)\right], \qquad \omega_{ij}^1 = \Lambda \frac{[W_i, W_j] \, (1\xi)^4 (ij)}{(1i)^2(1j)^2(\xi i)^2(\xi j)^2}.$$

Furthermore, the product of wavefunctions and measures can be expressed compactly as

$$\prod_{i=1}^n h(Z(\sigma_i)) \, \mathrm{D}\sigma_i = e^{i\mathcal{P}\cdot X} \, \mathrm{d}^2\sigma, \qquad \mathcal{P}_I^A = \sum_{i=1}^n W_{i\,I} \sigma_i^A, \qquad \mathrm{d}^2\sigma \equiv \prod_{i=1}^n \mathrm{d}^2\sigma_i.$$

We now note that the second term in the first line of (4.17) can be written as

$$\begin{aligned} &\int \mathrm{d}\mu \, X^2 \sum_i \psi_i^1 \left|\mathbb{H}_{12i}^{12i}\right| \, e^{i\mathcal{P}\cdot X} \mathrm{d}^2\sigma \\ &\qquad = \frac{1}{\Lambda} \int \mathrm{d}\mu \, X^2 \sum_i \left|\mathbb{H}_{12i}^{12i}\right| \left(\frac{(\xi 1)(\xi i)\sigma_1^A + (\xi 1)(1i)\xi^A}{(1i)^2(\xi i)^2}\right) \frac{\partial e^{i\mathcal{P}\cdot X}}{\partial \sigma_i^A} \mathrm{d}^2\sigma. \end{aligned}$$

But this means that we can integrate by parts to find:

$$\begin{aligned} &i \int \mathrm{d}\mu \, \sum_i \left|\mathbb{H}_{12i}^{12i}\right| \frac{(\xi 1) \, W_{i\,I}}{(1i)^2(\xi i)^2} \left[(\xi i) Z^I(\sigma_1) + (1i) Z^I(\xi)\right] e^{i\mathcal{P}\cdot X} \mathrm{d}^2\sigma \\ &\quad = -\frac{1}{\Lambda} \int \mathrm{d}\mu \, X^2 e^{i\mathcal{P}\cdot X} \sum_i \frac{\partial}{\partial \sigma_i^A} \left(\left|\mathbb{H}_{12i}^{12i}\right| \frac{(\xi 1)(\xi i)\sigma_1^A + (\xi 1)(1i)\xi^A}{(1i)^2(\xi i)^2}\right) \mathrm{d}^2\sigma \\ &\quad = -\frac{1}{\Lambda} \int \mathrm{d}\mu \, X^2 e^{i\mathcal{P}\cdot X} \sum_{i,j} \left|\mathbb{H}_{12ij}^{12ij}\right| \frac{[W_i, W_j](1\xi)^4(ij)}{(1i)^2(1j)^2(\xi i)^2(\xi j)^2} \mathrm{d}^2\sigma \\ &\quad = -\frac{1}{\Lambda} \int \mathrm{d}\mu \, X^2 \sum_{i,j} \omega_{ij}^1 \left|\mathbb{H}_{12ij}^{12ij}\right| e^{i\mathcal{P}\cdot X} \mathrm{d}^2\sigma. \end{aligned}$$

with the third line following after symmetrizing over $(i \leftrightarrow j)$ and several applications of the Schouten identity.

Hence, we see that following an integration by parts, the second term in (4.17) cancels the third term. A similar calculation demonstrates that the fourth and fifth terms also cancel with each other. We are therefore able to reduce our formula for the amplitude to one with only two terms:

$$\mathcal{M}_{n,0} = \frac{1}{\Lambda} \int_{\mathscr{M}_{n,1}} \mathrm{d}\mu \, \left[(X^2)^2 \left|\mathbb{H}_{12}^{12}\right| + \sum_{i,j,k,l} \omega_{ij}^1 \omega_{kl}^2 \left|\mathbb{H}_{12ijkl}^{12ijkl}\right|\right] \prod_{m=1}^n h(Z(\sigma_m)) \, \mathrm{D}\sigma_m \, + (1 \leftrightarrow 2), \tag{4.19}$$

where we have restored arbitrary twistor wavefunctions and homogeneous coordinates. Beyond the obvious improvement in terms of simplicity, this new representation of the amplitude is also useful for investigating the flat space-limit of $\mathcal{M}_{n,0}$.

A basic property that (4.17) or (4.19) must have is the correct behaviour under $\Lambda \to 0$. In particular, $\mathcal{M}_{n,0}$ should limit onto Hodges' formula for the MHV amplitude in this flat-space limit. In the language of $\mathcal{N} = 4$ supergravity, Hodges' formula is [12][10]:

$$\mathcal{M}_{n,0}(\Lambda = 0) = \int_{\mathscr{M}_{n,1}} \mathrm{d}\mu \frac{(12)^2}{(1i)^2(2i)^2} \left|\mathbb{H}^{12i}_{12i}\right| \prod_{j=1}^{n} h(Z(\sigma_j))\, \mathrm{D}\sigma_j. \tag{4.20}$$

Now, the conformal/Einstein gravity correspondence should allow us to extract the Einstein MHV amplitude from (4.19) even in the $\Lambda \to 0$ limit. If we work with the dual twistor wavefunctions, it is easy to see that we only need to consider

$$\lim_{\Lambda\to 0} \mathcal{M}_{n,0} = \lim_{\Lambda\to 0} \frac{1}{\Lambda} \int_{\mathscr{M}_{n,1}} \mathrm{d}\mu\, (X^2)^2 \left|\mathbb{H}^{12}_{12}\right|\, e^{i\mathcal{P}\cdot X} \mathrm{d}^2\sigma\, + (1 \leftrightarrow 2), \tag{4.21}$$

since the second term in (4.19) will go like $O(\Lambda)$ in the limit. However, in [17] it was shown that (4.21) was equal to Hodges' formula, and precisely the same methods can be used here to give the same result. We can therefore conclude that our expression for the MHV amplitude (4.19) has the desired behaviour:

$$\lim_{\Lambda\to 0} \mathcal{M}_{n,0} = \int_{\mathscr{M}_{n,1}} \mathrm{d}\mu \frac{(12)^2}{(1i)^2(2i)^2} \left|\mathbb{H}^{12i}_{12i}\right| \prod_{j=1}^{n} h(Z(\sigma_j))\, \mathrm{D}\sigma_j. \tag{4.22}$$

While the flat-space limit is an easy check on the validity of our formula, there is another more non-trivial property which a correct formula must have: it must be independent of the reference spinor $\xi \in \mathbb{CP}^1$. This entered the definition of the perturbative iteration in (4.4) due to the ambiguity in defining $\bar{\partial}_\sigma^{-1}$ on forms of positive degree. Hence, the choice of ξ is equivalent to a gauge choice of propagator for our Feynman diagram formalism on $\mathbb{CP}^1$; by (4.5) a variation in ξ should correspond to a diffeomorphism on the projective spinor bundle $\mathbb{PS}$. In other words, observables such as $\mathcal{M}_{n,0}$ should be independent of the reference spinor.

An obvious way of demonstrating this is to consider the infinitesimal variation generated by the derivative $\mathrm{d}_\xi = \mathrm{d}\xi^A \frac{\partial}{\partial \xi^A}$. The calculation of $\mathrm{d}_\xi \mathcal{M}_{n,0}$ is a lengthy but relatively straightforward procedure which is carried out in Appendix C; the final result is that

$$\mathrm{d}_\xi \mathcal{M}_{n,0} = \int_{\mathscr{M}_{n,1}} \frac{\mathrm{d}^{8|8}X}{\mathrm{vol}\ \mathrm{GL}(2,\mathbb{C})} \frac{\partial}{\partial X^{IA}} V^{IA} = 0, \tag{4.23}$$

where V^{IA} are the components of a smooth vector field on $\mathscr{M}_{n,1}$. The fact that $\mathrm{d}_\xi \mathcal{M}_{n,0}$ vanishes as a total divergence is in accordance with the claim that a variation in ξ should correspond to a diffeomorphism with respect to our coordinates on the spinor bundle $\mathbb{PS}$, and proves that (4.17), (4.19) is a well-defined formula for the amplitude.

[10]Note that there are many equivalent representations of this formula, we have simply presented the one which connects most directly to our conformal gravity arguments.

4.5 Twistor-string-like formula

We conclude our exposition of the twistor formula for the MHV amplitude with cosmological constant by noting that it can be manipulated into a format which is highly suggestive of a twistor-string origin. Skinner's $\mathcal{N}=8$ twistor-string is the first example of a theory which treats Einstein supergravity *directly* with twistor methods [15]. As a string theory, it is anomaly free for any genus worldsheet and is known to produce the complete tree-level S-matrix of $\mathcal{N}=8$ supergravity on a flat background. Furthermore, the worldsheet theory is perfectly well-defined for a non-simple infinity twistor, so in principle it should also be able to produce (after truncation to $\mathcal{N}=4$ supersymmetry) the same twistor space formulae we have derived here.

Unfortunately, it is not currently known how to compute meaningful worldsheet correlators of gravitational vertex operators with a cosmological constant in Skinner's twistor-string (beyond three-points). The issues which arise are the failure for the correlators to be independent of the position of picture changing operators as well as reference spinors (analogous to $\xi \in \mathbb{CP}^1$); this indicates that the correlators are not gauge invariant with respect to the worldsheet degrees of freedom. These problems could stem from any number of sources, including an incomplete understanding of the full spectrum of vertex operators for the theory, or the worldsheet Feynman rules when $\Lambda \neq 0$. Hence, it seems natural to ask if our formulae for $\mathcal{M}_{n,0}$ could shed any light on this twistor-string calculation.

Initially, it appears that the structure of $\mathcal{M}_{n,0}$ is a long way off from something we might expect from the twistor-string. If we use dual twistor wavefunctions (4.18), then (4.19) takes the form

$$\mathcal{M}_{n,0} = \frac{1}{\Lambda}\int \frac{\mathrm{d}^{8|8}X}{\mathrm{vol\;GL}(2,\mathbb{C})}\left[(X^2)^2\left|\mathbb{H}^{12}_{12}\right| + \sum_{i,j,k,l}\omega^1_{ij}\omega^2_{kl}\left|\mathbb{H}^{12ijkl}_{12ijkl}\right|\right] e^{i\mathcal{P}\cdot X}\,\mathrm{d}^2\sigma, \tag{4.24}$$

so the leading contribution (i.e., with no contact structure deformations) for the MHV amplitude in $\mathcal{N}=4$ supergravity is a twice-reduced determinant. These two reductions correspond to the two negative helicity graviton multiplets of the amplitude.

However, in Skinner's twistor-string the fundamental object is a *thrice* reduced determinant (just as it is in the Hodges formula). In the context of $\mathcal{N}=8$ supergravity, all external states are in the same multiplet so there is no preference based on helicity; the three reductions instead correspond to building a top-degree form on the space of fermionic automorphisms of the worldsheet [15]. A bit of manipulation (essentially equivalent to the computations required to extract Hodges' formula from (4.24) in the $\Lambda \to 0$ limit [17]) shows that we can also get our formula for $\mathcal{M}_{n,0}$ into a thrice-reduced determinant form.

Focusing on the first term in (4.24), note that we can represent each factor of X^2 by a differential 'wave operator' acting on $e^{i\mathcal{P}\cdot X}$:

$$X^2 \to \Box = \frac{I_{IJ}}{(12)}\frac{\partial}{\partial W_{1\,I}}\frac{\partial}{\partial W_{2\,J}}. \tag{4.25}$$

Doing this allows us to re-write the twice-reduced contribution to $\mathcal{M}_{n,0}$ as

$$\frac{1}{\Lambda}\int\frac{\mathrm{d}^{8|8}X}{\mathrm{vol\;GL}(2,\mathbb{C})}\mathrm{d}^2\sigma\;\left|\mathbb{H}^{12}_{12}\right|\;\Box^2 e^{i\mathcal{P}\cdot X} = \frac{1}{\Lambda}\int\frac{\mathrm{d}^2\sigma}{\mathrm{vol\;GL}(2,\mathbb{C})}\left|\mathbb{H}^{12}_{12}\right|\;\Box^2\delta^{8|8}(\mathcal{P}). \tag{4.26}$$

On the support of this delta-function, we know that the matrix $\mathbb{H}$ has co-rank three [12, 13] so we can integrate by parts once with respect to $\frac{\partial}{\partial W_2}$ to give

$$-\frac{1}{\Lambda}\int \frac{\mathrm{d}^2\sigma}{\mathrm{vol\ GL}(2,\mathbb{C})}\frac{\partial}{\partial W_{2\,J}}\left|\mathbb{H}^{12}_{12}\right|\frac{I_{IJ}}{(12)}\frac{\partial}{\partial W_{1\,I}}\square\delta^{8|8}(\mathcal{P})$$

$$=-\int \frac{\mathrm{d}^2\sigma}{\mathrm{vol\ GL}(2,\mathbb{C})}\sum_i \frac{(\xi 2)^2}{(12)(i2)(\xi i)^2}\left|\mathbb{H}^{12i}_{12i}\right|\ W_i\cdot\frac{\partial}{\partial W_1}\square\delta^{8|8}(\mathcal{P}).$$

Once again, the support of the delta-function indicates that we can take $W_i\cdot\frac{\partial}{\partial W_1}=\sigma_1\cdot\frac{\partial}{\partial\sigma_i}$, and then integrate by parts once again with respect to $\mathrm{d}^2\sigma_i$. This leaves us with

$$\begin{aligned}&\int \frac{\mathrm{d}^2\sigma}{\mathrm{vol\ GL}(2,\mathbb{C})}\sum_i \frac{(12)^2}{(1i)^2(2i)^2}\left|\mathbb{H}^{12i}_{12i}\right|\ \square\delta^{8|8}(\mathcal{P})\\ &\quad+\int \frac{\mathrm{d}^2\sigma}{\mathrm{vol\ GL}(2,\mathbb{C})}\sum_{i,j}\left(\frac{(\xi 2)^2(1\xi)(ji)+(\xi 2)^2(1j)(\xi i)}{(12)(i2)(ji)(\xi i)(\xi j)^2}\right)\mathbb{H}_{ij}\ \left|\mathbb{H}^{12ij}_{12ij}\right|\ \square\delta^{8|8}(\mathcal{P}).\end{aligned}\quad(4.27)$$

The contribution from the second line can be further simplified by noting that the summation entails symmetrization, term-by-term, in both $1\leftrightarrow 2$ and $i\leftrightarrow j$. A straightforward calculation involving several applications of the Schouten identity allows us to reduce this to

$$\int \frac{\mathrm{d}^2\sigma}{\mathrm{vol\ GL}(2,\mathbb{C})}\sum_{i,j}\left(\frac{(\xi 1)^2(i2)(j2)+(\xi 2)^2(i1)(j1)}{(1i)(2i)(1j)(2j)(\xi i)(\xi j)}\right)\mathbb{H}_{ij}\ \left|\mathbb{H}^{12ij}_{12ij}\right|\ \square\delta^{8|8}(\mathcal{P}).$$

Upon using the symmetry of $i\leftrightarrow j$ and the basic properties of determinants, we are finally left with an expression for the amplitude with thrice-reduced determinants:

$$\begin{aligned}\mathcal{M}_{n,0}=\int_{\mathscr{M}_{n,1}}\mathrm{d}\mu\,\Bigg[&X^2\sum_{i,j}\left(\frac{(\xi 1)^2(i2)(j2)+(\xi 2)^2(i1)(j1)}{(1i)(2i)(1j)(2j)(\xi i)(\xi j)}\right)\left|\mathbb{H}^{12i}_{12j}\right|\\ &X^2\sum_i\frac{(12)^2}{(1i)^2(2i)^2}\left|\mathbb{H}^{12i}_{12i}\right|+\frac{1}{\Lambda}\sum_{i,j,k,l}\omega^1_{ij}\omega^2_{kl}\left|\mathbb{H}^{12ijkl}_{12ijkl}\right|\Bigg]\prod_{m=1}^{n}h(Z(\sigma_m))\,\mathrm{D}\sigma_m,\end{aligned}\quad(4.28)$$

where we have reverted to arbitrary twistor wavefunctions. Note that not only does (4.28) have the desired thrice-reduced determinants, but it also features Vandermonde factors in the coordinates σ_i which are known to arise in the context of twistor-string theory. Of course, actually deriving this formula from Skinner's twistor-string remains an important task to which we hope to turn in future work. In particular, it is not immediately clear how the final term (with a six-times reduced determinant) might arise from the twistor-string theory.

5 Discussion & Conclusion

In this paper we have derived a twistorial formula for the MHV scattering amplitude of Einstein gravity with a cosmological constant. This builds on the many recent advances in understanding the tree-level S-matrix of gravity on a flat background, and demonstrates yet another application for twistor methods. The key ingredients were the conformal/Einstein gravity correspondence, the twistor action for conformal gravity, and the $\mathbb{CP}^1$ Feynman diagram calculus we used to extract the amplitude. This approach leaves many interesting open directions for future research, and we conclude by discussing some of them here. Of particular interest is the potential to derive a *MHV formalism* for conformal–and hence Einstein–gravity using the twistor action, but we also consider the possibility of studying non-minimal CSG on twistor space as well as how the twistorial formula presented here could be translated into a well-defined physical observable in momentum space.

5.1 The CSW gauge and the MHV formalism

One of the important applications of the twistor action for $\mathcal{N}=4$ SYM is that it leads to a derivation of the MHV formalism [7] for Yang-Mills by virtue of an axial gauge choice [19, 20]. The key benefit of the gauge choice is that it exploits the integrability of the self-dual sector, essentially trivializing it by knocking out the non-linear terms in the self-dual part of the action so that the only vertices are those arising from the non-local part of the action. The existence of an MHV formalism for gravity remains controversial [21, 22].[11] Nevertheless, we can still carry out the axial gauge-fixing procedure for the twistor action of conformal gravity, which leads to a twistorial definition of a MHV formalism. Upon restricting to Einstein states, this induces a twistorial MHV formalism for Einstein gravity. We outline this argument here, leaving a more complete treatment and (hopefully) a translation to a momentum space formalism for the future.

Let us work with the $\mathcal{N}=0$ twistor action of (3.13). A choice of gauge in our gravitational context is a choice of coordinates together with a choice of gauge for the Dolbeault representative g. In order to do this, we choose a reference twistor denoted Z_* and the key idea is to require that the lines through Z_* in the flat background are also holomorphic lines for the deformed twistor space $\mathbb{P}\mathscr{T}$. This implies that the $(0,1)$-form part of f^α vanishes on restriction to these lines. Similarly, the gauge freedom for g_α is chosen so that the $(0,1)$-form part of g_α vanishes on restriction to any of these lines. This has the effect of reducing the cubic term in the self-dual part of the twistor action (3.8) to zero so that the only vertices are those that arise from the expansion of S_2. Section 4 indicates that this will generate MHV vertices upon restricting to Einstein states.

The other simplification that arises is in the propagator $\Delta(Z,Z')$. After fixing the axial gauge, the kinetic portion of the twistor action is just

$$S^{\mathrm{kin}}[g,f]=\int_{\mathbb{P}\mathscr{T}}\mathrm{D}^3Z\wedge g_\alpha\wedge\bar{\partial}f^\alpha,$$

so finding the propagator corresponds to inverting the $\bar{\partial}$-operator on twistor space as well as incorporating the remaining freedom in g_α, f^β into its tensor structure.

For the scalar portion of the propagator serving as a Green's current for $\bar{\partial}$ on $\mathbb{P}\mathscr{T}$, we can take our cue from the twistor action of Yang-Mills theory [20] where the kinetic

[11] It is worth mentioning the recent work of [71], which proposes an MHV-like formalism based on delta-function relaxation in a Grassmannian representation of the gravitational amplitudes [14, 72].

operator is also $\bar{\partial}$. In this case, the scalar part of the propagator is essentially a delta function which restricts the two field points Z, Z' to lie on a line through the reference twistor Z_*; on this line, it produces the Cauchy kernel for $\bar{\partial}$.

In the bosonic, gravitational context at hand, the appropriate objects to consider are

$$\Delta_k(Z, Z') \equiv \bar{\delta}^2_{0,k,-k-4}(Z_*, Z, Z') = \int_{\mathbb{C}^2} \frac{\mathrm{d}s}{s^{1+k}} \frac{\mathrm{d}t}{t^{-3-k}} \bar{\delta}^4(Z_* + sZ + tZ'). \tag{5.1}$$

These enforce the projective collinearity of their arguments, and are of weight k and $-k-4$ in Z and Z' respectively. To see this, recall the behavior of the Cauchy kernel on $\mathbb{C}$:

$$\bar{\delta}(z) = \delta(x)\delta(y)\,\mathrm{d}\bar{z} = \frac{1}{2\pi i}\bar{\partial}\left(\frac{1}{z}\right), \qquad z = x + iy,$$

which is supported at the origin. In (5.1), we simply take

$$\bar{\delta}^4(Z) = \frac{1}{2\pi i}\bigwedge_{\alpha=0}^{3} \bar{\partial}\frac{1}{Z^\alpha}.$$

The parameter integrals over $\mathrm{d}s$ and $\mathrm{d}t$ reduce this to a projective current with the appropriate weights. It can be demonstrated that the Δ_k obey the axial gauge condition up to potential anomalies resulting from this gauge choice which can be removed by working on an appropriate choice of $\mathbb{P}\mathscr{T}$ [20].

Thus for the propagator between $f^\alpha(Z)$ and $g_\beta(Z')$ the scalar portion of the propagator should be $\Delta_1(Z, Z')$. However, we also need to fix the gauge freedoms in the f^α and g_β by imposing $\partial_\alpha f^\alpha = 0$ and $Z^\alpha g_\alpha = 0$. Since we are on a projective twistor space and the freedom in f^α corresponds to adding multiples of Z^α, we only really need to deal with the condition on g_β. This can be accounted for with the tensor structure of the propagator, leaving us with the full propagator

$$\Delta^\alpha_\beta(Z, Z') := \delta^\alpha_\beta \Delta_1(Z, Z') - \frac{1}{4} Z^\alpha \partial'_\beta \Delta_0(Z, Z'), \tag{5.2}$$

so that $Z'^\beta \Delta^\alpha_\beta = 0$ (up to an irrelevant anomaly proportional to the reference twistor).

The Feynman rules in twistor space are obtained by making diagrams out of the MHV vertices for conformal gravity and gluing them together with propagators. Unlike the Yang-Mills case, the MHV vertices are already quite complicated. For conformal gravity, they can be obtained as above using states defined relative to the choice of infinity twistors, but where $I^{\alpha\beta}$ and $I_{\alpha\beta}$ are independent. The expressions given above for the MHV vertex (before any contraction $I^{\alpha\beta}I_{\beta\gamma} = \Lambda\delta^\alpha_\gamma$ is used) determine the general conformal gravity expression by a perturbiner argument. This follows from the invariance under permutations of the positive-helicity states and the negative-helicity states. Thus one can set $I^{\alpha\beta} = \sum_i \epsilon_i I^{\alpha\beta}_i$, expand to first order in each ϵ_i and take the coefficient of $\prod \epsilon_i$ and similarly for the two occurrences of $I_{\alpha\beta}$. We will not develop this any further as the primary focus here is to deduce an MHV formalism for Einstein gravity from that for conformal gravity.

The MHV formalism should reduce to one for Einstein gravity, at least at tree level, by applying the conformal/Einstein gravity correspondence. This entails restricting to

Einstein wave functions with cosmological constant Λ and dividing the overall expression by Λ. In the MHV vertex, all occurrences of the infinity twistor will now be standard ones and the Einstein formulae given above will be valid with the caveat that they are understood in a Dolbeault format to be valid *off-shell* and to be compatible with the formulae for the propagator. On reduction to Einstein forms for $f^\alpha = I^{\alpha\beta}\partial_\beta h$ and $g_\beta = I_{\beta\gamma}Z^\gamma \tilde{h}$, we note that the kinetic part of the twistor action reduces to

$$\int_{\mathbb{P}\mathscr{T}} \mathrm{D}^3 Z \wedge g_\alpha \wedge \bar{\partial} f^\alpha = 2\Lambda \int_{\mathbb{P}\mathscr{T}} \mathrm{D}^3 Z \wedge \tilde{h} \wedge \bar{\partial} h,$$

using $I_{\alpha\beta}I^{\beta\gamma} = \Lambda\delta_\alpha^\gamma$ and the Euler homogeneity relation for h. This is in accordance with (3.22). Thus the propagator reduces to

$$\Delta^{\mathrm{Ein}}(Z, Z') = \frac{1}{2\Lambda}\Delta_2(Z, Z'). \tag{5.3}$$

At least in the first instance, this indicates that the MHV formalism for Einstein gravity which is produced by this procedure is based on the MHV vertices for conformal gravity $\mathcal{C}_{n,0} = \Lambda\mathcal{M}_{n,0}$ restricted to Einstein states. Here, $\mathcal{M}_{n,0}$ should be understood as (4.17) with the $(0,1)$-form wavefunctions allowed to be off-shell so that $\mathcal{M}_{n,0}$ is extended to a *vertex*.[12] Thus, for an N^kMHV tree amplitude we will need to sum diagrams with $k+1$ such vertices and k propagators and then divide by the overall factor of Λ required by the conformal/Einstein gravity correspondence. These diagrams will be built from $k+1$ $\mathcal{C}_{n,0}$ vertices together with k Δ_2 propagators and a factor of $\Lambda^{-(k+1)}$. In terms of purely Einstein building blocks, we will have a diagrams constructed out of $k+1$ $\mathcal{M}_n$ vertices and k propagators given by Δ_2.

Although this is sufficient to develop formulae for gravity amplitudes in twistor space along the lines of [20] for Yang-Mills, there is much work to be done to make contact with momentum space formulae particularly for $\Lambda = 0$ where the factors of $\Lambda^{-(k+1)}$ will need to be cleared. This will of course be reasonably straightforward if we can extend the formula (4.28) for the amplitude to one for the vertex.

5.2 Non-minimal twistor actions

The key tool in deriving (4.17) was the minimal $\mathcal{N} = 4$ CSG twistor action. While we do not expect Maldacena's argument to apply to non-minimal $\mathcal{N} = 4$ conformal supergravities, it is nevertheless interesting to ask if a twistor action principle can be found. We outline here a proposal for how a twistor action describing a particular version of non-minimal $\mathcal{N} = 4$ CSG due to Berkovits and Witten [4]. While we do not attempt to prove that our twistor action corresponds to this theory, we argue that its perturbation theory will produce all of the expected tree-level scattering amplitudes. Of course, there are unresolved questions as to whether such a theory is well-defined at the quantum level [49, 50], but all of our considerations here will be classical.

[12] However, it is worth noting that the arguments which derived (4.19) from (4.17) were on-shell in character, so it is unclear as to whether they can be extended to the off-shell context required for vertices. This is particularly the case for the subsequent formula (4.28) based on $n-3$-determinants.

Non-minimal versions of $\mathcal{N}=4$ CSG are highly non-unique: arbitrary analytic functions can couple the scalar φ to the conformal gravitons of the theory. This can also be captured at the level of a chiral superspace action. In the minimal case, we saw that the action (3.18) served to define a chiral superspace action in terms of $\mathcal{W}$. However, since $\mathcal{W}$ has conformal weight zero, an action of the form

$$S[\mathcal{W}] = \int_M \mathrm{d}\mu\, F(\mathcal{W}) + \int_{\bar{M}} \mathrm{d}\bar{\mu}\, \overline{F(\mathcal{W})},$$

where $\bar{M}$ is the anti-chiral super-manifold, will be conformal and supersymmetric for *any* holomorphic function F. While $F(\mathcal{W}) = \mathcal{W}^2$ corresponds to the minimal theory, other choices clearly lead to interactions between the scalars and conformal gravitons. For instance, $F(\mathcal{W}) = \mathcal{W}^3$ will clearly give a Lagrangian term $\varphi\Psi^{ABCD}\Psi_{ABCD}$.

The twistor-string theory of Berkovits and Witten appears to correspond to a very particular choice of non-minimal $\mathcal{N}=4$ CSG, with holomorphic function $F(\mathcal{W}) = \mathrm{e}^{2\mathcal{W}}$ [4]. We refer to this as Berkovits-Witten CSG, or BW-CSG for short. As a classical $\mathcal{N}=4$ theory, it is easy to distinguish BW-CSG from the minimal theory by looking at its scattering amplitudes. In the twistor-string theory for BW-CSG one finds a degree zero three-point amplitude of the form [4, 16, 48]:

$$\int \mathrm{D}^{3|4}Z \wedge \left(\partial_K f_1^I \partial_I f_2^J \partial_J f_3^K - \partial_J f_1^I \partial_K f_2^J \partial_I f_3^K\right). \tag{5.4}$$

Applying the Penrose transform, it is easy to see that this amplitude corresponds to a term $\bar{\varphi}\widetilde{\Psi}^{A'B'C'D'}\widetilde{\Psi}_{A'B'C'D'}$ in the space-time action.

Similarly, at degree one, there are amplitudes with an arbitrary number of g-insertions; at three-points, this provides the parity conjugate of (5.4). The n-point version of this amplitude is clearly generated by the chiral part of the space-time action:

$$\int_M \mathrm{d}\mu\, \exp\left(\mathcal{W}(x,\theta)\right) = \sum_{n=2}^{\infty} \int_{M^0} \mathrm{d}\mu^0\, \varphi^{n-2}\, \Psi^{ABCD}\Psi_{ABCD} + \cdots,$$

where $\mathrm{d}\mu^0$ denotes the measure on the bosonic body M^0. Parity invariance demands that we therefore have n-point analogues of (5.4), coming from the anti-chiral part of the space-time action.

Let us try to find a corresponding twistor action: our strategy is to proceed by requiring the twistorial theory to produce the tree-level scattering amplitudes of BW-CSG. To begin, we note that BW-CSG still has an anti-MHV three point amplitude (like the minimal theory); this comes from the self-dual twistor action we had before:

$$S_1[g,f] = \int_{\mathbb{P}\mathscr{T}} \mathrm{D}^{3|4}Z \wedge g_I \wedge \left(\bar{\partial} f^I + [f,f]^I\right). \tag{5.5}$$

Similarly, the twistorial version of $\int \mathrm{d}\mu\, \mathrm{e}^{\mathcal{W}}$ is an easy generalization of

$$S^{\mathrm{chiral}}[g,f] = \int_M \mathrm{d}\mu\, \exp\left(\int_X g\right). \tag{5.6}$$

If we expand in fermionic variables, it is clear that on space-time this is the chiral portion of the action

$$S^{\mathrm{chiral}} \sim \int \mathrm{d}\mu^0 \ \exp(\varphi) \ \Psi^{ABCD}\Psi_{ABCD} + \cdots ,$$

as expected.

We still need to obtain the parity conjugates of the amplitudes generated by (5.6). Consider a holomorphic Chern-Simons theory on the tangent bundle $T_{\mathbb{P}\mathscr{T}}$:

$$S^{\mathrm{hCS}}[g,f] = \int_{\mathbb{P}\mathscr{T}} \mathrm{D}^{3|4}Z \wedge \mathrm{tr}\left(f\wedge\bar{\partial}f + \frac{2}{3} f\wedge f\wedge f \right) \tag{5.7}$$

Clearly, the cubic term in this action leads to the three-point amplitude (5.4) of BW-CSG. The quadratic term in (5.5) leads to the $g-f$-propagator (5.2), so we can tie any number of $\overline{\mathrm{MHV}}$-vertices onto (5.4) to form a n-point amplitude which has all f external states. These all-f amplitudes form the parity-conjugate set to the all-g amplitudes generated by (5.6).

Hence, we conjecture that the twistor action

$$S^{\mathrm{BW-CSG}}[g,f] = S_1[g,f] + S^{\mathrm{hCS}}[g,f] - \varepsilon^2 S^{\mathrm{chiral}}[g,f], \tag{5.8}$$

should be (classically) equivalent to the non-minimal $\mathcal{N}=4$ CSG of Berkovits and Witten. Of course, our argument relies entirely upon the fact that (5.8) has the same tree amplitudes as BW-CSG. Furthermore, it is rather unfortunate that the anti-chiral portion of the space-time action is encoded only implicitly (i.e., we do not have an explicit $\exp(\bar{\mathcal{W}})$ term on twistor space). In a sense, this is to be expected because parity invariance is often obscured in twistor space [73].

5.3 Physical observables

Throughout this paper, we have referred to $\mathcal{M}_{n,0}$ as a 'scattering amplitude' for general relativity on a background with cosmological constant. As pointed out in the introduction, we have adopted this nomenclature for convenience only: the notion of a physically observable scattering amplitude on de Sitter space is not even well-defined. While the final formulae we obtain for $\mathcal{M}_{n,0}$ in (4.17), (4.19) are on twistor space, and hence make mathematical sense for arbitrary twistorial wavefunctions, it is useful to have a brief discussion of how these can be interpreted as physical observables (i.e., expressions in momentum space).

In our expressions for $\mathcal{M}_{n,0}$, there are two ingredients about which we have been (deliberately) vague: the nature of the wavefunctions to be used, and the integration over the moduli space $\mathscr{M}_{n,1}$. *A priori*, the twistor wavefunctions are required only to be $(0,1)$-form cohomology classes with the appropriate weights as dictated by the Penrose transform. We often utilized the dual twistor states (4.18) for calculational purposes in Section 4, but these do not directly produce momentum space expressions. In the case of flat-space, or in Yang-Mills theory, one obtains momentum space expressions for scattering amplitudes by using *momentum eigenstates* (c.f., [9]), with a choice of four-momentum $k_{AA'} = p_A\tilde{p}_{A'}$

$$h(Z(\sigma),k) = \int_{\mathbb{C}} \frac{\mathrm{d}t}{t^{1+w}} \bar{\delta}(t\lambda_A - p_A)\ \mathrm{e}^{t[\mu\tilde{p}]}, \tag{5.9}$$

where $w = -6$ for a negative helicity graviton and $w = 2$ for a positive helicity graviton.

Of course, these are rather un-natural from the point of view of de Sitter geometry, since there are no four-momenta for the de Sitter group. Furthermore, these eigenstates are singular on a finite light cone and don't recognize the infinity of global de Sitter space. This is because they are most natural in Minkowski space, so using them in the de Sitter context corresponds to working with wavefunctions adapted to the affine de Sitter slicing of (2.7).

Similarly, in (4.17), (4.19) we need to choose a contour for the integral over $\mathrm{d}^8 X$ which corresponds to the real slice of space-time. In other words, this moduli integral can be thought of as an integral over the scattering background itself, and after fixing the $\mathrm{GL}(2,\mathbb{C})$-freedom in the measure, really acts as a $\mathrm{d}^4 x$ integral. Hence, using the eigenstates (5.9) and integrating over the full real slice in the affine coordinates corresponds to computing a $\mathscr{I}^-$ to $\mathscr{I}^+$ scattering process in the affine patch (2.7). For instance, in this set-up we would find [16]

$$\mathcal{M}_{3,0} = \frac{\langle 12\rangle^6}{\langle 23\rangle^2 \langle 31\rangle^2} \left(2 - \Lambda \Box_k\right) \delta^4\left(\sum_{i=1}^{3} k_i\right). \tag{5.10}$$

While this is not a *physical* observable (because no asymptotic observer can integrate over all of de Sitter space), it is well-defined mathematically and can be classed as a 'meta-observable' in the sense of [18, 36]. Further, it limits onto the definition of the scattering amplitude when $\Lambda \to 0$ and manifests the de Sitter isometries through the operator $\Box_k$ acting on the momentum-conserving delta-function. Using this prescription for $\mathcal{M}_{n,0}$ will produce an operator of leading order $\Box_k^{n-2}$.

To obtain an actual physical observable, one should use twistor eigenstates which are explicitly adapted to de Sitter space, and choose the contour of integration in $\mathscr{M}_{n,1}$ to correspond to a physically observable region of dS_4. These eigenstates can be defined by using a spinor-helicity formalism adapted to the spatial three-slices of de Sitter space, and the integration contour can be chosen in congruence with the in-in formalism prescriptions which have been used to calculate the non-Gaussianities in the gravitational bispectrum from inflation (c.f., [74, 75]). We hope to address these issues in much more detail in a future work.

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A Generating Functionals for MHV Amplitudes

In this appendix, we prove a concrete example of the conformal/Einstein gravity correspondence for generating functionals of tree-level MHV amplitudes. The final result amounts to (2.13) in the text, and was first derived in [37].

We will exploit the chiral formulation of general relativity [76]: for a general space-time M with metric specified by a tetrad of 1-forms $\mathrm{d}s^2 = \epsilon_{AB}\epsilon_{A'B'}e^{AA'} \otimes e^{BB'}$, the basic variables are three ASD 2-forms:

$$\Sigma^{AB} = e^{A'(A} \wedge e^{B)}_{A'},$$

and the ASD spin connection Γ_{AB}. With a cosmological constant Λ, the action of general relativity is:

$$S[\Sigma, \Gamma] = \frac{1}{\kappa^2} \int_M \left(\Sigma^{AB} \wedge F_{AB} - \frac{\Lambda}{6} \Sigma^{AB} \wedge \Sigma_{AB} \right), \tag{A.1}$$

where

$$F_{AB} = \mathrm{d}\Gamma_{AB} + \Gamma_A^C \wedge \Gamma_{BC} \tag{A.2}$$

is the curvature of the ASD spin connection. This action produces two field equations, to which we append a third (the condition that Σ^{AB} be derived from a tetrad) [24]:

$$\mathrm{D}\Sigma^{AB} = 0, \tag{A.3}$$

$$F_{AB} = \Psi_{ABCD}\Sigma^{CD} + \frac{\Lambda}{3}\Sigma_{AB}, \tag{A.4}$$

$$\Sigma^{(AB} \wedge \Sigma^{CD)} = 0. \tag{A.5}$$

Here, D is the covariant derivative with respect to the ASD spin connection:

$$\mathrm{D}\Sigma^{AB} = \mathrm{d}\Sigma^{AB} + 2\Gamma^{(A}_{C} \wedge \Sigma^{B)C}.$$

Following [35], we can express a tree-level MHV amplitude as the classical scattering of two negative helicity gravitons off a SD background space-time, which (perturbatively) encodes the remaining positive helicity gravitons. For a SD background, we have $\Psi_{ABCD} = 0$, so (A.4) can be solved for Σ in terms of F while (A.3), (A.5) result in an algebraic condition on the curvature of the ASD spin connection. To be precise, a SD solution (Σ_0, Γ_0) obeys [77]:

$$\Sigma_0^{AB} = \frac{3}{\Lambda} F_0^{AB}, \tag{A.6}$$

$$F_{0(AB} \wedge F_{0\,CD)} = 0. \tag{A.7}$$

Now consider small perturbations away from this SD background of the form $\Sigma = \Sigma_0 + \sigma_0$, $\Gamma = \Gamma_0 + \gamma$. This results in a set of linearized field equations:

$$\mathrm{D}_0\sigma^{AB} = -2\gamma^{(A}_{C} \wedge \Sigma_0^{B)C}, \tag{A.8}$$

$$\mathrm{D}_0\gamma_{AB} = \psi_{ABCD}\Sigma_0^{CD} + \frac{\Lambda}{3}\sigma_{AB}, \tag{A.9}$$

$$\sigma^{(AB} \wedge \Sigma_0^{CD)} = 0, \tag{A.10}$$

where D_0 is the covariant derivative with respect to the background ASD spin connection Γ_0. It is fairly easy to see that the field ψ_{ABCD} corresponds to a linearized ASD Weyl spinor propagating on the SD background (Σ_0, Γ_0) [37].

Our goal is now to formalize the picture of an MHV amplitude in terms of linearized solutions propagating on a SD background. If $\mathcal{S}$ is the space of solutions to the full field equations (A.3)-(A.5), then solutions to the linearized equations (A.8)-(A.10) are a vector space V corresponding to the fiber of $T\mathcal{S}$ over the SD solution (Σ_0, Γ_0). Now, a linearized SD solution is fully characterized by the ASD spin connection, since

$$\sigma_{AB} = \frac{3}{\Lambda}\mathrm{D}_0\gamma_{AB}, \qquad \mathrm{D}_0\gamma^{(AB} \wedge F_0^{CD)} = 0. \tag{A.11}$$

This allows us to define the SD portion of V as

$$V^+ = \left\{(\sigma, \gamma) \in V \,:\, \mathrm{D}_0\gamma^{(AB} \wedge F_0^{CD)} = 0\right\},$$

and a corresponding V^- by the quotient map in the short exact sequence:

$$0 \longrightarrow V^+ \hookrightarrow V \longrightarrow V^- \longrightarrow 0.$$

In particular, this means we have

$$V^- \equiv V/V^+ = \{(\sigma,\gamma) \in V\} \,/\, \left\{\gamma \,:\, \mathrm{D}_0\gamma^{(AB} \wedge F_0^{CD)} = 0\right\}.$$

The space of solutions $\mathcal{S}$ comes equipped with a natural symplectic form ω given by the boundary term in the action [78]:

$$\omega = \frac{1}{\kappa^2}\int_C \delta\Sigma^{AB} \wedge \delta\Gamma_{AB}, \tag{A.12}$$

where C is a Cauchy surface in M (when $\Lambda > 0$, there is always a slicing where $C \cong S^3$ topologically) and δ is the exterior derivative on $\mathcal{S}$. It is straightforward to show that ω is independent of the choice of Cauchy surface and descends to a symplectic form on $\mathcal{S}/\mathrm{Diff}_0^+(M)$ [37].

This symplectic form induces an inner product between points in the linearized solution space V: for $h_i, h_j \in V$ we take

$$\langle h_i | h_j \rangle = -\frac{i}{\kappa^2}\int_C \sigma_j^{AB} \wedge \gamma_{i\,AB}. \tag{A.13}$$

An important fact about this inner product (which is obvious in the $\Lambda = 0$ setting, c.f., [35]) is that it annihilates the SD sector:

Lemma A.1 *Let $h_i, h_j \in V^+$ on the SD background with (Σ_0, Γ_0). Then $\langle h_i|h_j\rangle = 0$, or equivalently: for all $h_i \in V^+$, $\langle h_i|\cdot\rangle|_{V^+} = 0$.*

Proof: The inner product is skew-symmetric under interchange of h_i and h_j, so

$$\langle h_i | h_j \rangle = -\frac{i}{2\kappa^2}\int_C \left(\sigma_j^{AB} \wedge \gamma_{i\,AB} - \sigma_i^{AB} \wedge \gamma_{j\,AB}\right).$$

Suppose $h_j \in V^+$; then (A.9) implies that $\mathrm{D}_0\gamma_{j\,AB} = \frac{\Lambda}{3}\sigma_{j\,AB}$. In the $\Lambda = 0$ limit, the ASD spin connection is trivial $\mathrm{D}_0 \to \mathrm{d}$, so $\gamma_j^{AB}|_{\Lambda=0} = 0$, and we can write $\gamma_j^{AB} = \Lambda\nu_j^{AB}$ for some array of space-time 1-forms ν_i^{AB}. With this representation, the linearized SD field equation gives $\sigma_{j\,AB} = 3\mathrm{D}_0\nu_{j\,AB}$, and the inner product becomes:

$$-\frac{i}{2\kappa^2}\int_C \left(3\mathrm{d}\nu_j^{AB}\wedge\gamma_{i\,AB} + 6\Gamma^{(A}_{0\;C}\wedge\nu_j^{B)C}\wedge\gamma_{i\,AB} - \sigma_i^{AB}\wedge\gamma_{j\,AB}\right)$$
$$= \frac{i}{2\kappa^2}\int_C \left(3\nu_j^{AB}\wedge \mathrm{D}_0\gamma_{i\,AB} - \sigma_i^{AB}\wedge\gamma_{j\,AB}\right),$$

where the second line follows by integration by parts and a re-arranging of index contractions. Once again using $\gamma_{j\,AB} = \Lambda\nu_{j\,AB}$, we have:

$$\langle h_i|h_j\rangle = \frac{i}{2\kappa^2}\int_C \nu_j^{AB}\wedge(3\mathrm{D}_0\gamma_{i\,AB} - \Lambda\sigma_{i\,AB}) = \frac{3i}{2\kappa^2}\int_C \nu_j^{AB}\wedge\psi_{i\,ABCD}\Sigma_0^{CD},$$

using (A.9) for h_i. Hence, if $h_i \in V^+$ then $\psi_{i\,ABCD} = 0$ and the inner product vanishes. $\Box$

Note that lemma A.1 confirms that the all-positive helicity and $(-+\cdots+)$ amplitudes of general relativity vanish even with a cosmological constant in play. In the first case, we see that the SD field equations are integrable since their solutions are characterized by a single algebraic relation (A.7). In the second case, the fact that the inner product annihilates the SD sector ensures that scattering with only a single negative helicity graviton is also trivial.

We can use this inner product to define ASD solutions at the boundary of our M as in [35]: take a one-parameter family of Cauchy hypersurfaces $C_t \to \mathscr{I}^{\pm}$ as $t\to\pm\infty$. Then we say that $h_j = (\sigma_j, \gamma_j)$ is ASD at $\mathscr{I}^{\pm}$ if

$$\lim_{t\to\pm\infty}\int_{C_t}\sigma_j^{AB}\wedge\gamma_{i\,AB} = 0 \qquad \text{for all } h_i = (\sigma_i,\gamma_i)\in V^-. \tag{A.14}$$

Now we want to build the generating functional for the MHV amplitudes, which measure the probability for a pure ASD state at $\mathscr{I}^-$ to propagate across a SD background M and evolve into a SD state at $\mathscr{I}^+$. Hence, we take the incoming state to be $h_1|_{\mathscr{I}^-} \in V^-$. Since the inner product annihilates the SD sector, we need to compute the inner product between h_1 and some other state $h_2|_{\mathscr{I}^+} \in V^-$ at the future conformal boundary $\mathscr{I}^+$:[13] This gives the generating functional for the MHV amplitudes as

$$I^{\mathrm{GR}}[1^-,2^-,M^+] = \langle h_2|h_1\rangle = -\frac{i}{\kappa^2}\int_{\mathscr{I}^+}\sigma_1^{AB}\wedge\gamma_{2\,AB}. \tag{A.15}$$

This form of the generating functional is not particularly illuminating because the role of the SD background M is extremely implicit. However, we can manipulate (A.15) into a format which is explicitly in terms of an integral over the entire background space-time.

Proposition A.1 *The amplitude $\langle h_n|h_1\rangle$ is given by the formula:*

$$I^{\mathrm{GR}}[1^-,2^-,M^+] = \frac{i}{\kappa^2}\int_M\left(\Sigma_0^{AB}\wedge\gamma_{1\,A}^{\;C}\wedge\gamma_{2\,CB} - \frac{\Lambda}{3}\sigma_1^{AB}\wedge\sigma_{2\,AB}\right), \tag{A.16}$$

where M is a SD background space-time described by (Σ_0, Γ_0).

[13] As mentioned in the text, this corresponds to a 'meta-observable' since we integrate over the entire space-like surface $\mathscr{I}^+$.

Proof: Recall that $\partial M = \mathscr{I}^+ - \mathscr{I}^-$, so Stokes' theorem gives

$$-\frac{i}{\kappa^2}\int_{\mathscr{I}^+} \sigma_1^{AB}\wedge\gamma_{2\,AB} = -\frac{i}{\kappa^2}\int_M \left(\mathrm{d}\sigma_1^{AB}\wedge\gamma_{2\,AB} + \sigma_1^{AB}\wedge\mathrm{d}\gamma_{2\,AB}\right) - \frac{i}{\kappa^2}\int_{\mathscr{I}^-} \sigma_1^{AB}\wedge\gamma_{2\,AB}.$$

Now, the second term on the right vanishes, since $h_1 \in V^-$ at $\mathscr{I}^-$. Using the linearized field equations (A.8), (A.9) it follows that

$$\begin{aligned}
\mathrm{d}\sigma_1^{AB} &= -2\gamma_{1\,C}^{(A}\wedge\Sigma_0^{B)C} - 2\Gamma_{0\,C}^{(A}\wedge\sigma_1^{B)C}, \\
\mathrm{d}\gamma_{2\,AB} &= \psi_{2\,ABCD}\Sigma_0^{CD} + \frac{\Lambda}{3}\sigma_{2\,AB} - 2\Gamma_{0\,C(A}\wedge\gamma_{2\,B)}^{C},
\end{aligned}$$

and the generating functional becomes

$$\begin{aligned}
\frac{i}{\kappa^2}\int_M \Big(\Sigma_0^{AB}\wedge\gamma_{1\,A}^{C}\wedge\gamma_{2\,CB} + \sigma_1^{AB}\wedge\Gamma_{0\,A}^{C}\wedge\gamma_{2\,CB} + \sigma_1^{AB}\wedge\Gamma_{0\,CA}\wedge\gamma_{2\,B}^{C} \\
-\frac{\Lambda}{3}\sigma_1^{AB}\wedge\sigma_{2\,AB} - \sigma_1^{AB}\wedge\psi_{2\,ABCD}\Sigma_0^{CD}\Big).
\end{aligned}$$

The last term vanishes due to the linearized field equation (A.10) and the fact that $\psi_{ABCD} = \psi_{(ABCD)}$, while the second and third terms cancel after restructuring the spinor indices.

All that remains is to check that (A.16) has the correct gauge invariance: if one of the ASD states is pure gauge, the amplitude must vanish. Suppose that h_1 is pure gauge: $\psi_{1\,ABCD} = 0$. By (A.11), we know that $\frac{\Lambda}{3}\sigma_1^{AB} = \mathrm{D}_0\gamma_1^{AB}$, and integrating by parts in (A.16) gives

$$I^{\mathrm{GR}}[1^-, 2^-, M^+]|_{\psi_1=0} = \frac{i}{\kappa^2}\int_M \left(\Sigma_0^{AB}\wedge\gamma_{1\,A}^{C}\wedge\gamma_{2\,CB} + \gamma_1^{AB}\wedge\mathrm{D}_0\sigma_{2\,AB}\right) - \int_{\partial M}\gamma_1^{AB}\wedge\sigma_{2\,AB}.$$

The boundary term vanishes at $\mathscr{I}^+$ since $h_2|_{\mathscr{I}^+} \in V^-$, and also at $\mathscr{I}^-$ since h_1 is pure gauge. This leaves us with the bulk terms, which can be evaluated using the linearized field equation (A.8) for h_2:

$$\begin{aligned}
&\int_M \left(\Sigma_0^{AB}\wedge\gamma_{1\,A}^{C}\wedge\gamma_{2\,CB} + \gamma_1^{AB}\wedge\mathrm{D}_0\sigma_{2\,AB}\right) \\
&\qquad = \int_M \left(\Sigma_0^{AB}\wedge\gamma_{1\,A}^{C}\wedge\gamma_{2\,CB} - 2\gamma_1^{AB}\wedge\gamma_{2\,C(A}\wedge\Sigma_{0\,B)}^{C}\right) = 0,
\end{aligned}$$

with the final equality following after re-arranging contractions on spinor indices. □

The final step is to obtain the conformal/Einstein gravity correspondence for this generating functional. Upon restricting to Einstein scattering states, it is obvious that the generating functional in conformal gravity with two negative helicity gravitons and a SD background is given by the second term in (2.4):

$$I^{\mathrm{CG}}[1^-, 2^-, M^+] = \frac{2i}{\varepsilon^2}\int_M \mathrm{d}\mu\;\psi_1^{ABCD}\psi_{2\,ABCD}, \tag{A.17}$$

where M is again the SD background which encodes the $n-2$ remaining gravitons. By the conformal/Einstein gravity correspondence, we should be able to relate I^{CG} to I^{GR} on-shell (i.e., by apply the field equations of general relativity), and this is indeed the case [37].

Proposition A.2 *On-shell,* $I^{\mathrm{GR}}[1^-, 2^-, M^+] = -\frac{3\varepsilon^2}{\Lambda\kappa^2} I^{\mathrm{CG}}[1^-, 2^-, M^+]$.

Proof: (A.17) is equivalent to

$$I^{\mathrm{CG}}[1^-, 2^-, M^+] = \frac{i}{\varepsilon^2} \int_M \psi_1^{ABCD} \Sigma_{0\ CD} \wedge \psi_{2\ ABEF} \Sigma_0^{EF}.$$

Using the linearized field equation (A.9) for h_2, this becomes

$$\frac{i}{\varepsilon^2} \int_M \psi_1^{ABCD} \Sigma_{0\ CD} \wedge \left(\mathrm{D}_0 \gamma_{2\ AB} - \frac{\Lambda}{3} \sigma_{2\ AB} \right).$$

Integrating by parts in the first term gives

$$- \int_M \mathrm{D}_0 \psi_1^{ABCD} \Sigma_{0\ CD} \wedge \gamma_{2\ AB} + \int_{\partial M} \psi_1^{ABCD} \Sigma_{0\ CD} \wedge \gamma_{2\ AB} = \int_{\partial M} \psi_1^{ABCD} \Sigma_{0\ CD} \wedge \gamma_{2\ AB},$$

since ψ_1 is a linearized Weyl spinor. In the second term, a combination of both field equations (A.9) for h_1 and (A.8) for h_2 as well as integration by parts leaves

$$-\frac{2\Lambda}{3} \int_M \gamma_1^{AB} \wedge \gamma_{2\ C(A} \wedge \Sigma_{0\ B)}^{C} + \frac{\Lambda^2}{9} \int_M \sigma_1^{AB} \wedge \sigma_{2\ AB} - \frac{\Lambda}{3} \int_{\partial M} \gamma_1^{AB} \wedge \sigma_{2\ AB}.$$

Combining both terms gives:

$$\begin{aligned} I^{\mathrm{CG}}[1^-, 2^-, M^+] &= \frac{i}{\varepsilon^2} \left(-\frac{2\Lambda}{3} \int_M \gamma_1^{AB} \wedge \gamma_{2\ C(A} \wedge \Sigma_{0\ B)}^{C} + \frac{\Lambda^2}{9} \int_M \sigma_1^{AB} \wedge \sigma_{2\ AB} \right) \\ &\quad - \frac{i}{\varepsilon^2} \left(\int_{\partial M} \psi_1^{ABCD} \Sigma_{0\ CD} \wedge \gamma_{2\ AB} - \frac{\Lambda}{3} \int_{\partial M} \gamma_1^{AB} \wedge \sigma_{2\ AB} \right) \\ &= -\frac{\Lambda\kappa^2}{3\varepsilon^2} I^{\mathrm{GR}}[1^-, 2^-, M^+] + \text{boundary terms}. \end{aligned}$$

The proof is complete if we can show that the boundary terms vanish. Applying (A.9) to the first of these terms leaves us

$$\text{boundary terms} \sim \int_{\partial M} \mathrm{D}_0 \gamma_1^{AB} \wedge \gamma_{n\ AB} - \frac{\Lambda}{3} \int_{\partial M} \gamma_n^{AB} \wedge \sigma_{1\ AB} - \frac{\Lambda}{3} \int_{\partial M} \gamma_1^{AB} \wedge \sigma_{n\ AB},$$

with the second and third terms cancelling due to skew symmetry in h_1, h_n. Finally,

$$\begin{aligned} \int_{\partial M} \mathrm{D}_0 \gamma_1^{AB} \wedge \gamma_{n\ AB} &= \int_{\mathscr{I}^+} \mathrm{D}_0 \gamma_1^{AB} \wedge \gamma_{n\ AB} - \int_{\mathscr{I}^-} \mathrm{D}_0 \gamma_1^{AB} \wedge \gamma_{n\ AB} \\ &= - \int_{\mathscr{I}^+} \gamma_1^{AB} \wedge \mathrm{D}_0 \gamma_{n\ AB} - \int_{\mathscr{I}^-} \mathrm{D}_0 \gamma_1^{AB} \wedge \gamma_{n\ AB} = 0, \end{aligned}$$

by the fact that $h_1|_{\mathscr{I}^-} \in V^-$ and $h_n|_{\mathscr{I}^+} \in V^-$, as required. □

Note that the result of this proposition is in precise agreement with the prefactors predicted by Anderson's theorem in (2.11).

B Local Twistor Formalism

This appendix reviews the local twistor formalism and applies it to several issues from the text, proving and clarifying claims made there. We set out the basic formalism in the bosonic category; generalizations to supersymmetric twistor spaces are obvious. On $\mathbb{P}\mathscr{T}$, the twistor coordinates $Z^{\alpha}(x,\sigma)$ are abstract on until they are pulled back to the spinor bundle $\mathbb{PS}$; to get a concrete coordinate basis on the curved twistor space, we must use the *local twistor bundle*.

Let M be a four-manifold satisfying the conditions of theorem 1, and $\mathbb{P}\mathscr{T}$ be its associated twistor space. Local twistors are defined at points $x \in M$ as the fibers of the complex rank four bundle:

$$\begin{array}{ccc} Z^{\underline{\alpha}} = (\lambda_A, \mu^{A'}) & \longrightarrow & \mathbb{LT} \\ & & \big\downarrow \\ & & M \end{array}$$

Let $\mathbf{t} \in T_x M$ be a vector at x; then the infinitesimal variation of the local twistor bundle in the direction of $\mathbf{t}$ is [39]

$$\nabla_{\mathbf{t}} Z^{\underline{\alpha}}(x) = \left(t^{BB'}\nabla_{BB'}\lambda_A + i t^{BB'} P_{ABA'B'}\mu^{A'},\ t^{BB'}\nabla_{BB'}\mu^{A'} + i t^{BA'}\lambda_B \right), \tag{B.1}$$

where the tensor P_{ab} is the Schouten tensor:

$$P_{ABA'B'} = \Phi_{ABA'B'} - \Lambda\epsilon_{AB}\epsilon_{A'B'},$$

with $\Phi_{ABA'B'}$ the trace-free portion of the Ricci tensor. This local twistor transport along the vector $\mathbf{t}$ defines a *local twistor connection* on $\mathbb{LT}$ whose curvature can be computed by considering

$$i\left(\nabla_{\mathbf{t}}\nabla_{\mathbf{u}} - \nabla_{\mathbf{u}}\nabla_{\mathbf{t}} - \nabla_{[\mathbf{t},\mathbf{u}]}\right) Z^{\underline{\beta}} = Z^{\underline{\alpha}} F_{\underline{\alpha}}^{\underline{\beta}}(\mathbf{t},\mathbf{u}). \tag{B.2}$$

In the case where M is a SD background (as in theorem 1), this curvature is given by [39]

$$F_{\underline{\alpha}}^{\underline{\beta}}(\mathbf{t},\mathbf{u})\Big|_{M^+} = t_D^{C'} u^{DD'} \begin{pmatrix} 0 & \nabla^{A}_{A'}\widetilde{\Psi}^{B'A'}_{C'D'} \\ 0 & i\widetilde{\Psi}^{B'}_{C'D'A'} \end{pmatrix},$$

where $\widetilde{\Psi}_{A'B'C'D'}$ is the SD Weyl spinor of M. Hence, we see that on a SD background M, the local twistor bundle $\mathbb{LT}$ is half-flat, so the Ward transform applies [58] to give a rank four bundle $\mathbb{T}^{\underline{\alpha}} \to \mathbb{P}\mathscr{T}$ on twistor space [79]. Abusing terminology, we also refer to this bundle $\mathbb{T}^{\underline{\alpha}} \to \mathbb{P}\mathscr{T}$ as the 'local twistor bundle.'

By choosing a holomorphic frame $H^{\underline{\alpha}}_{\alpha}$ for $\mathbb{T}^{\underline{\alpha}}$, we can assign meaning to tensors on $\mathbb{P}\mathscr{T}$ [80]. For example, consider a tensor $f^{\alpha\cdots}_{\beta\cdots} \in H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(n-2))$ for $n<0$. Contracting with the holomorphic frame converts this to a $(0,1)$-form valued section of $\mathbb{T}^{\underline{\alpha}\cdots}_{\underline{\beta}\cdots} \otimes \mathcal{O}(n-2)$, to which we can apply the Penrose transform, obtaining a field on M:

$$\int_X \lambda_{A_1}\cdots\lambda_{A_n} f^{\underline{\alpha}\cdots}_{\underline{\beta}\cdots} \wedge \tau = \Gamma^{\underline{\alpha}\cdots}_{\underline{\beta}\cdots A_1\cdots A_n}.$$

The space-time field will obey a zero-rest-mass field equation

$$\nabla^{A_1A'}\Gamma^{\underline{\alpha}\cdots}_{\underline{\beta}\cdots A_1\cdots A_n}=0,$$

where the covariant derivative acts via the local twistor connection since the holomorphic frame on $\mathbb{T}^{\underline{\alpha}}$ corresponds to a covariantly constant frame on $\mathbb{LT}\to M$. From now on, we will drop the underline notation, and assume that the distinction between concrete and local twistor indices is clear from the context.

As an example, consider how ∇ acts on a space-time field with a single twistor index, say $\Gamma^{\beta}_{C}=(\Phi_{BC},\Psi^{B'}_{C})$ From (B.1), it follows that the covariant derivative acts as

$$\nabla^{AA'}\Gamma^{\beta}_{C}=\begin{pmatrix}\nabla^{AA'}\Phi_{BC}\\ \nabla^{AA'}\Psi^{B'}_{C}\end{pmatrix}+\begin{pmatrix}0 & iP^{AA'}_{BB'}\\ i\epsilon^{AB}\epsilon^{A'B'} & 0\end{pmatrix}\begin{pmatrix}\Phi_{BC}\\ \Psi^{B'}_{C}\end{pmatrix}. \tag{B.3}$$

Similar rules for dual twistor indices as well as higher-rank tensors can be derived or looked up in [39], and their space-time gauge freedom is fixed by computing the Penrose transform of $Z^{\gamma}f^{\alpha\cdots}_{\beta\cdots}$ and then imposing $Z^{\beta}f^{\alpha\cdots}_{\beta\cdots}=0$ [80].

From (B.3), we can see that the local twistor connection acts as $\nabla=\mathrm{D}+\mathscr{A}$, where D is the usual space-time covariant derivative and $\mathscr{A}\in\Omega^1_M(\mathfrak{psl}(4,\mathbb{C}))$ is the connection 1-form. This suggests that we can consider the local twistor bundle as a $\mathrm{PSL}(4,\mathbb{C})$-gauge bundle over space-time.

Theorem 3 (Merkulov [26]) *The local twistor bundle $\mathbb{LT}\to M$ is a $\mathrm{PSL}(4,\mathbb{C})$ gauge bundle with gauge-covariant derivative $\nabla=\mathrm{D}+\mathscr{A}$ given by* (B.3)*. Furthermore, this connection has curvature $F\in\Omega^2_M(\mathfrak{psl}(4,\mathbb{C}))$, and which is equal to* (B.2) *when M is self-dual.*

On twistor space, the chiral half of the fundamental fields of $\mathcal{N}=4$ CSG are meant to be encoded in the Lagrange multiplier $g\in H^{0,1}(\mathbb{P}\mathscr{T},\Omega^1)$, which defines the space-time chiral superfield $\mathcal{G}(x,\theta)$. In particular, we want each term in the expansion

$$g=g^0+\chi^a g^{-1}_a+\cdots\frac{\chi^4}{4!}g^{-4},$$

to correspond to a space-time field with the correct conformally invariant zero-rest-mass field equation via the Penrose transform. Since the CSG background is curved, we must use the local twistor formalism to operationalize this Penrose transform.

We begin with the leading term in the expansion, $g^0\in H^{0,1}(\mathbb{P}\mathscr{T},\Omega^1)$. The Penrose transform of this object was first described in [80]: write $g^0=a_\alpha \mathrm{d}Z^\alpha$ for $a_\alpha\in H^{0,1}(\mathbb{P}\mathscr{T},\mathcal{O}(-1))$. Picking a particular conformal frame, the Penrose transform gives:

$$\Gamma_{\alpha B'}=\begin{pmatrix}\Psi^{A}_{B'}\\ \Phi_{A'B'}\end{pmatrix}=\int_X\tau\wedge\frac{\partial a_\alpha}{\partial\mu^{B'}},\qquad \nabla^{BB'}\Gamma_{\alpha B'}=0. \tag{B.4}$$

Using the local twistor connection, the z.r.m. equations of (B.4) can be written on space-time as:

$$\begin{cases}\nabla^{BB'}\Psi^{A}_{B'}-i\epsilon^{BA}\Phi^{B'}_{B'}=0\\ \nabla^{BB'}\Phi_{A'B'}=0\end{cases},$$

while the Penrose transform of $Z^\alpha a_\alpha$ gives the conditions $\nabla_{BB'}\Psi^B_{A'} - i\epsilon_{B'A'}\Phi^{B'}_{B'} = 0$ and $\Phi^{B'}_{B'} = 0$. This means that we can write $\Psi_{AA'} = \Box\varphi$, and the content of (B.4) is reduced to $\Box^2\varphi = 0$, as desired.

Similar procedures are applied to the remaining components. For example, $g^{-4} = g_\alpha \mathrm{d}Z^\alpha$, should encode the conformal graviton. The Penrose transform gives:

$$\Gamma_{\delta ABC} = \begin{pmatrix} G^D_{ABC} \\ \gamma_{D'ABC} \end{pmatrix} = \int_X \tau \wedge \lambda_A\lambda_B\lambda_C g_\delta, \qquad \nabla^{AA'}\Gamma_{\delta ABC} = 0. \tag{B.5}$$

Recalling that ∇ acts on $\Gamma_{\delta ABC}$ via the local twistor connection,

$$\begin{cases} \nabla^{AA'}G^D_{ABC} - i\gamma^{A'D}_{BC} = 0 \\ \nabla^{AA'}\gamma_{D'ABC} - i\Phi^{AA'}_{DD'}G^D_{ABC} = 0 \end{cases}.$$

The gauge-fixing ensures that $G_{ABCD} = G_{(ABCD)}$, and this leaves us with

$$\left(\nabla^{AA'}\nabla^D_{D'} + \Phi^{ADA'}_{D'}\right) G_{ABCD} = 0, \tag{B.6}$$

which is the required Bach equation.

An identical procedure will give the following equations for the remaining components:

$$g^{-1}_a \Rightarrow \qquad \Box\nabla_{BB'}\psi^B_a - i\nabla_{AA'}\left(\Phi^{AA'}_{CB'}\psi^C_a\right) = 0, \tag{B.7}$$

$$g^{-2}_{ab} \Rightarrow \qquad \left(\nabla_{AA'}\nabla_{BB'} + \Phi_{ABA'B'}\right) T^{AB}_{ab} = 0, \tag{B.8}$$

$$g^{-3\,a} \Rightarrow \left(\nabla_{BD'}\nabla^{AA'}\nabla_{CC'} + \Phi^{AA'}_{CC'}\nabla_{BD'}\right)\eta^{a\,D'}_{AC} = 0. \tag{B.9}$$

These correspond to the spinor, ASD tensor, and conformal gravitino of $\mathcal{N} = 4$ CSG, respectively.

C Independence of the Reference Spinor

In this appendix we explicitly compute the infinitesimal variation $\mathrm{d}_\xi\mathcal{M}_{n,0}$. This is easiest if we use the representation of $\mathcal{M}_{n,0}$ given by (4.17); the proof of ξ-independence can also be accomplished using (4.19), but requires a bit more finesse.

We can compute the variation directly from (4.17) by using the basic property of determinants: $\mathrm{d}_\xi|\mathbb{H}| = \mathrm{tr}[\mathrm{adj}(\mathbb{H})\mathrm{d}_\xi\mathbb{H}]$. This leads to (ignoring irrelevant overall factors):

$$\begin{aligned} \mathrm{d}_\xi\mathcal{M}_{n,0} = \int_{\mathscr{M}_{n,1}} \mathrm{d}\mu\, \Bigg[&\sum_i \left|\mathbb{H}^{12i}_{12i}\right| \left((X^2)^2\, \mathrm{d}_\xi\mathbb{H}_{ii} + X^2\, \mathrm{d}_\xi\psi^1_i\right) \\ &+ \sum_{i,j} \left|\mathbb{H}^{12ij}_{12ij}\right| \left(X^2\, \psi^1_i\, \mathrm{d}_\xi\mathbb{H}_{jj} + X^2\, \mathrm{d}_\xi\omega^1_{ij} + \mathrm{d}_\xi\psi^1_i\, \psi^2_j + \psi^1_i\, \mathrm{d}_\xi\psi^2_j\right) \\ &+ \sum_{i,j,k} \left|\mathbb{H}^{12ijk}_{12ijk}\right| \left(X^2\, \omega^1_{ij}\, \mathrm{d}_\xi\mathbb{H}_{kk} + \psi^1_i\, \psi^2_j\, \mathrm{d}_\xi\mathbb{H}_{kk} + \mathrm{d}_\xi\psi^1_i\, \omega^2_{jk} + \psi^1_i\, \mathrm{d}_\xi\omega^2_{jk}\right) \\ &+ \sum_{i,j,k,l} \left|\mathbb{H}^{12ijkl}_{12ijkl}\right| \left(\psi^1_i\, \omega^2_{jk}\, \mathrm{d}_\xi\mathbb{H}_{ll} + \mathrm{d}_\xi\omega^1_{ij}\, \omega^2_{kl} + \omega^1_{ij}\, \mathrm{d}_\xi\omega^2_{kl}\right) \\ &+ \sum_{i,j,k,l,m} \left|\mathbb{H}^{12ijklm}_{12ijklm}\right|\, \omega^1_{ij}\, \omega^2_{kl}\, \mathrm{d}_\xi\mathbb{H}_{mm} \Bigg] \prod_{s=1}^n h(Z(\sigma_s))\, \mathrm{D}\sigma_s\, + (1 \leftrightarrow 2). \end{aligned} \tag{C.1}$$

To study $\mathrm{d}_\xi\mathcal{M}_{n,0}$, we need the individual variations which appear in (C.1). These are easily obtained by working with the dual twistor wavefunctions (4.18); after a bit of algebra (including the Schouten identity) we find

$$\mathrm{d}_\xi\mathbb{H}_{ii} = 2\sum_{j=1}^{n}\frac{[W_i, W_j](j\xi)}{(i\xi)^3}\mathrm{D}\xi = 2\frac{[W_i, \mathcal{P}\cdot\xi]}{(i\xi)^3}\mathrm{D}\xi, \tag{C.2}$$

$$\mathrm{d}_\xi\psi_i^1 = 2i\,\Lambda\frac{Z(\xi)\cdot W_i}{(i\xi)^3}\mathrm{D}\xi, \qquad \mathrm{d}_\xi\omega_{ij}^1 = 2\Lambda\frac{[W_i, W_j](ij)(1\xi)^3}{(1i)^2(1j)^2(i\xi)^3(j\xi)^3}\left[(j\xi)(i1)+(i\xi)(j1)\right]\mathrm{D}\xi. \tag{C.3}$$

We will now use these facts to show that $\mathrm{d}_\xi\mathcal{M}_{n,0}$ is a total divergence with respect to the moduli coordinates X^{JA}, and hence vanishes.

We can proceed order-by-order with respect to the sums appearing in (C.1). For instance, the integrand of the first line is

$$-2i\sum_i\left|\mathbb{H}^{12i}_{12i}\right|\left(i(X^2)^2\frac{[W_i, \mathcal{P}\cdot\xi]}{(i\xi)^3} - 2\Lambda\, X^2\frac{Z(\xi)\cdot W_i}{(i\xi)^3}\right)e^{i\mathcal{P}\cdot X}.$$

But upon inspection, this takes the form of a total divergence:

$$-\,2i\frac{\partial}{\partial X^{JA}}\left[(X^2)^2\,e^{i\mathcal{P}\cdot X}\sum_i\left|\mathbb{H}^{12i}_{12i}\right|\;I^{IJ}\frac{W_{i\,I}\xi^A}{(i\xi)^3}\right] \tag{C.4}$$

The key observation is that (for all terms contributing to $\mathrm{d}_\xi\mathcal{M}_{n,0}$) X-dependence only appears through explicit powers of X^2, the wavefunction factor of $e^{i\mathcal{P}\cdot X}$, ψ_i^1, or $\mathrm{d}_\xi\psi_i^1$. Applying this philosophy to the rest of (C.1), we can show that line-by-line it is equal to a total divergence.

If we refer to the contribution of (C.4) as the 'third-order' contribution (counting the number of rows and columns missing from the determinant factor), then divergences at each order are given as follows: At fourth-order,

$$-\,2i\frac{\partial}{\partial X^{JA}}\left[X^2\,e^{i\mathcal{P}\cdot X}\sum_{i,j}\left|\mathbb{H}^{12ij}_{12ij}\right|\;\psi_i^1\;I^{IJ}\frac{W_{j\,I}\xi^A}{(j\xi)^3}\right]+(1\leftrightarrow 2). \tag{C.5}$$

At fifth-order:

$$-\,2i\frac{\partial}{\partial X^{JA}}\left[e^{i\mathcal{P}\cdot X}\sum_{i,j,k}\left|\mathbb{H}^{12ijk}_{12ijk}\right|\left(X^2\omega_{ij}^1 - \psi_i^1\psi_j^2\right)I^{IJ}\frac{W_{k\,I}\xi^A}{(k\xi)^3}\right]\;(1\leftrightarrow 2). \tag{C.6}$$

At sixth-order:

$$-\,2i\frac{\partial}{\partial X^{JA}}\left[e^{i\mathcal{P}\cdot X}\sum_{i,j,k,l}\left|\mathbb{H}^{12ijkl}_{12ijkl}\right|\;\psi_i^1\;\omega_{jk}^2\;I^{IJ}\frac{W_{l\,I}\xi^A}{(l\xi)^3}\right]+(1\leftrightarrow 2). \tag{C.7}$$

At seventh-order, we only have a single term:

$$2\int\mathrm{d}\mu\sum_{i,j,k,l,m}\left|\mathbb{H}^{12ijklm}_{12ijklm}\right|\;\omega_{ij}^1\;\omega_{kl}^2\frac{[W_m, \mathcal{P}\cdot\xi]}{(m\xi)^3}\;e^{i\mathcal{P}\cdot X}\mathrm{d}^2\sigma.$$

After using the GL$(2,\mathbb{C})$-freedom to fix the scale and position of σ_1 and σ_2, we can simply perform the remaining $\mathrm{d}^{8|8}X$ integral (since $\omega_{ij}^{1,2}$ is independent of X), leaving:

$$2\int \mathrm{d}^2\sigma\ \delta^{8|8}(\mathcal{P}) \sum_{i,j,k,l,m} \left|\mathbb{H}^{12ijklm}_{12ijklm}\right|\ \omega_{ij}^1\ \omega_{kl}^2 \frac{[W_m, \mathcal{P}\cdot\xi]}{(m\xi)^3} = 0. \tag{C.8}$$

So the seventh-order contribution to $\mathrm{d}_\xi\mathcal{M}_{n,0}$ vanishes simply due to momentum conservation. Note that in the calculation of each of these divergences, care must be taken to symmetrize over all indices in the summation as well as $(1\leftrightarrow 2)$ in order to get the correct result.

Finally, we can combine (C.4)-(C.8) to see that

$$\mathrm{d}_\xi\mathcal{M}_{n,0} = \int_{\mathscr{M}_{n,1}} \frac{\mathrm{d}^{8|8}X}{\mathrm{vol}\ \mathrm{GL}(2,\mathbb{C})} \frac{\partial}{\partial X^{IA}} V^{IA} = 0. \tag{C.9}$$

This vanishing occurs because there are no ambiguities with respect to the compactification of the moduli space at degree one, and V^{IA} is smooth with respect to the X coordinates.

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